Cone 3-manifolds

Joan Porti

Abstract This is an overview on hyperbolic cone 3-manifolds, their deformation theory and their role in Thurston's orbifold theorem. We also describe the phenomena that may occur when deforming the cone angles, like cusp opening or collapses, under the assumption that the cone angles are less than π.

1 Introduction

Cone 3-manifolds are manifolds equipped with metrics of constant curvature that are singular at an embedded graph, and the singularity follows a specific conical structure. They can be obtained from 3-dimensional polyhedra of constant curvature by identifying their faces along isometries, thus the singular locus is contained in the 1-skeleton.

Cone 3-manifolds were considered by Thurston in his proof of the orbifold theorem. The underlying space of an orientable orbifold of constant curvature has a natural metric of cone manifold. The starting point in the proof of the orbifold theorem is another well known theorem of Thurston: the hyperbolic Dehn filling theorem. The proof of the hyperbolic Dehn filling theorem provides cone manifolds with small cone angles; then the main strategy of the orbifold theorem is to increase those cone angles (until the angles determined by the topology of the orbifold) and to analyze the possible phenomena that may occur. This motivates the study of geometric properties of cone 3-manifolds, like their deformation theory or the convergence of sequences of cone 3-manifolds.

The first sketch, or program, of the proof of the orbifold theorem was the content of a preprint by Thurston in 1982 [48], see also [49]. Then the proof was completed by several contributors [3, 4, 14, 25, 45, 54]. There are some later results that give

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a more natural argument in some parts of the proof, like the local rigidity theorems surveyed here. The goal of this paper is not to provide a proof of the orbifold theorem, but to give an overview of some properties of cone 3-manifolds.

Kleiner and Lott have proved the orbifold theorem with Ricci flow on orbifolds [30], without using cone manifolds. Cone manifolds remain however an interesting geometric object, that may have an intuitive visualization. Besides, cone manifolds have applications other than the orbifold theorem: Hodgson and Kerckhoff use them in [27] to find a uniform upper bound on the number of non-hyperbolic Dehn fillings. The deformation theory of cone manifolds is used by Bromberg in the proof of the Bers density conjecture [10], by Brock and Bromberg in in a generalization of this conjecture [8], by Brock, Bromberg, Evans and Souto in the tameness conjecture [7], as well as by Bonahon and Otal [5] to study bending measured laminations.

It is worth mentioning that there are a lot of contributions on cone 3-manifolds that are not overviewed here. For instance, the many examples of deformations and volume computations of the Siberian school around Alexander D. Mednykh, as well as the pioneering examples from the long term collaboration between Mike Hilden, José Maria Montesinos Amilibia and Maite Lozano Imízcoz. Here we just mention a few examples from these authors.

This paper is organized as follows: Section 2 reviews the definition, basic constructions, and elementary properties of cone manifolds, focusing in dimensions 2 and 3. Section 3 is devoted to Thurston's hyperbolic Dehn filling theorem, that explains how cone 3-manifolds with small cone angles occur, and the natural questions that arise. Then Section 4 reviews local rigidity results, in particular the results that allow to deform cone angles. Section 5 is devoted to sequences of cone 3-manifolds, more precisely to the notions of convergence, a compactness theorem, a description of thin parts and their applications (eg. global rigidity), all for cone angles strictly less than π. Finally Section 6 is devoted to some examples, that illustrate previous sections and give examples of different phenomena that occur to cone manifolds, including some examples with cone angles larger than π.

2 Cone manifolds

In this section we give the definition and basic constructions of cone manifolds, focusing on dimensions two and three.

We start with the definition in dimension 2, with curvature $\kappa \in \mathbb{R}$. To describe the metric in constant curvature κ, consider the function

$$\mathrm{s}_k(r) = \begin{cases} \frac{\sin(r\sqrt{k})}{\sqrt{k}} & \text{if } \kappa > 0 \\ r & \text{if } \kappa = 0 \\ \frac{\sinh(r\sqrt{-k})}{\sqrt{-k}} & \text{if } \kappa < 0 \end{cases}$$

This function is the unique solution to the differential equation $s_k'' + \kappa s_k = 0$ with initial conditions $s_k(0) = 0$, $s_k'(0) = 1$. In the next definition, the local description of the metric in polar coordinates for a cone surface (1) is a modification of the metric of the plane of constant curvature, $ds^2 = dr^2 + s_k^2(r)d\theta^2$, with $r \in (0, r_0)$, $\theta \in [0, 2\pi]$.

Definition 1 A *cone surface* of constant curvature $\kappa \in \mathbb{R}$ is a surface equipped with a length distance, where the metric is locally described, in polar coordinates, by

$$ds^2 = dr^2 + \left(\frac{\alpha}{2\pi}\right)^2 s_k^2(r)d\theta^2, \qquad r \in (0, r_0),\ \theta \in [0, 2\pi], \tag{1}$$

where $\theta = 2\pi$ is identified to $\theta = 0$. The parameter $\alpha > 0$ is called the *cone angle* at the point with coordinate $r = 0$.

When $\alpha \neq 2\pi$, we say that the point is *singular*, or a *cone point*.

For $\alpha = 2\pi$ the metric is locally a Riemannian metric of constant curvature κ and the point is called *regular*.

In Equation (1), $r \in (0, r_0)$ is the radial coordinate and $\theta \in [0, 2\pi]$ is the angle parameter. Furthermore, when $\kappa > 0$ we require $r_0 \leq \frac{\pi}{\sqrt{\kappa}}$.

Notice that the metric in (1) can be changed to the standard metric by re-parameterizing and changing the domain of the coordinate θ:

$$ds^2 = dr^2 + s_k^2(r)d\theta^2, \qquad r \in (0, r_0),\ \theta \in [0, \alpha], \tag{2}$$

where $\theta = \alpha$ is identified to $\theta = 0$. Namely, we consider a sector of angle α in the space of constant curvature κ and we identify its sides by a rotation, Figure 1.

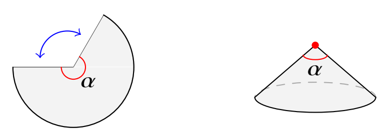


Fig. 1 A singular point of cone angle $\alpha < 2\pi$ is viewed as a cone (though the definition allows cone angle $\alpha > 2\pi$)

Example 1 Consider a triangle with angles $0 < \frac{\alpha_1}{2}, \frac{\alpha_2}{2}, \frac{\alpha_3}{2} < \pi$. It lies in a plane of constant curvature κ, with

$$\begin{cases} \kappa < 0 & \text{if } \frac{\alpha_1}{2} + \frac{\alpha_2}{2} + \frac{\alpha_3}{2} < \pi, \\ \kappa = 0 & \text{if } \frac{\alpha_1}{2} + \frac{\alpha_2}{2} + \frac{\alpha_3}{2} = \pi, \\ \kappa > 0 & \text{if } \frac{\alpha_1}{2} + \frac{\alpha_2}{2} + \frac{\alpha_3}{2} > \pi \ \ (\text{and } \frac{\alpha_i}{2} + \frac{\alpha_j}{2} < \frac{\alpha_k}{2} + \pi, \ \ \text{for } i \neq j \neq k \neq i). \end{cases}$$

We double the triangle with angles $\frac{\alpha_1}{2}$, $\frac{\alpha_2}{2}$, and $\frac{\alpha_3}{2}$ along its boundary: in this way we obtain a Riemannian metric on S^2 of constant curvature everywhere except at the vertices. Namely we obtain a cone surface with three cone points, of respective cone angles α_1, α_2 and α_3, subject to $\alpha_i + \alpha_j < \alpha_k + 2\pi$, for $i \neq j \neq k \neq i$. This example is called a *turnover* and it is denoted by $S^2(\alpha_1, \alpha_2, \alpha_3)$, see Figure 2, left.

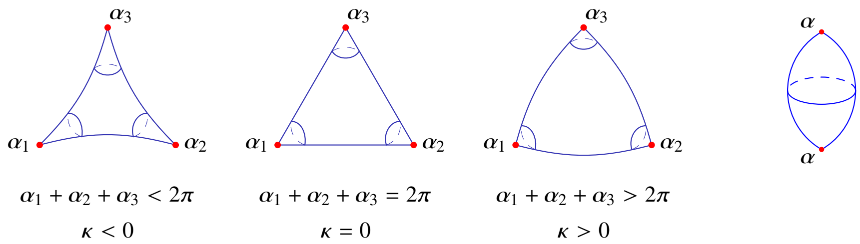


Fig. 2 On the left, three turnovers: cone surfaces $S^2(\alpha_1, \alpha_2, \alpha_3)$ of curvature κ (subject to $\alpha_i + \alpha_j < 2\pi + \alpha_k$, for $i \neq j \neq k \neq i$). On the right, cone surfaces $S^2(\alpha, \alpha)$ of curvature > 0.

Example 2 Consider a spherical bigon of angle $0 < \alpha < 2\pi$, namely the region of S^2 bounded by two meridians at angle α. In spherical coordinates it is the region with parameters $(\rho, \theta) \in (0, \pi) \times [0, \alpha]$ where ρ is the distance to the north pole and θ the longitude (and $\pi/2 - \rho$ the latitude). We identify the sides by a rotation. The result is a cone manifold with two cone points of angle α, that we denote by $S^2(\alpha, \alpha)$. See Figure 2 right. It is the spherical suspension of a circle of length α, namely with the metric

$$ds^2 = d\rho^2 + \sin^2(\rho) d\theta^2, \qquad \text{for } \rho \in (0, \pi) \text{ and } \theta \in [0, \alpha]/\alpha \sim 0.$$

The cone manifold $S^2(\alpha, \alpha)$ can be seen as the limit of $S^2(\alpha_1, \alpha_2, \alpha_3)$ when $\alpha_3 \to 2\pi$, because $|\alpha_1 - \alpha_2| \leq 2\pi - \alpha_3$.

The definition of cone manifold is inductive on the dimension. The construction uses the metric cone. Start with the topological cone: for a compact topological space X, consider the product $X \times [0, R)$ for some $R > 0$ and collapse $X \times \{0\}$ to a point (the *tip* of the cone), and denote the quotient by $X \times [0, R)/\sim$.

Definition 2 Let (X, d_X) be a metric space of diameter $\leq \pi$. The *metric cone* of constant curvature κ on X is the topological cone $X \times [0, R)/\sim$ (we require that $R < 2\pi/\sqrt{\kappa}$ when $\kappa > 0$) equipped with the distance d so that $(x_1, r_1), (x_2, r_2) \in X \times (0, R]$ and the tip of the cone $(*, 0)$ form a triangle isometric to the triangle in the plane of constant curvature κ with edge lengths $r_1, r_2, d((x_1, r_1), (x_2, r_2))$, and angle $d_X(x_1, x_2)$ at the tip $(*, 0)$. It is denoted by

$$\mathrm{Cone}_{R,\kappa}(X) = (X \times [0, R)/\sim, d).$$

The space X is called the *link* of the *tip* of the cone.

When the distance on X is provided by a Riemannian metric ds^2_X, then the metric on $\mathrm{Cone}_{R,\kappa}(X)$ is described by

$$ds^2 = dr^2 + \mathrm{s}^2_\kappa(r) ds^2_X.$$

Notice that in Definition 1 the local description of a cone surface is the metric cone of constant curvature over a circle.

Definition 3 A d-dimensional *cone manifold* of constant curvature κ is a metric length space C satisfying the following local property. For each $x \in C$ there exists a cone manifold $\mathrm{Link}(x)$ of curvature 1 homeomorphic to S^{d-1} such that a neighborhood of x is isometric to the metric cone of constant curvature κ on $\mathrm{Link}(x)$, $\mathrm{Cone}_{\varepsilon,\kappa}(\mathrm{Link}(x))$.

When the curvature κ is equal to 1 the cone manifold is called *spherical*, when $\kappa = 0$, *Euclidean*, and when $\kappa = -1$, *hyperbolic*.

Remark 1 We require that $\mathrm{Link}(x)$ is homeomorphic (not isometric) to a sphere S^{d-1}, so that C is topologically a manifold.

If we do not require $\mathrm{Link}(x)$ to be homeomorphic to a sphere, then we talk about *conifolds*, but we will not consider them here. The easiest example of conifold that is not a cone manifold is the cone on the projective plane.

Proposition 1 *The underlying space of an orientable orbifold of constant curvature and dimension* 2 *or* 3 *inherits naturally the structure of a cone manifold.*

Proof The underlying space of an orbifold of constant curvature is locally modeled on $\mathbb{X}^n_\kappa/\Gamma$, where $\mathbb{X}^n_\kappa$ is the space of constant curvature κ, and $\Gamma \subset \mathrm{SO}(n)$ is a finite group of isometries fixing a point.

By construction, there exists $\varepsilon > 0$ such that a neighborhood of a point x in the underlying space is isometric to $B(\tilde{x}, \varepsilon)/\Gamma$, where $B(\tilde{x}, \varepsilon)$ is a metric ball of radius $\varepsilon > 0$ in $\mathbb{X}^n_\kappa$. Notice that $B(\tilde{x}, \varepsilon)/\Gamma$ is the metric cone of radius $\varepsilon > 0$ on its link S^{n-1}/Γ. Since we assume orientability, for $n = 2$, S^1/Γ is homeomorphic to a circle and for $n = 3$, S^2/Γ is homeomorphic to a 2-sphere. □

The previous proposition holds in any dimension if we allow conifolds instead of cone manifolds, i.e. if we do not require the link to be homeomorphic to a circle.

Proposition 2 (Gluing polygons in dimension 2) *Let $P_1, \ldots, P_k \subset \mathbb{X}^2_\kappa$ be polygons of constant curvature κ. Assume that their edges $(P_i)_j$ are paired by isometries s_{ij}. Then the metric space obtained by identification along the isometries*

$$(P_1 \sqcup \cdots \sqcup P_k)/\!\sim_{s_{ij}}$$

is a cone surface.

The cone structure is easily constructed from matching the cones on the polygons. The key point is to prove that the link of each point is a circle; this follows from the classification of 1-dimensional manifolds (see also Theorem 6.7.6 in [42]).

Remark 2 Proposition 2 generalizes to dimension 3 if we can guarantee that links of equivalence classes of vertices are homeomorphic to spheres. This holds true for instance if cone angles of edges are $\leq 2\pi$, by Gauss-Bonnet (Proposition 3).

Proposition 2 is illustrated in Examples 1 and 2. By means of the Dirichlet polyhedron (below in Definition 5 and Proposition 5) we show that all cone manifolds can be constructed from Proposition 2.

Definition 4 On a cone d-manifold C, a point $x \in C$ is *singular* if its link is not *isometric* to the standard $(d-1)$-sphere S^{d-1}, and regular otherwise. The singular locus of C is denoted by Σ.

Remark 3 The singular locus Σ is a stratified subspace of codimension ≥ 2. In particular, for a cone surface, Σ is a discrete subset.

For a 2-dimensional cone manifold, we have a Gauss-Bonnet formula, see for instance [50, 34]:

Proposition 3 (Gauss-Bonnet formula for cone surfaces) *Let C^2 be a cone surface of constant curvature κ, with finite area and n cone points of respective cone angles $\alpha_1, \ldots, \alpha_n$. Then*

$$\kappa\,\mathrm{area}(C^2) + \sum_i (2\pi - \alpha_i) = 2\pi\chi(C^2),$$

where $\chi(C^2)$ denotes the Euler characteristic of the underlying surface.

It follows from the Gauss-Bonnet formula that if $\kappa = 1$ and the cone angles are $\leq \pi$, then there are at most three cone points. With some extra work, one can determine geometrically those cone manifolds:

Proposition 4 *Let C^2 be a spherical cone surface with cone angles $\leq \pi$. If C^2 is orientable, then it is one of the following:*

1. *A smooth sphere S^2.*
2. *$S^2(\alpha, \alpha)$, the spherical suspension of a circle.*
3. *$S^2(\alpha, \beta, \gamma)$, a turnover with $\alpha + \beta + \gamma > 2\pi$.*

If C^2 is not orientable, then it is the quotient of S^2 or $S^2(\alpha, \alpha)$ by the antipodal map, i.e. the projective plane with possibly a cone point, P^2 or $P^2(\alpha)$.

Furthermore, the isometry class of C^2 is determined by the cone angles (namely they are rigid).

This rigidity does not hold anymore for spherical cone manifolds with cone angles larger than π; consider for instance the double of a spherical quadrilateral. See [34] for a description of the moduli space of spherical cone surfaces.

From Proposition 4, as the link of a point is a spherical cone manifold, we have:

Corollary 1 *A 3-dimensional cone manifold with cone angles $\leq \pi$ is locally isometric to one of the following:*

1. *A smooth point (the cone of a smooth sphere).*
2. *A singular edge (the cone of $S^2(\alpha, \alpha)$).*
3. *A trivalent vertex of a singular graph (the cone of $S^2(\alpha, \beta, \gamma)$).*

In particular, the singular locus Σ is a disjoint union of circles and trivalent graphs, see Figure 3.

Furthermore, the isometry class of a neighborhood is determined by the cone angles.

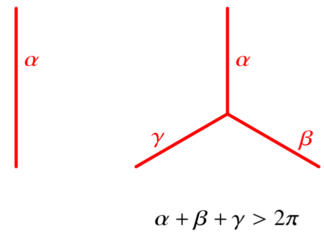


Fig. 3 The models of the singular locus Σ when cone angles are $\leq \pi$.

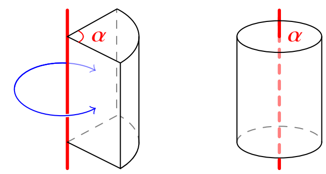


Fig. 4 Locally, a singular edge is the result of identifying the sides of a sector in the space of constant curvature by a rotation.

For a singular edge, the cone angle of the link at every point is also called the cone angle of the edge, see Figure 4.

Definition 5 Let C be a cone 3-manifold and $x \in C \setminus \Sigma$ a regular point. The *cut locus* of C centered at x is

$$\mathrm{Cut}_x = \{y \in C \mid y \in \Sigma \text{ or } y \text{ has at least 2 minimizing segments to } x\}.$$

The complement of the cut locus is the *Dirichlet domain* centered at $x \in C \setminus \Sigma$, $D_x = C \setminus \mathrm{Cut}_x$:

$$D_x = \{y \in C \setminus \Sigma \mid \text{there is a unique minimizing segment from } x \text{ to } y\}.$$

Proposition 5 *The Dirichlet domain embeds as a star-shaped domain in $\mathbb{X}^3_\kappa$, the space of constant curvature $\kappa \in \mathbb{R}$, and for $\kappa \leq 0$ its closure is a polyhedron.*

Furthermore, when the cone angles are $\leq \pi$, this Dirichlet polyhedron is convex.

This proposition helps to explain why the hypothesis on cone angles $\leq \pi$ is relevant for cone manifolds. The fact that the Dirichlet polyhedron is convex allows to reproduce arguments in Riemannian geometry in this context. We see examples in Section 5.

Before finishing this section, we state a result related to the following section.

Proposition 6 *Let C be a closed hyperbolic cone 3-manifold without singular vertices (i.e. Σ is a link). Then $|C| \setminus \Sigma$ is a hyperbolic manifold (namely, it admits a complete hyperbolic metric).*

Proof Deform the non-complete metric on $|C| \setminus \Sigma$ to a complete metric of variable negative curvature. Then one can show that it has the topological properties required for being hyperbolic (irreducible, atoroidal, and $\pi(|C| \setminus \Sigma)$ has no center) and apply geometrization for Haken manifolds. □

The complete structure on this proposition can be seen as a cone manifold structure of angle zero. This is better explained in the next section, by Thurston's hyperbolic Dehn filling.

Remark 4 Let C be a closed hyperbolic cone 3-manifold without singular vertices as in Proposition 6. Then the volume of the complete hyperbolic structure on $|C| \setminus \Sigma$ is larger than the volume of the cone 3-manifold C. The maximality of the volume is due to Gromov–Thurston–Goldman, and written by Dunfield [20]. More precisely, as explained in [20], Goldman notices in [21] that the proof of Mostow rigidity in Thurston's notes [47] applies to representations, a proof that Thurston attributes to Gromov.

3 Hyperbolic Dehn filling

Thurston's hyperbolic Dehn filling provides examples of hyperbolic cone three-manifolds, with small cone angles, and it is the starting point of the proof of the orbifold theorem. Those cone 3-manifolds are obtained by deforming cusped manifolds and then taking the metric completion.

We first consider a two-dimensional example:

Example 3 Start with a hyperbolic triangle with ideal vertices and double it along its boundary. This yields a planar hyperbolic surface with three cusps, that we call $S^2(0, 0, 0)$. Next consider triangles with finite vertex and angle $\alpha/2 > 0$ at every vertex, for α in a neighborhood of 0. By taking the double of the triangles along the boundary, we get a family of turnovers $S^2(\alpha, \alpha, \alpha)$ as in Example 1 and Figure 2. As triangles with small angles are deformations of ideal triangles, the turnovers $S^2(\alpha, \alpha, \alpha)$ are deformations of the cusped surface $S^2(0, 0, 0)$.

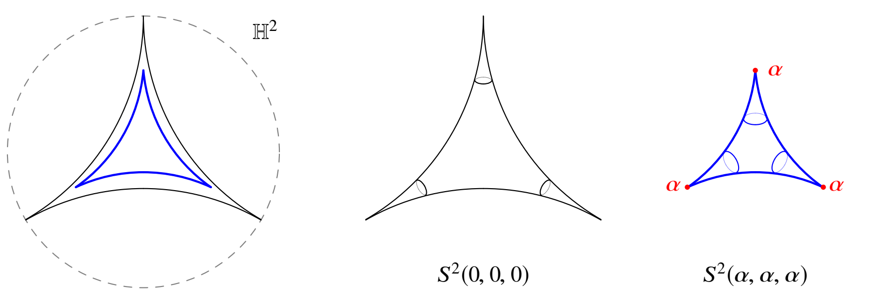


Fig. 5 Triangle of small angles as perturbation of the ideal triangle, with angles 0 (left). The double of the ideal triangle is the cusped surface $S^2(0, 0, 0)$ (center) and the double of the compact triangle is the turnover $S^2(\alpha, \alpha, \alpha)$ (right).

In dimension three, we first recall the topological description of filling. Consider a compact 3-manifold M^3 with boundary a 2-torus $\partial M^3 \cong T^2 \cong S^1 \times S^1$. Attach a solid torus $D^2 \times S^1$ (a product of a disc and a circle) to its boundary:

$$M^3 \cup_\varphi D^2 \times S^1 = \left(M^3 \sqcup D^2 \times S^1\right) / x \sim \varphi(x)$$

where $\varphi\colon \partial D^2 \times S^1 \to \partial M^3$ is a homeomorphism. The solid torus $D^2 \times S^1$ is called the *filling torus* and the curve $\{0\} \times S^1$ its *soul*.

The homeomorphism type of the Dehn filling depends only on the unoriented isotopy class of the curve $\varphi(\partial D^2 \times \{*\})$ in $\partial M^3 \cong T^2$, the filling meridian. In its turn, this unoriented isotopy class is determined by its homology class in $H_1(T^2, \mathbb{Z})$ up to sign, hence it may be described by a rational slope, an element of $\mathbb{Q} \cup \{\infty\}$, as follows. Fix a basis for the first cohomology group, namely an isomorphism $H_1(T^2, \mathbb{Z}) \cong \mathbb{Z}^2$; the filling meridian with homology class $\pm(p, q)$ via this isomorphism is described by the slope $p/q \in \mathbb{Q} \cup \{\infty\}$.

When the 3-manifold is a knot exterior in S^3, a Dehn filling on its exterior is called Dehn surgery on the knot.

Next we state the well known Thurston hyperbolic Dehn filling theorem in terms of cone manifolds. To simplify, we state it for only one cusp.

Theorem 1 (Thurston's generalized hyperbolic Dehn filling) *Let M^3 be a compact orientable 3-manifold with boundary a 2-torus. Assume its interior is hyperbolic.*

For every slope $q \in \mathbb{Q} \cup \{\infty\}$ there exists $\Theta_q > 0$, depending on the slope q and the manifold M^3, so that there is a family of cone manifold structures on the Dehn filling with slope q, with singular locus the soul of the filling torus, and with cone angles in the interval $(0, \Theta_q)$.

Furthermore, the number of slopes $q \in \mathbb{Q} \cup \{\infty\}$ such that $\Theta_q \leq 2\pi$ is finite.

Notice that when $\Theta_q > 2\pi$, Thurston's hyperbolic Dehn filling provides a honest hyperbolic three-manifold (e.g. with a metric with no singularities), and we recover the usual statement of Thurston's hyperbolic Dehn filling theorem. The last statement in Theorem 1 guarantees that almost all Dehn fillings are hyperbolic manifolds. In fact the statement is even more general. Thurston's proof provides a deformation space with a complex parameter. In this deformation space, the metric is non complete and its metric completion may yield a topological manifold (with singular metric or not) or a singular space, a so-called singularity of "generalized Dehn type". In this deformation space, the manifold Dehn filings are a countable set of points, joined by lines to the initial point, corresponding to the angle deformation of the cone manifolds.

Remark 5 Cone manifolds in Theorem 1 are constructed by deforming the complete metric structure and taking the metric completion. As in Example 3, we may view the metric at angle zero as the complete metric on the interior of M^3, hence the cone angle varies in $[0, \Theta_q)$.

Remark 6 In Theorem 1 one can replace 2π in the last sentence by any positive constant $C > 0$; the conclusion is that the number $\#\{q \in \mathbb{Q} \cup \{\infty\} \mid \Theta_q \leq C\}$ is finite. Of course this number depends on C, and a priori it depends also on M^3.

Theorem 2 *Let M^3 be a compact orientable 3-manifold with boundary a 2-torus and with hyperbolic interior. Then, for* every *slope $q \in \mathbb{Q} \cup \{\infty\}$, $\Theta_q \geq 2\pi/3$.*

This theorem is part of the proof of Thurston's orbifold theorem, and can be found in the different approaches to the proof [3, 4, 14, 25, 45, 54]. We discuss it later in Section 5. When $\Theta_q > 2\pi/n$ Theorem 1 yields a hyperbolic orbifold, with branching locus the soul of the filling torus, and branching index n. Thus, as corollary of Theorem 2:

Corollary 2 *Let M^3 be a compact orientable 3-manifold with boundary a 2-torus and with hyperbolic interior. For* every *slope $q \in \mathbb{Q} \cup \{\infty\}$, the orbifold with underlying space the q-Dehn filling, branching locus the soul of the filling torus, and branching index $n \geq 4$ is hyperbolic.*

The bound $n = 4$ is optimal: for instance the orbifold with underlying space the three-sphere, branching locus the figure eight knot and ramification 3 is Euclidean. Equivalently, there exists a Euclidean cone manifold structure on S^3, with singular locus the figure eight knot and cone angle $2\pi/3$.

If we focus on nonsingular Dehn fillings, then a natural question is to find a uniform bound on the number of Dehn fillings that are not hyperbolic. This has been found by Hodgson and Kerckhoff in [27]:

Theorem 3 (Hodgson–Kerckhoff) *Let M^3 be a compact orientable 3-manifold with boundary a 2-torus and with hyperbolic interior. Then $\Theta_q \leq 2\pi$ for at most 60 slopes $q \in \mathbb{Q} \cup \{\infty\}$, independently of M^3.*

The statement in [27] involves the so-called normalized length of a slope in the hororspherical torus. This torus has a natural Euclidean structure up to homotety, and one normalizes it so that it has area 1. Hodgson and Kerckhoff prove that for slopes q so that its normalized length in the horospherical torus is at least 7.515, we have $\Theta_q > 2\pi$. Besides the tools we describe here, one of the main innovations of Hodgson and Kerckhoff are infinitesimal harmonic deformations. They succeed in controlling the radius of a metric tube around the singular geodesic when deforming. We recall more results of [27] in Section 5.

In the proof of Theorem 2 there are two basic ingredients: deforming the structures by changing the cone angles and studying the limits of sequences. In Section 4 we describe how cone manifolds are deformed, and in Section 5 we analyze sequences of cone manifolds.

4 Local rigidity

In this section we overview results that allow to deform the cone angles of cone manifolds. Those are *local rigidity* results because they show that the multiangles are local parameters of the deformation space.

Given a cone manifold C, the *underlying manifold* is denoted by $|C|$. The topological pair formed by $(|C|, \Sigma)$ is called the *topological type*, where $\Sigma \subset |C|$ is the singular locus. The *meridians* are (conjugacy classes of) elements in the fundamental group $\pi_1(|C| \setminus \Sigma)$ represented by loops around the arcs and circles of Σ (that in $|C|$ bound a disc that intersects Σ in its center), Figure 6.

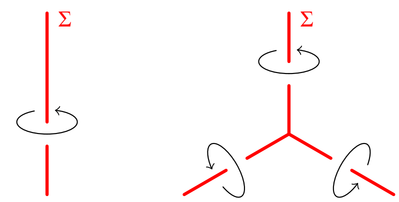


Fig. 6 Loops representing meridians of the singular locus in $\pi_1(|C| \setminus \Sigma)$

We are interested in deformations that preserve the topological type. The complement $|C| \setminus \Sigma$ inherits a non-singular hyperbolic metric that is not complete, whose metric completion is C. The incomplete structure on $|C| \setminus \Sigma$ has a holonomy representation

$$\mathrm{hol}_C \colon \pi_1(|C| \setminus \Sigma) \to \mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{PSL}(2, \mathbb{C})$$

that is unique up to conjugation. We consider the topology in the deformation space of C induced by the variety of representations up to conjugation

$$\hom(\pi_1(|C| \setminus \Sigma), \mathrm{PSL}(2, \mathbb{C}))/\mathrm{PSL}(2, \mathbb{C}).$$

Here we are using Ehresmann principle to say that deformations of structures are described by conjugacy classes of representations, cf. [13].

Notice that not all representations in $\hom(\pi_1(|C| \setminus \Sigma), \mathrm{PSL}(2, \mathbb{C}))/\mathrm{PSL}(2, \mathbb{C})$ close to the holonomy of the initial cone manifold correspond to the holonomy of a cone manifold structure: we must require that the holonomy of the meridians of Σ are rotations.

Theorem 4 (Hyperbolic local rigidity [26, 33, 51, 53]) *Let C be a compact orientable hyperbolic 3-manifold with topological type $(|C|, \Sigma)$. Then the deformation space with fixed topological type is locally parameterized by the cone angles (in particular it cannot be deformed without changing the cone angles).*

This theorem was first proved by Hodgson and Kerckhoff [26] when the singular locus Σ is a link. For arbitrary Σ but cone angles $\leq \pi$ (hence the singular locus is a trivalent graph) it was proved by Weiss [51], and the general case was proved independently by Montcouquiol-Mazzeo [33] and Weiss [53]. The approach of Hodgson–Kerckhoff and Weiss uses infinitesimal deformations as differential forms valued on the Lie algebra and their cohomology theory, though Mazzeo and Montcouquiol use the deformation theory of Einstein metrics.

The local rigidity theorem requires a fixed topological type $(|C|, \Sigma)$. This hypothesis is satisfied when the singular locus Σ is a manifold (there are no singular vertices) or when all cone angles are at most π. In general there are deformations that may change the singular locus: for instance a 4-valent vertex of Σ may split into two 3-valent vertices joined by a graph (this does not change the topology of $|C| \setminus \Sigma$). See [36].

Infinitesimal rigidity has been generalized by Bromberg to noncompact geometrically finite manifolds (without rank one cusps nor singular vertices):

Theorem 5 ([9]) *If C^3 is a geometrically finite cone-manifold without rank one cusps and if all cone angles are $\leq 2\pi$, then M is locally rigid relative to the cone angles and the conformal boundary.*

Remark 7 There is a stronger notion, *infinitesimal rigidity*, that implies local rigidity. In fact Theorems 4 and 5 are proved by establishing infinitesimal rigidity first.

When the cone angles are larger than 2π, infinitesimal rigidity does not hold. In a talk at the Third MSJ regional workshop in Tokyo in 1998 (devoted to the orbifold theorem), Casson gave an example of infinitesimally non-rigid hyperbolic cone 3-manifolds with singular vertices. Izmestiev has given further examples of infinitesimally non-rigid hyperbolic cone 3-manifolds, including examples without singular vertices. Furthermore, Izmestiev has provided examples that are not locally rigid in [28].

We conclude the section discussing spherical and Euclidean geometry.

Theorem 6 (Spherical local rigidity [51]) *Let C be a spherical cone 3-manifold with cone angles $\leq \pi$ and such that the topological pair $(|C|, \Sigma)$ is not Seifert fibered; then it is locally rigid.*

We say that the pair $(|C|, \Sigma)$ is Seifert fibered when $|C|$ is Seifert fibered and Σ consists of fibres. The Seifert fibered case has been discussed by Kolpakov in [32]. Essentially, it corresponds to the deformation space of the basis.

By means of polyhedra, Schlenker constructed examples of spherical cone 3-manifolds that are not locally rigid, with singular vertices and cone angles $\leq 2\pi$ [43]. Without singular vertices and allowing cone angles $\geq 2\pi$, non locally rigid spherical cone manifolds are found in [38].

Theorem 7 (Euclidean local rigidity [41]) *Let C be a closed orientable Euclidean cone 3-manifold with cone angles $\leq \pi$. If C is not an almost product, then in a neighborhood U of the space of multiangles there is a cone manifold structure with topological type $(|C|, \Sigma)$ with these angles. To determine the type of structure, there exists a smooth, properly embedded hypersurface $E \subset U$ consisting of multiangles of Euclidean cone structures that splits U into 2-connected components corresponding to multiangles of spherical and hyperbolic cone structures respectively, Figure 7.*

Furthermore, for each $\bar{\alpha} \in E$ the tangent space of E at $\bar{\alpha}$ is orthogonal to the vector of singular lengths $\bar{l}$.

Almost product means that it can be realized as a product $C^2 \times S^1$ divided by a finite group of isometries. For instance the cone manifold structure on S^3 with singular

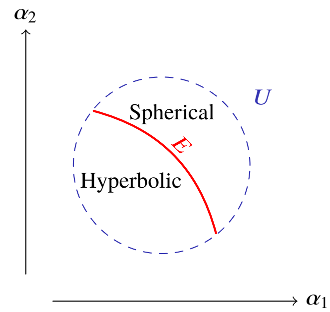


Fig. 7 The open set U in the space of multiangles and the hypersurface $E \subset U$ as in Theorem 7

locus the Borromean rings and cone angle π is an almost product. In Section 6 we describe the deformation space of the Borromean rings, as well as an example that illustrates Theorem 7.

The last result we review in this section is Schläfli's formula. It is named so because it can be established from the classical formula for the volume variation in a family of polyhedra of constant curvature (due to Schläfli for spherical tetrahedra). See for instance [24, 37].

Proposition 7 (Schläfli's formula) *Let C_t be a deformation of cone manifolds of constant curvature κ, for $t \in I$. Assume that it has fixed topological type $(|C|, \Sigma_C)$ and that it is of class C^1 (in the variety of representations of $|C| \setminus \Sigma_C$). Then the volume is differentiable and*

$$\kappa \frac{d\,\mathrm{Vol}(C_t)}{dt} = \frac{1}{2} \sum_e l_e \frac{d\alpha_e}{dt},$$

where the sum runs on the singular edges and circles e of Σ, l_e denotes the length and α_e, the cone angle at e.

A consequence of this formula is that, when cone angles increase, then the volume decreases for hyperbolic cone manifolds, but the volume increases for spherical cone manifolds. It also explains why the space of multiangles of Euclidean structures is perpendicular to the vector of singular lengths in Theorem 7.

5 Sequences of cone manifolds

After reviewing results that allow us to deform cone angles, we look for applications by considering sequences of cone manifolds with fixed topological type. We start with the notion of convergence and a compactness result in Subsection 5.1. Then, in Subsection 5.2 we analyze the thin part, in order to describe the possible limiting cone manifolds. Finally, applications are described in Subsections 5.3 and 5.4, by decreasing and increasing respectively the cone angles.

5.1 Compactness theorem

Let C be a compact cone 3-manifold of constant curvature κ. By definition, for every $x \in C$ a metric ball $B(x, \varepsilon)$ centered at x of radius $\varepsilon > 0$ is isometric to the cone (of curvature κ) of its link $\mathrm{Link}(x)$, see Definition 2, which is a spherical cone surface:

$$B(x, \varepsilon) \cong \mathrm{Cone}_{\kappa,\varepsilon}(\mathrm{Link}(x)).$$

This is called a *standard ball*.

Definition 6 The *injectivity radius* of C at x is

$$\mathrm{inj}(x) = \sup\{\delta > 0 \text{ such that } B(x, \delta) \text{ is standard ball in } C\}.$$

The *cone-injectivity radius* of C at x is

$$\mathrm{cinj}(x) = \sup\{\delta > 0 \text{ such that } B(x, \delta) \text{ is contained in a standard ball in } C\}.$$

Notice that in a compact cone manifold, a point x can be arbitrarily close to the singular locus, therefore its injectivity radius can be arbitrarily small, this is why Thurston defined the cone injectivity radius. The standard ball in the definition of cone injectivity radius does not need to be centered at x, in this way regular points arbitrarily close to the singular locus may have cone-injectivity radius away from zero. The definition of injectivity radius $\mathrm{inj}(x)$ can also be given in terms of the exponential map.

Let X and Y be metric spaces and $\varepsilon > 0$. We call a map $\phi : X \to Y$ a $(1 + \varepsilon)$*-bi-Lipschitz embedding* if

$$\frac{1}{1+\varepsilon} < \frac{d(\phi(x_1), \phi(x_2))}{d(x_1, x_2)} < (1 + \varepsilon)$$

holds for all $x_1 \neq x_2 \in X$.

Definition 7 (Geometric convergence) Let $(C_n, x_n)_{n\in\mathbb{N}}$ be a sequence of pointed cone-3-manifolds. We say that the sequence (C_n, x_n) converges *geometrically* to a pointed cone-3-manifold (C_∞, x_∞) if for every $R > 0$ and $\varepsilon > 0$ there exists $N = N(R, \varepsilon) \in \mathbb{N}$ such that for all $n \geq N$ there is a $(1 + \varepsilon)$-bi-Lipschitz embedding $\phi_n : B_R(x_\infty) \to C_n$ satisfying:

1. $d(\phi_n(x_\infty), x_n) < \varepsilon$,
2. $B(x_n, (1 - \varepsilon)R) \subset \phi_n(B(x_\infty, R))$, and
3. $\phi_n(B(x_\infty, R) \cap \Sigma_\infty) = \phi_n(B(x_\infty, R)) \cap \Sigma_n$.

If the C_n have curvature $\kappa_n \in \mathbb{R}$, then C_∞ has curvature $\kappa_\infty = \lim_{n\to\infty} \kappa_n$. The cone-angle along an edge of Σ_∞ is the limit of the cone-angles along the corresponding edge in Σ_n. Notice also that part of the singular locus of the approximating cone-3-manifolds may disappear at the limit by going to infinity.

Theorem 8 (Compactness) *Let $(C_n, x_n)_{n\in\mathbb{N}}$ be a sequence of pointed cone-3-manifolds with curvature $\kappa_n \in [-1, 1]$ and cone-angles $\leq \pi$. Suppose that for some $\rho > 0$, $\operatorname{inj}(x_n) > \rho$. Then (possibly after passing to a subsequence) the sequence (C_n, x_n) converges geometrically to a pointed cone-3-manifold (C_∞, x_∞) with curvature $\kappa_\infty = \lim_{n\to\infty} \kappa_n$.*

There are two remarks to be made:

- Firstly, we fix a lower bound $\rho > 0$ on the injectivity radius of the base point x_n, not the cone-injectivity radius. We can use the cone injectivity radius if we fix a lower bound away from zero for the cone angles.
- Secondly, Theorem 8 is analogous to a well known compactness theorem for sequences of pointed Riemannian manifolds with bounded sectional curvature and injectivity radius at the base point bounded away from zero. One of the main steps is to establish a uniform lower bound on the cone-injectivity radius at every point in balls $B(x_n, R)$, depending only on ρ and R. This uses the hypothesis on cone angles $\leq \pi$, see Proposition 5.

In view of applications we consider sequences of cone manifolds with fixed topological type $(|C|, \Sigma)$ and with bounded volume. To analyze the limits, we need to understand non compact hyperbolic cone manifolds with finite volume. In particular their thin or cone-thin parts.

5.2 Cone-thin part

For a non-singular hyperbolic 3-manifold, Margulis theorem yields a description of the set of points with injectivity radius less than a uniform constant μ_3, called the Margulis constant. Those are either cusps or tubular neighborhoods of short geodesics. Besides the cone manifold version of cusps and tubes, we still need another model to describe regions with small injectivity radius, called *necks*.

Let $S^2(\alpha, \beta, \gamma)$ be a turnover, with constant curvature -1, 0 or $+1$ according to the sign of $\alpha + \beta + \gamma - 2\pi$, Example 1 and Figure 2. View it as the double of a triangle $T = T(\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2})$ and consider the following constructions:

- When $\alpha+\beta+\gamma < 2\pi$, view the triangle $T = T(\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2})$ in a totally geodesic plane $\mathbb{H}^2 \subset \mathbb{H}^3$. Consider

  $$N_R(T) = \{x \in \mathbb{H}^3 \mid d(x, \mathbb{H}^2) \leq R \text{ and } \operatorname{pr}(x) \in T\}$$

  where $\operatorname{pr} : \mathbb{H}^3 \to \mathbb{H}^2$ denotes the orthogonal projection, see Figure 8.
  A *neck* of radius R over $S^2(\alpha, \beta, \gamma)$ is the double of $N_R(T)$ along $\partial T \times [-R, R]$. In the smooth part, the metric is written locally as

  $$ds^2 = dt^2 + \cosh^2(t)\left(dr^2 + \sinh^2(r)d\theta^2\right),$$

where $\left(dr^2{+}\sinh^2(r)d\theta^2\right)$ is the hyperbolic metric on the smooth part of $S^2(\alpha,\beta,\gamma)$ and $t\in[-R,R]$ is the signed distance to the turnover.
The boundary of a neck consists of two umbilical turnovers of curvature $-\cosh^{-2}(R)$.

- When $\alpha+\beta+\gamma=2\pi$, view the triangle $T=T(\frac{\alpha}{2},\frac{\beta}{2},\frac{\gamma}{2})$ in a horosphere $\mathcal{H}$ centered at an ideal point center$(\mathcal{H})\in\partial_\infty\mathbb{H}^3$. Consider

$$N_\infty(T)=\{x\in\mathbb{H}^3\mid x \text{ lies in a geodesic from } T \text{ to center}(\mathcal{H})\},$$

see Figure 8. A *cusp* with horospherical cross-section $S^2(\alpha,\beta,\gamma)$ is the double of $N_\infty(T)$ along $\partial T\times[0,+\infty)$. In the smooth part, the metric is locally written as

$$ds^2=dt^2+e^{-2t}\left(dr^2+r^2d\theta^2\right),$$

where $\left(dr^2+r^2d\theta^2\right)$ is the Euclidean metric on the smooth part of $S^2(\alpha,\beta,\gamma)$ and $t\in(0,+\infty)$ is the distance to the turnover.
The boundary of a cusp is an umbilical Euclidean turnover.

Those necks and cusps are naturally foliated by umbilical cone surfaces, that are turnovers. Here we have considered hyperbolic and Euclidean turnovers. For spherical turnovers the corresponding region foliated by umbilical turnovers is a standard ball.

Cusps have always small cone injectivity radii. Hyperbolic necks may have small cone injectivity radius. If we fix a lower bond on the cone angle, small cone injectivity radius at necks only occurs when $\alpha+\beta+\gamma$ approaches π.

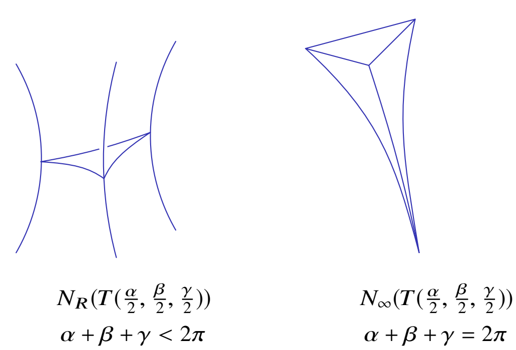


Fig. 8 The models whose double are a neck (left) and a cusp (right).

Theorem 9 (Cone-thin part [3]) *For $D>0$ and $0<\alpha\le\beta<\pi$ there exists $\rho=\rho(D,\alpha,\beta)>0$ such that the following holds: If C is an orientable cone-3-manifold (without boundary) of constant curvature $\kappa\in[-1,0)$ with cone-angles $\in[\alpha,\beta]$ and $\operatorname{diam}(X)\ge D$, then the set of points $\{x\in C\mid \operatorname{cinj}(x)<\rho\}$ is contained in the disjoint union of:*

1. *Tubular neighborhoods of (perhaps singular) closed geodesics.*

2. *Cusps with horospherical cross-section a 2-torus or a Euclidean turnover*
3. *Necks on a hyperbolic turnover.*

Here are some remarks about Theorem 9:

- The theorem assumes that the diameter is larger than a positive constant $D > 0$. In fact there are hyperbolic cone manifolds with small cone-injectivity radius everywhere, but they have small diameter: they correspond to sequences of hyperbolic cone manifolds that collapse to a point.
- The theorem does not hold when we allow cone angles close to π: we show in Section 6 sequences of hyperbolic cone manifolds that Hausdorff converge to a two-dimensional cone manifold, hence with a positive lower bound of the diameter.
- Notice that the necks describe the only way two singular edges can approach, under the assumptions that cone angles are bounded above away from π and that the diameter is bounded below away from zero.

Remark 8 In [3] a stronger version of this theorem is proved, with the description of points with injectivity radius less than some constant, i.e. including regular points close to the singularity. This includes cones over turnovers, that have a large cone injectivity radius but small injectivity radius at all regular points. One of the conclusions is that the boundary of the components of the thin part includes a point with large injectivity radius.

Theorem 9 and the stronger statement in Remark 8 need a careful analysis to construct, from short loops, foliations by umbilical surfaces. Theorem 9 can also be proved from the classification of non-compact Euclidean cone manifolds with cone angles less than π.

Next we give two applications of Theorem 9. Notice that the boundary of the neighborhoods of small cone-injectivity radius contains always a point with large injectivity radius. Thus we have:

Corollary 3 (thickness) *There exists $r = r(D, \alpha, \beta) > 0$ such that if C is as in Theorem 9, then C contains an embedded smooth standard ball of radius r.*

Corollary 4 (finiteness) *Let C be as in Theorem 9 and suppose in addition that $vol(C) < \infty$. Then C has finitely many ends and all of them are (smooth or singular) cusps with compact horospherical cross-sections.*

5.3 Decreasing cone angles: global rigidity

Definition 8 We say that a hyperbolic cone 3-manifold C is *globally rigid* if, when C' is a hyperbolic cone manifold with the same topological type, $(|C'|, \Sigma_{|C'|}) \cong (|C|, \Sigma_{|C|})$, and the same cone angles, then C' is isometric to C.

Theorem 10 (Hyperbolic global rigidity [31, 52]) *Hyperbolic cone manifolds with cone angles less than π are globally rigid.*

This theorem was first proved by Kojima in [31] when there are no singular vertices, using the local rigidity theorem of Hodgson and Kerckhoff, available at that time, and the case with vertices was proved by Weiss in [52], after he had proved local rigidity when there are vertices.

Here is a sketch of the proof. Assume first that C has no singular vertices, i.e. that Σ is a link. The proof consists in decreasing the cone angles, until one reaches a hyperbolic orbifold. The angles can be decreased by local rigidity, and one has to analyze the limits to prove that the space of angles realized by a hyperbolic cone structure on $(|C|, \Sigma)$ is not only open but closed. We consider sequences of cone manifold structures with decreasing cone angles. The volume of these sequences increases (by Schläfli's formula), in particular the diameter is bounded below by $D > 0$. Furthermore the volume of C is bounded above by the volume of the complete structure on $|C| \setminus \Sigma$, Remark 4. As the diameter is bounded below by $D > 0$, by the compactness theorem (Theorem 8) the sequence of cone manifolds converges to a finite volume hyperbolic manifold C_∞. If C_∞ is compact, then it has the same topological type as C_n, which means that we can continue decreasing the angles. If C_∞ is non compact, then one uses the finiteness theorem (Corollary 4) and a topological argument to get a contradiction with the opening of cusps.

When C has singular vertices, then one has to take into account that some of the singular vertices can go to infinity, i.e. the cone on a spherical turnover becomes a cusp with horospherical cross-section a turnover.

Once one reaches cone angles that are $2\pi/n$, the argument concludes from Mostow–Prasad rigidity on orbifolds: the structure on the orbifold is unique, and, by local rigidity, the path to reach it is also unique.

In the spherical case, Weiss establishes also global rigidity by increasing cone angles; here Mostow–Prasad is replaced by a rigidity theorem in the spherical case due to de Rham:

Theorem 11 (Spherical global rigidity [52]) *Non Seifert fibered spherical cone manifolds with cone angles less than π are globally rigid.*

From Theorems 7, 10 and 11, we get:

Theorem 12 (Euclidean global rigidity [41]) *Let C be a closed orientable Euclidean cone 3-manifold with cone angles $\leq \pi$. If C is not an almost product, then C is globally rigid (up to homoteties).*

Furthermore, for every multiangle $\bar\alpha \in (0, \pi)^q$ there exists a unique cone manifold structure of constant curvature in $\{-1, 0, 1\}$ on C with those cone angles:

- *If all cone angles of C are π, then every point in $(0, \pi)^q$ is the multiangle of a hyperbolic cone structure on C.*
- *If some of the cone angles is $< \pi$, then the subset $E \subseteq (0, \pi)^q$ of multiangles of Euclidean cone structures is a smooth, properly embedded hypersurface that splits $(0, \pi)^q$ into 2 connected components, corresponding to multiangles of spherical and hyperbolic cone structures respectively.*

Sequences of cone manifolds without singular vertices can also be analyzed by controlling the radius of a metric tube around the singular locus, so the singular locus does not crosses itself. This is the technique of Hodgson and Kerckhoff to prove Theorem 3, and by decreasing the cone angle they also prove the following theorem for short geodesics (cf. Proposition 6):

Theorem 13 ([27]) *Let M^3 be a closed hyperbolic 3-manifold and γ a geodesic in M^3 of length less than 0.111; then there exists a family of hyperbolic cone structures on M^3 with singular locus γ and cone angle in $[0, 2\pi]$ (the cone angles decrease from 2π, the non-singular metric, to 0, the complete structure on $M^3 \setminus \gamma$).*

This has applications in Kleinian groups [7, 8, 10].

5.4 Increasing cone angles

Next we discuss sequences of cone manifolds with increasing cone angles. We assume that the cone angles are bounded above away from π.

Theorem 14 *Let C_n be a sequence of compact hyperbolic cone manifolds with fixed topological type and increasing cone angles that are bounded above by $\eta < \pi$. Then, up to a subsequence, there are three possibilities:*

- *It converges geometrically (Definition 7) to a compact hyperbolic cone manifold with the same topological type.*
- *It converges geometrically to a hyperbolic cone manifold C_∞ of finite volume with cusps, each cusp with horospherical cross-section a turnover (singular cusps opening).*
- *It collapses to a point and, after rescaling, it converges geometrically to a Euclidean cone manifold.*

The idea of the proof is to apply the compactness theorem (Theorem 8) and the finiteness theorem (Corollary 4). More precisely, if the diameter of C_n stays bounded below away from zero, then we apply the compactness theorem, and the limit C_∞ is a manifold of finite volume (the deformation decreases the volume by Schläfli's formula). Furthermore we can get rid of the case where C_∞ has some nonsingular cusp by a topological argument on Dehn fillings. Hence all cusps of C_∞ are singular, and they have horospherical cross-section a turnover. This yields the first two items of the conclusion of the theorem. The remaining case occurs when the diameter of C_n converges to zero: then the cone manifold collapses to a point. In this case we rescale by the diameter, so that the curvature converges to zero. We apply again the compactness theorem and we get convergence to a compact Euclidean cone manifold (of diameter one).

Recall that Theorem 2 says that, for an orientable hyperbolic manifold with a single cusp M^3 and for any slope q, we have $\Theta_q \geq \frac{2\pi}{3}$, i.e. the cone manifold is hyperbolic for cone angles $\alpha \in (0, \frac{2\pi}{3})$. With all the results we have reviewed, we can sketch its proof.

Sketch of the proof of Theorem 2

Assume that for some slope q, $\Theta_q < \frac{2\pi}{3}$ and, seeking a contradiction, consider a sequence of angles $\alpha_n < \Theta_q$ converging to Θ_q. Apply Theorem 14; then there are three possibilities. The first one is that the sequence converges to a compact hyperbolic manifold with the same topological type. In this case the cone angle can be increased by the local rigidity theorem and we get a contradiction by the definition of Θ_q. The second case of Theorem 14 is that a singular cusp opens, with horospherical cross-section a Euclidean turnover. But a Euclidean turnover has at least one cone angle $\geq \frac{2\pi}{3}$. Therefore this case does not occur because $\Theta_q < \frac{2\pi}{3}$. The third case is that the sequence of cone manifolds collapses to a Euclidean cone manifold with cone angle Θ_q. Since $\Theta_q < \frac{2\pi}{3}$, the Euclidean cone manifold is not an almost product. By Theorem 7 the cone angle can be increased to be spherical. Then, by Weiss's theorem (Theorem 11 and its proof), or by Theorem 12, the cone manifold with cone angle $\frac{2\pi}{3}$ is spherical. As the cone angle is $\frac{2\pi}{3}$ and M^3 (the smooth part) is hyperbolic, then the spherical orbifold is not Seifert fibered. Finally, we look at the classification of Dunbar of spherical orbifolds that are not Seifert fibered [19], and we reach a contradiction.

This finishes the sketch of the proof of Theorem 2.

6 Examples

In this section we discuss a few examples of deformations of cone manifolds, possibly with cone angles π or larger.

6.1 Hyperbolic two-bridge knots and links

A two-bridge knot or link is the result of gluing two tangles (a tangle is the pair formed by a ball with two unknotted arcs), Figures 9 and 10. Such a link is either a torus link or hyperbolic. See [11] for details.

We next discuss the *canonical tunnels*. The arcs in the tangle may be joined by a third arc, called tunnel, so that they form a letter H shape, Figure 9. These arcs are indeed tunnels: the exterior of the union of the link and any of the tunnels is a handlebody. These tunnels are geodesic in the complete hyperbolic structure [1], see also [2] for hyperbolic cone manifold structures singular along the tunnels. Here we show that the tunnels play a role in the limit of spherical cone structures.

The double covering of S^3 branched along a two-bridge link L is a lens space (it is the union of two solid tori, the double covering of the balls branched along the tangles). Hence the orbifold on S^3 with ramification locus L and branching index 2 is spherical.

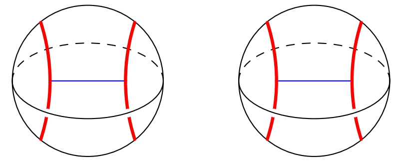


Fig. 9 Two tangles, with the arcs and the canonical tunnels. The union along the boundaries yields a 2-bridge link of one or two components.

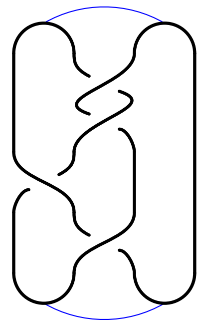


Fig. 10 The figure eight knot as a two bridge knot. The canonical tunnels are represented by a blue thin line

Proposition 8 ([39]) *Let L be a hyperbolic 2-bridge knot or link. There exists* $\alpha_{\mathrm{Euc}} \in [\frac{2\pi}{3}, \pi)$ *such that* S^3 *has a cone manifold structure with singular locus L and cone angle* α *(the same cone angle in both components if it is a link) of the following type:*

- *hyperbolic for* $\alpha \in (0, \alpha_{\mathrm{Euc}})$,
- *Euclidean for* $\alpha = \alpha_{\mathrm{Euc}}$,
- *spherical for* $\alpha \in (\alpha_{\mathrm{Euc}}, 2\pi - \alpha_{\mathrm{Euc}})$.

Furthermore, when $\alpha \to 2\pi - \alpha_{\mathrm{Euc}}$ *the singular locus intersects itself transversely along two points (the length of the canonical tunnels converges to zero) and the cone manifold converges to the spherical suspension of a sphere with four cone points of cone angle* $2\pi - \alpha_{\mathrm{Euc}}$.

From Theorem 1 the cone manifold is hyperbolic for angles in the interval $(0, \frac{2\pi}{3})$. Furthermore, as it is spherical for angle $\alpha = \pi$, it has to become Euclidean at some angle $\alpha_{\mathrm{Euc}} \in [\frac{2\pi}{3}, \pi)$, by [4, Appendix A]. By Theorem 12 it is spherical for $\alpha \in (\alpha_{\mathrm{Euc}}, \pi]$. The spherical structures with cone angles $(\pi, 2\pi - \alpha_{\mathrm{Euc}})$ are constructed in [39], using the symmetry of the variety of representations of $\pi_1(S^3 \setminus K)$ in SU(2), as SU(2) $\times$ SU(2) is the universal covering of SO(4). In [35] the explicit example of the figure-eight knot is explained.

Notice that for the figure eight knot $\alpha_{\mathrm{Euc}} = \frac{2\pi}{3}$. From Dunbar's classification or Euclidean orbifolds [18], form any other 2-bridge knot or link $\alpha_{\mathrm{Euc}} > \frac{2\pi}{3}$.

For links we may consider different cone angles on each component, Theorem 12 applies. We describe it with one example, the Whitehead link.

Example 4 Consider the cone manifold structures on S^3 with singular locus the Whitehead link, and cone angles α and β (Proposition 8 assumes $\alpha = \beta$). Cone manifold structures have been described by several authors, for instance Shmatkov [44]. Here we follow [41].

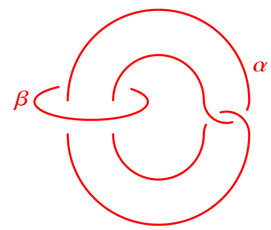


Fig. 11 The Whitehead link.

For $(\alpha, \beta) \in [0, \pi)^2$ there exists a cone manifold structure on S^3 with singular locus the Whitehead link and angles α, and β according to Figure 12.

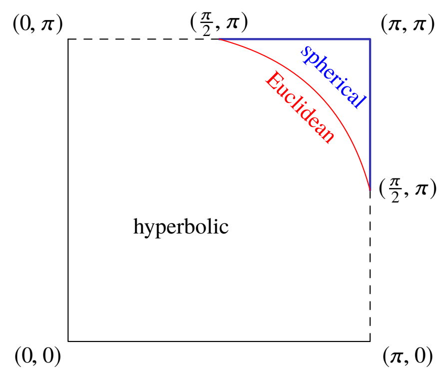


Fig. 12 The kind of geometric structures on the Whitehead link according to cone angles α and β

The curve of Euclidean cone manifolds is described by

$$x^6y^2 - 2x^4y^4 + 2x^4y^2 + x^2y^6 + 2x^2y^4 - 11x^2y^2 + 32 \\ - 48x^2 - 48y^2 + 24y^4 + 24x^4 - 4x^6 - 4y^6 = 0. \quad (3)$$

where $x = \pm 2\cos(\alpha/2)$ and $y = \pm 2\cos(\beta/2)$. Here is an explanation of Equation (3). The fundamental group of a two bridge link exterior $S^3 - L$ is generated by two elements μ_1 and μ_2, that are represented by meridians. The variety of $\mathrm{SL}(2, \mathbb{C})$-characters of $\pi_1(S^3 - L)$ is an affine surface in $\mathbb{C}^3$, with coordinates $x([\rho]) = \mathrm{trace}(\rho(\mu_1))$, $y([\rho]) = \mathrm{trace}(\rho(\mu_2))$ and $z([\rho]) = \mathrm{trace}(\rho(\mu_1\mu_2))$, for every conjugacy class (or character) of a representation $\rho\colon \pi_1(S^3 - L) \to \mathrm{SL}(2, \mathbb{C})$. Then the curve (3) is the discriminant of the projection of the variety of characters to the plane with coordinates (x, y), intersected with $\mathbb{R}^2$.

For fixed $\beta < \pi$, when $\alpha \to \pi^-$:

- for $\beta < \pi/2$ the cone manifold collapses to a two-dimensional hyperbolic cone manifold with boundary.
- for $\beta = \pi/2$ it collapses to a point (the corresponding orbifold has Nil geometry, see [46] and [38]).

- $\beta \in (\pi/2, \pi]$, the limit is a spherical cone 3-manifold.

This is because the double branched covering along one of the components of the Whitehead link is again S^3, and the other component lifts to a torus link. This assertion can be extrapolated to general hyperbolic links with two bridges, but the limits $\alpha \to \pi^-$ depend on the geometry of the partial double covering.

6.2 Montesinos links

Montesinos links are links $L \subset S^3$ such that the double covering of S^3 branched along L is Seifert fibered, and the fibration is transverse to the branching locus. For instance, 2-bridge links are Montesinos. The Seifert fibration of the double covering induces an orbifold Seifert fibration of the orbifold structure on S^3 with ramification locus L and ramification index 2, see [6] or [11]. The orbifold basis of this fibration is a 2-orbifold, with underlying space a polygon P_L, mirror edges and corner reflectors (corresponding to rational tangles). The polygonal 2-orbifold is geometric: the polygon P_L can be realized in a plane of constant curvature (the angles being π/n for a corner reflector of order $2n$, hence determined by the topology of the link). For a 2-bridge link, P_L is a spherical bigon. For the link L in Figure 13, P_L is a hyperbolic quadrilateral. Notice that when P_L has more than three vertices, then the 2-orbifold has a nontrivial Teichmüller space.

Proposition 9 *Let $L \subset S^3$ be a hyperbolic Montesinos link. Consider the cone manifold $C(\alpha)$ with underlying space S^3, branching locus L and cone angle α. Let P_L be the polygonal basis of the orbifold Seifert fibration:*

- *If P_L is spherical, then there exists an angle $\alpha_E \in [\frac{2\pi}{3}, \pi)$ so that $C(\alpha)$ is hyperbolic for $\alpha \in [0, \alpha_E)$, Euclidean for $\alpha = \alpha_E$ and spherical for $\alpha \in (\alpha_E, \pi]$.*
- *Otherwise $C(\alpha)$ is hyperbolic for $\alpha \in [0, \pi)$.*

Furthermore, when P_L is hyperbolic, as $\alpha \to \pi^-$, $C(\alpha)$ Hausdorff converges to the polygon with minimal perimeter among all polygons with given angles.

In the spherical case, the discussion is the same as for two-bridge links. Furthermore, if a collapse occurs before π then P_L must be spherical.

The assertion on the hyperbolic case is proved in [40], including the minimal perimeter of the polygon P_L with given angles.

When P_L is Euclidean, the orbifold has naturally a Nil or Euclidean structure. In the Nil case, for $\alpha > \pi$ the cone manifold $C(\alpha)$ becomes spherical [38]. When P_L is hyperbolic, the natural way to continue the deformations is by means of anti-de Sitter structures [15].

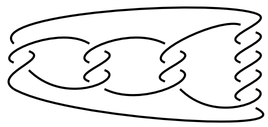


Fig. 13 Example of Montesinos knot. When $\alpha \to \pi$, the corresponding hyperbolic cone manifold $C(\alpha)$ collapses to a hyperbolic quadrilateral with angles $\frac{\pi}{2}$, $\frac{\pi}{3}$, $\frac{\pi}{3}$ and $\frac{\pi}{5}$.

6.3 A cusp opening

Fix three angles $\alpha, \beta, \gamma \in (0, \pi)$ subject to

$$\alpha + \frac{\gamma}{2} < \pi, \qquad \beta + \frac{\gamma}{2} < \pi.$$

By Andreev's theorem, there exists a truncated hyperbolic tetrahedron with angles α and β at opposite edges, and $\frac{\gamma}{4}$ at the remaining 4 edges. The truncation triangles are totally geodesic and perpendicular to the sides of the tetrahedron, so that we can view the polyhedron as a hyperbolic tetrahedron with vertices outside the hyperbolic space (in the de Sitter sphere). See Figure 14.

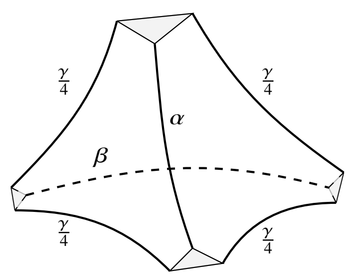


Fig. 14 The truncated hyperbolic tetrahedron.

To construct a cone manifold identify the faces of the tetrahedron by rotations along the edges of angles α and β. After the identification, the four edges of angles $\frac{\gamma}{4}$ correspond to a single equivalence class. We obtain in this way a cone manifold with totally geodesic boundary consisting of two turnovers $S^2(\alpha, \alpha, \gamma)$ and $S^2(\beta, \beta, \gamma)$, with underlying space $S^2 \times [0, 1]$, and singular locus three arcs of cone angles α, β and γ as in Figure 15.

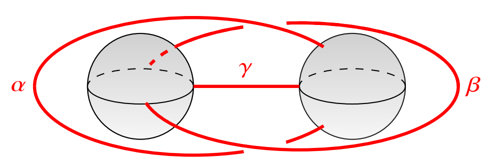


Fig. 15 The cone manifold after side pairings of the tetrahedron in Figure 14

Notice that when $2\alpha + \gamma = 2\pi$ or when $2\beta + \gamma = 2\pi$, some of the exterior vertices of the truncated tetrahedron in Figure 14 become ideal (i.e. the truncation triangles go to infinity, to an ideal vertex). This means that the corresponding totally geodesic boundary component goes to infinity and the end becomes a cusp, with horospherical cross-section a turnover.

If we furthermore assume $\alpha = \beta$, then we may identify one boundary component with the other by an isometry (turnovers are rigid). In this way we get a family of closed hyperbolic cone manifolds with an embedded totally geodesic turnover when $2\alpha + \gamma \leq 2\pi$, that develops a cusp with horospherical cross-section a turnover when $2\alpha + \gamma \to 2\pi$. This example can be found in [25], see Figure 16.

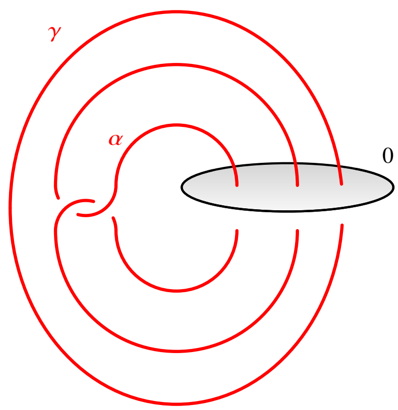


Fig. 16 Surgery description of [25], due to Hodgson. When $2\alpha + \gamma < 2\pi$ the turnover is totally geodesic, and when $2\alpha + \gamma \to 2\pi$ it converges to a horospherical turnover.

6.4 Borromean rings

Next we are interested in cone manifold structures on S^3 with singular locus the Borromean rings. Those have been described by many authors, starting by Thurston in his notes [47] for the Euclidean structures, and including for instance [23, 22]. To my knowledge, the different degenerations of hyperbolic structures at angle π are first described in Hodgson's thesis [24], and they are also in [14].

The building block for the hyperbolic cone manifold structures is the Lambert cube. For $\alpha, \beta, \gamma \in (0, \pi)$, the hyperbolic Lambert cube $\mathcal{L}(\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2})$ is a hyperbolic cube with three dihedral angles $\frac{\alpha}{2}$, $\frac{\beta}{2}$, and $\frac{\gamma}{2}$, as in Figure 18, and all other angles right. By Andreev's theorem, it exists and is unique. Its name comes from its faces, that are Lambert quadrilaterals, Figure 19. The hyperbolic Lambert cube has been considered by several authors, see for instance [12, 17, 29].

We consider eight copies of the Lambert cube $\mathcal{L}(\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2})$, after duplicating it three times, to obtain a polyhedron as in Figure 20. We identify faces of this polyhedron by side pairings along rotations as indicated in Figure 20, so that we get a hyperbolic

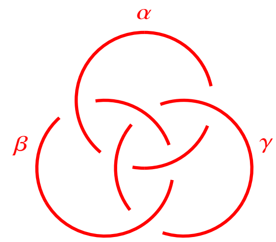


Fig. 17 The Borromean rings. They are the singular locus of a hyperbolic cone manifold structure on S^3 with cone angles $\alpha, \beta, \gamma \in [0, \pi)$.

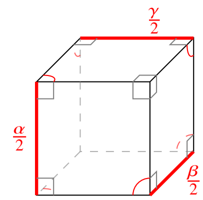


Fig. 18 (A Euclidean representation of) the hyperbolic Lambert cube $\mathcal{L}(\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2})$, with three dihedral angles $\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2} \in (0, \frac{\pi}{2})$, the other dihedral angles are $\pi/2$.

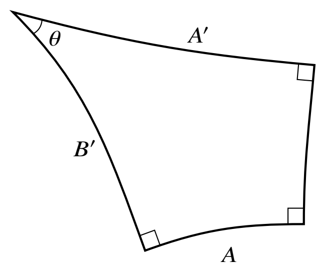


Fig. 19 A Lambert quadrilateral with angle $\theta \in (0, \frac{\pi}{2})$. Edges A and B can be arbitrarily short. For a given $\theta \in (0, \frac{\pi}{2})$ the length of A' and B' is bounded below away from zero.

cone structure on S^3 with singular locus the Borromean rings and cone angles α, β and γ, as explained in Thurston's notes [47].

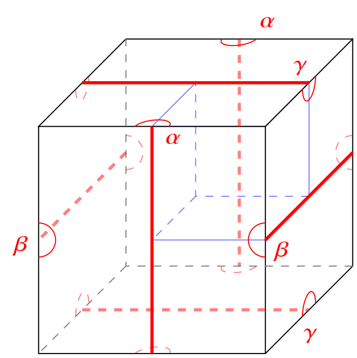


Fig. 20 Eight copies of the Lambert cube, after duplicating it three times. We identify the pentagonal faces by rotations along the red axis we obtain the cone manifold $\mathcal{B}(\alpha, \beta, \gamma)$ of Proposition 10.

Thus we have:

Proposition 10 *For every multiangle* $(\alpha, \beta, \gamma) \in [0, \pi)^3$ *there exists a hyperbolic cone structure* $\mathcal{B}(\alpha, \beta, \gamma)$ *on* S^3 *with singular locus the Borromean rings and cone angles* α, β *and* γ.

When some of the angles are zero, we just replace the corresponding edge in the Lambert cube by an ideal point. Notice that Andreev's Theorem applies to the polyhedron of Figure 20, but the computations are easier for the Lambert cube.

Next we ask what happens when some angles converge to π. We do not give the explicit formulas, we just mention that the results below on the limits of Lambert cubes $\mathcal{L}(\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2})$ can be determined from the formulas in [12, 17, 29].

First assume that all angles converge to π.

Lemma 1 *When* $\alpha \to \pi^-$, *then the Lambert cube* $\mathcal{L}(\frac{\alpha}{2}, \frac{\alpha}{2}, \frac{\alpha}{2})$ *converges to a point, and after rescaling it converges to a Euclidean cube.*

More precisely, if $\alpha, \beta, \gamma \to \pi^-$ *and the ratios* $\frac{\pi-\alpha}{\pi-\beta}$ *and* $\frac{\pi-\alpha}{\pi-\gamma}$ *converge to positive reals, then* $\mathcal{L}(\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2})$ *converges to a point and, after rescaling it converges to a right rectangular prism.*

Corollary 5 *When* $\alpha, \beta, \gamma \to \pi^-$, *and if* $\frac{\pi-\alpha}{\pi-\beta}$ *and* $\frac{\pi-\alpha}{\pi-\gamma}$ *converge to positive real numbers, then* $\mathcal{B}(\alpha, \beta, \gamma)$ *collapses to a point. Furthermore, after rescaling* $\mathcal{B}(\alpha, \beta, \gamma)$ *converges to a Euclidean orbifold.*

This Euclidean orbifold is an almost product and Theorem 7 does not apply.

Next assume that one of the angles remains constant.

Lemma 2 *Fix* $\alpha \in (0, \pi)$. *The Hausdorff limit of the Lambert cube* $\mathcal{L}(\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2})$ *when* $\beta, \gamma \to \pi^-$ *is a, possibly degenerate, Lambert quadrilateral (a hyperbolic quadrilateral with three right angles and a fourth angle* $\alpha/2$*, Figure 21), provided that* $\frac{\pi-\beta}{\pi-\gamma}$ *converges in* $[0, +\infty]$.

Furthermore, any (possibly degenerate) Lambert quadrilateral of angle $\frac{\alpha}{2}$ *is realized as a limit, depending on the limit of* $\frac{\pi-\beta}{\pi-\gamma}$.

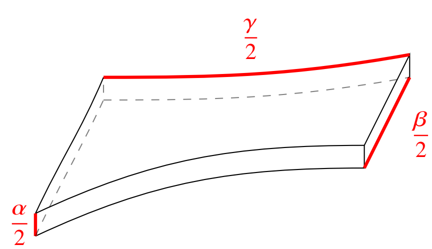


Fig. 21 A Lambert cube collapsing to a quadrilateral, when $\beta/2$ and $\gamma/2$ approach $\pi/2$. Four of the Lambert quadrilaterals on the boundary collapse to segments.

By a possibly degenerate Lambert quadrilateral we mean a triangle with an ideal vertex and two finite vertices, of angles $\frac{\pi}{2}$ and $\frac{\alpha}{2}$.

Again Lemma 2 is proved using the formulas for Lambert cubes and quadrilaterals. It is useful to have in mind the following remark, to know what edge lengths can converge to zero:

Remark 9 Given $\theta \in (0, \frac{\pi}{2})$, a Lambert quadrilateral is determined by an angle θ and the length of any of the edges, Figure 19. Allowing degenerate Lambert quadrilaterals, the length of an edge takes any value in the interval:

- $[\operatorname{arccosh}(1/\sin(\theta)), +\infty]$, if the edge is adjacent to the vertex of angle θ;
- $[0, +\infty]$, if the edge is disjoint from to the vertex of angle θ.

From Lemma 2, by gluing two Lambert quadrilaterals of angle $\frac{\alpha}{2}$ we have:

Corollary 6 *For fixed $\alpha \in (0, \pi)$, when $\beta, \gamma \to \pi^-$ and $\frac{\pi-\beta}{\pi-\gamma}$ converges in $[0, +\infty]$, then $\mathcal{B}(\alpha, \beta, \gamma)$ Hausdorff converges to a (possibly degenerate) hyperbolic cone surface with boundary and corners, a bigon with right angles and a cone point α in the interior, Figure 22 (or Figure 24 for the degenerate case).*

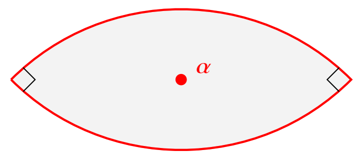


Fig. 22 A cone surface that is the limit when $\beta, \gamma \to \pi^-$. The singular components with angles β and γ converge to the segments in the boundary.

Next we fix two angles $\alpha, \beta \in (0, \pi)$ and look at the limit when $\gamma \to \pi^-$. We describe the behavior of its six sides. It can be computed that:

- The sides that are Lambert quadrilaterals of angle $\gamma/2$ collapse to a segment.
- The sides that are Lambert quadrilaterals of angle $\alpha/2$ or $\beta/2$ converge to ideal triangles.

In particular four of the edge lengths converge to zero, four of them converge to infinity, and the remaining four have a non-vanishing finite limit. This can be visualized by a "long" Lambert cube as in Figure 23.

Lemma 3 *For fixed $\alpha, \beta \in (0, \pi)$, when $\gamma \to \pi^-$ the diameter of $\mathcal{L}(\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2})$ converges to infinity. There are choices of base points so that the pointed Hausdorff limit is either an ideal triangle of angle $\frac{\alpha}{2}$, an ideal triangle of angle $\frac{\beta}{2}$, or a line. See Figure 23.*

Two phenomena occur simultaneously when $\gamma \to \pi^-$. On the one hand, there is a cusp opening, whose horospherical cross section is a sphere with 4 cone points $S^2(\pi, \pi, \pi, \pi)$ (corresponding to the middle quadrilateral in Figure 23) that separates the cone manifold in two components see Figure 25. On the other hand, each one of these pieces collapses to a hyperbolic cone surface with boundary and finite area, Figure 24. The end of this surface is the quotient of a cusp by an involution, and corresponds to a collapse of the Euclidean cone manifold $S^2(\pi, \pi, \pi, \pi)$ to a segment.

Corollary 7 *For fixed $\alpha, \beta \in (0, \pi)$, when $\gamma \to \pi^-$, $\mathcal{B}(\alpha, \beta, \gamma)$ develops a cusp with horospherical cross-section $S^2(\pi, \pi, \pi, \pi)$, that separates $\mathcal{B}(\alpha, \beta, \gamma)$ in two pieces that collapse to cone surfaces as in Figure 24.*

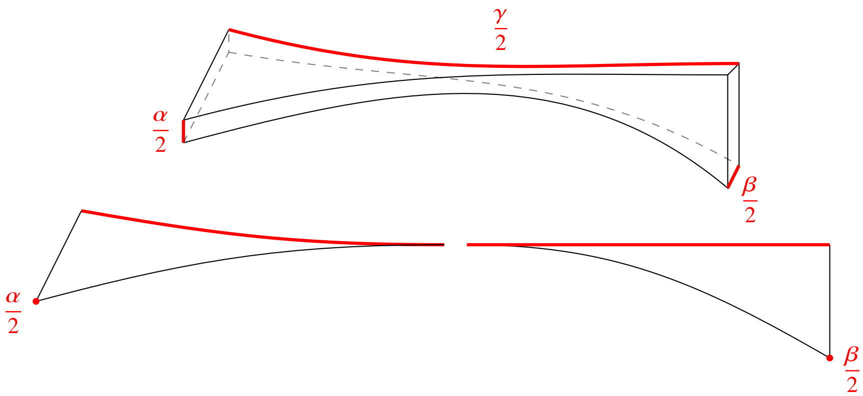


Fig. 23 A "long" Lambert cube, when $\gamma/2$ approaches $\pi/2$ (top) and the limiting ideal triangles (bottom).

Fig. 24 One of the cone surfaces that appear when $\gamma \to \pi^-$ (the other is obtained by replacing α by β).

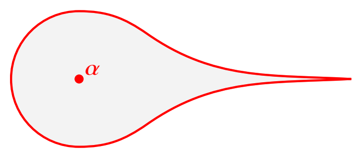
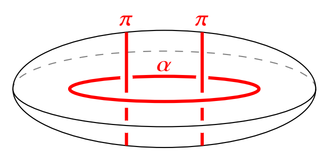


Fig. 25 One of the components after splitting $\mathcal{B}(\alpha, \beta, \pi)$ along the Euclidean cone 2-manifold $S^2(\pi, \pi, \pi, \pi)$. It is Seifert fibered over the surface of Figure 24.



6.5 Borromean rings revisited: spherical structures

Next we consider cone angles $\geq \pi$. For dihedral angles between $\pi/2$ and π, the Lambert cube is spherical, and it has been studied for instance by Díaz [17] and Derevnin and Mednykh [16].

Proposition 11 ([17]) *For $\alpha, \beta, \gamma \in (\pi, 2\pi)$:*

- *The Lambert cube $\mathcal{L}(\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2})$ with dihedral angles $\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2}$ is spherical and rigid.*
- *S^3 admits a unique spherical structure with singular locus the Borromean rings and cone angles (α, β, γ), $\mathcal{B}(\alpha, \beta, \gamma)$.*

Now we look at the spherical Lambert cube when some dihedral angles approach $\pi/2$ (hence some of the cone angles of $\mathcal{B}(\alpha, \beta, \gamma)$ converges to π).

Lemma 4 *When $\gamma \to \pi^+$ and $\alpha, \beta > \pi$ remain constant, $\mathcal{L}(\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2})$ Hausdorff converges to a spherical tetrahedron with right angles, except at two opposite edges, that have angles $\alpha/2 - \pi/2$ and $\beta/2 - \pi/2$, Figure 26.*

In Lemma 4, the edge with dihedral angle $\alpha/2 - \pi/2$ is the result of merging two edges, one with dihedral angle $\alpha/2$ and another one with a right angle, hence its dihedral angle is $\left(\frac{\alpha}{2} + \frac{\pi}{2}\right) - \pi$.

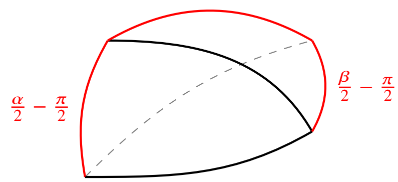


Fig. 26 The tetrahedron in Lemma 4, with the dihedral angles (when they are not right). The length of an edge is the dihedral angle of the opposite edge, thus $l_\alpha = \frac{\beta}{2} - \frac{\pi}{2}$, $l_\beta = \frac{\alpha}{2} - \frac{\pi}{2}$, and $l_\gamma = \frac{\pi}{2}$.

When two of the cone angles converge to π, we have a collapse similar to the hyperbolic case:

Lemma 5 *When $\beta, \gamma \to \pi^+$ and $\alpha > \pi$ remains constant, $\mathcal{L}(\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2})$ Hausdorff converges to a spherical Lambert quadrilateral of angle $\frac{\alpha}{2}$, provided that the ratio $\frac{\beta-\pi}{\gamma-\pi}$ converges in $[0, +\infty]$.*

Furthermore, any (possibly degenerate) Lambert quadrilateral of angle $\frac{\alpha}{2}$ is realized as a limit, according to the limit of the ratio $\frac{\beta-\pi}{\gamma-\pi}$.

Finally, the case where all cone angles converge to π^- is similar to the hyperbolic case.

Lemma 6 *When $\alpha, \beta, \gamma \to \pi^+$, and the ratios $\frac{\alpha-\pi}{\beta-\pi}$ and $\frac{\alpha-\pi}{\gamma-\pi}$ converge to positive real numbers, then $\mathcal{L}(\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2})$ converges to a point. After rescaling, it converges to a right rectangular prism.*

The translation of the results on Lambert cubes to cone manifolds is the following:

Corollary 8 *1. When $\gamma \to \pi^+$, and $\alpha, \beta > \pi$ remain constant, the Hausdorff limit of $\mathcal{B}(\alpha, \beta, \gamma)$ is S^3 with a singular locus as in Figure 27. The singular components of angle α and β intersect the component of angle π and are folded to a segment with cone angle $2\alpha - 2\pi$ and $2\beta - 2\pi$ respectively.*
2. When $\beta, \gamma \to \pi^+$, $\alpha > \pi$ remains constant and the ratio $\frac{\beta-\pi}{\gamma-\pi}$ converges in $[0, +\infty]$, then $\mathcal{B}(\alpha, \beta, \gamma)$ converges to a cone surface as in Figure 22, possibly degenerate (if the cone point goes to the boundary).
3. When $\alpha, \beta, \gamma \to \pi^+$, and the ratios $\frac{\beta-\pi}{\gamma-\pi}$ and $\frac{\alpha-\pi}{\gamma-\pi}$ converge in $(0, +\infty)$, then $\mathcal{B}(\alpha, \beta, \gamma)$ Hausdorff converges to a point, and after rescaling it converges to a Euclidean orbifold.

We can also consider limits when the cone angles α, β or γ approach 2π; the Hausdorff limits of the spherical Lambert cube $\mathcal{L}(\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2})$ are described in Figure 28.

We describe the limits of the cone manifold in the following remark.

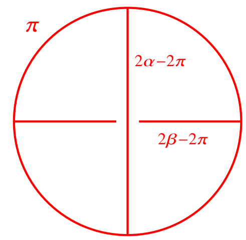


Fig. 27 Singular locus of the limit of $\mathcal{B}(\alpha, \beta, \gamma)$ when $\gamma \to \pi^+$.

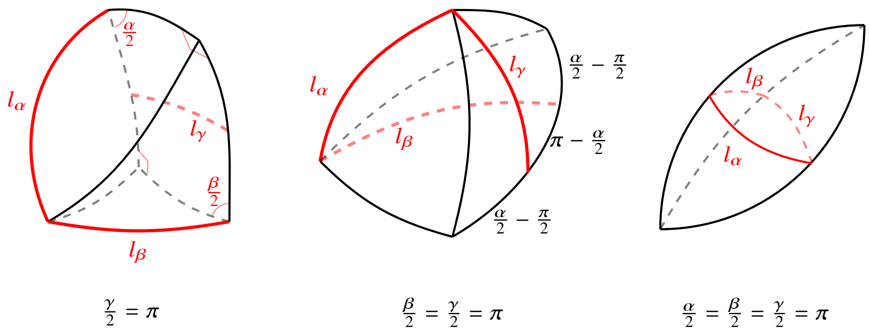


Fig. 28 The Hausdorff limit of the spherical Lambert cube when some of the dihedral angles converge to π.

Remark 10 When $\alpha \to 2\pi^-$, $\beta \to \beta_0 \in (\pi, 2\pi]$, and $\gamma \to \gamma_0 \in (\pi, 2\pi]$, $\mathcal{B}(\alpha, \beta, \gamma)$ Hausdorff converges to the spherical suspension over a cone surface S. The first singular geodesic converges to a geodesic in S, and, at the limit, the other singular components intersect at the tips of the suspension.

Notice that we allow the limit β_0 or γ_0 to equal 2π. The suspension structure of the remark is obtained from doubling the cones of the Lambert cubes in Figure 28. Hence the Hausdorff limit of $\mathcal{B}(\alpha, \beta, \gamma)$ has a suspension structure for each cone angle that becomes 2π.

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