Integral Geometry of pairs of lines and planes

Julià Cufí, Eduardo Gallego and Agustí Reventós

Abstract In this paper we present some results obtained previously in [3, 4, 5, 6] related to convex sets in the plane and in the space.

In the plane we deal with Hurwitz's inequality, which provides an upper bound of the isoperimetric deficit of a convex set K in terms of the area of the evolute of the boundary of K. We improve this inequality finding strictly positive lower bounds for the Hurwitz's deficit, these bounds involving the visual angle of the boundary of K. In a different look we provide a unified approach that encompasses some integral formulas for functions of the visual angle of a compact convex set due to Crofton, Hurwitz and Masotti. Also we interpret these formulas from the point of view of Integral Geometry of pairs of lines.

In the space we deal with integrals of invariant measures of pairs of planes, expressing some of these integrals in terms of functions of the visual dihedral angle of the convex set. As a consequence of our results we evaluate the deficit in a Crofton-type inequality due to Blaschke.

Key words: convex set, visual angle, invariant measure, constant width, dihedral visual angle
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1 Introduction

Here we present some results obtained previously in [3, 4, 5, 6] related to convex sets in the plane and in the space. For the case of the plane our contribution is strongly related to the celebrated work by Hurwitz [10] in which he introduced the use of Fourier series to deal with geometrical problems, some of them related to the visual angle of a convex set. In the case of the space Fourier series must be substituted by spherical harmonics and the visual angle by the dihedral visual angle. In both cases we adopt the point of view of Integral Geometry according to Santaló [12].

The results in section 3 are related to the the classical isoperimetric inequality

$$\Delta = L^2 - 4\pi F \geq 0,$$

where L is the length of a simple closed plane curve Γ enclosing a region of area F. In the case that Γ bounds a convex set K, Hurwitz ([10]) established a kind of reverse isoperimetric inequality, namely

$$L^2 - 4\pi F \leq \pi |F_e|, \tag{1}$$

where F_e is the algebraic area ($F_e \leq 0$) enclosed by the evolute of Γ. Moreover equality holds in (1) if and only if Γ is a circle or a curve parallel to an astroid.

We improve Hurwitz's inequality (1) finding strictly positive lower bounds for the *Hurwitz deficit* $\pi|F_e| - \Delta$. These bounds involve the *visual angle* of K from a point P, that is the angle between the two tangents from P to the boundary of K (see Theorem 1). In the constant-width case we prove, in Theorem 2, the inequality $L^2 - 4\pi F \leq \frac{4}{9}\pi|F_e|$.

Before addressing what we do in section 4 let us remember that in 1868 Crofton showed ([2]) the well known formula

$$\int_{P\notin K} (\omega - \sin\omega)\, dP = \frac{L^2}{2} - \pi F, \tag{2}$$

where $\omega = \omega(P)$ is the visual angle of K from the point P.

Later on, Hurwitz in 1902 [10], considered again the integral of some functions of the visual angle. In particular he gave a new proof of the Crofton formula using the Fourier series of the radius of curvature ρ of the boundary ∂K. He also stablished the equality

$$\int_{P\notin K} \sin^3\omega\, dP = \frac{3}{4}L^2 + \frac{1}{4}\pi^2\gamma_2^2 \tag{3}$$

whith $\gamma_k^2 = \alpha_k^2 + \beta_k^2$, where α_k and β_k are the Fourier coefficients of ρ.

In 1955 Masotti ([11]) considered a Crofton-type formula computing

$$\int_{P\notin K} (\omega^2 - \sin^2\omega)\, dP$$

in terms of the area of K, the length of ∂K and the Fourier coefficients of the radius of curvature of ∂K. Santaló in 1976 ([12, I.4.5]) gave lower and upper bounds for the above integral.

In subsection 4.1 we provide a unified approach that encompasses the previous results and allows us to obtain new integral formulas for functions of the visual angle. The basic tool is the integral formula given in Theorem 3. Using this theorem we obtain integral formulas for any power of $\sin\omega$ (Theorem 5) and for the function $\omega^m - \sin^m \omega$ (see equation (19)).

In 4.2 we deal with a general type of integral formulas of the visual angle including those we have just commented above, from the point of view of Integral Geometry according to Crofton and Santaló [12]. The purpose is twofold: to provide an interpretation of these formulas in terms of integrals of functions with respect to the canonical measure in the space of pairs of lines and to give new simpler proofs of them (see Propositions 7, 8 and 9).

The main goal of part 5 is to study integrals of invariant measures with respect to euclidean motions in the euclidean space $\mathbb{E}^3$, extended to the set of pairs of planes meeting a compact convex set. To carry out this objective we express these integrals in terms of functions of the dihedral visual angle of the convex set from a line and integrate them with respect to an invariant measure in the space of lines. The main tool we use are spherical harmonics. In this sense Theorem 8 plays an important role. Then we assign to any invariant measure on the space of pairs of planes an appropriate function of the dihedral visual angle of a given convex set. The integral of this function with respect to the measure on the space of lines gives the integral of the above measure extended to those planes meeting the convex set (see Theorem 9). In subsection 5.2 we relate this last result to Blaschke's work [1, p. 75] in Theorem 10.

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2 Preliminaries

Support function

A set $K \subset \mathbb{R}^n$ is *convex* if it contains the complete segment joining every two points in the set. We shall consider nonempty compact convex sets. The *support function* of K is defined as

$$p_K(u) := \sup\{\langle x, u\rangle \,:\, x \in K\} \qquad \text{for} \quad u \in \mathbb{R}^n.$$

For a vector u in the unit sphere S^{n-1} the number $p_K(u)$ is the signed distance of the support hyperplane to K with outer normal vector u from the origin. The distance is negative if and only if u points into the open half-space containing the origin (cf. [13]).

In the case of the plane we shall denote by $p(\varphi)$ the 2π-periodic function obtained by evaluating $p_K(u)$ on $u = (\cos\varphi, \sin\varphi)$. Note that ∂K is the envelope of the one parametric family of lines given by $x\cos\varphi + y\sin\varphi = p(\varphi)$. When p is a C^2 function the radius of curvature $\rho(\varphi)$ of ∂K is given by $p(\varphi) + p''(\varphi)$. Then, convexity is equivalent to $p(\varphi) + p''(\varphi) \geq 0$. We say that a C^2 support function p defines a *strictly convex* set if $p(\varphi) + p''(\varphi) > 0$ for every value of φ.

It can be seen that the length L of ∂K and the area F of K are given by

$$L = \int_0^{2\pi} p\, d\varphi, \quad F = \frac{1}{2}\int_0^{2\pi} p(p + p'')d\varphi.$$

In general, a one parameter family of lines $x\cos t + y\sin t = f(t)$, where f is a differentiable function, defines a curve in the plane. In this setting the curve is not necessarily closed nor convex. When a curve $\gamma(t)$, $a \leq t \leq b$, is defined as the envelope of a family of lines of this type, for a function f of class C^2, we say that $f(t)$ is the *generalized support function* of the curve. The *area with multiplicities* swept by the radius vector of the curve is given by

$$F = \frac{1}{2}\int_a^b f(f + f'')dt, \tag{4}$$

as a simple computation shows.

Let $p(\varphi)$ be the support function of a strictly convex set K. Then $p_r(\varphi) = p(\varphi) + r$ defines for each real r a *parallel curve* to ∂K. If the origin is in the interior of K then p is a strictly positive function. If $r > 0$ the function p_r corresponds to the outer parallel set at distance r. When $r < 0$ the curve given by p_r is not necessarily convex (this is the case when $|r| > \min(\rho)$, ρ being the radius of curvature of ∂K).

A special type of convex sets are those of *constant width*, that is those convex sets whose orthogonal projection on any direction have the same length w. In terms of the support function p of K, constant width means that $p(\varphi) + p(\varphi + \pi) = w$. Expanding p in Fourier series

$$p(\varphi) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\varphi) + b_n \sin(n\varphi), \tag{5}$$

one obtains that K has constant width if and only if $a_n = b_n = 0$ for all even $n > 0$.

The *Steiner point* of K is defined by the vector-valued integral

$$s(K) = \frac{1}{\pi}\int_0^{2\pi} p(\varphi)N(\varphi)\, d\varphi,$$

where $N(\varphi) = (\cos\varphi, \sin\varphi)$ denotes the normal vector to ∂K. The Steiner point is rigid motion equivariant; this means that $s(gK) = gs(K)$ for every rigid motion g.

It is known that $s(K)$ lies in the interior of K (cf. [8, p. 56]). In terms of the Fourier coefficients of $p(\varphi)$ given in (5) the Steiner point is $s(K) = (a_1, b_1)$. Hence, taking the Steiner point as a new origin, one has

$$p(\varphi) = a_0 + \sum_{n\geq 2} (a_n \cos n\varphi + b_n \sin n\varphi)\,.$$

Measure of lines in the plane

We denote by $\mathcal{A}_{2,1}$ the set of straight lines in the plane. For each straight line $G \in \mathcal{A}_{2,1}$ that does not pass through the origin let P be the point of G at a minimum distance from the origin. We take as coordinates for G the polar coordinates (p, φ) of the point P, with $p > 0$ and $0 \leq \varphi < 2\pi$. Notice that p and φ can also be seen as functions in this space of lines, and we shall write $p(G), \varphi(G)$ for the corresponding coordinates of the straight line G.

The invariant measure in the set of lines of the plane not containing the origin is given by a constant multiple of

$$dG = dp \wedge d\varphi. \tag{6}$$

In fact this measure is, except for a constant factor, the only one invariant under Euclidean motions (see [12], Section I.3.1). In the space of ordered pairs of lines $\mathcal{A}_{2,1} \times \mathcal{A}_{2,1}$ we consider the canonical measure $dG_1 \wedge dG_2$. For every function $\tilde{f}(G_1, G_2)$ we can consider the measure $\tilde{f}(G_1, G_2)\, dG_1 \wedge dG_2$. We prove in Proposition 5 that this measure is invariant under Euclidean motions if and only if $\tilde{f}(G_1, G_2) = f(\varphi(G_2) - \varphi(G_1))$ with f an even π-periodic function on $\mathbb{R}$.

Spherical harmonics

Let us recall that a *spherical harmonic* of degree n on the unit sphere S^2 is the restriction to S^2 of an harmonic homogeneous polynomial of degree n. It is known that every continuous function on S^2 can be uniformly approximated by finite sums of spherical harmonics (see for instance [8]).

More precisely, the function $p(u)$ can be written in terms of spherical harmonics as

$$p(u) = \sum_{n=0}^{\infty} \pi_n(p)(u), \tag{7}$$

where $\pi_n(p)$ is the projection of the support function p on the vector space of spherical harmonics of degree n. An orthogonal basis of this space is given in terms of the longitude θ and the colatitude φ in S^2 by

$$\{\cos(j\theta)(\sin\varphi)^j\, P_n^{(j)}(\cos\varphi), \quad \sin(j\theta)(\sin\varphi)^j\, P_n^{(j)}(\cos\varphi): \quad 0 \le j \le n\}$$

where $P_n^{(j)}$ denotes the jth derivative of the nth Legendre polynomial P_n (cf. [8]).

It can be seen that $\pi_0(p) = \mathcal{W}/2 = M/4\pi$ where $\mathcal{W} = 1/4\pi \int_{S^2} w(u)\, du$ is the *mean width* of K, and M is the *mean curvature* of K. Moreover $\pi_1(p) = \langle s(K), \cdot\rangle$ where $s(K)$ denotes the Steiner point of K (cf. [8, p. 182]). It is clear that $\pi_0(p)$ is invariant under euclidean motions and that $\pi_1(p)$ is not. It is known that $\pi_n(p)$ is invariant under translations for every $n \ne 1$ (cf. [13, p. 5]). One can easily check that K has constant width if and only if $\pi_n(p) = 0$ for $n \ne 0$ even.

3 Lower bounds for the Hurwitz deficit

In order to study the Hurwitz deficit $\pi|F_e| - \Delta$ of a convex set K we introduce the *Wirtinger deficit* W_q of a C^1 function $q(\varphi)$ of period 2π,

$$W_q = \int_0^{2\pi} (q'^2 - q^2)\, d\varphi.$$

Note that by (4), $W_q = -2F$ where F is the area with multiplicities enclosed by the curve defined by the generalized support function q.

Recall that Wirtinger's inequality (see [8]) states that if $\int_0^{2\pi} q(\varphi)d\varphi = 0$, then $W_q \ge 0$. In particular we always have $W_{q'} \ge 0$. In [3] we give a relationship between the Wirtinger and Hurwitz deficits:

Proposition 1 *Let K be a compact strictly convex set of area F bounded by a curve $\Gamma = \partial K$ of class C^2 and length L. Let p be the support function of K and let F_e be the area with multiplicities enclosed by the evolute of Γ. Then*

$$\pi|F_e| - \Delta = \frac{\pi}{2}(W_{q'} - 4W_q)$$

where $q(\varphi) = p(\varphi) - \frac{L}{2\pi}$ and $\Delta = L^2 - 4\pi F$.

Remark 1 Let F be the area enclosed by the curve with generalized support function the 2π-periodic function q, and let F_e be the area enclosed by the evolute of this curve, both areas counted with multiplicities. The equalities $W_{q'} = -2F_e$ and $W_q = -2F$ give

$$\frac{1}{2}(W_{q'} - W_q) = F - F_e.$$

Thus, for closed curves with positive curvature, we have

$$F - F_e = \frac{1}{2}\int_0^{2\pi} (q + q'')^2\, d\varphi = \frac{1}{2}\int_0^L \rho\, ds \tag{8}$$

where $\rho = q + q''$ is the radius of curvature and L the length of the curve. We have used the relation $ds = \rho\, d\varphi$. Equality (8) for the case of simple closed curves that bound a strictly convex domain was proved in [10].

Next Lemma compares the Wirtinger deficit of a given function with that of its derivative. The proof follows the standard pattern of the proof of Wirtinger inequality using Fourier series.

Lemma 1 *Let $q = q(\varphi)$ a 2π-periodic C^2 function. Then*

$$W_{q'} \geq 4W_q + \frac{2}{\pi}\left(\int_0^{2\pi} q\, d\varphi\right)^2 \geq 0.$$

Moreover the first inequality is an equality if and only if

$$q(\varphi) = a_0 + a_1 \cos\varphi + b_1 \sin\varphi + a_2 \cos 2\varphi + b_2 \sin 2\varphi,$$

for some constants $a_0, a_1, b_1, a_2, b_2 \in \mathbb{R}$.

Remark that the first inequality in Lemma 1 improves Wirtinger's inequality for the derivative of 2π-periodic functions.

As a consequence of Proposition 1 one obtains the well known Hurwitz's inequality (1).

We proceed now to find a lower bound for the Hurwitz deficit $\pi|F_e| - \Delta$ so improving (1). If

$$p(\varphi) = a_0 + \sum_{n\geq 1} a_n \cos n\varphi + b_n \sin n\varphi$$

is the Fourier series of the support function of K, it is known that the quantities $c_n^2 = a_n^2 + b_n^2$, for $n \geq 2$, are invariant under the group of plane motions.

Consider ω the *visual angle* of ∂K from P and let dP be the area measure. Writing

$$I_n = \int_{P\notin K}\left(-2\sin(\omega) + \frac{n+1}{n-1}\sin(n-1)\omega - \frac{n-1}{n+1}\sin(n+1)\omega\right) dP,$$

it is proved in [10][1] that

$$I_n = L^2 + (-1)^n \pi^2 (n^2 - 1) c_n^2,\ n \geq 2 \tag{9}$$

L being the length of the boundary of K. For instance, if $n = 2$ one gets

$$\frac{4}{3}\int_{P\notin K} \sin^3\omega\, dP = L^2 + 3\pi^2 c_2^2. \tag{10}$$

Moreover, this visual angle satisfies the Crofton formula (see [10])

[1] There is a misprint with the sign in Hurwitz's paper. Moreover the c_n coefficients appearing in (9) are different from those in Hurwitz's paper because the latter correspond to the Fourier series of the curvature radius function.

$$\int_{P\notin K}(\omega-\sin\omega)dP=\frac{L^2}{2}-\pi F. \tag{11}$$

We can prove now the following result involving (9) and (11).

Theorem 1 ([3])

Let K be a compact strictly convex set of area F bounded by a curve $\Gamma=\partial K$ of class C^2 and length L. Let F_e be the area with multiplicities enclosed by the evolute of Γ and let Δ be the isoperimetric deficit. Then

$$\pi|F_e|-\Delta\geq\frac{5}{4}L^2+5\int_{P\notin K}(\omega-\sin\omega-\frac{2}{3}\sin^3\omega)dP. \tag{12}$$

The right hand side of this inequality is a strictly positive quantity except when $\pi|F_e|-\Delta=0$ in which case it also vanishes.

Proof It can be seen that

$$\pi|F_e|-\Delta=\frac{\pi}{2}(W_{q'}-4W_q)=\frac{\pi}{2}\left(4\int_0^{2\pi}q^2d\varphi-5\int_0^{2\pi}q'^2d\varphi+\int_0^{2\pi}q''^2d\varphi\right),$$

where $q(\varphi)=p(\varphi)-L/2\pi$, and $p(\varphi)$ is the support function of K with respect to the Steiner point.

In terms of the Fourier coefficients of p

$$\pi|F_e|-\Delta=\frac{\pi^2}{2}\sum_{n\geq3}(n^4-5n^2+4)c_n^2.$$

Observe now that, for $n\geq3$, we have $n^4-5n^2+4\geq5(n^2-1)$, with equality only for $n=3$. Therefore

$$\begin{aligned}\pi|F_e|-\Delta&\geq\frac{5\pi^2}{2}\sum_{n\geq3}(n^2-1)c_n^2=\frac{5\pi^2}{2}\left(\sum_{n\geq2}(n^2-1)c_n^2-3c_2^2\right)\\&=\frac{5}{4}L^2-5\pi F-\frac{15\pi^2}{2}c_2^2=\frac{15}{4}L^2-5\pi F-\frac{10}{3}\int_{P\notin K}\sin^3\omega\,dP.\end{aligned} \tag{13}$$

Using Crofton's formula (11), the last expression can be written as

$$\frac{5}{4}L^2+5\int_{P\notin K}(\omega-\sin\omega-\frac{2}{3}\sin^3\omega)dP$$

and the inequality in the theorem is proved. Moreover, the sum $\sum_{n\geq3}(n^2-1)c_n^2$ in (13) vanishes if and only if $c_n=0$ for $n\geq3$ as well as $\pi|F_e|-\Delta$. □

The equality in Theorem 1 is considered in the following result

Proposition 2 *Equality in* (12) *holds if and only if for the compact strictly convex set K one of the following assertions holds:*

a) K is a disk or it is bounded by a curve parallel to an astroid.
b) K is bounded by a curve parallel to a Steiner curve (deltoid).
c) K is parallel to the Minkowski sum of compact sets of the above types.

Although Hurwitz's inequality (1) cannot be improved for general convex domains, it is possible to obtain a stronger inequality for convex sets of constant width. In fact we have

Theorem 2 ([3])

Let K be a compact strictly convex set of constant width and area F bounded by a curve $\Gamma = \partial K$ of class C^2 and length L. Let F_e be the area with multiplicities of the evolute of Γ. Then

$$L^2 - 4\pi F \leq \frac{4}{9}\pi|F_e|. \tag{14}$$

Equality holds if and only if Γ is a circle or a curve parallel to a Steiner curve at distance $L/2\pi$.

We also obtain an inequality better than (14) in terms of the visual angle see [3, Theorem 5.3].

4 Integral formulas for the visual angle

As seen in the previous section the view of a convex set from an exterior point gives interesting geometric information about this set. Now we study some aspects of the visual angle of a convex set.

4.1 On Crofton's and Hurwitz's formulas.

In this subsection we provide a unified approach for some integral formulas for functions of the visual angle of a convex set due to Crofton, Hurwitz and Massoti obtaining also new integral formulas for this kind of functions. The basic tool is the integral formula given in

Theorem 3 ([4])

Let K be a compact convex set with boundary of class C^2 and let L be the length of ∂K. Let $c_k^2 = a_k^2 + b_k^2$ where a_k, b_k are the Fourier coefficients of the support function of K. Then, for every continuous function of the visual angle $f(\omega)$ on $[0, \pi]$ such that $f(\omega) = O(\omega^3)$, as ω tends to zero, one has

$$\int_{P\notin K} f(\omega)\,dP$$
$$= \left(\int_0^{\pi} \frac{f(\omega)(1+\cos\omega)^2}{\sin^3\omega}\,d\omega\right)\frac{L^2}{2\pi} + \pi\sum_{k\geq 2}\left(\int_0^{\pi}\frac{f(\omega)h_k(\omega)}{\sin^3\omega}\,d\omega\right)c_k^2,$$

where h_k, for $k \geq 2$, are the universal functions given in (15).

Proof For each point $P \notin K$ let φ be the angle at the origin formed by the normal to one of the tangents from P to ∂K with the x axis and ω the visual angle from P; the pair (φ, ω) can be considered as a system of coordinates in $\mathbb{R}^2 \setminus K$.

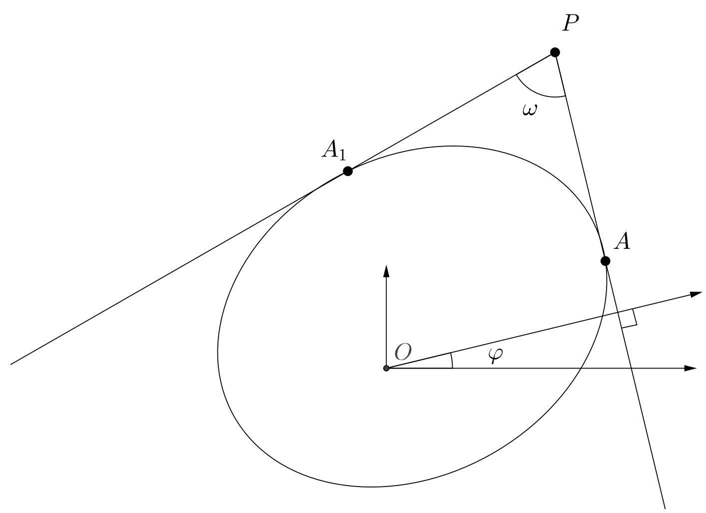

Fig. 1 The visual angle ω.

We shall denote by A, A_1 the contact points of the tangents from P to ∂K, and by $p = p(\varphi)$ the support function of K with respect an origin O inside K (see Figure (1)). Let us write $T = PA$ and $T_1 = PA_1$.

The area element dP of $\mathbb{R}^2 \setminus K$ is given by

$$dP = \frac{TT_1}{\sin\omega}\,d\varphi \wedge d\omega.$$

This expression of the area element, introduced by Crofton in [2], appears also in [12, I.2.2].

Hence, the integral on $\mathbb{R}^2 \setminus K$ of a suitable function of the visual angle $f(\omega)$ is given by

$$\int_{P\notin K} f(\omega)\,dP = \int_0^{\pi}\int_0^{2\pi}\frac{f(\omega)}{\sin\omega}TT_1\,d\varphi\,d\omega = \int_0^{\pi}\frac{f(\omega)}{\sin\omega}\left(\int_0^{2\pi}TT_1\,d\varphi\right)d\omega.$$

Now we will write the product TT_1 in terms of the Fourier coefficients of $p(\varphi)$ given in (5), and the Fourier coefficients of $p_1(\varphi) := p(\pi + \varphi - \omega)$ given by

$$p_1(\varphi) = a_0 + \sum_{k>0}(A_k \cos k\varphi + B_k \sin k\varphi)$$

which are related to the coefficients of $p(\varphi)$ by

$$\begin{aligned} A_k &= (-1)^{k+1}(-a_k \cos k\omega + b_k \sin k\omega), \\ B_k &= (-1)^{k+1}(-a_k \sin k\omega - b_k \cos k\omega). \end{aligned}$$

Substituting these Fourier series in the expressions of T and T_1, a straightforward but long calculation gives

$$\int_0^{2\pi} TT_1 \, d\varphi = \frac{1}{\sin^2 \omega}\left(\frac{L^2}{2\pi}(1 + \cos\omega)^2 + \pi \sum_{k>0} c_k^2 h_k(\omega)\right),$$

where $c_k^2 = a_k^2 + b_k^2$ and

$$\begin{aligned} h_k(\omega) = 2\cos\omega + (-1)^{k+1}\big(&-\cos k\omega(1 + \cos^2\omega) \\ &- 2k \sin k\omega \sin\omega \cos\omega + k^2 \cos k\omega \sin^2\omega\big). \end{aligned} \tag{15}$$

This ends the proof. □

If we consider the area $F(\omega)$ enclosed by the locus C_ω of the points from which the convex set K is viewed under the same angle ω and introduce the functions

$$g_k(\omega) = 1 + \frac{(-1)^k}{2}\left((k+1)\cos(k-1)\omega - (k-1)\cos(k+1)\omega\right),$$

the previous Theorem can be stated as

Proposition 3 *With the same hypothesis of Theorem 3 we have*

$$\int_{P\notin K} f(\omega)\, dP = -\left[f(\omega)F(\omega)\right]_{0_+}^{\pi_-} + \frac{L^2}{2\pi}M(f) + \pi \sum_{k\geq 2} \beta_k(f) c_k^2,$$

where

$$M(f) = \int_0^\pi \frac{f'(\omega)}{1 - \cos\omega}\, d\omega \quad \textit{and} \quad \beta_k(f) = \int_0^\pi \frac{f'(\omega) g_k(\omega)}{\sin^2\omega}\, d\omega. \tag{16}$$

As an application of Proposition 3 we can easily prove Crofton's formula

$$\int_{P\notin K} (\omega - \sin\omega)\, dP = -\pi F + \frac{L^2}{2}. \tag{17}$$

Indeed $M(\omega - \sin\omega) = \pi$ and $\beta_k(\omega - \sin\omega) = \int_0^\pi g_k(x)/(1+\cos(x))\,dx = 0$, as can be easily seen integrating by parts and using elementary trigonometric identities. Since

$$-\lim_{\omega\to\pi} f(\omega)F(\omega) + \lim_{\omega\to 0} f(\omega)F(\omega) = -\pi F$$

the formula follows.

Also, from Proposition 3 we obtain a new proof (see [4]) of the classical equality of Hurwitz given in (9).

In [11] Masotti gives without proof a Crofton type formula evaluating $\int_{P\notin K}(\omega^2 - \sin^2\omega)\,dP$. We derive here Masotti's formula from Proposition 3. To this end consider the function $f(\omega) = \omega^2 - \sin^2\omega$ which clearly satisfies the hypothesis of Theorem 3, and let us compute $[f(\omega)F(\omega)]_0^\pi$, $M(f)$ and the integrals $\int_0^\pi f'(\omega)\cos(j\omega)\,d\omega$, for j integer. We have

$$\lim_{\omega\to\pi} f(\omega)F(\omega) = \pi^2 F$$

and

$$\lim_{\omega\to 0} f(\omega)F(\omega) = \lim_{\omega\to 0}\frac{\omega^2 - \sin^2\omega}{\sin^2\omega}\left(\frac{L^2}{2\pi}(1+\cos\omega) + \pi\sum_{k\geq 2} c_k^2 g_k(\omega)\right) = 0,$$

since the term inside the parentheses is bounded. Hence, $[f(\omega)F(\omega)]_0^\pi = \pi^2 F$.

On the other hand

$$\begin{aligned} M(f) = \int_0^\pi \frac{f'(\omega)}{1-\cos\omega}\,d\omega &= \int_0^\pi \frac{2\omega - \sin(2\omega)}{1-\cos\omega}\,d\omega \\ &= \left[\sin^2(\omega/2) - 3\cos^2(\omega/2) - 2\omega\cot(\omega/2)\right]_0^\pi = 8. \end{aligned}$$

Moreover, for $j \neq 2$,

$$\int_0^\pi f'(\omega)\cos(j\omega)\,d\omega = \int_0^\pi (2\omega - \sin(2\omega))\cos(j\omega)\,d\omega$$

$$= \left[\frac{2}{j^2}(\cos(j\omega) + j\omega\sin(j\omega)) - \frac{\cos((j-2)\omega)}{2(j-2)} + \frac{\cos((j+2)\omega)}{2(j+2)}\right]_0^\pi = \frac{8(1-(-1)^j)}{j^2(j^2-4)},$$

and

$$\int_0^\pi (2\omega - \sin(2\omega))\cos(2\omega)\,d\omega = 0.$$

It follows that

$$\sum_{j=1,\,\mathrm{odd}}^{k-1}\int_0^\pi f'(\omega)\cos(j\omega)\,d\omega = \sum_{j=1,\,\mathrm{odd}}\frac{16}{j(j^2-4)} = \frac{4k^2}{1-k^2}.$$

Summing up we obtain

Theorem 4 (Masotti, [11])

Let K be a compact convex set of area F with boundary of class C^2 and length L. Let $c_k^2 = a_k^2 + b_k^2$ where a_k, b_k are the Fourier coefficients of the support function of K. Then

$$\int_{P \notin K} (\omega^2 - \sin^2 \omega)\, dP = -\pi^2 F + \frac{4L^2}{\pi} + 8\pi \sum_{k \geq 2,\, even} \left(\frac{1}{1-k^2} \right) c_k^2. \tag{18}$$

Moreover the equality

$$\int_{P \notin K} (\omega^2 - \sin^2 \omega)\, dP = -\pi^2 F + \frac{4L^2}{\pi}$$

holds if and only if the compact convex set K has constant width.

Now we compute the integral of $\sin^m(\omega)$ for integer values of m greater than 3. The case $m = 3$, due to Hurwitz was given in (10). We have the following theorem.

Theorem 5 ([4])

Let K be a compact convex set with boundary of class C^2 and length L. Write $c_k^2 = a_k^2 + b_k^2$ where a_k, b_k are the Fourier coefficients of the support function of K. Then, for $m \geq 3$,

$$\int_{P \notin K} \sin^m \omega\, dP = M(\sin^m \omega) \frac{L^2}{2\pi} + \frac{m!\pi^2}{2^{m-1}(m-2)} \sum_{k \geq 2,\, even} \frac{(-1)^{\frac{k}{2}+1}(k^2-1)}{\Gamma(\frac{m+1+k}{2})\Gamma(\frac{m+1-k}{2})} c_k^2,$$

where $M(\sin^m(\omega))$ comes from (16)*. For m odd the index k in the sum runs only from* 2 *to $m-1$.*

In the special case of convex sets of constant width we get

Proposition 4 *Let K be a compact convex of constant width with boundary of class C^2 and length L. Then, for $m \geq 3$,*

$$\int_{P \notin K} \sin^m \omega\, dP = \frac{\pi\, m!}{2^{m-1}(m-2)\Gamma(\frac{m+1}{2})^2} \frac{L^2}{2\pi}.$$

We also consider the integral

$$\int_{P \notin K} (\omega^m - \sin^m \omega)\, dP.$$

For $m = 1$ and $m = 2$ these are the integrals appearing in Crofton's formula (17) and in the Masotti integral formula (18), respectively. For the general case, we obtain from Proposition 3

$$\int_{P\notin K} (\omega^m - \sin^m \omega)\, dP = -\pi^m F + M_m \frac{L^2}{2\pi} + \pi \sum_{k\geq 2} \beta_k c_k^2, \tag{19}$$

where $M_m = M(\omega^m - \sin^m \omega)$ and $\beta_k = \beta_k(\omega^m - \sin^m \omega)$ are given in (16).

We are able to prove that $\beta_k \leq 0$ for $k \geq 2$. So we get the following inequality.

Theorem 6 ([4])

Let K be a compact convex set with boundary of class C^2, area F and length of the boundary L, and let $\omega = \omega(P)$ be the visual angle from the point P. Then

$$\int_{P\notin K} (\omega^m - \sin^m \omega)\, dP \leq -\pi^m F + M_m \frac{L^2}{2\pi}, \qquad m \geq 1,$$

where $M_m = \displaystyle\int_0^\pi \frac{(\omega^m - \sin^m \omega)'}{1-\cos\omega}\, d\omega$. *Equality holds only for circles.*

4.2 Measure of pairs of lines in the plane.

The classical proof of Crofton's formula (2) comes from the study of the measure of pairs of lines intersecting a convex set. In order to obtain new formulas involving the visual angle we consider the measures $\tilde{f}(G_1, G_2)dG_1 \wedge dG_2$ for every function $\tilde{f}(G_1, G_2)$ defined on the space of ordered pair of lines $\mathcal{A}_{2,1} \times \mathcal{A}_{2,1}$. We want now to characterize when these measures are invariant under Euclidean motions. Using the polar coordinate $\varphi(G)$ introduced in (6) we have

Proposition 5 *The measure $\tilde{f}(G_1, G_2)dG_1 \wedge dG_2$ is invariant under the group of Euclidean motions if and only if $\tilde{f}(G_1, G_2) = f(\varphi(G_2) - \varphi(G_1))$ with f an even π-periodic function on $\mathbb{R}$.*

Proof Let (p_i, φ_i) be the coordinates of G_i and define the function g by $g(p_1, \varphi_1, p_2, \varphi_2) = \tilde{f}(G_1, G_2)$. The invariance of the measure is equivalent to the equality $g(p_1, \varphi_1, p_2, \varphi_2) = g(p_1', \varphi_1', p_2', \varphi_2')$ for each Euclidean motion sending the pair of lines with coordinates $(p_1, \varphi_1, p_2, \varphi_2)$ to the pair of lines with coordinates $(p_1', \varphi_1', p_2', \varphi_2')$. First of all let us show that g does not depend on p_1, p_2. In fact, for every straight line $G = G(p, \varphi)$ and an arbitrary $a > 0$ there is a parallel line to G with coordinates (a, φ). Given two straight lines $G_1 = G(p_1, \varphi_1), G_2 = G(p_2, \varphi_2)$ and two numbers $a_1, a_2 > 0$ let G_1' and G_2' be the corresponding parallel lines with coordinates $(a_1, \varphi_1), (a_2, \varphi_2)$. Performing the translation that sends the point $G_1 \cap G_2$ to the point $G_1' \cap G_2'$ we have that $g(p_1, \varphi_1, p_2, \varphi_2) = g(a_1, \varphi_1, a_2, \varphi_2)$ and so g does not depend on p_1 and p_2.

Given now the line $G(p, \varphi)$ if we perform, for instance, the translation given by the vector $-(p + \epsilon)(\cos\varphi, \sin\varphi)$, $\epsilon > 0$, the translated line has coordinates $(\epsilon, \varphi + \pi)$. Therefore the function g must be π-periodic with respect to the arguments φ_1, φ_2. Due to the invariance under rotations it follows that $g(p_1, \varphi_1, p_2, \varphi_2) =$

$g(p_1, 0, p_2, \varphi_2 - \varphi_1)$ and so $g(p_1, \varphi_1, p_2, \varphi_2) = f(\varphi_2 - \varphi_1) = f(\varphi(G_2) - \varphi(G_1))$ with f a π-periodic function. Finally the invariance under symmetries implies that f is an even function.

Conversely it is clear that if f is an even π-periodic function then the measure $f(\varphi(G_2) - \varphi(G_1))dG_1 \wedge dG_2$ is invariant under Euclidean motions. □

Our goal is now to integrate measures, not necessarily invariant under Euclidean motions, over the set of pairs of lines meeting a compact convex set K. We shall consider measures of the form $\tilde{f}(G_1, G_2)dG_1 \wedge dG_2 = f(\varphi(G_2) - \varphi(G_1))dG_1 \wedge dG_2$ with f a continuous 2π-periodic function. Notice that $\varphi(G_2) - \varphi(G_1)$ gives one of the two angles between the lines G_1 and G_2. In Theorem 7 we give a formula to compute the integral of the above measures in terms of both the Fourier coefficients of f and of the support function of K.

Theorem 7 ([5])

Let K be a compact convex set with C^1 boundary of length L. Let f be a 2π-periodic continuous function on $\mathbb{R}$ with Fourier expansion

$$f(\varphi) = \sum_{n \geq 0} A_n \cos(n\varphi) + B_n \sin(n\varphi).$$

Then

$$\int_{G_i \cap K \neq \emptyset} f(\varphi(G_2) - \varphi(G_1))\, dG_1 dG_2 = A_0 L^2 + \pi^2 \sum_{n \geq 1} A_n c_n^2,$$

with $c_n^2 = a_n^2 + b_n^2$ where a_n, b_n are the Fourier coefficients of the support function p of K.

The original proof of Crofton's formula, via Integral Geometry, involves a measure on the space of pairs of lines. The aim now is to interpret the formulas of Masotti, powers of sine, and Hurwitz in [4] in terms of integrals of measures in the space of pairs of lines.

The classical proof of Crofton's formula is based on the change of variables in the space of ordered pairs of lines given by

$$(p_1, \varphi_1, p_2, \varphi_2) \longrightarrow (P, \alpha_1, \alpha_2)$$

where P is the intersection point of the two straight lines and $\alpha_i \in [0, \pi]$ are the angles which determine the directions of the lines. With these new coordinates, proceeding as in [12, I.4.3] one has

$$dG_1 \wedge dG_2 = |\sin(\alpha_2 - \alpha_1)|\, d\alpha_1 \wedge d\alpha_2 \wedge dP.$$

Using this change of variables we have

Proposition 6 *Let f be an even π-periodic continuous function on $\mathbb{R}$, and let H be the C^2 function on $[0,\pi]$ satisfying the conditions $H''(x) = f(x)\cdot\sin(x)$, $x \in [0,\pi]$, and $H(0) = H'(0) = 0$. Then one has*

$$\int_{G_i \cap K \neq \emptyset} f(\varphi(G_2) - \varphi(G_1))\, dG_1 dG_2 = 2H(\pi)F + 2\int_{P \notin K} H(\omega)\, dP. \tag{20}$$

Using this Proposition we give the announced interpretation of the following formulas

Crofton's formula

Taking $H(x) = x - \sin(x)$ it follows that $f = 1$ in Proposition 6 and since $H(\pi) = \pi$ using (20) we get

Proposition 7 *The following equality holds.*

$$\int_{G_i \cap K \neq \emptyset} dG_1 dG_2 = 2\pi F + 2\int_{P \notin K} (\omega - \sin\omega)\, dP.$$

Masotti's formula

Taking $H(x) = x^2 - \sin^2(x)$ one gets $H''(x)/\sin(x) = 4\sin(x)$. So the function $f(x) = 4|\sin(x)|$, $x \in \mathbb{R}$, satisfies the hypothesis of Proposition 6 and equation (20) gives

Proposition 8 *The following equality holds*

$$2\int_{G_i \cap K \neq \emptyset} |\sin(\varphi(G_2) - \varphi(G_1))| dG_1 dG_2 = \pi^2 F + \int_{P \notin K} (\omega^2 - \sin^2\omega)\, dP.$$

Powers of sine formula

Finally, in an analogous way we can interpret the integral of any power of the sine of the visual angle. Effectively for $H(x) = \sin^m(x)$ it follows that

$$H''(x)/\sin(x) = m(m-1)\sin^{m-3}(x) - m^2 \sin^{m-1}(x).$$

So taking $f(x) = m(m-1)|\sin^{n-3}(x)| - m^2|\sin^{m-1}(x)|$ the hypotheses of Proposition 6 are satisfied and by (20) we have

Proposition 9 *The following equality holds*

$$2\int_{P\notin K} \sin^m(\omega)\, dP =$$
$$= \int_{G_i\cap K\neq\emptyset} \Big(m(m-1)|\sin^{m-3}(\varphi(G_2)-\varphi(G_1))| - m^2|\sin^{m-1}(\varphi(G_2)-\varphi(G_1))|\Big)\, dG_1 dG_2.$$

An interesting consequence of our results is that when f is a π-periodic function, according to Proposition 6, the integral $\int_{G_i\cap K\neq\emptyset} f(\varphi(G_2)-\varphi(G_1))\, dG_1 dG_2$ is a linear combination of integrals extended outside K of the functions of the visual angle $H_k(\omega)$, where

$$H_k(x) = \frac{1}{2(k^2-1)}\left(f_k(x) + 2(\sin x - x)\right), \quad k \geq 2,$$

and $H_1(x) = (1/8)(2x - \sin(2x))$, $f_k(\omega)$ being the functions of Hurwitz given by

$$f_k(\omega) = -2\sin\omega + \frac{k+1}{k-1}\sin((k-1)\omega) - \frac{k-1}{k+1}\sin((k+1)\omega), \quad k \geq 2.$$

Summarizing, it appears that the functions of Crofton and Hurwitz are some kind of *basis* for the integral of any π-periodic function with respect to the measure $dG_1 \wedge dG_2$ over the set of pairs of lines meeting a given compact convex set.

5 Integral formulas for the dihedral angle

So far we have been working with pairs of lines in the plane. From now on we will consider pairs of planes in the space. The aim of this section is to write the integral of an isometry-invariant measure over the pairs of planes meeting a convex set K as an integral of an appropriate function of the *dihedral visual angle*.

5.1 Invariant measures in the set of ordered pairs of planes in the space

We consider measures in the space $\mathcal{A}_{3,2} \times \mathcal{A}_{3,2}$ of pairs of planes in $\mathbb{E}^3$ of the form $m_{\tilde f} := \tilde f(E_1, E_2) dE_1 \wedge dE_2$ with dE_i the normalized isometry-invariant measures in $\mathcal{A}_{3,2}$ as considered in [12]. We want to study which functions $\tilde f$ give an isometry-invariant measure, that is a measure $m_{\tilde f}$ satisfying $m_{\tilde f}(B) = m_{\tilde f}(gB)$ for every euclidean motion g. For instance, it is known that for a given compact convex set K one has $\int_{E\cap K\neq\emptyset} dE = M$. So when $\tilde f(E_1, E_2) = 1$ we have

$$\int_{K\cap E_i\neq\emptyset} dE_1 dE_2 = M^2 = 4\pi^2\mathcal{W}^2, \tag{21}$$

where M and $\mathcal{W}$ are the mean curvature and the mean width of K, respectively. A first result in this direction is

Proposition 10 *The measure given by $\tilde{f}(E_1, E_2)dE_1 \wedge dE_2$ in $\mathcal{A}_{3,2} \times \mathcal{A}_{3,2}$ is invariant under isometries of $\mathbb{E}^3$ if and only if $\tilde{f}(E_1, E_2) = f(\langle u_1, u_2 \rangle)$ where $\pi(E_i)^\perp = \operatorname{span}\{u_i\}$, $i = 1, 2$ and $f : [-1, 1] \to \mathbb{R}$ is an even measurable function.*

Let K be a compact convex set in the euclidean space $\mathbb{E}^3$. According to equality (21) it is a natural question to evaluate

$$\int_{E_i \cap K \neq \emptyset} \tilde{f}(E_1, E_2) dE_1 dE_2,$$

where $\tilde{f}(E_1, E_2)dE_1 \wedge dE_2$ is an isometry-invariant measure on $\mathcal{A}_{3,2} \times \mathcal{A}_{3,2}$. This can be done in terms of the coefficients of the expansion of the support function of K in spherical harmonics and the coefficients of the Legendre series of the measurable even function $f : [-1, 1] \to \mathbb{R}$ such that $\tilde{f}(E_1, E_2) = f(\langle u_1, u_2 \rangle)$.

The following result is a special case, with a different notation, of Theorem 5 in [9], whose proof is based on the Funk-Hecke Theorem ([8, p. 98]).

Theorem 8 *Let K be a compact convex set with support function p given in terms of spherical harmonics by (7). Let $\tilde{f}(E_1, E_2)dE_1\, dE_2$ be an isometry-invariant measure on $\mathcal{A}_{3,2} \times \mathcal{A}_{3,2}$ and $f : [-1, 1] \to \mathbb{R}$ an even measurable function such that $\tilde{f}(E_1, E_2) = f(\langle u_1, u_2 \rangle)$ where $\pi(E_i)^\perp = \operatorname{span}\{u_i\}$, $i = 1, 2$. Then*

$$\int_{E_i \cap K \neq \emptyset} \tilde{f}(E_1, E_2) dE_1\, dE_2 = \frac{\lambda_0}{4\pi} M^2 + \sum_{n=1}^{\infty} \lambda_{2n} \|\pi_{2n}(p)\|^2, \tag{22}$$

where $\lambda_{2n} = 2\pi \int_{-1}^{1} f(t) P_{2n}(t)\, dt$ with P_{2n} the Legendre polynomial of degree $2n$.

Recall that when K is a convex set of constant width $\mathcal{W}$ one has $\pi_n(p) = 0$ for $n \neq 0$ even. Therefore, in this case,

$$\int_{E_i \cap K \neq \emptyset} f(\langle u_1, u_2 \rangle)\, dE_1\, dE_2 = \frac{\lambda_0}{4\pi} M^2 = \lambda_0 \pi \mathcal{W}^2.$$

The measure $dE_1\, dE_2$ in the space of pairs of planes in $\mathbb{E}^3$ can be written according to Santaló (cf. [12], section II.12.6) as

$$dE_1 \wedge dE_2 = \sin^2(\alpha_2 - \alpha_1)\, d\alpha_1\, d\alpha_2\, dG, \tag{23}$$

where α_i are the angles of E_i about G.

Introducing the *visual dihedral angle* of a convex set K from a line G not meeting K as the angle $\omega = \omega(G)$, $0 \leq \omega \leq \pi$, between the half-planes through G tangent to K and using (23) we can prove the following

Theorem 9 ([6])

Let K be a compact convex set and let $f : [-1,1] \longrightarrow \mathbb{R}$ be an even continuous function. Let H be the C^2 function on $[-\pi,\pi]$ satisfying

$$H''(x) = f(\cos(x))\sin^2(x), \quad -\pi \leq x \leq \pi, \quad H(0) = H'(0) = 0.$$

Then

$$\int_{E_i \cap K \neq \emptyset} f(\langle u_1, u_2 \rangle)\, dE_1\, dE_2 = \pi H(\pi) F + 2 \int_{G \cap K = \emptyset} H(\omega)\, dG. \tag{24}$$

where u_i are unit normal vectors to the planes E_i, $\omega = \omega(G)$ is the dihedral visual angle from the line G and F is the area of the boundary of K.

5.2 Crofton's formula in the space

In Blaschke's work [1, p. 75] the following Crofton-Herglotz formula is given

$$\int_{G \cap K = \emptyset} (\omega^2 - \sin^2 \omega)\, dG = 2M^2 - \frac{\pi^3 F}{2}. \tag{25}$$

We can easily recover (25) from Theorem 9. In fact considering $f(t) = 1$ one gets $H(x) = (x^2 - \sin^2 x)/4$ and equality (24) gives

$$M^2 = \int_{E_i \cap K \neq \emptyset} dE_1\, dE_2 = \frac{1}{4}\pi^3 F + \frac{1}{2} \int_{G \cap K = \emptyset} (\omega^2 - \sin^2 \omega)\, dG.$$

Formula (25) reveals the significance of the function of the dihedral visual angle $\omega^2 - \sin^2 \omega$. One can ask what role the function $\omega - \sin \omega$ does it play; this function, interpreting ω as the visual angle in the plane, is significant thanks to Crofton's formula (2).

In [1, p. 85] Blaschke shows that

$$\int_{G \cap K = \emptyset} (\omega - \sin \omega) dG = \frac{1}{4} \int_{u \in S^2} L_u^2 du - \frac{\pi^2}{2} F, \tag{26}$$

where L_u is the length of the boundary of the projection of K on $\mathrm{span}\{u\}^\perp$.

It can be easily seen that $\int_{u \in S^2} L_u du = 2\pi M$ and from this equality, applying Schwarz's inequality, one gets

$$\int_{u \in S^2} L_u^2 du \geq \pi M^2. \tag{27}$$

Introducing (27) into (26) one obtains

$$\int_{G\cap K=\emptyset}(\omega-\sin\omega)dG \geq \frac{\pi}{4}(M^2-2\pi F), \tag{28}$$

an inequality given by Blaschke.

As a consequence of Theorem 9 we can now evaluate the deficit in both inequalities (27) and (28).

Theorem 10 ([6])

Let K be a compact convex set with support function p, area of its boundary F and mean curvature M. Let L_u be the length of the boundary of the projection of K on $\mathrm{span}\{u\}^{\perp}$ *and let $\omega=\omega(G)$ be the dihedral visual angle of K from the line G. Then*

1. $\displaystyle\int_{u\in S^2} L_u^2 du = \pi M^2 + 4\pi\sum_{n=1}^{\infty}\frac{\Gamma(n+1/2)^2}{\Gamma(n+1)^2}\|\pi_{2n}(p)\|^2,$

2. $\displaystyle\int_{G\cap K=\emptyset}(\omega-\sin\omega)dG = \frac{\pi}{4}(M^2-2\pi F)+\pi\sum_{n=1}^{\infty}\frac{\Gamma(n+1/2)^2}{\Gamma(n+1)^2}\|\pi_{2n}(p)\|^2,$

whith $\pi_{2n}(p)$ the projection of the support function p of K on the vector space of spherical harmonics of degree $2n$.

Moreover equality holds both in (27) *and* (28) *if and only if K is of constant width.*

Proof We consider $f(t)=1/\sqrt{1-t^2}$. For this function the corresponding H in Theorem 9 is $H(x)=|x|-|\sin x|$. Applying equality (24) and Theorem 8 with the corresponding λ_{2n}'s given by

$$\lambda_{2n} = 2\pi\int_{-1}^{1} f(t)P_{2n}(t)dt = 2\pi\frac{\Gamma(n+1/2)^2}{\Gamma(n+1)^2}$$

(cf. [7], 7.226), item *2* follows. Equality in item *1* is a consequence of item *2* and (26).

The statement about equality in (27) and (28) is a consequence of the fact that K is of constant width if and only if $\pi_{2n}(p)=0$ for $n\neq 0$. □

In [6, Proposition 6.2] we decompose the integral of an inavriant measure of pairs of planes as

$$\int_{E_i\cap K\neq\emptyset} f(\langle u_1,u_2\rangle)\, dE_1\, dE_2 = a_0M^2+\sum_{m=2}^{\infty}a_{2m}\int_{G\cap K=\emptyset}\sin^{2m}\omega\, dG,$$

where u_i are normal vectors to the planes E_i, ω the dihedral visual angle from the line G, F denotes the area of the boundary of K and certain coefficients a_{2n} depending on f. From this (cf. [6, Proposition 6.3]) one sees that

$$\int_{E_i\cap K\neq\emptyset} f(\langle u_1,u_2\rangle)\, dE_1\, dE_2 = a_0M^2+\sum_{m=2}^{\infty}a_{2m}\int_{E_i\cap K\neq\emptyset}h_m(\langle u_1,u_2\rangle)\, dE_1\, dE_2,$$

where

$$h_m(t) = m(2mt^2 - 1)(1 - t^2)^{m-2}, \ m > 1.$$

So we have exhibited a simple family of polynomial functions that are in some sense a basis for the integrals in Theorem 8. In fact every invariant integral can be written as an infinite linear combination of the integrals of the invariant measures given by the polynomials h_m.

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