Designing a workshop on noticing aspects of mathematical discourse for teaching angles

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This study is part of a broader project aimed at researching mathematics teachers' noticing of mathematical discourse aspects in content teaching. This report aims to present and discuss the design of a two-hour workshop for secondary-school mathematics teachers on noticing three aspects of mathematical discourse for teaching angles. We consider elements of the Mathematical Discourse in Instruction (MDI) framework, teachers' noticing processes of identifying, interpreting, and deciding, and students' learning challenges. The workshop includes introductory documents on students' learning challenges about angles, and on mathematical vocabulary, mathematical explanations, and graphical examples for teaching angles. We also design a document that includes professional tasks for identifying, interpreting, and deciding on those aspects of the mathematical discourse in secondary-school mathematics teaching that support students' understanding of angles.

Keywords: Secondary-school mathematics teachers, noticing processes, mathematical vocabulary, mathematical explanations, graphical examples, teaching of angles.

Introduction

Recent research on mathematical discourse in content teaching and in mathematics teacher education has focused on the study of *naming* and *explaining* (Planas, 2021), *explaining* and *exemplification* (González-Forte et al., 2022) and *exemplification* and *diagrams use* (Adler & Pournara, 2020; Ratnayake et al., 2022; Mwadzaangati et al., 2022). Mathematics teachers generally use these aspects, amongst and in interaction with others, in their mathematics teaching. In the ongoing PhD investigation of the first author, the pervasiveness of these aspects and their didactic potential in teaching are assumed (Planas, 2021). In all this, mathematical vocabulary and mathematical explanations, in the context of verbal language, have a communicative nature other than diagrams and exemplification that mixes linguistic and graphical aspects.

Regarding work with mathematics teachers, Planas et al. (2022) claim that word names into/and explanatory sentences "in teacher talk can prevent or diminish learning challenges shown to be persistent across school ages, individual learners and classroom settings" (p. 318). In the context of pre-service mathematics teachers' education, González-Forte et al. (2022) examine variation as a practical dimension of exemplification, which is understood as "giving sentences with encoded variations of content-related elements oriented towards reducing content learning challenges" (p. 317). Adler and Pournara (2020) write that the "example set informed by principles of variation is a core component of our professional development work with teachers" (p. 329). Ratnayake et al. (2022) state that examples and graphical representations are inextricably linked in the teaching of geometry. Their proposal on the use of examples in teaching relates to the use of diagrams. We will refer to this aspect as graphical examples. Those authors complement this use of examples with explanatory communication in the *Mathematical Discourse in Instruction* (MDI) framework (Adler & Ronda, 2015), and this is an inspiring idea for the PhD work reported in this paper.

In-service teachers training on noticing aspects of mathematical discourse and on mathematical explanations with mathematical vocabulary and graphical examples are not common in our institutional context of teacher education. We have thus designed a two-hour workshop for secondary-school mathematics teachers on noticing these aspects of mathematical discourse. The research question addressed in this paper is as follows: *How can a two-hour workshop on noticing aspects of mathematical discourse for teaching angles be designed for secondary-school mathematics teachers?* In the next section, we introduce the theoretical framework and then present and discuss the workshop's design. We finally discuss an answer to the research question.

Theoretical framework

Aspects of mathematical discourse in teaching

Adler and Ronda (2015) drew on the MDI framework to highlight aspects that mediate mathematics teaching. This framework is based on Vygotskian sociocultural stances and has been empirically developed with inspiration from mathematics teaching practices. The later operationalisation of MDI (called *Mathematics Teaching Framework* (MTF)) is aimed at working with mathematics teachers to facilitate reflection on mathematics teaching practices. These frameworks include four interacting components: *exemplification*, *explanatory talk* (explanatory communication in MTF), *learner participation*, and the *object of learning* (lesson goal in MTF).

Exemplification and explanatory talk are closely interrelated, and examples are at the heart of explanatory talk; specifically, an example is "a particular case of a larger class from which one can generalize" (Zodik & Zaslavsky, 2008, p. 165, as cited in Adler & Ronda, 2015, p. 239). However, examples are not necessarily enough in terms of accomplishing explaining in teaching. Explanatory talk consists of naming and legitimating "what is focused on and talked about, that is, related examples and task" (p. 241). These components have been reinterpreted across mathematical domains in subsequent MDI-inspired work. Mwadzaangati et al. (2022) address exemplification that includes examples (geometric figures and their attributes), tasks and representations. Ratnayake et al. (2022) focus on exemplification and diagrams as a way of accessing and describing teachers' knowledge. In her account of explaining, Planas (2021) draws on the Hallidayan idea of lexical meaning with which certain linguistic forms are used in classroom talk. She speaks of lexicalisation as the elaboration and use of sentences with the potential to make mathematical meanings precise in content teaching.

In the PhD investigation, *mathematical vocabulary* is understood as the set of words (or phrases) in a mathematical register used to name objects, symbols, properties, procedures, and practices in mathematics. By "mathematics register", we refer to "the meanings that belong to the language of mathematics (…), and that a language must express if it is being used for mathematical purposes" (Halliday, 1978, p. 195). This understanding is close to Planas et al. (2022), where naming consists of giving "a word name from mathematical content registers" (p. 317), and to the naming in the MDI's explanatory talk. Like in González-Forte et al. (2022), *mathematical explanation* refers to a practical dimension of lexicalisation, and is viewed as lexical elaboration consisting of sentences with mathematical vocabulary that communicate mathematical meanings and relationships between meanings. The value of explanation lies in the possibility of favouring the communication of meanings, as is the case of explanatory talk in the MDI. We understand *graphical examples* in the

same way as Mwadzaangati et al. (2022) understand diagrams: "in geometry, a diagram can be viewed as both an example and a representation" (p. 220). In graphical examples, the relationship between an example and its graphical representation is emphasised. These authors indicate that patterns of variation are required in geometry and refer to dimensions of variation and range of change. In the current study, the variations particularly include the connection of examples of angles to everyday situations, and the representation of non-prototypical examples of angles.

In this report, mathematical vocabulary, mathematical explanations, and graphical examples are inspired by the MDI, specifically in relation to naming, explanatory talk and representations, respectively. In the PhD investigation, these three aspects of mathematical discourse are defined by considering some theoretical aspects of other MDI-inspired works. Mathematical explanations can integrate mathematical vocabulary and graphical examples to support students' understanding of mathematical content. These aspects in teaching are essential *per se* as well as to each other. In this regard, Schleppegrell (2007) points out that "the written language, the mathematics symbolic statements, the visual representation, and the oral language work together to construct meaning as the teacher and students interact in discussing the problem" (p. 142).

Teachers' noticing processes of mathematical discourse aspects for teaching

Van Es and Sherin (2002) proposed a *Learning to Notice* framework with three noticing processes: 1) identifying important events in a classroom situation; 2) using contextual knowledge to reason about these events; 3) making connections between these events and related principles of teaching and learning. These authors suggest that it is crucial for teachers to notice relevant situations in the classroom to improve their teaching practice and support students' learning. We focus on aspects of mathematical discourse in teaching and on how teachers identify these aspects, interpret them, and decide on forms of change and improvement under specific circumstances.

In the PhD investigation, the work with mathematics teachers starts with considering students' challenges in the learning of angles to identify, interpret and decide on aspects of mathematical discourse in teaching aimed at overcoming these challenges. Specialised literature on challenges of secondary-school students, when learning angles, indicates that many students tend to understand angles as static objects (Mitchelmore & White, 2000), and to confuse the angle with its measure, and even with the unit of measure (Tanguay & Venant, 2016). For mathematics teaching to encourage the mathematical communication of angles as dynamic and with non-measurable properties, the three processes of noticing considered are as follows: 1) identifying mathematical explanations with mathematical vocabulary and graphical examples in the teaching of angles, 2) interpreting the potential effects of these aspects of the mathematical discourse on the students' learning of angles, and 3) deciding on mathematical explanations with mathematical vocabulary and graphical examples with the potential to overcome such challenges in the students' learning of angles. We wanted teachers in the workshop to identify mathematical vocabulary, mathematical explanations, and graphical examples in fictional situations of teaching angles; to interpret how all these aspects of mathematical discourse together can mediate the learning of angles; and to decide on mathematical explanations with mathematical vocabulary and graphical examples to support the teaching of angles.

Designing the workshop

In this report, the focus is on the design of professional tasks for a mathematics teachers' workshop, in which aspects of the mathematical discourse in the teaching of angles are the objects of noticing. Ruthven (2015) explains that a task design framework should address both, the task and its acting. He suggests that the design should include a selection of the following elements: 1) a template for phasing the task activity, 2) criteria for devising a productive task, 3) ways of organising the task environment, and 4) guidance for managing key task variables. In our workshop, these elements are interrelated because we use documents to guide the phasing and organising of the tasks. The criteria for devising a productive task and the guidance for managing key task variables relate to the analysis of the mathematical discourse aspects within the noticing processes.

A two-hour workshop was designed and carried out based on documents. The language used for the design of the documents was Catalan, but Spanish and Catalan were used for the implementation of the workshop and the discussion of the documents and tasks. The design process was expected to involve a small group of up to five secondary-school mathematics teachers in Barcelona. The documents were based on research in mathematics education, including literature on challenges in learning angles and on mathematical discourse aspects in teaching. These documents include professional tasks to notice aspects of mathematical discourse in fictional situations of teaching angles. The first author will present two introductory documents and a document with professional tasks for the teachers to be done individually and then in a group. The introductory documents illustrate: the challenges of angles as dynamic (Mitchelmore & White, 2000) and as including more than measure (Tanguay & Venant, 2016), as well as mathematical vocabulary, mathematical explanations, and graphical examples for the teaching of angles. The challenges considered for the design of the workshop are based on the mathematics education literature. Teachers engaged in the professional tasks to notice mathematical explanations with mathematical vocabulary and graphical examples. The tasks included questions to enact processes of identifying, interpreting and deciding on aspects of mathematical discourse for teaching angles (van Es & Sherin, 2002). Participation in these tasks should allow teachers to notice aspects of mathematical discourse that seemingly evoke common teaching and teaching talk in their classrooms.

In the broader project, other workshops have been conducted and are in process with different groups of secondary-school teachers, teacher educators and mathematical contents. A first significant difference with some other workshops is that, in these workshops the teaching and students' data presented in the professional tasks are from former projects in school classrooms, and hence the mathematical explanations are not fictional, and the learning challenges are not only research-based. A second significant difference is the exclusive focus on mathematical-linguistic practices of naming and explaining of the other workshops, compared to the interest in a broader view of mathematical discourse by including exemplifying and graphical examples in the current workshop. These differences are in part justified by the conditions for carrying out the workshops (e.g., the second author used real classroom data in her workshops because she had been working with this data in past projects), and by the place of diagrams and graphical representations in the teaching of geometry and in relation to the examples and exemplification in the MDI. The first author decided to carry out a study that would include graphical aspects as a complement to the linguistic aspects of the project.

The introductory documents

The introductory documents were designed to illustrate aspects of teaching and learning angles in secondary-school mathematics. Document 1 relates to student learning and presents two challenges from the literature. It also includes languages of students that can suggest biased reasoning. Some of the languages of the students who refer to the angle as static include, e.g., "An angle is formed by the intersection of two straight lines", and "An angle is the space between two rays with the same origin". Students who tend to communicate their thinking about angles as static, and in general all students, can benefit from teaching that communicates angles as static and dynamic.

Document 2 illustrates mathematical vocabulary, mathematical explanations, and graphical examples that can be used in the teaching of angles to support students' learning. This document has four parts. The first part presents mathematical vocabulary for the teaching of angles (e.g., words such as angle or rotation, and phrases such as angular amplitude or dynamic angle). The second part presents mathematical explanations with examples and the potential to support learning angles as dynamic (e.g., "An angle is the result of a turn or rotation movement; for example, the handles of the clock demarcate angles when turning"), as non-metric inclination (e.g., "Angles also indicate the inclination of an object with respect to another; for example, the inclination of the Pisa Tower with respect to the surface"), or as dynamic and non-metric (e.g., "The angle width formed by a light ray and its reflection on a mirror varies if you vary the position of the ray source"). The third part presents graphical examples connected with the mathematical explanations and representing angles in everyday situations, with diagrams of a clock or the Pisa Tower (see Figure 1). These representations include mathematical symbols to denote arcs or segments.

Figure 1: English version of extracts of Document 2

The last part of the Document 2 presents a fictional dialogue between a teacher and two students starting with the question, "What do crossroads and the handles of a clock have in common?". The dialogue shows mathematical vocabulary, mathematical explanations, and graphical examples of angles in teaching. The following is an English version of the last turns of the dialogue, with mathematical vocabulary in bold and explanations underlined:

23 Teacher: Un **angle** també es pot determinar a partir de **moviments de rotació**
*An **angle** can also be determined from **rotational movements***
o **girs**. En obrir una porta, **girem** la maneta i anem veient **angles**.
*or **turns**. When we open a door, we **turn** the handle and keep seeing **angles.***

24 Student: Aleshores, construïm angles amb semirectes que s'intersequen i amb girs?
So, do we construct angles with intersecting rays and with turns?

25 Teacher: Sí. És important comprendre que hi ha **angles estàtics**, per exemple,
*Yes, it is important to understand that there are **static angles**, for example*
als carrers que s'intersequen, i **angles dinàmic**, en els girs de

*in the crossroads, and **dynamic angles**, in the **turns** of the*
la maneta de la porta i de les manetes del rellotge. Podem parlar de tots
door handle and in the handles of the clock. We can talk about all
aquests angles sense mesurar la seva amplitud.
*these **angles** without **measuring** their **amplitude**.*

The document with the professional tasks

In the second part of the workshop, two professional tasks were proposed about the uses of mathematical explanations with mathematical vocabulary and graphical examples in the teaching of angles. The first task is related to a question of a student to her teacher after reading a definition and a classification of angles in the textbook. The textbook extract is provided alongside the question, "If an angle is the portion of the plane between two straight lines, why do we take a different portion in the concave angle? The outer one". Embedded in the question is the meaning of "between" in the everyday register and compared to the meaning in the mathematical register of school plane geometry. Figure 2 shows an English version of an extract of the first task.

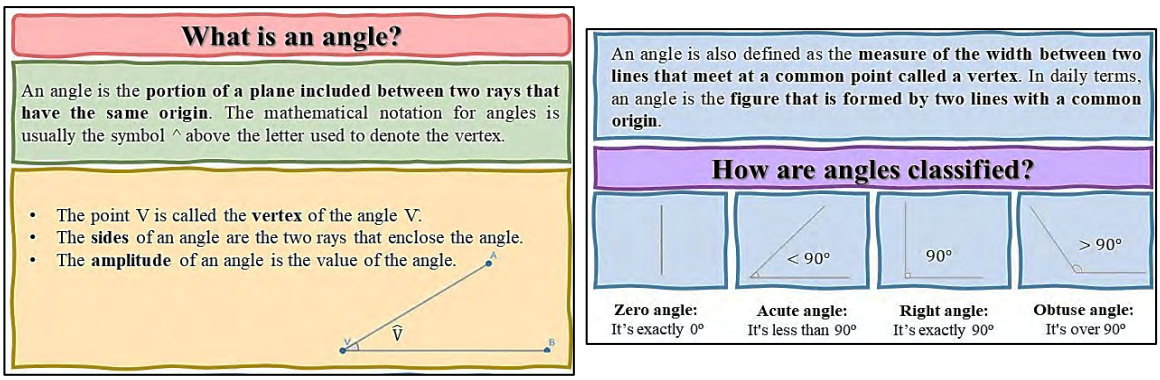

Figure 2: English version of an extract of the first task

The second task is a fictional dialogue between the teacher and two students. This dialogue begins with the presentation of a definition of an angle as: "the figure formed by two straight lines that have in common their origin and are located on different straight lines". Embedded in this definition is the contrast with the angle whose measure is zero. In the simulated teaching, the mathematical explanations and graphical examples do not particularly support the student's wider learning of angles since the communicated meanings are limited to static and metric properties. In the design, we used ambiguous words, redundant or unclear explanations and prototypical graphical examples, and we did not mark mathematical vocabulary and explanations because we wanted the teachers to notice them. Below, we reproduce the final turns of the dialogue with everyday or ambiguous words (e.g., stripe), an unclear explanation that does not suggest the angle as rotation or turn, and a graphical example that is limited with respect to clarifying the student's understanding.

14 Teacher: Per què dius que no hi hauria angle?
Why do you say there would be no angle?

15 Student: Perquè no hi ha dues rectes diferents.
Because there are not two different straight lines.

16 Teacher: L'angle de zero graus està format per dues rectes, però una està a sobre
The zero-degree angle is formed by two stripes, but one is above
de l'altra. Fixeu-vos en aquest dibuix de l'angle de zero graus.
the other. Look at this drawing of the zero-degree angle.

Teachers in the workshop were asked questions for prompting noticing (van Es & Sherin, 2002), e.g., (*Identifying*) What mathematical vocabulary, mathematical explanations, and graphical examples are provided to the student in teaching? (*Interpreting*) What effects could all these aspects of mathematical discourse have on the students' learning of angles? (*Deciding)* What mathematical explanations with mathematical vocabulary and graphical examples could be used in teaching to respond to the student?

Approaching an answer to the research question

In this final section, we approach an answer to the research question: *How can a two-hour workshop on noticing aspects of mathematical discourse for teaching angles be designed for secondary-school mathematics teachers?* A workshop on noticing aspects of mathematical discourse for secondary-school mathematics teachers can be designed from the interrelationship of some of the elements suggested by Ruthven (2015). The structure and phasing of the documents are fundamental elements in the design and implementation of the workshop. The distinction between introductory documents and documents with professional tasks makes it possible to define the role of the teachers and teacher educators in the different phases of the workshop. The introductory documents serve as a lens to facilitate teachers' noticing when solving the professional tasks, and the tasks with fictional situations and questions that ask them to identify, interpret and decide on aspects of mathematical discourse in the classroom are essential to the aims of the PhD investigation. The review of theoretical frameworks (MDI, noticing processes and learning challenges) favours the design of productive tasks. For the workshop discussed in this report, the consideration of graphical examples and the grounding in fictional data are important variations from these other workshops. These variations are reflected at the level of design of professional tasks and bring differences in the implementation thinking. Finally, the implementation of the workshop requires the management of different variables, as discussed in the presentation of the documents.

The PhD investigation is currently in the data analysis phase. The implementation was carried out with a group of five secondary-school mathematics teachers from Barcelona. The data consisted of written records of individual answers and audio recordings of group discussions. Content and discourse analysis tools are being used to characterise the processes of noticing aspects of mathematical discourse in teaching angles. We hope to share the results of this analysis and research soon.

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