Chapter 1
Topology of singular foliation germs in $\mathbb{C}^2$

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Abstract In this article we give an overview on the topology of singularities of holomorphic foliation germs in $\mathbb{C}^2$. We describe several results of the authors on the topology of the leaves and the structure of the leaf space. We state criteria of topological conjugacy for any two foliation germs. These are based on the key notion of monodromy of a singular foliation, a topological invariant of geometric and dynamic nature. After a historical introduction, we focus on the simplest invariant sets (separatrices, separators and dynamical components) and we compare them to geometric blocks classical in the study of the topology of 3-dimensional manifolds. Subsequently, we introduce the notion of foliated connectedness, used in proving the incompressibility property of the leaves of the foliation, which plays a crucial role in the definition of the monodromy. We describe the ideas of the proofs of the main theorems leading to the topological classification of generic foliations that are generalized curves. Finally, we give an algebraic description of topological moduli spaces and we state the existence of complete families, with minimal redundancy

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given by an explicit action of a countable group on the finite dimensional parameter space.

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Contents

1.1 Introduction

The object of this survey is to give an overview on the topology of singularities of holomorphic *foliation germs* on $(\mathbb{C}^2, 0)$. Such a foliation germ $\mathcal{F}$ can be defined by a differential form $\omega = a(x, y)dx + b(x, y)dy$ or equivalently by a vector field $X = b(x, y)\partial_x - a(x, y)\partial_y$, with $a(x, y)$ and $b(x, y)$ elements of the ring $\mathcal{O}_{\mathbb{C}^2,0}$ of germs of holomorphic functions vanishing at the origin of $\mathbb{C}^2$, without common divisor. A *leaf* of the foliation $\mathcal{F}_U$ that represents the germ $\mathcal{F}$ on an open neighborhood U of 0 on which $a(x, y)$ and $b(x, y)$ are defined and holomorphic, is a maximal immersed Riemann surface in $U \setminus \{0\}$ that is solution of the equation $\omega = 0$, or equivalently that is tangent to X. Two foliation germs $\mathcal{F}$ and $\mathcal{G}$ are *topologically equivalent* (or *C^0-conjugated*) if there is a homeomorphism between two open neighborhoods U and V of 0 that sends any leaf of $\mathcal{F}_U$ to a leaf of $\mathcal{G}_V$. This notion is an equivalence relation on the set of all foliation germs, and the final objective of a topological classification would be to obtain a list of foliation germs containing an element of each topological class, with minimal redundancy.

Special leaves L of $\mathcal{F}_U$ are those such that $L \cup \{0\}$ is a closed analytic curve in U; they are topologically characterized by the property of being closed in $U \setminus \{0\}$. However for the germ $\mathcal{F}$ this notion takes a rigorous sense only as germ of curve at 0, called *separatrix* of $\mathcal{F}$. Indeed there are germs $\mathcal{F}$ for which any leaf of $\mathcal{F}_U$ contains a separatrix (as punctured curve germ) and there are leaves of $\mathcal{F}_U$ containing infinitely many separatrices, see Remark 1.2.10 and Figure 1.4.

We begin with a historical approach to the topological study of singularities of foliation germs. The first result on the topology of separatrices is the famous paper [1] (1856) of C. Briot and J.C. Bouquet where the authors, after introducing the notion of Newton's polygon of a foliation germ, establish an algorithm giving the Puiseux pairs of any separatrix, thus determining their topological type. Later A. Seidenberg establishes in [40] (1968) that for a given foliation germ, the number of topological types of separatrices is finite. For this he proves that after a finite number of blowing-ups over the origin, the foliation is locally defined, at each singular point on the exceptional divisor, by a vector field whose linear part is diagonalizable with a non-zero eigenvalue. After supplementary blowing-ups, each singular point becomes *reduced*, i.e. the quotient of eigenvalues is not a positive rational number. A simple proof of this result is obtained by A. van den Essen [44], [27, Appendice I] (1979) and F. Cano [2] (1986). The existence of a separatrix is established in 1982 by C. Camacho and P. Sad in [7]. Alternative proofs of this result are given later by J. Cano [3] (1997), M. Toma [43] (1999).

The simplest reduced foliation germs are given by 1-differential forms ω with linear part $\omega_L = \lambda_1 y dx + \lambda_2 x dy$, with $\lambda_1, \lambda_2 \in \mathbb{C}^*$ and $\lambda_1/\lambda_2 \notin \mathbb{R}$. They are studied by H. Poincaré and H. Dulac [10] (1904) proving the existence of local holomorphic coordinates for which $\omega = \omega_L$, i.e. $\mathcal{F}$ is *linearizable*. All these foliation germs belong to the same topological class, see §1.5.1.1. The topological classification of reduced foliation germs given by a 1-form ω with linear part $pydx + qxdy$ with $p, q \in \mathbb{N}^*$, called *resonant saddles*, is made by C. Camacho and P. Sad in [6] (1982), who obtain a countable set of topological classes and show that the topological type is determined by a finite jet of ω. The remaining reduced foliation germs are the *saddle-nodes*, that are given by a differential form ω with $\omega_L = ydx$, whose topological classification is made by P.M. Elizarov in [11] (1990), and the non-linearizable foliation germs given by ω with $\omega_L = \lambda_1 ydx + \lambda_2 xdy$ and $\lambda_1/\lambda_2 \in \mathbb{R} \setminus \mathbb{Q}$. In this last case the foliation germs which are linearizable form a topological class, and the classification problem of those which are non-linearizable is equivalent to that of the topological classification of Cremer biholomorphisms of one variable.

For non-reduced foliation germs, C. Camacho, A. L. Neto and P. Sad highlight in [4] (1984) an important class of foliation germs: the so called *generalized curves* for which no saddle-node singularity appear after reduction. They prove the topological invariance of the *Milnor number* of any foliation germ (defined as the complex codimension of the ideal of $\mathcal{O}_{\mathbb{C}^2,0}$ generated by the coefficients of the 1-form with isolated singularity defining the foliation germ). Using this invariant they characterize topologically generalized curves. They also prove that generalized curves have same reduction as their separatrices, but this topological property does not characterize them [29] (2004). Few topological results are known for non-generalized curves, see §1.4.2.

After these results and in the 1990's the study of the topology of non-reduced foliation germs was "local" in the sense of a search for invariants under "small C^0-trivial deformations" defined by families of germs of differential forms parametrized by a germ of space. An important step in this study is the topological rigidity property of non solvable groups of biholomorphism germs in one variable proved by A.A. Shcherbakov [41] (1984) and completed by I. Nakai [31] (1994). This one makes it possible to put new (generic) assumptions on the differential form defining a foliation germ (the non-solvability of some holonomy groups), that induce transverse holomorphy properties on any homeomorphism conjugating such foliations germs, see §1.6.4. Under rigidity assumptions D. Cerveau and P. Sad prove in [8] (1986) that the holonomy of the exceptional divisor is invariant under C^0-trivial deformation for generic foliation germs reduced by a single blowing-up; they also conjecture that this holds for any pair of topologically conjugated generalized curves not necessarily contained in a C^0-trivial deformation. The classification of generic logarithmic foliations established by E. Paul in [34] (1989) shows that the rigidity assumption is necessary, see Theorem 1.6.7. In fact, it is not sufficient to consider the holonomies to obtain a complete topological classification. Other continuous topological invariants under C^0-trivial deformations exist and are completely specified by two of us in [28] (1997) for generalized curves satisfying some additional assumptions.

The object of this paper is to describe, giving ideas of the proofs, results obtained by the authors on the topology of leaves, the structure of leaf space and criteria of conjugacy for any two foliation germs not necessarily contained in a C^0-trivial deformation. They are based on the construction of the monodromy, a new topological invariant of geometric and dynamical nature. All these results lead to prove, for two generalized curves satisfying some generic assumptions, that the existence of a conjugating homeomorphism between neighborhoods of the origin of $\mathbb{C}^2$ is equivalent to the existence of a conjugating homeomorphism between neighborhoods of the exceptional divisors after reduction of singularities. Clearly this property (Excellence Theorem 1.7.1) gives a positive answer to the Cerveau-Sad conjecture. It is also the starting point of a series of three works [23, 24, 25] (2020-2022) that enables us to carry out the topological classification of generic generalized curves, which we summarize in §1.7.2.

This paper is structured as follows:

- §1 We study the simplest invariant sets of foliation germs: separatrices and separators. Separators appear when there are nodal singularities or non-invariant irreducible components of the exceptional divisor of the reduction of the foliation germ. They divide any small neighborhood of $0 \in \mathbb{C}^2$ into disjoint invariant sets that we call dynamical components. We compare it with the decomposition into conical geometric blocks of the complement of the separatrices obtained using 3-manifolds theory [16].
- §2 The key topological property of the leaves of a generalized curve is their incompressibility in the complement of a suitable union of separatrices called appropriate curve (Incompressibility Theorem 1.3.1). The introduction of the foliated connectedness notion and a foliated Van Kampen Theorem allow to prove incompressibility thanks to a foliated block decomposition of the space guided by the decomposition into geometric blocks.
- §3 We discuss an example that shows the necessity of non-isolated separatrices in an appropriate curve.
- §4 We first give examples of the use of the ends of leaves space in the analytical study of a reduced foliation. We then highlight the complex structure of the leaf space of the foliation induced by $\mathcal{F}_U$ on the universal covering of $U \setminus S$, for U in a suitable system of neighborhoods of an appropriate curve S. This allows to introduce the notion of monodromy of a foliation germ as an action on the leaf spaces system of the fundamental group of the complement of S. Together with the Camacho-Sad indices, the monodromy is a complete topological invariant, see Classification Theorem 1.5.16.
- §5 We describe different types of dynamical components that permit to state optimal assumptions for topological invariance of the Camacho-Sad indices at the singular points in the exceptional divisor of the reduction.
- §6 We explain the proof of Excellence Theorem 1.7.1 and we state the topological classification of generalized curves.

1.2 Separatrices and separators

A consequence of the long exact sequence associated to the Milnor fibration [30] of a germ of function with isolated singularity is that the fundamental group of a regular level set injects into the fundamental group of the complement of the singular fiber (**incompressibility property**). In the more general setting of generalized curves, we will see that incompressibility property of the leaves in the complement of a suitable invariant curve (containing the isolated separatrices) also holds.

1.2.1 Graph decomposition of the complement of a germ of curve

Let $(S,0)$ be a germ of analytic curve S at the origin of $\mathbb{C}^2$ with reduced equation $f=0$. It is well known that the **full Milnor tube of** S

$$\mathbb{T}^*_{\varepsilon,\eta} := \mathbb{T}_{\varepsilon,\eta}\backslash S\,, \quad \text{with} \quad \mathbb{T}_{\varepsilon,\eta} := \{|f(z)|\leq\eta, \|z\|\leq\varepsilon\}\subset\mathbb{C}^2\,, \quad 0<\eta\ll\varepsilon\ll 1\,,$$

is a retract by deformation of the complement of the curve in the ball $\{\|z\|\leq\varepsilon\}$ and also that it is a cone on the **empty Milnor tube**

$$\delta\mathbb{T}_{\varepsilon,\eta} := \{|f(z)|=\eta, \|z\|\leq\varepsilon\},$$

i.e. there is a homeomorphism $\psi : \mathbb{T}^*_{\varepsilon,\eta}\xrightarrow{\sim}\delta\mathbb{T}_{\varepsilon,\eta}\times(0,1]$ such that $\psi(z)=(z,1)$ if $z\in\delta\mathbb{T}_{\varepsilon,\eta}$. The structure of $\delta\mathbb{T}_{\varepsilon,\eta}$ as graphed 3-manifold is described by various authors [45, 14, 15] and we adopt here the presentation of [16, Theorems 1.2.3 and 1.2.6], see also [20, §3.2]:

There is a finite collection $\mathscr{T}$ of 2-tori embedded in $\delta\mathbb{T}_{\varepsilon,\eta}$, unique up to isotopy, that decomposes $\delta\mathbb{T}_{\varepsilon,\eta}$ into incompressible compact submanifolds with boundary K_α, each of them being endowed with a (unique) Seifert fibration $\mathscr{S}_\alpha$, such that if $T\in\mathscr{T}$ is the intersection $K_\alpha\cap K_\beta$, then

1. *T is incompressible in both K_α and in K_β, and it is not homotopic to a connected component of the boundary of $\delta\mathbb{T}_{\varepsilon,\eta}$,*
2. *in T the regular fibers of $\mathscr{S}_\alpha$ and of $\mathscr{S}_\beta$ are not homotopic.*

We will describe a way to obtain the collection $\mathscr{T}$ using the desingularization map $E_S : M_S\to\mathbb{C}^2$ of S. We recall that the **dual graph A_C of a curve C with normal crossings**, is a graph whose vertices are the irreducible components (compact or not) of this curve, two vertices being joined by an edge if and only if the corresponding irreducible components of the curve intersect. For simplicity we will denote by A_S the dual graph $\mathsf{A}_{\mathcal{D}_S}$ of the **total transform**

$$\mathcal{D}_S := E_S^{-1}(S)\,.$$

Clearly A_S is a tree. We equip it with the metric for which the edges have length equal to 1. The **valence** of a vertex is the number of edges attached to it. A **chain** in A_S is a geodesic of A_S joining two vertices of valence at least three called **end vertices of a chain**, all the other vertices being of valence 2. A **branch of** A_S is a geodesic that joins a vertex of valence at least three called **attaching vertex** to a vertex of valence one, called **end vertex**, the remaining vertices being of valence 2. It is called **dead branch** if its end vertex corresponds to a compact irreducible component of $\mathcal{D}_S$ and **simple branch** otherwise. The union of the irreducible components corresponding to the vertices of a dead or simple branch of A_S is called a **dead** or **simple branch of** $\mathcal{D}_S$**.** Similarly, a **chain of** $\mathcal{D}_S$ is the union of the irreducible components corresponding to the vertices of valence 2 of a chain of A_S or the unique intersection point $D \cap D'$ of a chain reduced to $\overset{D}{\bullet} - \overset{D'}{\bullet}$.

Now, for each chain C in A_S let us choose an edge e_C. It corresponds to the intersection point of two irreducible components of $\mathcal{D}_S$. Let us also choose local holomorphic coordinates (u_C, v_C) at this point such that $u_C v_C = 0$ is a local equation of $\mathcal{D}_S$. The real analytic hypersurface $\{|u_C| = 1\}$ intersects transversally the "Milnor tube" $E_S^{-1}(\delta\mathbb{T}_{\varepsilon,\eta})$ along a 2-torus T_C, with $0 < \eta \ll \varepsilon \ll 1$. After identifying $E_S^{-1}(\delta\mathbb{T}_{\varepsilon,\eta})$ with $\delta\mathbb{T}_{\varepsilon,\eta}$ via E_S, we set

$$\mathscr{T} = \{T_C\}_{C \in \mathrm{chain}(\mathsf{A}_S)}$$

where $\mathrm{chain}(\mathsf{A}_S)$ is the set of all chains in A_S. Similarly, the collection of real hypersurfaces $(\{|u_C| = 1\})_{C \in \mathrm{chain}(\mathsf{A}_S)}$ induces a **geometric block decomposition** of $E_S^{-1}(\mathbb{T}^*_{\varepsilon,\eta}) \simeq \mathbb{T}^*_{\varepsilon,\eta}$, each **geometric block** being a real four-dimensional submanifold with boundary and corners and we have:

(B1) each block is incompressible in $\mathbb{T}^*_{\varepsilon,\eta}$,

(B2) any intersection of two blocks is a boundary component of each block, is incompressible in each block and is homeomorphic to a coreless full torus $\mathbb{D}^* \times \mathbb{S}^1$,

(B3) the closure in $E_S^{-1}(\mathbb{T}_{\varepsilon,\eta})$ of each block $\mathcal{B}$ contains a unique irreducible component D of $\mathcal{D}_S$ with at least three singular points of $\mathcal{D}_S$ and we denote $\mathcal{B}$ by $\mathcal{B}_D$; conversely for any vertex of A_S with valence at least three, there is a block $\mathcal{B}$ such that $\mathcal{B} = \mathcal{B}_D$,

(B4) given a block $\mathcal{B}_D$, over the open set $D^\star \subset D$ such that $D \setminus D^\star$ is a union of disjoint discs centered at singular points of $\mathcal{D}_S$ in D, the Seifert fibration of $E_S^{-1}(\delta\mathbb{T}_{\varepsilon,\eta}) \cap \mathcal{B}$ coincides with the Hopf fibration of D; moreover the exceptional fibers consist in one Hopf fiber for each irreducible component of $\mathcal{D}_S$ that corresponds to an end vertex of a dead branch of A_S with attaching vertex D.

1.2.2 Separatrices

Now, and in all the sequel we fix a germ of singular foliation $\mathcal{F}$ at the origin of $\mathbb{C}^2$ defined by a germ of holomorphic differential form ω with 0 as isolated singular point.

A **separatrix** of $\mathcal{F}$ is a germ of **invariant** irreducible holomorphic curve, i.e. any parametrization of it $\gamma : (\mathbb{C}, 0) \to \mathbb{C}^2$ satisfies $\gamma^*\omega \equiv 0$. Classically, a germ of a curve C is a separatrix if and only if there is an open neighborhood U of 0 and a closed leaf L of the regular foliation defined by ω on $U \setminus \{0\}$, such that $C \cap (U \setminus \{0\}) = L$. According to Camacho-Sad existence theorem [7], the number of separatrices of a foliation is strictly positive. It may be infinite, but the number of their topological classes is finite by Seidenberg's reduction theorem [40]. This one states the existence of a unique proper map called **reduction map of** $\mathcal{F}$

$$E_{\mathcal{F}} : M_{\mathcal{F}} \to \mathbb{C}^2$$

obtained by a minimal sequence of blowing-up maps with finite centers, such that the germ of the foliation $\mathcal{F}^\sharp := E_{\mathcal{F}}^{-1}\mathcal{F}$ at any point of the **exceptional divisor**

$$\mathcal{E}_{\mathcal{F}} := E_{\mathcal{F}}^{-1}(0)$$

is either regular or defined in suitable local coordinates (u, v) by a **reduced** differential form

$$(\lambda v + \cdots)du - (u + \cdots)dv \quad \text{with} \quad \lambda \in \mathbb{C} \setminus \mathbb{Q}_{>0}\,, \tag{1.1}$$

moreover, at any point of a **dicritical component** (i.e. non $\mathcal{F}^\sharp$-invariant) D of $\mathcal{E}_{\mathcal{F}}$, the foliation $\mathcal{F}^\sharp$ is regular and transverse to D.

We recall that a foliation germ $\mathcal{F}$ is called **generalized curve** if there is no **saddle-node** singularity of $\mathcal{F}^\sharp$, i.e. given by a differential form (1.1) with $\lambda = 0$. For this type of foliations $E_{\mathcal{F}}$ coincides with the minimal composition of blowing-ups that desingularizes all separatrices of $\mathcal{F}$, see [4].

A separatrix C will be called **dicritical separatrix** if its **strict transform** $\overline{E_{\mathcal{F}}^{-1}(C \setminus \{0\})}$ through $E_{\mathcal{F}}$ meets a dicritical component of $\mathcal{E}_{\mathcal{F}}$, otherwise C is called **isolated separatrix**. The **isolated separatrices curve** is the germ of the curve union of all isolated separatrices. We denote it, resp. its total transform by the reduction map of $\mathcal{F}$, by

$$S_{\mathcal{F}} \subset (\mathbb{C}^2, 0), \quad \text{resp.} \quad S_{\mathcal{F}}^\sharp := E_{\mathcal{F}}^{-1}(S_{\mathcal{F}})\,. \tag{1.2}$$

These notions can be topologically characterized. Indeed, a separatrix is dicritical if and only if it is contained in a non-constant equireducible family of invariant analytic curve germs. On the other hand, classically for families of germs of curves in $(\mathbb{C}^2, 0)$, equireducibility and topological triviality are equivalent notions.

The topological information contained in the dicritical and isolated separatrices may be organized into a global combinatorial object which will also be a topological invariant of the foliation.

Definition 1.2.1 The **dual graph** $\mathsf{A}_{\mathcal{F}}$ **of** $\mathcal{F}$ is the dual graph of the normal crossings curve $S^{\sharp}_{\mathcal{F}}$.

Notice that the desingularization map $E_{S_{\mathcal{F}}}$ of $S_{\mathcal{F}}$ can be different from the reduction of singularities of $\mathcal{F}$ and in this case $\mathsf{A}_{\mathcal{F}}$ is not the dual graph $\mathsf{A}_{S_{\mathcal{F}}}$ of $E^{-1}_{S_{\mathcal{F}}}(S_{\mathcal{F}})$, see Figure 1.1.

Proposition 1.2.2 *Any homeomorphism germ* $\phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ *that conjugates two generalized curves* $\mathcal{F}$ *and* $\mathcal{G}$, $\phi(\mathcal{F}) = \mathcal{G}$, *defines an isomorphism*

$$\mathsf{A}_{\phi} : \mathsf{A}_{\mathcal{F}} \to \mathsf{A}_{\mathcal{G}}$$

between the dual graphs of $\mathcal{F}$ *and* $\mathcal{G}$, *such that if* D *is a vertex corresponding to the strict transform of an irreducible component* C *of* $S_{\mathcal{F}}$, *then* $\mathsf{A}_{\phi}(D)$ *is the vertex of* $\mathsf{A}_{\mathcal{G}}$ *corresponding to the strict transform of* $\phi(C)$. *This isomorphism sends dicritical components into dicritical components and is compatible with the intersection forms, i.e. we have*

$$\mathsf{A}_{\phi}(D) \cdot \mathsf{A}_{\phi}(D') = D \cdot D'$$

for any vertices D, D' *of* $\mathsf{A}_{\mathcal{F}}$ *considered as irreducible components of* $S^{\sharp}_{\mathcal{F}}$. *Moreover the correspondence* $(\mathcal{F}, \phi) \mapsto (\mathsf{A}_{\mathcal{F}}, \mathsf{A}_{\phi})$ *is a functor from the category of generalized curves and topological conjugacies, to the category of graphs and isomorphisms of graphs.*

Let us give an idea of the proof. A consequence of the results in [4] is that when $\mathcal{F}$ is a generalized curve, the reduction map $E_{\mathcal{F}}$ coincides with the desingularization map of a suitable curve germ $C_{\mathcal{F}}$, obtained by adding to $S_{\mathcal{F}}$ a pair of non-isolated separatrices for each dicritical component. Moreover the reduction map of $\mathcal{G} = \phi(\mathcal{F})$ also coincides with the desingularization map of $\phi(C_{\mathcal{F}})$. A classical Zariski theorem [46] associates to ϕ a morphism

$$\phi_{\bullet} : \mathsf{A}_{C_{\mathcal{F}}} \to \mathsf{A}_{\phi(C_{\mathcal{F}})}$$

between the dual trees of the total transforms of $C_{\mathcal{F}}$ and $\phi(C_{\mathcal{F}})$ by the reduction map $E_{C_{\mathcal{F}}} = E_{\mathcal{F}}$. More specifically $\phi_{\bullet}$ can be obtained by applying the following theorem to $X = C_{\mathcal{F}}$, $Y = \phi(C_{\mathcal{F}})$ and $\varphi = \phi$:

Theorem 1.2.3 ([20, Theorem A]) *Let* $\varphi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ *be a germ of homeomorphism that sends a germ of curve* X *onto a germ of curve* Y. *Then there exists a germ of homeomorphism* $\psi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ *and a germ of homeomorphism along the exceptional divisors of the desingularizations of* X *and* Y

$$\Psi : (M_X, \mathcal{E}_X) \to (M_Y, \mathcal{E}_Y)$$

such that

1. *Ψ lifts ψ, i.e. $E_Y \circ \Psi = \psi \circ E_X$;*
2. *for any irreducible component C of X we have $\varphi(C) = \psi(C)$;*
3. *for small enough Milnor balls B for X and $B' \supset \psi(B) \cup \varphi(B)$ for Y, the group morphisms $\pi_1(B \setminus X) \to \pi_1(B' \setminus Y)$ induced by φ and ψ are equal up to composition by an inner automorphism.*

The homeomorphism germ Ψ induces a graph morphism $\mathsf{A}_{C_{\mathcal{F}}} \to \mathsf{A}_{\phi(C_{\mathcal{F}})}$ between the dual trees of the total transforms of $C_{\mathcal{F}}$ and $\phi(C_{\mathcal{F}})$ by their desingularization maps, sending the vertex corresponding to an irreducible component D into the vertex corresponding to the irreducible component $\Psi(D)$ and satisfying $\Psi(D) \cdot \Psi(D') = D \cdot D'$ for any pair of vertices D, D' of $\mathsf{A}_{C_{\mathcal{F}}}$. In fact, it results from the proof of this theorem that this graph-morphism does not depend on the choice of Ψ, hence we can denote it by $\phi_\bullet$. Since $E_{\mathcal{F}} = E_{C_{\mathcal{F}}}$, we have the inclusion

$$S^\sharp_{\mathcal{F}} \subset E^{-1}_{C_{\mathcal{F}}}(C_{\mathcal{F}})$$

and $\mathsf{A}_{\mathcal{F}}$ is a subgraph of $\mathsf{A}_{C_{\mathcal{F}}}$, similarly $\mathsf{A}_{\mathcal{G}}$ is a subgraph of $\mathsf{A}_{\phi(C_{\mathcal{F}})}$. These subgraphs are obtained by deleting the vertices corresponding to dicritical separatrices. Using the fact that ϕ sends dicritical separatrices of $\mathcal{F}$ to dicritical separatrices of $\mathcal{G}$, we see that $\phi_\bullet$ sends $\mathsf{A}_{\mathcal{F}}$ onto $\mathsf{A}_{\mathcal{G}}$, inducing the isomorphism A_ϕ between these graphs stated in Proposition 1.2.2.

As we will see in §1.4.1, there are dicritical foliations $\mathcal{F}$ for which the incompressibility property of the leaves in the complement of $S_{\mathcal{F}}$ in any neighborhood, does not hold. In other words, the fundamental group Γ of the complement of $S_{\mathcal{F}}$ in a Milnor ball may be too small to contain the fundamental group of any leaf. For this reason we will increase Γ by adding to $S_{\mathcal{F}}$ some dicritical separatrices.

Definition 1.2.4 A germ of an invariant curve containing all the isolated separatrices of $\mathcal{F}$ and whose strict transform by the reduction of $\mathcal{F}$ meets any dicritical component D with $\mathrm{card}(D \cap \mathrm{Sing}(\mathcal{E}_{\mathcal{F}})) = 1$, will be called **$\mathcal{F}$-appropriate curve germ**.

Proposition 1.2.5 *Any homeomorphism germ $\phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ that conjugates two generalized curves $\mathcal{F}$ and $\mathcal{G}$, $\phi(\mathcal{F}) = \mathcal{G}$, necessarily transforms any dicritical, resp. isolated separatrix of $\mathcal{F}$, resp. $\mathcal{F}$-appropriate curve germ, into a dicritical, resp. isolated separatrix of $\mathcal{G}$, resp. $\mathcal{G}$-appropriate curve germ.*

Proof Using Proposition 1.2.2, it suffices to see that the image by ϕ of a $\mathcal{F}$-appropriate curve S is a $\mathcal{G}$-appropriate curve. For this we may perform the previous analysis with $X = S$ and $Y = \phi(S)$. We similarly obtain a graph isomorphism $\mathsf{A}_{S^\sharp} \to \mathsf{A}_{\phi(S)^\sharp}$ from the dual graph of $S^\sharp := E^{-1}_{\mathcal{F}}(S)$ to that of $\phi(S)^\sharp := E^{-1}_{\mathcal{G}}(\phi(S))$, that sends dicritical components into dicritical components and sends $\mathsf{A}_{\mathcal{F}} \subset \mathsf{A}_{S^\sharp}$ into $\mathsf{A}_{\mathcal{G}} \subset \mathsf{A}_{\phi(S)^\sharp}$. We easily deduce that $\phi(S)$ is $\mathcal{G}$-appropriate. □

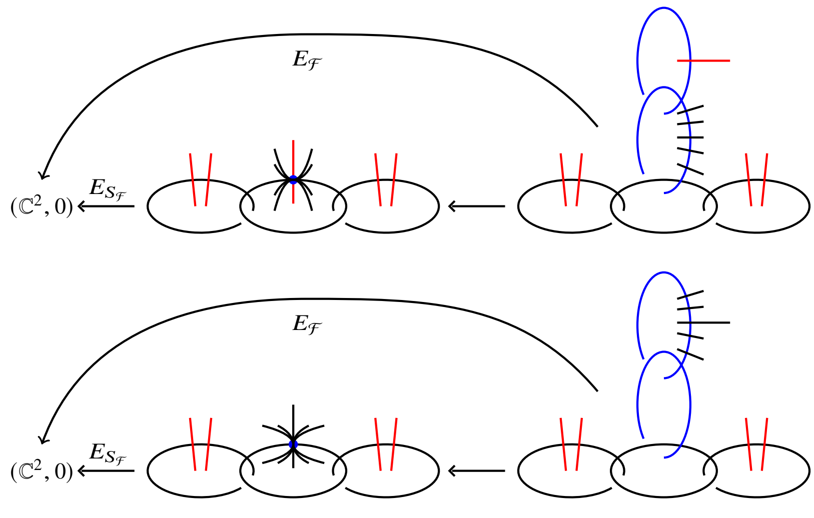


Fig. 1.1 In both cases $E_{S_{\mathcal{F}}} \neq E_{\mathcal{F}}$ due to a non-reduced singularity in the central divisor of $E_{S_{\mathcal{F}}}$ (in blue). In the first one $S_{\mathcal{F}}$ (in red) is $\mathcal{F}$-appropriate and in the second one is not.

1.2.3 Separators and dynamical decomposition

One says that the germ of the reduced foliation $\mathcal{F}^\sharp$ at a singular point p in the exceptional divisor $\mathcal{E}_{\mathcal{F}}$ is a **node** or a **nodal singularity**, when it is defined in suitable local coordinates (u, v) by a differential form (1.1) with $\lambda \in \mathbb{R}_{>0} \setminus \mathbb{Q}$. Then there are exactly two invariant curve germs at this point; these are regular, with normal crossings and they contain the germ of the exceptional divisor.

Other invariant sets can be constructed using the existence of **local linearizing coordinates** (u, v) in which $\mathcal{F}^\sharp$ is given by the differential form $\lambda v du - u dv$. For any small enough ball $B := \{|u|^2 + |v|^2 < r\}$, any leaf of the foliation $\mathcal{F}^\sharp$ restricted to $B \setminus \{uv = 0\}$ is contained and dense in a real invariant hypersurface defined by an equation $|v| = c|u|^\lambda$, $c \in \mathbb{R}^*$. More generally any subset

$$S_{c_1,c_2} := \{c_1|u|^\lambda \leq |v| \leq c_2|u|^\lambda\} \subset B, \quad \text{with} \quad 0 < c_1 \leq c_2 < \infty,$$

is **invariant**, i.e. it is a union of leaves, and it divides B in two invariant connected components. For this reason we call **nodal separator of $\mathcal{F}^\sharp$ at** p, any intersection of S_{c_1,c_2} with a neighborhood of $\mathcal{E}_{\mathcal{F}}$.

Another type of invariant set can be constructed using a dicritical irreducible component D of $\mathcal{E}_{\mathcal{F}}$. For this we first choose a small enough compact tubular neighborhood T of D so that $\mathcal{F}^\sharp$ restricted to T is a locally trivial fibration. Then we choose open conformal discs $D_s \subset D$ centered at the singular points $s \in D$ of the exceptional divisor $\mathcal{E}_{\mathcal{F}}$, such that $\overline{D_s} \cap \overline{D_{s'}} = \emptyset$ for $s \neq s'$. The union of the leaves of the foliation $\mathcal{F}^\sharp$ restricted to T

that meet no disc D_s is a $\mathcal{F}^\sharp$-invariant set which we call **dicritical separator of $\mathcal{F}^\sharp$ at D**.

The intersection of a neighborhood of $0 \in \mathbb{C}^2$ with the image by $E_{\mathcal{F}}$ of a nodal or dicritical separator of $\mathcal{F}^\sharp$ will be called **nodal** or **dicritical separator of** $\mathcal{F}$. A **nodal separatrix of** $\mathcal{F}$ is a separatrix whose strict transform by $E_{\mathcal{F}}$ passes through a nodal singularity of $\mathcal{F}^\sharp$. C. Camacho and R. Rosas prove that every invariant set contains a separatrix or a nodal separator, see [5, Theorem 1].

Theorem 1.2.6 ([38, Theorem 1] and [39, Theorem 1.3]) *The image of a nodal separatrix, resp. nodal separator germ, resp. dicritical separator germ of $\mathcal{F}$ by a germ of homeomorphism that conjugates $\mathcal{F}$ to a germ of foliation $\mathcal{G}$, is a nodal separatrix, resp. nodal separator germ, resp. dicritical separator germ of $\mathcal{G}$.*

Let us consider now a collection of separators of $\mathcal{F}$

$$\mathcal{S} = (\mathcal{S}_\alpha)_{\alpha \in \mathcal{A}}, \qquad \mathcal{A} = \mathcal{E}_{\mathcal{F}}^{\mathrm{dic}} \cup \mathcal{N},$$

where $\mathcal{N}$ is the collection of the nodal singular points of $\mathcal{F}^\sharp$, $\mathcal{E}_{\mathcal{F}}^{\mathrm{dic}}$ is the collection of the dicritical components of $\mathcal{E}_{\mathcal{F}}$ and for each $\alpha \in \mathcal{A}$, $\mathcal{S}_\alpha$ is the image by $E_{\mathcal{F}}$, in a closed ball, of a chosen closed nodal or dicritical separator of $\mathcal{F}^\sharp$ at α. We will say that $\mathcal{S}$ is a **complete system of separators for** $\mathcal{F}$. In any small enough closed ball B_r of radius $r \leq R$, every $\mathcal{S}_\alpha$ is a cone, as foliated manifold, over its intersection with the sphere ∂B_r. This results from the transversality of the leaves contained in $\mathcal{S}_\alpha$ to any sphere ∂B_r, $0 < r \leq R$. We will say that B_R is a **Milnor ball for** $\mathcal{S}$.

Definition 1.2.7 Let B be a Milnor ball for a complete system $\mathcal{S}$ of separators for $\mathcal{F}$. We call $\mathcal{F}$-**dynamical component of** B defined by $\mathcal{S}$, any connected component of $B \setminus \cup_{\alpha \in \mathcal{A}} \mathcal{S}_\alpha$. We call **dynamical component of** $S_{\mathcal{F}}^\sharp$, see (1.2), the closure in $S_{\mathcal{F}}^\sharp$ of any connected component of $S_{\mathcal{F}}^\sharp \setminus (\mathcal{E}_{\mathcal{F}}^{\mathrm{dic}} \cup \mathcal{N})$, see Figure 1.2. The dual graph of a dynamical component of $S_{\mathcal{F}}^\sharp$ is called **dynamical component of** $\mathsf{A}_{\mathcal{F}}$, see Definition 1.2.1.

By construction if $\mathcal{D}$ is a $\mathcal{F}$-dynamical component of B, then the intersection

$$\mathcal{E}_{\mathcal{D}} = \overline{E_{\mathcal{F}}^{-1}(\mathcal{D})} \cap \overline{S_{\mathcal{F}}^\sharp \setminus \mathcal{E}_{\mathcal{F}}^{\mathrm{dic}}}$$

of is a dynamical component $S_{\mathcal{F}}^\sharp$. The **dual graph** $\mathsf{A}_{\mathcal{D}}$ **of** $\mathcal{E}_D$ can also be characterized as a connected component of the graph obtained by removing from the dual graph $\mathsf{A}_{\mathcal{F}}$ of $S_{\mathcal{F}}^\sharp$:

(a) the edges $\langle D, D' \rangle$ such that $D \cap D'$ is a nodal singularity of $\mathcal{F}^\sharp$,

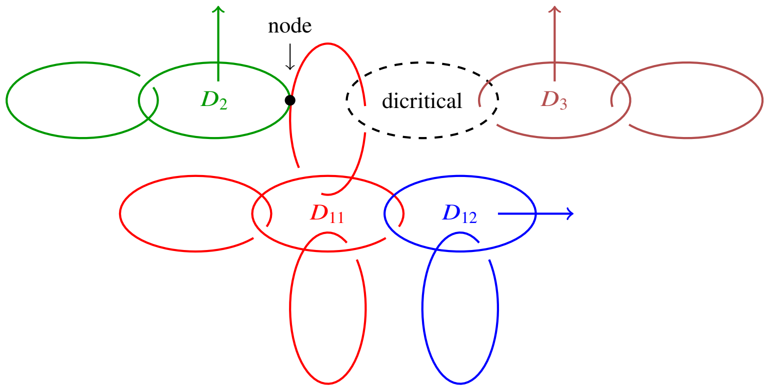


Fig. 1.2 In this schematic representation of the divisor $S^\sharp_{\mathcal{F}}$ we have three dynamical components: the green and black ones and on the other hand the union of the red and blue components. Each color correspond to a geometric block (in the sense of §1.2.1) containing a single irreducible component of valence ≥ 3. The non-compact components of $S^\sharp_{\mathcal{F}}$ are denoted by arrows.

(b) the vertices D corresponding to a dicritical component of $\mathcal{E}_{\mathcal{F}}$ and all the edges containing these vertices.

Notice that a dynamical component $\mathcal{E}_{\mathcal{D}}$ of $S^\sharp_{\mathcal{F}}$ can be reduced to a single irreducible component of $S^\sharp_{\mathcal{F}}$: we have $\mathcal{E}_{\mathcal{D}} = C$ when C is a non-compact irreducible component of $S^\sharp_{\mathcal{F}}$ that intersects $\mathcal{E}_{\mathcal{F}}$ at a nodal singular point of $\mathcal{F}^\sharp$, in other words, when C is the strict transform by $E_{\mathcal{F}}$ of a nodal separatrix.

The remarkable property of dynamical components is the following refinement of the Camacho-Sad existence theorem, given by L. Ortiz-Bobadilla, E. Rosales-González and S.M. Voronin [32, Strong Camacho-Sad Separatrix Theorem] and completed in [22, Lemma 1.9] for dicritical foliations.

Theorem 1.2.8 *Any dynamical component of the total transform $S^\sharp_{\mathcal{F}}$ of the separatrices curve $S_{\mathcal{F}}$ either contains a non-nodal isolated separatrix, or is reduced to a single nodal separatrix.*

We will highlight now that dynamical components of $S^\sharp_{\mathcal{F}}$ can be characterized as traces on $S^\sharp_{\mathcal{F}}$ of "minimal" invariant sets.

The foliation $\mathcal{F}$ being defined on a ball B let $A \subset A'$ be two subsets of B, resp. of $E^{-1}_{\mathcal{F}}(B)$. We will denote by $\mathrm{sat}(A, A')$ the union of the connected components of $L \cap A'$ for every leaf L of $\mathcal{F}$, resp. of $\mathcal{F}^\sharp$, meeting A. We also denote by $\overline{\mathrm{sat}}(A, A')$ the closure of $\mathrm{sat}(A, A')$ in B, resp. in $E^{-1}_{\mathcal{F}}(B)$. For any subset A in $\mathbb{C}^2$ let us write

$$A^\sharp := E^{-1}_{\mathcal{F}}(A).$$

Following the ideas developed in [5] one can prove:

Proposition 1.2.9 *Let B be a Milnor ball for $S_{\mathcal{F}}$ and let $(W_i)_i$ be a fundamental system of neighborhoods of $S_{\mathcal{F}}$ in B. Let Δ be an embedded disc in $B \setminus \{0\}$ transverse to $\mathcal{F}$, meeting the isolated separatrices $S_{\mathcal{F}}$ at a single point p, and define $\Delta_i = \Delta \cap W_i$. Then*

1. *$\bigcap_i \operatorname{sat}(\Delta_i^\sharp, B^\sharp)$ is the dynamical component of $S_{\mathcal{F}}^\sharp$ meeting $\Delta^\sharp$;*
2. *there exist two complete systems of separators $\mathcal{S} = (\mathcal{S}_\alpha)_{\alpha\in\mathcal{A}}$ in B and $\mathcal{S}' = (\mathcal{S}'_\alpha)_{\alpha\in\mathcal{A}}$ in a ball $B' \subset B$ such that*
   - *B is a Milnor ball for $\mathcal{S}$, B' is a Milnor ball for $\mathcal{S}'$ and $\mathcal{S}_\alpha \cap B' \subset \mathcal{S}'_\alpha$,*
   - *if W_i is small enough then $\operatorname{sat}(\Delta_i, B)$ is contained in one $\mathcal{F}$-dynamical component of B defined by $\mathcal{S}$ and contains one $\mathcal{F}$-dynamical component of B' defined by $\mathcal{S}'$.*

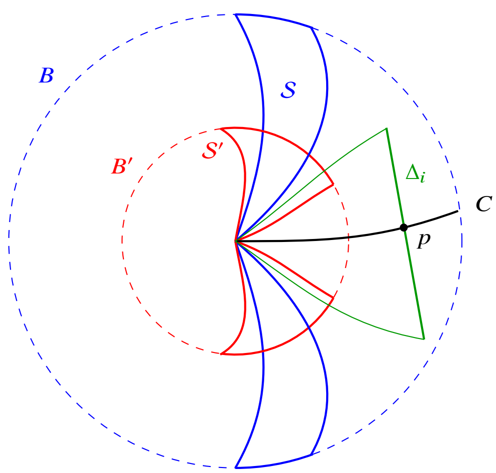


Fig. 1.3 Illustration of Proposition 1.2.9, C is an irreducible component of $S_{\mathcal{F}}$.

Remark 1.2.10 As a consequence of Proposition 1.2.9, when $\mathcal{F}^\sharp$ has no nodal singularities and $\mathcal{E}_{\mathcal{F}}$ contains dicritical components then $U = E_{\mathcal{F}}\big(\operatorname{sat}(\mathcal{E}_{\mathcal{F}}^{\mathrm{dic}}, B^\sharp) \cup S_{\mathcal{F}}^\sharp\big)$ is a neighborhood of 0 in $\mathbb{C}^2$. Thus, for every leaf $L \subset U$ of $\mathcal{F}$ there is a separatrix contained (as germ of curve) in $L \cup \{0\}$. If in addition $\mathcal{E}_{\mathcal{F}}$ contains an invariant irreducible component with infinite holonomy group then there are leaves $L \subset U$ of $\mathcal{F}$ containing infinitely many separatrices, see Figure 1.4. □

Assertion 1 in Proposition 1.2.9 does not imply that the decomposition of $\mathsf{A}_{\mathcal{F}}$ into dynamical components is a topological invariant of $\mathcal{F}$.

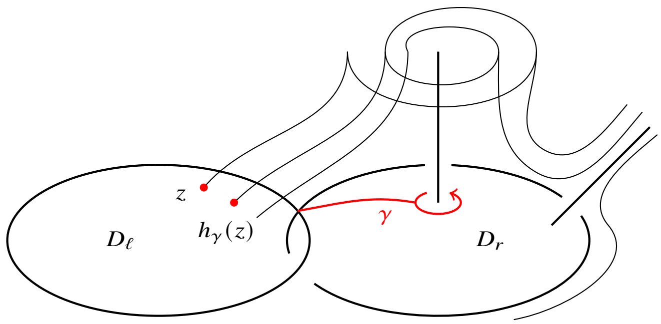


Fig. 1.4 The left irreducible component D_ℓ of $\mathcal{E}_\mathcal{F}$ is dicritical. The right irreducible component D_r is invariant and its projective holonomy group is infinite. The leaf of $\mathcal{F}^\sharp$ passing through the point $z \in D_\ell$ meets D_ℓ infinitely many times and consequently its image by $E_\mathcal{F}$ is a leaf of $\mathcal{F}$ containing infinitely many separatrices as germs of curves through 0.

Question 1.2.11 If $\phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ is a homeomorphism germ conjugating a generalized curve $\mathcal{F}$ to another generalized curve $\mathcal{G}$, does the graph morphism A_ϕ defined in Proposition 1.2.2 send any dynamical component of $\mathsf{A}_\mathcal{F}$ into a dynamical component of $\mathsf{A}_\mathcal{G}$? □

It can be easily deduced from assertion 1 in Proposition 1.2.9 that the answer is positive for dynamical components without nodal singularities:

Corollary 1.2.12 *Let $\mathcal{D}$ be the closure of a connected component of $\mathcal{E}_\mathcal{F} \setminus \mathcal{E}_\mathcal{F}^{\mathrm{dic}}$ and let $\mathsf{A}_\mathcal{D}$ be the dual graph of $\mathcal{D}$. If $\phi : \mathcal{F} \to \mathcal{G}$ is a topological conjugacy between generalized curves then $\mathsf{A}_\phi(\mathsf{A}_\mathcal{D})$ is the dual graph of the closure of a connected component $\mathcal{D}'$ of $\mathcal{E}_\mathcal{G} \setminus \mathcal{E}_\mathcal{G}^{\mathrm{dic}}$. Moreover if $\mathcal{D}$ does not contain nodal singular points of $\mathcal{F}^\sharp$ then $\mathcal{D}'$ does not contain nodal singular points of $\mathcal{G}^\sharp$.*

Moreover, as we will see later, under some Krull-generical assumptions on the differential form defining $\mathcal{F}$, Question 1.2.11 has also a positive answer.

1.3 Incompressibility of leaves

The goal of this chapter is to bound for any generalized curve the complexity of the topology of the leaves as immersed Riemann surfaces. This is given by the following result:

Theorem 1.3.1 (Incompressibility Theorem) *Let $\mathcal{F}$ be a generalized curve, S a $\mathcal{F}$-appropriate curve in a Milnor tube $\mathbb{T}$ for S, and C a non-necessarily irreducible closed analytic curve in $\mathbb{T}$ such that $C \setminus \{0\}$ is smooth and transverse to $\mathcal{F}$. Then there exists a fundamental system $(U_n)_{n\in\mathbb{N}}$, $U_{n+1} \subset U_n$, of open neighborhoods of S in $\mathbb{T}$ such that for each $n \in \mathbb{N}$ we have:*

1. *the inclusion map $U_n \hookrightarrow \mathbb{T}$ induces an isomorphism between the fundamental groups of $U_n \setminus S$ and $\mathbb{T} \setminus S$;*
2. *for each leaf L of the foliation induced by $\mathcal{F}$ on $U_n \setminus S$, the inclusion map $L \hookrightarrow U_n \setminus S$ induces an injective morphism $\pi_1(L) \hookrightarrow \pi_1(U_n \setminus S)$;*
3. *any connected component $C' \subset q^{-1}(C)$ of the pull-back of $C \cap U_n$ by the universal covering map $q : \tilde{U}_n \to U_n \setminus S$ is an embedded open disc such that, for each leaf L' of the foliation induced by $q^{-1}\mathcal{F}$ on $\tilde{U}_n$, we have* $\mathrm{card}(L' \cap C') \leq 1$.

This result was obtained by two of us in [19] for generalized curves under the additional assumptions of non-dicriticity and the absence, after reduction, of singularities that are simultaneously non-resonant and non-linearizable. The non-dicriticity assumption was removed in [22] by introducing the notion of appropriate curve, cf. Definition 1.2.4. Finally in [42] L. Teyssier extended the result under the sole generalized curve assumption; he also proved the necessity of this hypothesis by constructing non-dicritical foliations which are non-generalized curves possessing compressible leaves in the complement of their unique appropriate curve, cf. §1.4.2.

In the following two paragraphs we will describe the steps of the proof of Theorem 1.3.1.

1.3.1 Foliated connectedness and a foliated Van Kampen Theorem

To study the incompressibility property of the leaves of $\mathcal{F}$ in the complement of a $\mathcal{F}$-appropriate curve S, the main tool is the notion of $\mathcal{F}$-connectedness introduced in [19]. It allows to localize this global property of each leaf, by transforming it into a local property of the foliations induced by $\mathcal{F}$ on each block of a suitable decomposition of the space. Each block of this decomposition will be contained in a geometric block of the decomposition of a Milnor tube of S described in §1.2.1.

Definition 1.3.2 Let $A \subset B$ be subsets of a manifold M endowed with a regular foliation $\mathcal{G}$. We say that A **is strictly $\mathcal{G}$-connected in** B and we denote $A \overset{\mathcal{G}}{\hookrightarrow} B$ or $B \overset{\mathcal{G}}{\hookleftarrow} A$, if the following properties are fulfilled:

1. A **is incompressible**[1] **in** B,
2. A **is $\mathcal{G}$-connected in** B, i.e. for any leaf L of $\mathcal{G}$, if a path $\beta : [0, 1] \to L \cap B$ is homotopic in B to a path $\alpha : [0, 1] \to A$, then β is also homotopic in $L \cap B$ to a path $\gamma : [0, 1] \to L \cap A$ and the paths α and γ are homotopic in A.

A remarkable property of this notion is that if the set A is reduced to a single point, A is strictly $\mathcal{G}$-connected in B if and only if the leaf L containing this point is incompressible in B. Another trivial but useful property is the transitivity of this relation:

$$\left(A \overset{\mathcal{G}}{\hookrightarrow} B \text{ and } B \overset{\mathcal{G}}{\hookrightarrow} C\right) \Longrightarrow A \overset{\mathcal{G}}{\hookrightarrow} C\,. \tag{1.3}$$

[1] i.e. the inclusion $A \subset B$ induces monomorphisms $\pi_1(A, a) \hookrightarrow \pi_1(B, a)$ for every $a \in A$.

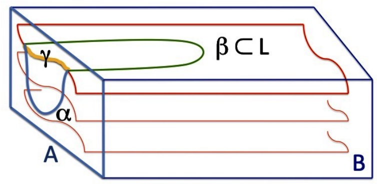


Fig. 1.5 Illustration of A being $\mathcal{G}$-connected in B.

Thus, to prove the incompressibility property of leaves stated in Theorem 1.3.1, it suffices to construct a filtration in the Milnor tube $\mathcal{T}_{\varepsilon,\eta} := E_{\mathcal{F}}^{-1}(\mathbb{T}_{\varepsilon,\eta})$ of $S^\sharp = E_{\mathcal{F}}^{-1}(S)$

$$W_0 \subset W_1 \subset \cdots \subset W_m \ \subset \ \mathcal{T}_{\varepsilon,\eta}\,, \tag{1.4}$$

satisfying:

(a) any leaf of $\mathcal{F}^\sharp$ restricted to $W_0^* := W_0 \setminus S^\sharp$ is incompressible in W_0^*,

(b) if we denote $W_i^* := W_i \setminus S^\sharp$, then we have $W_{i-1}^* \overset{\mathcal{F}^\sharp}{\hookrightarrow} W_i^*$, $i = 1, \ldots, m$,

(c) W_m is an open set in $\mathcal{T}_{\varepsilon,\eta}$, containing a Milnor tube $\mathcal{T}_{\varepsilon,\eta'}$ of $S^\sharp$ and the inclusions $\mathcal{T}_{\varepsilon,\eta'} \setminus S^\sharp \subset W_m^* \subset \mathcal{T}_{\varepsilon,\eta} \setminus S^\sharp$ induce isomorphisms between their fundamental groups.

In order to obtain property (b) the method developed in [19] is to construct a sequence $(V_j)_{j=0}^m$ of real four dimensional submanifolds of $\mathcal{T}_{\varepsilon,\eta}$ with boundary, that we call **foliated blocks**, such that after setting

$$W_i := V_0 \cup \cdots \cup V_i\,, \quad i = 0, \ldots, m\,, \tag{1.5}$$

each pair (W_{i-1}^*, V_i^*), where $V_i^* := V_i \setminus S^\sharp$, $i = 1, \ldots, m$, satisfies the relation

$$W_{i-1}^* \overset{\mathcal{F}^\sharp}{\hookleftarrow} W_{i-1}^* \cap V_i^* \overset{\mathcal{F}^\sharp}{\hookrightarrow} V_i^*\,.$$

The following **foliated Van Kampen Theorem** gives then the desired relation

$$W_{i-1}^* \overset{\mathcal{F}^\sharp}{\hookrightarrow} W_{i-1}^* \cup V_i^* = W_i^*\,.$$

Theorem 1.3.3 ([19, Théorème 1.3.1]) *Let (A, B) be a pair of connected subsets of a manifold M endowed with a regular foliation $\mathcal{G}$ of class C^1. Suppose that $A \cap B$ is a (non-necessary connected) transversely orientable submanifold of M, that is transverse to $\mathcal{G}$, and that $A \cup B$ is a neighborhood of $A \cap B$. Then we have:*

$$\left(A \overset{\mathcal{G}}{\hookleftarrow} A \cap B \overset{\mathcal{G}}{\hookrightarrow} B\right) \Longrightarrow A \overset{\mathcal{G}}{\hookrightarrow} A \cup B\,.$$

Notice that the above implication is a corollary of the classical Van Kampen Theorem when we only consider the incompressibility property 1 in Definition 1.3.2.

1.3.2 Construction of foliated blocks

For each irreducible component D of the total transform $S^\sharp$ of the $\mathcal{F}$-appropriate curve S in Theorem 1.3.1, we define K_D as the complement in D of disjoint conformal discs centered at the points of $\mathrm{Sing}(S^\sharp) \cap D$ that do not belong to any dead branch. The foliated blocks V_j used in (1.5) to construct the exhaustion $(W_i)_{i=0}^m$ are of three types:

- **Foliated Seifert blocks** $\mathscr{B}_D$ that are contained in geometric blocks of the form $\mathcal{B}_D$ described in (B3) of §1.2.1, the inclusion map inducing an isomorphism $\pi_1(\mathscr{B}_D \setminus S^\sharp) \simeq \pi_1(\mathcal{B}_D \setminus S^\sharp)$. Moreover, $\mathscr{B}_D$ is a neighborhood of the union of K_D with all the dead branches of $S^\sharp$ meeting D.
- **Foliated collar blocks** $\mathscr{B}_C$ that are neighborhoods of either

  (a) a set K_D with D a non-dicritical irreducible component of valence 2 in a chain of $S^\sharp$, or
  (b) a set K_D with D the strict transform of an irreducible component of S, or
  (c) a set $K_s = S^\sharp \cap \{|u|, |v| \leq 1\}$ where (u, v) are local coordinates centered at a singular point s of $S^\sharp$ belonging to a chain and $uv = 0$ is a local equation of $S^\sharp$.

  The name foliated collar is justified by the fact that there is a homeomorphism $\varphi : \mathbb{D}^* \times \mathbb{S}^1 \times (0,1) \xrightarrow{\sim} \mathring{\mathscr{B}}_C \setminus S^\sharp$ such that $\varphi(\{(z,\theta)\} \times (0,1))$ is contained in a leaf of $\mathcal{F}^\sharp$, where $\mathring{\mathscr{B}}_C$ denotes the interior of $\mathscr{B}_C$.
- **Gluing blocks** that are neighborhoods of K_D with D a dicritical component of $S^\sharp$.

Moreover, if $\mathscr{B}$ and $\mathscr{B}'$ are foliated blocks such that $\mathscr{B} \cap \mathscr{B}' \neq \emptyset$, then this intersection is a common connected component of $\partial\mathscr{B}$ and of $\partial\mathscr{B}'$, satisfying

$$\mathscr{B} \setminus S^\sharp \overset{\mathcal{F}^\sharp}{\hookleftarrow} (\mathscr{B} \cap \mathscr{B}') \setminus S^\sharp \overset{\mathcal{F}^\sharp}{\hookrightarrow} \mathscr{B}' \setminus S^\sharp.$$

Each block is built by an induction process, from the common boundary component of one of the previously constructed blocks.

The boundary components of each block are all sets of **suspension type**. This means that they are constructed as follows. First we fix a submersion ρ defined on a neighborhood Ω of an invariant irreducible component D of $S^\sharp$, that is equal to the identity map on D, and we also fix a simple loop $\gamma : [0,1] \to D \setminus \mathrm{Sing}(S^\sharp)$ and a holomorphic coordinate

$$z : \rho^{-1}(\gamma(0)) \xrightarrow{\sim} \{|z| \leq 1\} \subset \mathbb{C}, \qquad z(\gamma(0)) = 0\,.$$

Then we consider a closed conformal disc $\Delta \subset \rho^{-1}(\gamma(0))$ containing $\gamma(0)$ in its interior whose boundary $\partial\Delta$ is parametrized by a piecewise real analytic path δ. We assume Δ small enough so that for any point $m \in \Delta$, γ lifts through ρ to the leaf L_m of $\mathcal{F}^\sharp$ containing m, defining a (unique) path $\gamma_m : [0,1] \to L_m \cap \Omega$ with origin $\gamma_m(0) = m$, that satisfies $\rho \circ \gamma_m = \gamma$. The set

$$T_\Delta := \{\gamma_m(t) \mid m \in \Delta, 0 \leq t \leq 1\}$$

is called **set of suspension type relatively to Δ over γ**.

Let us highlight that the set $T^*_\Delta := T_\Delta \setminus S^\sharp$ may have a complex topology because the fundamental group of $\Delta \cup h_\gamma(\Delta)$ may be non-trivial, h_γ denoting the holonomy map $m \mapsto \gamma_m(1)$. To avoid this possibility we must ensure that Δ and $h_\gamma(\Delta)$ are star-shaped. To control this property we fix a parametrization $\delta(t)$ of $\partial\Delta$ and we define the **roughness** of Δ as the maximum $e_z(\Delta)$ of the angles between the tangents $\delta'_\pm(t)$ to $\partial\Delta$ at $\delta(t)$ and the tangent $i\delta(t)$ to the circle passing through this point with center $z = 0$. The star-shaped property is fulfilled when the roughness is $< \pi/2$. Moreover, as consequence of the smoothness of h_γ, the quotient $|e_z(\Delta)/e_z(h_\gamma(\Delta))|$ tends to 1 when the diameter of Δ tends to 0. Hence we have that T^*_Δ has the homotopy type of a 2-torus when the roughness and the diameter of Δ are small enough.

The inductive process to construct foliated blocks is based on the following existence property (cf. [19, Théorème 3.2.1]). For every suspension type set T over a loop $\gamma \subset \partial K$, with $K = K_D$ or $K = K_s$, there exists a foliated block $\mathscr{B}$ which is a neighborhood of K such that

- T is a connected component of $\partial\mathscr{B}$,
- every connected component T' of $\partial\mathscr{B}$ is a suspension type set whose roughness is controlled by that of T and it satisfies $T' \setminus S^\sharp \overset{\mathcal{F}^\sharp}{\hookrightarrow} \mathscr{B} \setminus S^\sharp$,
- the leaves of the restriction $\mathcal{F}^\sharp_{|\mathscr{B}\setminus S^\sharp}$ of $\mathcal{F}^\sharp$ to $\mathscr{B} \setminus S^\sharp$ are incompressible in $\mathscr{B} \setminus S^\sharp$,
- for any small enough Milnor tube $\mathbb{T}_{\varepsilon,\eta}$ of S we have $\pi_1(\mathscr{B} \setminus S^\sharp) \simeq \pi_1(W \cap E_S^{-1}(\mathbb{T}^*_{\varepsilon,\eta}))$, where W is a fixed neighborhood of K.

The order of construction of the blocks is given by first fixing a suitable exhaustion of $S^\sharp$ which will correspond to the exhaustion

$$W_0 \cap S^\sharp \subset W_1 \cap S^\sharp \subset \cdots \subset W_m \cap S^\sharp = S^\sharp.$$

determined by (1.4) at the end of the induction process.

1.4 Examples

1.4.1 Dicritical cuspidal singularity

We revisit Example 1.1 of [22]. Consider the dicritical foliation $\mathcal{F}$ in $(\mathbb{C}^2, 0)$ defined by the rational first integral $f(x, y) = \frac{y^2-x^3}{x^2}$ whose isolated separatrix set is the cusp $S = \{y^2-x^3 = 0\}$. We will first compute the fundamental group of the complement of S in a Milnor ball B, then we will show that not all the leaves of $\mathcal{F}$ are incompressible in $B \setminus S$, but by adding to S a non-isolated separatrix S' of $\mathcal{F}$ the incompressibility property will be fulfilled for all the leaves in $B \setminus (S \cup S')$.

1.4.1.1 Fundamental group of the complement of S

We consider the composition $E : M \to B$ of the three blowing-ups defining the minimal desingularization of S, that also coincides with the reduction map of the foliation. The exceptional divisor $\mathcal{E} = E^{-1}(0)$ has three irreducible components D_1, D_2, D_3 which we numerate according to the order that they appear in the blowing-up process. Thus, the corresponding self-intersections are

$$D_1 \cdot D_1 = -3\,, \quad D_2 \cdot D_2 = -2\,, \quad D_3 \cdot D_3 = -1\,.$$

The strict transform $\mathcal{S}$ of S only meets D_3 and consequently D_1 and D_2 are dead branches attached to D_3. In addition, D_1 is a dicritical component for $\mathcal{F}$ (in fact, it is totally transverse to $\mathcal{F}^\sharp = E^{-1}\mathcal{F}$). Let us illustrate the computation of the fundamental group of $B \setminus S \simeq M \setminus (\mathcal{E} \cup \mathcal{S})$. For each $i = 1, 2, 3$ we consider a tubular neighborhood U_i of D_i which is a disc fibration over $D_i \simeq \mathbb{P}^1$ of Chern class $k = D_i \cdot D_i$. For $i = 1, 2$, $U_i \cap (\mathcal{E} \cup \mathcal{S})$ consists in D_i and $r = 1$ fiber and for $i = 3$, $U_i \cap (\mathcal{E} \cup \mathcal{S})$ consists in D_i and $r = 3$ fibers. We can cover each

$$U = U_i = \varphi_x(\mathbb{D} \times \mathbb{D}^*) \cup \varphi_y(\mathbb{D} \times \mathbb{D}^*)$$

by two trivializing charts $\varphi_x(t, x)$, $\varphi_y(s, y)$ with transition functions $ts = 1$, $y = t^{-k}x$. We can assume that

$$U^* := U \setminus (\mathcal{E} \cup \mathcal{S}) = \varphi_x((\mathbb{D} \setminus \{t_1, \ldots, t_r\}) \times \mathbb{D}^*) \cup \varphi_y(\mathbb{D} \times \mathbb{D}^*).$$

We consider simple loops $\bar{\alpha}_j(u)$ in $\mathbb{D} \setminus \{t_1, \ldots, t_r\}$ of index 1 around t_j so that the product $\bar{\alpha}_0 = \bar{\alpha}_1 \cdots \bar{\alpha}_r$ is homotopic to $e^{2i\pi u}$. We define $\alpha_j(u) = \varphi_x(\bar{\alpha}_j(u), 1)$ and $\gamma(u) = \varphi_x(1, e^{2i\pi u})$. Then the loop

$$\alpha_0 \gamma^k \sim \varphi_x(e^{2i\pi u}, e^{2i\pi k u}) = \varphi_y(e^{-2i\pi u}, 1)$$

is null-homotopic in $\varphi_y(\mathbb{D} \times \mathbb{D}^*) \subset U^*$. By applying Seifert-Van Kampen's theorem we obtain that

$$\pi_1(U^*) = \langle \alpha_1, \ldots, \alpha_r, \gamma \mid \alpha_1 \cdots \alpha_r = \gamma^{-k}, [\alpha_j, \gamma] = 1 \rangle.$$

Particularizing this computation to each U_i we deduce that

$$\pi_1(U_1^*) = \langle a, c \mid c = a^3, [a, c] = 1 \rangle, \quad \pi_1(U_2^*) = \langle b, c \mid c = b^2, [b, c] = 1 \rangle$$

and

$$\pi_1(U_3^*) = \langle a, b, c, d \mid abd = c, [a, c] = [b, c] = [b, d] = 1 \rangle,$$

where the loops a, b, c, d are depicted in Figure 1.6. Notice that the loops a, b, c, d can be also defined as boundaries (meridians) of fibers of disc fibrations over D_1, D_2, D_3 and $\mathcal{S}$ respectively. Since any geometric realization of the dual tree of $\mathcal{E} \cup \mathcal{S}$ is contractible, a suitable embedding of it in $M \setminus (\mathcal{E} \cup \mathcal{S})$ described in [20, §3.1], can be chosen as a common "base point" of the loops a, b, c, d, cf. Figure 1.6.

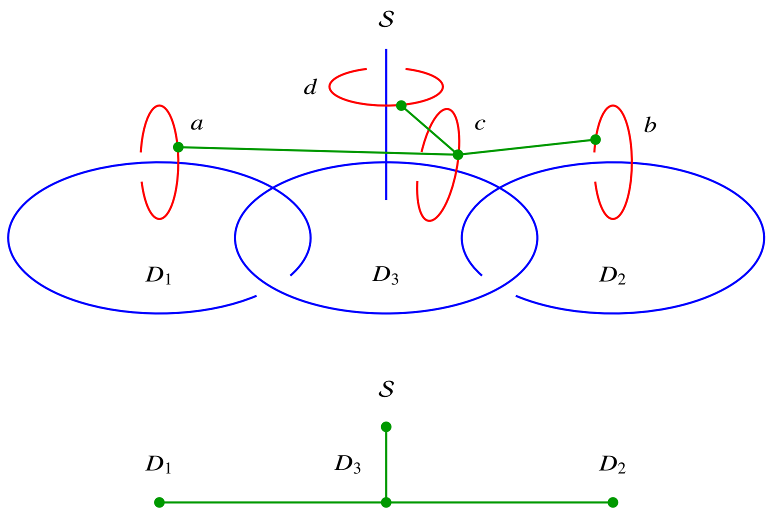


Fig. 1.6 The strict transform $\mathcal{S}$ of of the cusp $S = \{y^2 - x^3 = 0\}$, the total transform $S^\sharp = D_1 \cup D_2 \cup D_3 \cup \mathcal{S}$ with its dual tree, and the generators a, b, c, d of the fundamental group of its complement.

By applying again Seifert-Van Kampen's theorem we obtain the following presentation of the fundamental group of $M^* := M \setminus (\mathcal{E} \cup \mathcal{S})$:

$$\begin{aligned}\pi_1(M^*) &= \pi_1(U_1^*) *_{\pi_1(U_1^* \cap U_3^*)} \pi_1(U_3^*) *_{\pi_1(U_3^* \cap U_2^*)} \pi_1(U_2^*) \\ &= \langle a, b, c \mid a^3 = b^2 = c \rangle.\end{aligned}$$

This method can be applied to compute the fundamental group of the complement of any germ of curve (see [20, Proposition 3.3]).

1.4.1.2 Compressible leaves

Now, we shall show that there exist non-incompressible leaves of $\mathcal{F}$ inside $B \setminus S$. Looking at the situation after the first blow-up $E_1 : (M_1, D_1) \to (B, 0)$ in the chart (t, x) such that $(x, y) = E_1(t, x) = (x, tx)$ we have that

$$f_1(t, x) := f(x, tx) = t^2 - x$$

and we see that there are two types of leaves of $\mathcal{F}$: those that are near the isolated separatrix S, which are discs minus two points and the others which are diffeomorphic to $\mathbb{D}^*$. If L is a leaf of the first kind then $\pi_1(L)$ is a free group of rank 2. We claim that we can choose the generators λ, μ of $\pi_1(L)$ so that the morphism $\imath : \pi_1(L) \to \pi_1(M^*)$ induced by the inclusion is given by $\imath(\lambda) = a$ and $\imath(\mu) = b^{-1}ab$. It will then follow that

$$\imath(\lambda^3\mu^{-3}) = a^3 b^{-1} a^{-3} b = 1$$

and consequently L will not be incompressible in $B \setminus S \simeq M^*$.

At first, let $\varepsilon > 0$ be small enough so that $V := f_1^{-1}(\mathbb{D}_\varepsilon)$ is a retract by deformation of the image of $U_2 \cup U_3$ by the blowing-up $M \to M_1$ and the restriction of f_1 to $W := f_1^{-1}(\mathbb{S}^1_\varepsilon) \setminus D_1$ is a locally trivial C^∞-fibration over the circle $\mathbb{S}^1_\varepsilon$ of radius ε, whose fiber over ε is isomorphic to $F_\varepsilon := \mathbb{D} \setminus \{\pm\sqrt{\varepsilon}\}$. The pull-back of $f_1 : W \to \mathbb{S}^1_\varepsilon$ by the exponential map $[0,1] \to \mathbb{S}^1_\varepsilon$, $u \mapsto \varepsilon e^{2i\pi u}$, is trivial and there is a trivializing map

$$\tau : F_\varepsilon \times [0,1] \to W\,, \qquad (z,u) \mapsto (t,x) = (z e^{i\pi u}, (z^2 - \varepsilon) e^{2i\pi u})\,.$$

We consider the path $\beta : [0,1] \to F_\varepsilon \times [0,1]$ given by $\beta(u) = (0,u)$ projecting by τ onto the loop $(t,x) = (0, -\varepsilon e^{2i\pi u}) \in W \subset U_1$ which is a meridian of D_2, so that we can choose the generator b of $\pi_1(M^*)$ as the homotopy class of $\tau(\beta)$.

1.4.1.3 Appropiate curve

Now, let $z(u)$ be a simple loop in F_ε based in $z = 0$ having index +1 around $+\sqrt{\varepsilon}$ and index 0 around $-\sqrt{\varepsilon}$. We define $\alpha_i(u) = (z(u), i)$ for $i = 0, 1$. It is clear that α_1 is homotopic to $\beta\alpha_0\beta^{-1}$ in $F_\varepsilon \times [0,1]$. Hence their respective projections by τ, $\lambda = \tau(\alpha_1)$ and $\tau(\beta\alpha_0\beta^{-1}) = b\mu b^{-1}$ are also homotopic in W, where $\mu = \tau(\alpha_0)$. Notice that the loops λ, μ, which are contained in the leaf L passing through the point $(t,x) = (0, -\varepsilon)$, are meridians around D_1 so that we can choose the generator $a \in \pi_1(M^*)$ as the homotopy class of λ. Moreover, λ (resp. μ) turns around the point $(t,x) = (+\sqrt{\varepsilon}, 0) \in \bar{L} \cap D_1$ (resp. $(t,x) = (-\sqrt{\varepsilon}, 0) \in \bar{L} \cap D_1$). Hence the free fundamental group of L is generated by λ and μ whose images by $\imath : \pi_1(L) \to \pi_1(M^*)$ are a and $b^{-1}ab$ respectively.

Finally, if we remove the non-isolated separatrix $S' = \{x = 0\}$ then all the leaves of $\mathcal{F}$ are incompressible in $B \setminus (S \cup S')$. Indeed, using the description given in §1.4.1.1 we obtain that

$$\pi_1(U_1 \setminus (\mathcal{E} \cup \mathcal{S} \cup \mathcal{S}')) = \langle a, c, e \mid ce = a^3, [a,c] = [a,e] = 1 \rangle\,,$$

where e is a meridian of the strict transform $\mathcal{S}'$ of S'; consequently

$$\begin{aligned}\pi_1(M \setminus (\mathcal{E} \cup \mathcal{S} \cup \mathcal{S}')) = \langle a, b, c, d, e \mid ce = a^3, abd = c = b^2, \\ [a, e] = [c, a] = [c, b] = [c, d] = 1\rangle \\ = \langle a, b \mid [a, b^2] = 1\rangle \,.\end{aligned}$$

The quotient of this group by the normal subgroup $\langle b^2\rangle$ is the free product $\mathbb{Z}a * \mathbb{Z}_2\bar{b}$ where $\bar{b}$ is the class of b modulo $\langle b^2\rangle$. The composition morphism

$$\langle \lambda, \mu\rangle \to \langle a, b \mid [a, b^2] = 1\rangle \to \mathbb{Z}a * \mathbb{Z}_2\bar{b}\,,$$

sending the free generators λ and μ of $\pi_1(L)$ to a and $\bar{b}a\bar{b}$ respectively, is clearly injective and so is $\pi_1(L) \hookrightarrow \pi_1(M \setminus (\mathcal{E} \cup \mathcal{S} \cup \mathcal{S}'))$. Alternatively, it can be seen that W is a retract by deformation of $M \setminus (\mathcal{E} \cup \mathcal{S} \cup \mathcal{S}')$ and $\pi_1(L) = \pi_1(F_\varepsilon)$ injects into $\pi_1(M \setminus (\mathcal{E} \cup \mathcal{S} \cup \mathcal{S}')) = \pi_1(W)$ thanks to the long exact sequence of the fibration $F_\varepsilon \to W \xrightarrow{f_1} \mathbb{S}^1_\varepsilon$.

1.4.2 Foliations which are not generalized curves

Every saddle-node has a convergent separatrix (called strong) and a formal separatrix (called weak) that may be divergent. If all the saddle-nodes that appear in the minimal reduction of a foliation have the strong separatrix transverse to the exceptional divisor, the foliation is called **strongly presentable** in [42, Définition 1.2]. The general result of L. Teyssier [42, Théorème C] is that the properties 1 and 2 in Incompressibility Theorem 1.3.1 remain true for strongly presentable foliations and not only for generalized curves. Moreover, property 3 also remains true by choosing suitably the curve C in Theorem 1.3.1.

On the other hand, L. Teyssier [42, §8.2] exhibits examples of non-dicritical foliations $\mathcal{F}$ that are not strongly presentable for which the fundamental group of the leaves do not inject into the fundamental group of the complement of the separatrices curve $S_\mathcal{F}$. This is a consequence of the fact that the strong separatrix of some of the saddle-nodes appearing in the singularity reduction of $\mathcal{F}$ is contained in the exceptional divisor and the corresponding weak separatrix is divergent so that the fundamental group $\mathbb{Z} \oplus \mathbb{Z}$ of the complement of $S_\mathcal{F}$ is too small to contain the fundamental group (which is non-abelian free group) of a generic leaf (which is a non-compact Riemann surface).

Surprisingly, he also presents [42, §8.3] an example of a foliation reduced after one blowing-up in which all saddle-nodes have two convergent separatrices admitting compressible leaves in the complement of the separatrices. The construction is based on the existence of transverse curves (discovered by E. Paul) for which foliated connectedness property 3 of Theorem 1.3.1 fails.

1.5 Monodromy of singular foliations

The notion of monodromy of a singular foliation reflects the action of the fundamental group of the space on the “ends” of leaves or, more specifically, on the leaf spaces inverse system. Before we introduce this notion in §1.5.4, we first examine in §1.5.1 how for reduced foliations the structure of the leaf spaces contain all the useful informations to classify these foliations. In paragraph §1.5.2 we see that Incompressibility Theorem 1.3.1 allows to endow any leaf space with a (in general non-Haussdorff) complex manifold structure. In §1.5.4, after defining the notion of monodromy, we make explicit its relation to the holonomies of the invariant irreducible components of the exceptional divisor of the reduction, and to the extended holonomy defined in §1.5.3. That one corresponds in the case of foliations that can be reduced by a single blowing-up, to the notion introduced by L. Ortiz-Bobadilla, E. Rosales-González and S.M. Voronin in [33]. Finally in §1.5.6 we give the statement and an idea of the proof of Classification Theorem 1.5.16 in terms of monodromy and Camacho-Sad indices.

We shall suppose from now on that all singular foliations are generalized curves.

1.5.1 Ends of leaves space of reduced foliations

The analytic classification of reduced foliation singularities can be reformulated in the framework of ends of leaves space. Let us first recall the case of a saddle $\mathcal{F}$ defined by a 1-form

$$\omega = (1+\cdots)ydx - (\lambda+\cdots)xdy, \quad \text{with} \quad \lambda \notin \mathbb{Q}_{\geq 0}.$$

The axes $\{x = 0\}$ and $\{y = 0\}$ are the only separatrices and $S := \{xy = 0\}$ is a $\mathcal{F}$-appropriate curve. Assume that ω is holomorphic on the polydisc $P = \{|x| \leq 1,\ |y| \leq 1\}$ and let us fix a fundamental system of open neighborhoods $\mathcal{U}$ of $S \cap P$ in P, ordered by the inclusion. For any $U \in \mathcal{U}$ we denote by Q_U the leaf space of the foliation $\mathcal{F}_{|U\setminus S}$ induced by $\mathcal{F}$ on $U \setminus S$. They form **the inverse system of leaf spaces** together with the continuous maps

$$\rho_{VU} : Q_U \longrightarrow Q_V\,, \quad V \supset U\,, \quad U, V \in \mathcal{U}\,, \tag{1.6}$$

that send a leaf L_U of $\mathcal{F}_{|U\setminus S}$ to the leaf of $\mathcal{F}_{|V\setminus S}$ that contains L_U. The **space of ends of leaves of** $\mathcal{F}$ **along** $S \cap P$ is the inverse limit

$$\varprojlim_{U\in\mathcal{U}} Q_U := \{(L_U)_{U\in\mathcal{U}}\ ;\ L_U \in Q_U\ , V \supset U \Longrightarrow L_U \subset L_V\} \subset \prod_{U\in\mathcal{U}} Q_U\,,$$

which does not depend on the choice of the fundamental system $\mathcal{U}$. We distinguish three cases:

1.5.1.1 Poincaré type singularity $\lambda \notin \mathbb{R}$

We can construct $\mathcal{U} = (U_r)_{0<r<1}$ such that the intersections $U_r \cap \{|x| = 1\}$ are suspension type subsets relatively to the transverse discs $\Delta_r := \{x = 1, |y| < r\}$ over the loop $\gamma(s) = (e^{2i\pi s}, 0)$, cf. §1.3.2, and such that their saturations by $\mathcal{F}_{|U_r}$ are the whole U_r. Thus the space Q_{U_r} can be identified with the space of orbits of the holonomy $h : \Delta_r \xrightarrow{\sim} h(\Delta_r)$ of the foliation along γ, which is a linearizable contraction (or a dilatation). Hence Q_{U_r} is the elliptic curve

$$\mathbb{C}/(\mathbb{Z}2i\pi + \mathbb{Z}2i\pi\lambda) \simeq \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\lambda)$$

and the projections $\varprojlim_r Q_{U_r} \to Q_{U_r}$ are isomorphisms. Classically λ determines the analytic type of $\varprojlim_r Q_{U_r}$ and λ is the only analytic invariant of h and hence of $\mathcal{F}$ because it is linearizable.

Notice that all Poincaré type foliations are topologically conjugated, as we recall in Lemma 1.6.1. Indeed, the leaves are transverse to sufficiently small spheres so they are cones over the intersections with a sphere. The induced foliation on the sphere is a dimension one real foliation of Morse-Smale type with two closed leaves.

1.5.1.2 Non-linearizable resonant saddles

The topological classification of these foliations is given by three natural numbers $p, q, k \in \mathbb{N}$, $\gcd(p, q) = 1$. Indeed, by [6] any non-linearizable resonant saddle $\mathcal{F}$ is topologically conjugated to the foliation defined by

$$\omega_{p,q,k,\nu} = py(1 + (\nu - 1)(x^p y^q)^k)dx + qx(1 + \nu(x^p y^q)^k)dy$$

for any $\nu \in \mathbb{C}$, which is an analytic (and formal) invariant of $\mathcal{F}$ but not a topological one.

As in the previous case we can use a fundamental system $(U_r)_r$ of neighborhoods of the axes using transverse discs Δ_r which allows to identify again the leaf space Q_{U_r} to the space of orbits of the holonomy of the loop γ realized on Δ_r. The leaf space Q_{U_r} is again a Riemann surface, but in contrast to the previous case, is not Hausdorff. More precisely, Q_{U_r} has a structure of "chapelet de sphères" (string of spheres) in the terminology of J. Martinet and J.-P. Ramis, that is, the quotient of the disjoint union $\bigsqcup_{j\in\mathbb{Z}/k\mathbb{Z}}(S_j^- \cup S_j^+)$ of punctured spheres $S_j^+ = S_j^- = \overline{\mathbb{C}} \setminus \{0, \infty\}$, under the equivalence relation $m \sim \varphi_j^\star(m)$ induced by "gluing" biholomorphisms

$$\varphi_j^0 \;:\; S_j^+ \supset (V_j^+ \setminus \{0\}) \;\xrightarrow{\sim}\; (V_j^- \setminus \{0\}) \subset S_j^-$$

$$\varphi_j^\infty \;:\; S_j^- \supset (W_j^- \setminus \{\infty\}) \;\xrightarrow{\sim}\; (W_{j+1}^+ \setminus \{\infty\}) \subset S_{j+1}^+ ,$$

where $V_j^{\pm}$, resp. $W_j^{\pm}$, are open neighborhoods of 0, resp. of ∞, in $S_j^{\pm}$ and $j \in \mathbb{Z}/k\mathbb{Z}$, see for instance [26] or [18, p. 116].

The space of ends of leaves $\varprojlim_r Q_{U_r}$ is the Hausdorff non-connected Riemann surface $\bigsqcup_{j\in\mathbb{Z}/k\mathbb{Z}}(S_j^- \cup S_j^+)$ where k is the number of petals of the dynamics of the holonomy. To obtain a full analytic invariant of the foliation one needs to collect all the gluing data and consider the whole inverse system of leaf spaces $(Q_{U_r})_r$ introduced in (1.6). Classification results of Birkhoff, Écalle [18, Théorème 2.7.1] and Martinet-Ramis [26] can be reformulated in terms of their inverse systems:

Theorem 1.5.1 *Two germs of non-linearizable resonant saddles $\mathcal{F}$ and $\mathcal{F}'$ with the same topological invariants p, q, k and the same formal invariant ν are analytically conjugated if and only if there is an isomorphism between their inverse systems of leaf spaces.*

In the above statement the notion of **isomorphism of inverse systems** is as follows, see [9, §2.8]:

Definition 1.5.2 Let Top be the category of topological spaces and continuous maps. The category $\underleftarrow{\mathrm{Top}}$ is the category whose objects are inverse systems of topological spaces

$$\mathcal{A} := \big((A_\alpha)_\alpha\,, \quad (\rho_{\alpha',\alpha} : A_\alpha \to A_{\alpha'})_{\alpha\preccurlyeq\alpha'}\big)$$

and whose morphisms, that we call here **continuous germs**[2], are the elements of

$$\underleftarrow{\mathrm{Hom}}\big(\mathcal{A},\mathcal{B}\big) := \varprojlim_\beta \varinjlim_\alpha C^0(A_\alpha, B_\beta), \tag{1.7}$$

where $C^0 := \mathrm{Hom}_{\mathrm{Top}}$. Similarly $\underleftarrow{\mathbb{C}\text{-Man}} \subset \underleftarrow{\mathrm{Top}}$ is the subcategory of inverse systems of complex manifolds, its morphisms, that we call **holomorphic germs**, being the elements of $\varprojlim_\beta \varinjlim_\alpha \mathcal{O}(A_\alpha, B_\beta)$, with $\mathcal{O} = \mathrm{Hom}_{\mathbb{C}\text{-Man}}$.

Notice that for any β the system $(\mathcal{O}(A_\alpha, B_\beta))_\alpha$ is direct, while the system

$$\Big(\varinjlim_\alpha \mathcal{O}(A_\alpha, B_\beta)\Big)_\beta$$

is inverse. This explains the limits used in the definition of $\underleftarrow{\mathrm{Hom}}$.

Going back to Theorem 1.5.1, if $Q = (Q_{U_r})_r$ and $Q' = (Q_{U'_{r'}})_{r'}$ are systems of leaf spaces of resonant saddles $\mathcal{F}$ and $\mathcal{F}'$ then the invertible elements $\underleftarrow{\mathrm{Iso}}(Q,Q') \subset \underleftarrow{\mathrm{Hom}}(Q,Q')$ correspond to biholomorphisms between the ends of leaves spaces (Hausdorff "chapelets de sphères")

$$\Phi : \varprojlim Q \xrightarrow{\sim} \varprojlim Q'$$

[2] In [21] they are called pro-germs.

that commute with the gluing maps $(\varphi_j^0, \varphi_j^\infty)$. Therefore the set of automorphisms of the ends of leaves space $\varprojlim Q$ strictly contains the automorphisms of the inverse system Q.

1.5.1.3 Real saddles $\lambda \in \mathbb{R}_{<0}$.

For this type of singularities the space of ends of leaves $\varprojlim Q$ provides a criterion of linearizability.

Theorem 1.5.3 *A real saddle is linearizable if and only if its ends of leaves space is empty.*

Proof If $\lambda = -q/p \in \mathbb{Q}_{<0}$ and ω is conjugated to $pydx + qxdy$ then has $x^p y^q$ as first integral, no leaf accumulates on S and the space of ends of leaves $\varprojlim Q$ is empty. If $\lambda \in \mathbb{Q}_{<0}$ and ω is not linearizable then the previous description in §1.5.1.2 trivially shows that the ends of leaves space $\varprojlim Q$ is non-empty.

If $\lambda \in \mathbb{R}_{<0} \setminus \mathbb{Q}$ and the singularity is linear in the coordinates (x, y) then the leaves are contained in the closed real hypersurfaces $|x||y|^\lambda$ =constant. In the non-linearizable case Pérez-Marco [36] proves the existence of orbits of the holonomy accumulating to 0. Consequently there are leaves accumulating to the separatrices and $\varprojlim Q \neq \emptyset$. □

1.5.1.4 Non-reduced logarithmic singularities

Let $\mathcal{F}$ be a logarithmic foliation germ defined by the meromorphic form

$$\lambda_1 \frac{df_1}{f_1} + \cdots + \lambda_r \frac{df_r}{f_r}$$

with $\lambda_i/\lambda_j \notin \mathbb{R}$ if $i \neq j$. By [35, Corollaire 2.5] there exists a fundamental system of neighborhoods $\mathcal{U}$ of the origin such that leaf spaces Q_U, $U \in \mathcal{U}$, are all identified with

$$\mathbb{C}/(\mathbb{Z}2i\pi\lambda_1 + \cdots + \mathbb{Z}2i\pi\lambda_r) \simeq \mathbb{C}/(\mathbb{Z}\lambda_1 + \cdots + \mathbb{Z}\lambda_r),$$

which for $r \geq 3$ is not a topological manifold. However, the leaf space of the pull-back foliation $\tilde{\mathcal{F}}$ by the universal covering map $q : \tilde{U} \to U \setminus S$, $S = \{f_1 \cdots f_r = 0\}$, can be identified with $\mathbb{C}$. This example shows the interest for non-reduced singularities to rather consider the foliation induced on the universal covering space $\tilde{U}$.

1.5.2 Complex structure on leaf spaces

Let us suppose again that $\mathcal{F}$ is a generalized curve and let us fix a Milnor tube $\mathbb{T}$ of a $\mathcal{F}$-appropriate curve S. We respectively denote by

$$q : \tilde{\mathbb{T}} \to \mathbb{T} \setminus S, \quad \Gamma := \mathrm{Aut}_q(\tilde{\mathbb{T}}), \quad \tilde{\mathcal{F}} := q^{-1}\mathcal{F},$$

the universal covering of $\mathbb{T}\setminus S$, the deck transformation group and the pull-back of the foliation $\mathcal{F}$ on $\tilde{\mathbb{T}}$. We fix a fundamental system $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ of neighborhoods of S satisfying the properties stated in Incompressibility Theorem 1.3.1 and we consider the **leaf spaces inverse system of** $\tilde{\mathcal{F}}$, i.e. the inverse system of topological spaces

$$\mathcal{Q} := \{(Q_n)_{n\in\mathbb{N}},\ (Q_n \leftarrow Q_m)_{m<n}\}, \qquad Q_n := Q^{\tilde{\mathcal{F}}}_{\tilde{U}_n} \tag{1.8}$$

where $Q^{\tilde{\mathcal{F}}}_{\tilde{U}_n}$ denotes the leaf space of the foliation $\tilde{\mathcal{F}}_{|\tilde{U}_n}$ induced by $\tilde{\mathcal{F}}$ on

$$\tilde{U}_n := q^{-1}(U_n \setminus S)$$

We associate to any subset $A \subset q^{-1}(\mathbb{T} \setminus S)$ the following inverse system of topological spaces

$$(A, \infty) := \left\{\left(A \cap \tilde{U}_n\right)_{n\in\mathbb{N}},\ \left(A \cap \tilde{U}_m \hookleftarrow A \cap \tilde{U}_n\right)_{m<n}\right\}.$$

The maps

$$\tau_{m,n} : A \cap \tilde{U}_n \to Q_m, \quad m < n,$$

sending $p \in A \cap \tilde{U}_n$ to the leaf of $\tilde{\mathcal{F}}_{|\tilde{U}_m}$ passing through p, define a morphism

$$\tau_{_A} : (A, \infty) \to \mathcal{Q} \tag{1.9}$$

in the category $\underleftarrow{\mathrm{Top}}$, that we call **tautological morphism associated to** A.

Assertion 3 in Theorem 1.3.1 is not sufficient to cover the whole leaf space with coordinate charts in the dicritical case. For this reason we give a refinement of this property:

Lemma 1.5.4 *Let $\mathbb{T}$ be a Milnor tube of an appropriate curve S of a generalized curve $\mathcal{F}$. Then there exists a fundamental system $(U_n)_{n\in\mathbb{N}}$ of open neighborhoods of S in $\mathbb{T}$ satisfying assertions 1 and 2 of Theorem 1.3.1 and there exists in each U_n a closed curve Δ_n such that*

(i) any connected component $\tilde{\Delta}_n^\alpha \subset q^{-1}(\Delta_n \setminus S)$ of the preimage of $\Delta_n \setminus S$ by the universal covering map $q : \tilde{U}_n \to U_n \setminus S$ is an embedded open disc satisfying: for each leaf L of the foliation induced by $q^{-1}\mathcal{F}$ on $\tilde{U}_n$, we have $\mathrm{card}(L \cap \tilde{\Delta}_n^\alpha) \leq 1$;
(ii) $\mathrm{sat}(\Delta_n \setminus S,\ U_n \setminus S) = U_n \setminus S$.

Idea of the proof We consider the construction process of each U_n as a union of foliated blocks V_j described in §1.3.1 and §1.3.2. It follows from the construction of each V_j in [19] that we can choose a union $\check{\Delta}_{j,n} \subset V_j$ of disjoint embedded discs such that

$$(\check{\Delta}_{j,n} \setminus S^\sharp) \overset{\mathcal{F}^\sharp}{\hookrightarrow} (V_j \setminus S^\sharp) \quad \text{and} \quad \mathrm{sat}(\check{\Delta}_{j,n} \setminus S^\sharp, V_j \setminus S^\sharp) = V_j \setminus S^\sharp .$$

By transitivity of the foliated connectedness, see (1.3), the curve $\Delta_{j,n} := E_{\mathcal{F}}(\check{\Delta}_{j,n})$ satisfies $(\Delta_{j,n} \setminus S) \overset{\mathcal{F}^\sharp}{\leftrightarrow} (U_n \setminus S)$. This relation implies that in the universal covering of $U_n \setminus S$ each connected component of $q^{-1}(\Delta_{j,n})$ is an embedded disc that meets each leaf in at most one point. To end the proof we set $\Delta_n := \cup_j \Delta_{j,n}$. □

A direct consequence of Lemma 1.5.4 is that each leaf space Q_n of the foliation induced by $\tilde{\mathcal{F}} = q^{-1}\mathcal{F}$ on $\tilde{U}_n$ is endowed with an atlas[3], called here **distinguished atlas induced by** Δ_n. The charts of this atlas are the inverses of the injective open maps

$$\tilde{\Delta}_n^\alpha \hookrightarrow Q_n$$

sending $p \in \tilde{\Delta}_n^\alpha$ to the leaf of $\tilde{\mathcal{F}}_{|\tilde{U}_n}$ containing p. We thus obtain a structure of (not necessarily Hausdorff) Riemann surface on Q_n. We can see that this structure does not depend on the choice of Δ_n.

1.5.3 Extended Holonomy along geometric blocks of the foliation

After fixing a $\mathcal{F}$-appropriate curve S, let us now consider an invariant irreducible component D of the total transform $S^\sharp$ of S, that is not contained in a dead branch of $S^\sharp$. Let us also consider a smooth retraction defined on a tubular neighborhood Ω_D of D,

$$\rho_D : \Omega_D \to D,$$

that is a locally trivial disc fibration, holomorphic on a neighborhood of each singular point $s_1, \ldots, s_\nu$ of $\mathcal{F}^\sharp$ in D and such that the discs $\rho_D^{-1}(s_j)$ are in $S^\sharp$. Up to permutation, we suppose that $s_1, \ldots, s_r$, $1 \leq r \leq \nu$, are the singular points s_j which do not belong to any dead branch of $S^\sharp$. We choose small enough disjoint closed discs

$$D_{s_j} \subset D, \quad j = 1, \ldots, \nu,$$

centered at s_j so that ρ_D is holomorphic and trivial along D_{s_j}. We also suppose the Milnor tube $\mathbb{T}$ of S small enough so that the real hypersurfaces $\rho_D^{-1}(\partial D_{s_j})$ intersect transversally the boundary of $E_{\mathcal{F}}^{-1}(\mathbb{T})$ along a 2-torus contained in $E_{\mathcal{F}}^{-1}(\delta\mathbb{T})$. Then we consider the closure $\mathcal{B}_D$ of the connected component of the complement in $E_{\mathcal{F}}^{-1}(\mathbb{T})$ of $\cup_{j=1}^r \rho_D^{-1}(D_{s_j})$ that contains the complement of $\bigcup_{j=1}^r D_{s_j}$ in D, i.e.

$$K_D := D \setminus \bigcup_{j=1}^r \mathring{D}_{s_j} \ \subset\ \mathcal{B}_D \ \subset\ E_{\mathcal{F}}^{-1}(\mathbb{T}) \setminus \bigcup_{j=1}^r \rho_D^{-1}(\mathring{D}_{s_j}).$$

We fix a connected component $\tilde{\mathcal{B}}_D^\xi$ of $\tilde{\mathcal{B}}_D := q^{-1}(E_{\mathcal{F}}(\mathcal{B}_D) \setminus S)$ and we will denote by

$$\Gamma_\xi \subset \Gamma$$

[3] In fact, it is an "atlas" with values in $q^{-1}(\Delta_n)$.

the subgroup of deck transformations of the covering q leaving invariant $\tilde{\mathcal{B}}_D^{\xi}$. Using the particular presentation of the fundamental group of $\mathcal{B}_D \setminus S^{\sharp}_{\mathcal{F}}$ given in [20, Proposition 3.3] we can prove that

- *$\mathcal{B}_D \setminus S^{\sharp}$ is incompressible in $E_{\mathcal{F}}^{-1}(\mathbb{T}) \setminus S^{\sharp}$, consequently the restriction of q to $\tilde{\mathcal{B}}_D^{\xi}$ is the universal covering of $E_{\mathcal{F}}(\mathcal{B}_D \setminus S^{\sharp})$ and Γ_{ξ} can be canonically identified with the deck transformation group of this covering;*
- *$\mathcal{B}_D \cap S^{\sharp}$ contains all the dead branches of $S^{\sharp}$ meeting D;*
- *the restriction of ρ_D to $(\mathcal{B}_D \setminus S^{\sharp}) \cap \Omega_D$ extends to a smooth surjective map*

$$\pi_D : \mathcal{B}_D \setminus S^{\sharp} \to K_D \tag{1.10}$$

  that is a fibration in punctured discs, locally trivial, except when $r < v$ at the points $s_{r+1}, \ldots, s_v$; in this case the closure of $\pi_D^{-1}(s_j)$ is a closed disc transverse to the end component of the dead branch of $S^{\sharp}$ containing s_j, $j = r+1, \ldots, v$;
- *there is a retraction by deformation along the fibers of π_D of $\mathcal{B}_D \setminus S^{\sharp}$ to $\delta\mathcal{B}_D := \mathcal{B}_D \cap E_{\mathcal{F}}^{-1}(\delta\mathbb{T})$;*
- *the restriction of π_D to $\delta\mathcal{B}_D$ is a Seifert fibration whose exceptional fibers are the transverse intersections $\delta\mathcal{B}_D \cap \pi_D^{-1}(s_j)$, $j = r+1, \ldots v$, when $r < v$.*

Notations 1.5.5 Let us now fix a connected regular curve $\Sigma \subset \mathcal{B}_D$ that meets $D \subset S^{\sharp}$ at a regular point σ and is transverse to $\mathcal{F}^{\sharp}$. We denote by $\tilde{\Sigma}^j$, $j \in \mathcal{J}$, the connected components of $q^{-1}(E_{\mathcal{F}}(\Sigma)) \cap \tilde{\mathcal{B}}_D^{\xi}$. We assume that the fundamental system $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ satisfies assertion 3 in Theorem 1.3.1 taking $C = E_{\mathcal{F}}(\Sigma)$. Then each $\tilde{\Sigma}_n^j := \tilde{\Sigma}^j \cap \tilde{U}_n$ is an embedded disc and $\text{card}(L \cap \tilde{\Sigma}_n^j) \leq 1$ for each leaf L of $\tilde{\mathcal{F}}_{|\tilde{\mathcal{B}}_D^{\xi}}$. □

Remark 1.5.6 The holomorphic maps $\tau_n^j : \tilde{\Sigma}_n^j \hookrightarrow Q_n$ sending $p \in \tilde{\Sigma}_n^j$ into the leaf of $\tilde{\mathcal{F}}_{|\tilde{U}_n}$ passing through p, define the tautological morphism

$$\tau_{\tilde{\Sigma}^j} : (\tilde{\Sigma}^j, \infty) \to Q$$

introduced in (1.9). As τ_n^j are injective, $\tau_{\tilde{\Sigma}^j}$ is a monomorphism in the category $\underleftarrow{\mathbb{C}\text{-Man}}$. □

Proposition 1.5.7 ([21, Proposition 4.2.1]) *For any $i, j \in \mathcal{J}$ and large enough $n \in \mathbb{N}$, the images of the maps τ_n^i and τ_n^j have a non-empty intersection W_n^{ij}, and for $p \gg n \gg 1$ we have the following inclusions*

$$\tilde{\Sigma}_n^k \cap q^{-1}(U_p) \subset (\tau_n^k)^{-1}(W_n^{ij}), \qquad k = i, j\,.$$

The family of holomorphic maps

$$m_n^{ij} = (\tau_n^i)^{-1} \circ \tau_n^j : (\tau_n^j)^{-1}(W_n^{ij}) \to (\tau_n^i)^{-1}(W_n^{ij}), \quad n \in \mathbb{N}\,.$$

defines a morphism in the category $\underleftarrow{\mathbb{C}\text{-Man}}$

$$m^{ij} \in \mathrm{Hom}_{\underleftarrow{\mathbb{C}\text{-Man}}}\big((\tilde{\Sigma}^j,\infty),\ (\tilde{\Sigma}^i,\infty)\big)$$

such that $\tau_{\tilde{\Sigma}^i} \circ m^{ij} = \tau_{\tilde{\Sigma}^j}$ *and for any* $i, j, k \in \mathcal{J}$ *and any deck automorphism* $\varphi \in \Gamma_\xi$ *we have*

$$m^{ij} \circ m^{jk} = m^{ik}\,, \quad m^{ij} = (m^{ji})^{-1}\,, \quad \varphi \circ m^{ij} \circ \varphi^{-1} = m^{\varphi(i)\varphi(j)}\,, \tag{1.11}$$

where $\varphi(k) \in \mathcal{J}$ *is the index such that* $\varphi(\tilde{\Sigma}^k) = \tilde{\Sigma}^{\varphi(k)}$.

We easily deduce from relations (1.11) that the map

$$\mathcal{H}^i : \Gamma_\xi \longrightarrow \underleftarrow{\mathrm{Aut}}(\tilde{\Sigma}^i,\infty)\,, \quad \gamma \mapsto \mathcal{H}^i(\gamma) := m^{i\gamma(i)} \circ \gamma\,,$$

is well defined and is a group morphism. We call it the **extended holonomy of $\mathcal{F}$ along $\mathcal{B}_D$ over** $\tilde{\Sigma}^i$ (see Figure 1.7).

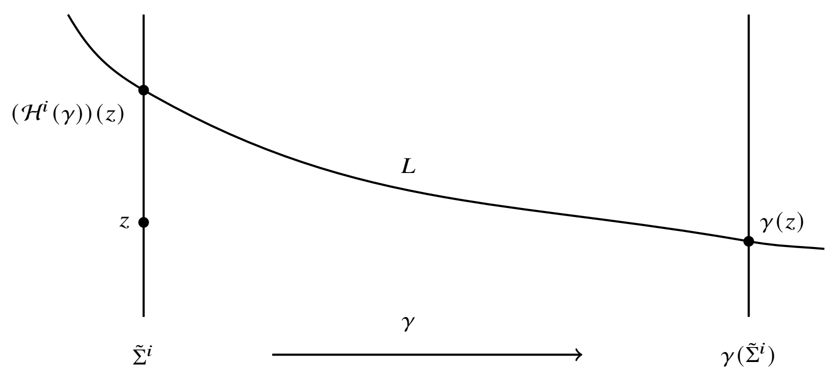


Fig. 1.7 The extended holonomy transformation $\mathcal{H}^i(\gamma)$: the deck transformation γ is represented as an horizontal translation and L is the leaf of $\tilde{\mathcal{F}} = q^{-1}\mathcal{F}$ containing $\gamma(z)$.

1.5.4 Monodromy representation of a singular foliation

Keeping the notations of previous subsection, let us consider now the group

$$\underleftarrow{\mathrm{Aut}}(Q) \subset \underleftarrow{\mathrm{Hom}}(Q, Q)$$

of invertible morphisms of $\underleftarrow{\mathrm{Hom}}(Q,Q)$ in the category $\underleftarrow{\mathrm{Top}}$, see (1.7). Any deck transformation $\gamma \in \Gamma$ keeps invariant the foliation $\tilde{\mathcal{F}}$, it induces an automorphism of each leaf space $Q_{\tilde{U}_n}$, $n \in \mathbb{N}$, and thus defines an automorphism $\gamma_* \in \underleftarrow{\mathrm{Aut}}(Q)$.

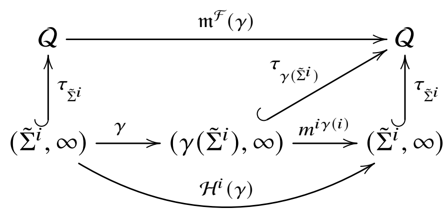
Definition 1.5.8 The map

$$\mathfrak{m}^{\mathcal{F}} : \Gamma \to \underleftarrow{\mathrm{Aut}}(Q)\,, \quad \gamma \mapsto \gamma_*\,,$$

is a group morphism which we call **monodromy representation** of $\mathcal{F}$.

In fact the values of $\mathfrak{m}^{\mathcal{F}}$ are in the group of holomorphic automorphisms of Q.

The commutative diagram below shows that $\mathcal{H}^i(\gamma)$ can be interpreted as the "expression of $\mathfrak{m}^{\mathcal{F}}(\gamma)$ in the chart $\tilde{\Sigma}^i$"



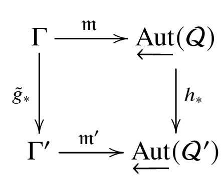
(1.12)

Now for another foliation $\mathcal{F}'$, let us fix $\mathbb{T}'$, S', Δ', $\check{\Delta}'$, $\mathcal{U}' := (U'_n)_n$, $q' : \tilde{\mathbb{T}}' \to \mathbb{T}' \setminus S'$, D', $\tilde{\mathcal{B}}'^{\xi'}_{D'}$, Σ', σ' and let us denote by $\tilde{\mathcal{F}}'$, Γ', Q', $(\tilde{\Sigma}'^j_n)_{j\in\mathcal{J}'}$, $\Gamma'_{\xi'}$, the analogous elements for $\mathcal{F}'$ that we have considered for $\mathcal{F}$. We will also denote more simply by $\mathfrak{m}$ and $\mathfrak{m}'$ the monodromy representations $\mathfrak{m}^{\mathcal{F}}$ and $\mathfrak{m}^{\mathcal{F}'}$ of $\mathcal{F}$ and $\mathcal{F}'$, and by $\mathcal{H}^i$ and $\mathcal{H}'^{i'}$ their extended holonomies along $\mathcal{B}_D$ and $\mathcal{B}'_{D'}$, over $\tilde{\Sigma}^i$ and $\tilde{\Sigma}'^{i'}$.

A C^0**-conjugacy**[4] **between the monodromies** $\mathfrak{m}$ and $\mathfrak{m}'$ is a pair $(\tilde{g}_*, h)$ formed by

(i) a group morphism $\tilde{g}_* : \Gamma \to \Gamma'$, $\gamma \mapsto \tilde{g}\circ\gamma\circ\tilde{g}^{-1}$, defined by a lifting $\tilde{g} : \tilde{\mathbb{T}} \to \tilde{\mathbb{T}}'$ of a homeomorphism $g : (\mathbb{T}, S) \to (\mathbb{T}', S')$,
(ii) an isomorphism $h : Q \to Q'$ in the category $\underleftarrow{\mathrm{Top}}$

such that the following diagram is commutative:



(1.13)

[4] In [21] this corresponds to the notion of geometric conjugacy.

where $\tilde{g}_*(\gamma) := \tilde{g} \circ \gamma \circ \tilde{g}^{-1}$ and $h_*(\psi) := h \circ \psi \circ h^{-1}$. According to Theorem 1.2.3, without modifying the first element $\tilde{g}_*$, we can always choose the homeomorphism g so that its lifting through the reduction maps $E_{\mathcal{F}}$ and $E_{\mathcal{F}'}$ extends homeomorphically to the exceptional divisor $\mathcal{E}_{\mathcal{F}}$ and this extension is holomorphic at the singular points of $S^\sharp := E_{\mathcal{F}}^{-1}(S)$; we then say that g is an **excellent conjugacy between** S **and** S'.

Definition 1.5.9 We will say that a **conjugacy** $(\tilde{g}_*, h)$ **between** $\mathfrak{m}$ **and** $\mathfrak{m}'$ is **excellent** if g is excellent and h is **excellent** in the following sense: there is a union K of nodal and dicritical separators of $\mathcal{F}$ such that $h = \varprojlim_n \varinjlim_m h_{mn}$ and, for large enough m, n, the C^0-maps $h_{mn} : Q_m \to Q'_n$ are holomorphic at each point $L \in Q_m$ that is a leaf not intersecting K.

1.5.5 Monodromy vs Holonomy conjugacies

A **holomorphic conjugacy between the extended holonomies** $\mathcal{H}^i$ and $\mathcal{H}'^{i'}$ along $\mathcal{B}_D$ and $\mathcal{B}'_{D'}$ over $\tilde{\Sigma}^i$ and $\tilde{\Sigma}'^{i'}$ is a pair $(\tilde{g}_*, \tilde{\varphi})$ formed by a group morphism $\tilde{g}_* : \Gamma_\xi \to \Gamma'_{\xi'}$ such that:

1. g is an excellent conjugacy between S and S' whose lifting $g^\sharp$ through $E_{\mathcal{F}}$ and $E_{\mathcal{F}'}$ sends D onto D',
2. $\tilde{g}$ is a lifting of g through q and q' such that for large enough $n \gg p$, $\tilde{g}(\tilde{\Sigma}^i_n)$ is contained in $\tilde{\Sigma}'^{i'}_p$ and consequently the induced by g group morphism $\tilde{g}_* : \Gamma \to \Gamma'$ sends Γ_ξ onto $\Gamma'_{\xi'}$,
3. $\tilde{\varphi} : (\tilde{\Sigma}^i, \infty) \to (\tilde{\Sigma}'^{i'}, \infty)$ is a lifting of a biholomorphism germ $\varphi : (\Sigma, \sigma) \to (\Sigma', \sigma')$ and for any $\gamma \in \Gamma_\xi$ the following diagram is commutative:

$$\begin{array}{ccc} (\tilde{\Sigma}^i, \infty) & \xrightarrow{\tilde{\varphi}} & (\tilde{\Sigma}'^{i'}, \infty) \\ \downarrow{\scriptstyle \mathcal{H}^i(\gamma)} & & \downarrow{\scriptstyle \mathcal{H}'^{i'}(\tilde{g}_*(\gamma))} \\ (\tilde{\Sigma}^i, \infty) & \xrightarrow{\tilde{\varphi}} & (\tilde{\Sigma}'^{i'}, \infty) \, . \end{array} \tag{1.14}$$

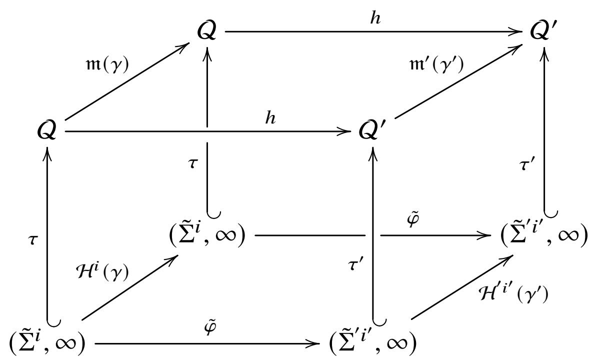
With Notations 1.5.5 we have:

Proposition 1.5.10 *Let us consider a C^0-conjugacy $(\tilde{g}_*, h)$ between the monodromies $\mathfrak{m}$ and $\mathfrak{m}'$ of generalized curves $\mathcal{F}$ and $\mathcal{F}'$. Let $\tilde{\varphi} : (\tilde{\Sigma}^i, \infty) \to (\tilde{\Sigma}'^{i'}, \infty)$ be a lifting of a biholomorphism germ $\varphi : (\Sigma, \sigma) \to (\Sigma', \sigma')$ that is **compatible with** h in the sense that the following diagram*

$$\begin{array}{ccc} (\tilde{\Sigma}^i, \infty) & \xrightarrow{\tilde{\varphi}} & (\tilde{\Sigma}'^{i'}, \infty) \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau'} \\ Q & \xrightarrow{h} & Q' \end{array} \tag{1.15}$$

is commutative, with τ*, resp.* τ'*, denoting the tautological morphisms* $\tau_{\tilde{\Sigma}^i}$*, resp.* $\tau_{\tilde{\Sigma}'^{i'}}$ *defined in (1.9). Assume also that* g *is excellent,* $g^\sharp(D) = D'$ *and* $\tilde{g}$ *satisfies* $\tilde{g}(\tilde{\Sigma}^i_n) \subset \tilde{\Sigma}'^{i'}_n$*. Then* $(\tilde{g}_*, \tilde{\varphi})$ *is a holomorphic conjugacy between the extended holonomies of* $\mathcal{F}$ *and* $\mathcal{F}'$ *along* $\mathcal{B}_D$ *and* $\mathcal{B}'_{D'}$ *over* $\tilde{\Sigma}^i$ *and* $\tilde{\Sigma}'^{i'}$*.*

Proof For any connected component $\tilde{\Sigma}^i$ of $\tilde{\Sigma}$, let us consider the following diagram where $\tilde{\Sigma}'^{i'}$ is the image of $\tilde{\Sigma}^i$ by $\tilde{\varphi}$, $\gamma \in \Gamma_\xi$ and $\gamma' := \tilde{\varphi}_*(\gamma) \in \Gamma'_{\xi'}$.



(1.16)

According to the $\tilde{\varphi}$-compatibility relation (1.15) and the extended holonomy relations (1.12), all lateral squares of this cubic diagram are commutative. We need to prove the commutativity of the bottom square (1.14). For this, first notice that we have the following equivalence:

$$\tilde{\varphi} \circ \mathcal{H}^i(\gamma) = \mathcal{H}'^{i'} \circ \tilde{\varphi} \iff \tau' \circ \tilde{\varphi} \circ \mathcal{H}^i(\gamma) = \tau' \circ \mathcal{H}'^{i'} \circ \tilde{\varphi} \tag{1.17}$$

because τ' is a monomorphism, see Remark 1.5.6. But the commutativity of the lateral squares gives:

$$\tau' \circ \tilde{\varphi} \circ \mathcal{H}^i(\gamma) = h \circ \mathfrak{m}(\gamma) \circ \tau \quad \text{and} \quad \tau' \circ \mathcal{H}'^{i'} \circ \tilde{\varphi} = \mathfrak{m}'(\gamma') \circ h \circ \tau\,.$$

Therefore the two equivalent equalities in (1.17) are also equivalent to

$$h \circ \mathfrak{m}(\gamma) \circ \tau = \mathfrak{m}'(\gamma') \circ h \circ \tau\,.$$

But that equality is satisfied because the top square in diagram (1.16) is commutative thanks to the conjugacy of monodromies relation (1.13). □

With the same notations we also have:

Proposition 1.5.11 *Let* $(\tilde{g}_*, \tilde{\varphi})$ *be a holomorphic conjugacy between the extended holonomies* $\mathcal{H}^i$ *and* $\mathcal{H}'^{i'}$ *of generalized curves* $\mathcal{F}$ *and* $\mathcal{F}'$ *along* $\mathcal{B}_D$ *and* $\mathcal{B}_{D'}$ *over* $\tilde{\Sigma}^i$ *an* $\Sigma'^{i'}$ *respectively. Then the holonomy representations* $H_D^{\mathcal{F}}$ *and* $H_{D'}^{\mathcal{F}'}$ *of the foliations* $\mathcal{F}^\sharp$ *and* $\mathcal{F}'^\sharp$ *along* D *and* D' *are holomorphically conjugated, more precisely the following diagram is commutative:*

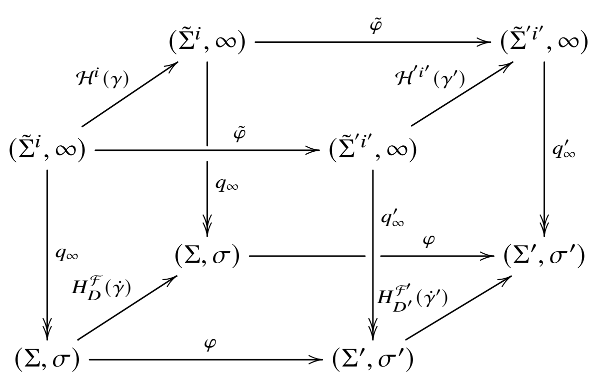
$$\begin{array}{ccc} \pi_1(D^*,\sigma) & \xrightarrow{H_D^{\mathcal{F}}} & \mathrm{Diff}(\Sigma,\sigma) \\ {\scriptstyle g_*^\sharp}\downarrow & & \downarrow{\scriptstyle \varphi_*} \\ \pi_1(D'^*,\sigma') & \xrightarrow{H_{D'}^{\mathcal{F}'}} & \mathrm{Diff}(\Sigma',\sigma') \end{array} \tag{1.18}$$

with $g_*^\sharp$ induced by the restriction of $g^\sharp$ to $D^* := D \setminus \mathrm{Sing}(\mathcal{F}^\sharp)$ onto $D'^* := D' \setminus \mathrm{Sing}(\mathcal{F}'^\sharp)$ and $\varphi_*(f) := \varphi \circ f \circ \varphi^{-1}$.

Idea of the proof We can assume that Σ is the (regular) fiber over $\sigma \in D \setminus \mathrm{Sing}(\mathcal{F}^\sharp)$ of the fibration $\pi_D : \mathcal{B}_D \setminus S^\sharp \to K_D$ previously described, see (1.10). For any $\gamma \in \Gamma_\xi$, thanks to Proposition 1.5.7 there is a path α in a leaf of $\tilde{\mathcal{F}}_{|\tilde{\mathcal{B}}_D^\xi}$ with endpoints in $\tilde{\Sigma}^i$ and $\gamma(\tilde{\Sigma}^i)$; when $r < v$ we can choose α generic enough so that the image of $E_{\mathcal{F}}^{-1} \circ q \circ \alpha$ meets no exceptional fiber $\pi_D^{-1}(s_j)$, $j = r+1, \ldots, v$. The particular presentation of the fundamental group of $\mathcal{B}_D \setminus S^\sharp$ given in [21, §4.1], allows to check that

- *the class $\dot{\gamma}$ of the loop $\pi_D \circ E_{\mathcal{F}}^{-1} \circ q \circ \alpha$ in the quotient group $\pi_1(D^*,\sigma)/\ker(H_D^{\mathcal{F}})$ does not depend on the choice of the path α but only on γ;*
- *the map $\Gamma_\xi \to \pi_1(D^*,\sigma)/\ker(H_D^{\mathcal{F}})$, $\gamma \mapsto \dot{\gamma}$, is surjective and $H_D^{\mathcal{F}}(\dot{\gamma})$ is well defined.*

For $\gamma' = \tilde{g}_*(\gamma) \in \Gamma'_{\xi'}$ we similarly define $\dot{\gamma}' \in \pi_1(D'^*,\sigma')/\ker(H_{D'}^{\mathcal{F}'})$. Let q_∞, resp. q'_∞, be the morphism in the category $\underleftarrow{\mathbb{C}\text{-Man}}$ induced by the restrictions of the covering map q, resp. q', to $q^{-1}(U_n)$, resp. $q'^{-1}(U'_n)$, $n \in \mathbb{N}$. It follows from the construction of $\dot{\gamma}$ and $\dot{\gamma}'$ that all lateral squares of the following cubic diagram are commutative:



The top square is commutative by hypothesis. We deduce that the bottom square is also commutative by proceeding as in the proof of Proposition 1.5.10 after noting that q_∞ and q'_∞ are epimorphisms in the category $\underleftarrow{\mathbb{C}\text{-Man}}$. □

Finally, as a consequence of Propositions 1.5.10 and 1.5.11 we obtain:

Theorem 1.5.12 *[21, Theorem 4.3.1] Let $(\tilde{g}_*, h)$ be a conjugacy between the monodromies of generalized curves $\mathcal{F}$ and $\mathcal{F}'$, let Σ and $\Sigma' := g^\sharp(\Sigma)$ be transverse* *sections to invariant irreducible components $D \subset \mathcal{E}_{\mathcal{F}}$ and $D' := g^\sharp(D) \subset \mathcal{E}_{\mathcal{F}'}$. Let $\tilde{\varphi} : \tilde{\Sigma} \xrightarrow{\sim} \tilde{\Sigma}'$ be a lifting of a biholomorphism $\varphi : \Sigma \to \Sigma'$, that is compatible with h. Then the holonomies $H_D^{\mathcal{F}}$ and $H_{D'}^{\mathcal{F}'}$ of $\mathcal{F}^\sharp$ and $\mathcal{F}'^\sharp$ along D and D' are holomorphically conjugated by φ, i.e. diagram (1.18) is commutative.*

1.5.6 Classification Theorem

We consider again two germs of foliations $\mathcal{F}$ and $\mathcal{F}'$ at $0 \in \mathbb{C}^2$ that are generalized curves. We keep the previous notations: in a Milnor tube $\mathbb{T}$, resp. $\mathbb{T}'$, of an appropriate curve S for $\mathcal{F}$, resp. S' for $\mathcal{F}'$; we again denote by $q : \tilde{\mathbb{T}} \to \mathbb{T} \setminus S$ and $q' : \tilde{\mathbb{T}}' \to \mathbb{T}' \setminus S'$ the universal covering maps and by Γ and Γ' the deck transformation groups of these coverings; $\mathcal{Q}$, resp. $\mathcal{Q}'$, denotes the leaf spaces inverse system for $\mathcal{F}$, resp. $\mathcal{F}'$, defined as in (1.8) by a fundamental system $\mathcal{U} = (U_n)_n$, resp. $\mathcal{U}' = (U'_n)_n$, of open neighborhoods of S, resp. of S', that satisfies the properties stated in Theorem 1.3.1.

Definition 1.5.13 Given two subsets $V \subset \mathbb{T}$ and $V' \subset \mathbb{T}'$ whose closures meet S and S' respectively, we call **realization on the germs** (V, S) **and** (V', S') **of a conjugacy** $(\tilde{g}_*, h)$ between monodromies $\mathfrak{m}$ and $\mathfrak{m}'$ of $\mathcal{F}$ and $\mathcal{F}'$, the data $(\psi, \tilde{\psi})$ of a germ of homeomorphism $\psi : (V, S) \to (V', S')$ and a $\underleftarrow{\mathrm{Top}}$-morphism

$$\tilde{\psi} : (\tilde{V}, \infty) \longrightarrow (\tilde{V}', \infty)\,, \quad (\tilde{V}, \infty) := (q^{-1}(V \cap U_n))_n\,, \quad (\tilde{V}', \infty) := (q'^{-1}(V' \cap U'_n))_n\,,$$

that lifts ψ, i.e. $q' \circ \tilde{\psi} = \psi \circ q$, and satisfies the commutativity of the following diagrams

$$\begin{array}{ccc} (\tilde{V}, \infty) & \xrightarrow{\tilde{\psi}} & (\tilde{V}', \infty) \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau'} \\ \mathcal{Q} & \xrightarrow{h} & \mathcal{Q}' \end{array} \qquad\qquad \begin{array}{ccc} \Gamma_{\tilde{V}} & \xrightarrow{\tilde{\psi}_*} & \Gamma'_{\tilde{V}'} \\ {\scriptstyle \iota}\uparrow & & \uparrow{\scriptstyle \iota'} \\ \Gamma & \xrightarrow{\tilde{g}^*} & \Gamma' \end{array}$$

where $\Gamma_{\tilde{V}}$ and $\Gamma'_{\tilde{V}'}$ are the deck transformation groups of the (not necessarily connected) coverings $q^{-1}(V) \to V$ and $q'^{-1}(V') \to V'$ induced by q and q'.

When ψ lifts through the reduction of singularities maps as a homeomorphism germ

$$\psi^\sharp : (\hat{V}, S^\sharp) \longrightarrow (\hat{V}', S'^\sharp)\,, \qquad E_{\mathcal{F}'} \circ \psi^\sharp = \psi \circ E_{\mathcal{F}}\,,$$

between the strict transforms $\hat{V} := \overline{E_{\mathcal{F}}^{-1}(V) \setminus S^\sharp}$ and $\hat{V}' := \overline{E_{\mathcal{F}'}^{-1}(V') \setminus S'^\sharp}$ of V and V' we will also say that $(\psi^\sharp, \tilde{\psi})$ is a **realization of** $(\tilde{g}_*, h)$ **on the germs** $(\hat{V}, S^\sharp)$ **and** $(\hat{V}', S'^\sharp)$.

Remark 1.5.14 If V contains an open set U and $(\psi, \tilde{\psi})$ realizes on (V, S) and (V', S') a conjugacy between the monodromies of $\mathcal{F}$ and $\mathcal{F}'$, then the restriction of ψ to U is a conjugacy between the restriction of $\mathcal{F}$ to U and $\mathcal{F}'$ to $\psi(U)$. □

Definition 1.5.15 We call **complete $\mathcal{F}$-transversal curve** in a Milnor ball B for $S_{\mathcal{F}}$, any finite union Δ of disjoint connected embedded discs in $B\setminus\{0\}$ that are transverse to $\mathcal{F}$ and such that any dynamical component of $S^\sharp_{\mathcal{F}}$ meets $E^{-1}_{\mathcal{F}}(\Delta)$ at a single point (necessarily belonging to the strict transform of an isolated separatrix of $\mathcal{F}$).

Theorem 1.2.8 assures the existence of a complete $\mathcal{F}$-transversal curve by choosing a non-nodal isolated separatrix in each dynamical component of $S^\sharp_{\mathcal{F}}$ and a small embedded disc transverse to their image by $E_{\mathcal{F}}$.

Theorem 1.5.16 ([20, Classification Theorem]) *Let us fix a complete $\mathcal{F}$-transversal curve Δ, resp. complete $\mathcal{F}'$-transversal curve Δ'. Assume that there exists a C^0-conjugacy $(\tilde{g}_*, h)$ between the monodromy representations $\mathfrak{m}$ and $\mathfrak{m}'$ of generalized curves $\mathcal{F}$ and $\mathcal{F}'$ and let*

$$\tilde{\varphi}:\tilde{\Delta}\to\tilde{\Delta}',\quad \tilde{\Delta}:=q^{-1}(\Delta),\quad \tilde{\Delta}':=q'^{-1}(\Delta'),$$

be a lifting through the covering maps q and q' of a germ of biholomorphism $\varphi:(\Delta,S_{\mathcal{F}})\to(\Delta',S_{\mathcal{F}'})$. Let us assume that $(\varphi,\tilde{\varphi})$ is a realization of the conjugacy $(\tilde{g}_*,h)$ on the germs $(\Delta,S_{\mathcal{F}})$ and $(\Delta',S_{\mathcal{F}'})$ and that

1. *the image by φ of any connected component of Δ meeting an isolated separatrix C of $\mathcal{F}$ is the connected component of Δ' meeting $g(C)$,*
2. *the Camacho-Sad index of $\mathcal{F}^\sharp$ along the strict transform $\mathscr{C}$ of any isolated separatrix C of $\mathcal{F}$, is equal to that of $\mathcal{F}'^\sharp$ along the strict transform $\mathscr{C}'$ of $g(C)$, i.e.*

$$\mathrm{CS}(\mathcal{F}^\sharp,\mathscr{C},s)=\mathrm{CS}(\mathcal{F}'^\sharp,\mathscr{C}',s'),\qquad \{s\}:=\mathscr{C}\cap\mathcal{E}_{\mathcal{F}},\quad \{s'\}:=\mathscr{C}'\cap\mathcal{E}_{\mathcal{F}'}.$$

Then there exists a homeomorphism germ $\Phi:(\mathbb{C}^2,0)\to(\mathbb{C}^2,0)$ that conjugates $\mathcal{F}$ to $\mathcal{F}'$ and lifts through the reduction maps, to a homeomorphism germ:

$$\Phi^\sharp:(M_{\mathcal{F}},S^\sharp_{\mathcal{F}})\to(M_{\mathcal{F}'},S^\sharp_{\mathcal{F}'}),\quad E_{\mathcal{F}'}\circ\Phi^\sharp=\Phi\circ E_{\mathcal{F}},\quad \Phi^\sharp(\mathcal{F}^\sharp)=\mathcal{F}'^\sharp.$$

Moreover $\Phi^\sharp$ is holomorphic at each non-nodal singular point of $\mathcal{F}^\sharp$ and is transversally holomorphic at each point of $S^\sharp_{\mathcal{F}}$ that is regular for $\mathcal{F}^\sharp$ and does not belong to a dicritical component of $\mathcal{E}_{\mathcal{F}}$.

Notice that the holomorphy of φ on the complete $\mathcal{F}$-transversal curve Δ implies that h is necessarily excellent in the sense of Definition 1.5.9.

Definition 1.5.17 A homeomorphism Φ that satisfies the conclusions of Theorem 1.5.16 will be called **excellent conjugacy** between $\mathcal{F}$ and $\mathcal{F}'$ and we will say that $\mathcal{F}$ and $\mathcal{F}'$ are **C^{ex}-conjugated**.

Remark 1.5.18 Any C^{ex}-conjugacy Φ between $\mathcal{F}$ and $\mathcal{F}'$ defines an excellent conjugacy between the monodromies of $\mathcal{F}$ and $\mathcal{F}'$ for which $(\Phi,\tilde{\Phi})$ is a realization on neighborhoods of the origin. □

Idea of the proof With some additional assumptions this theorem is proven in [21] for foliations with a single dynamical component. A proof for the general statement is given in [23, Proof of Theorem 11.4, Steps (iii)-(v)]. The conjugacy is independently constructed along each dynamical component of $S^\sharp_{\mathcal{F}}$, their gluing on separators being made in a final step.

We divide any dynamical component $\mathcal{D}$ of $S^\sharp_{\mathcal{F}}$ in **elementary pieces** that are either (small pieces) connected components of the intersection of $S^\sharp_{\mathcal{F}}$ with small closed polydiscs centered at the singular points of $\mathcal{F}^\sharp$, or (big pieces) the closure of the connected components of the complement in $\mathcal{D}$ of the union of all small pieces. Over each big piece K we fix a locally trivial smooth retraction-fibration $\rho_K : \Omega_K \to K$ defined on an open neighborhood Ω_K of K whose fibers are discs transverse to $\mathcal{F}^\sharp$. We can suppose that ρ_K is holomorphic at each point of the boundary ∂K of K. We can construct an exhaustion of $\mathcal{D}$ by connected closed sets,

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_k = \mathcal{D}\,, \qquad Z_j = Z_{j-1} \cup K_j\,,$$

where Z_0 is the big piece whose image by $E_{\mathcal{F}}$ is contained in the (unique) isolated separatrix in $\mathcal{D}$ meeting Δ, and the closure K_j of $Z_j \setminus Z_{j-1}$ is an elementary piece intersecting Z_{j-1}. Without loss of generality we suppose that g is excellent and that $g^\sharp$ is holomorphic on a neighborhood of any small elementary piece. As we will see in §1.6.2 the Camacho-Sad equalities on the isolated separatrices in hypothesis 2 induce the equality

$$\mathrm{CS}(\mathcal{F}^\sharp, D, s) = \mathrm{CS}(\mathcal{F}'^\sharp, g^\sharp(D), g^\sharp(s)) \tag{1.19}$$

for any irreducible component D of $S^\sharp_{\mathcal{F}}$ and any singular point $s \in D$ of $\mathcal{F}^\sharp$. Therefore $g^\sharp$ sends any dynamical component of $S^\sharp_{\mathcal{F}}$ onto a dynamical component of $S^\sharp_{\mathcal{F}'}$.

We also have the exhaustion

$$Z'_0 \subsetneq Z'_1 \subsetneq \ldots \subsetneq Z'_k = \mathcal{D}'\,, \qquad Z'_i := g^\sharp(Z_i)\,,$$

of the dynamical component $\mathcal{D}' := g^\sharp(\mathcal{D})$ of $S^\sharp_{\mathcal{F}'}$. Up to performing isotopies [20, Theorem 2.9 and Definition 2.5] we can also suppose that for any big piece $K \subset \mathcal{D}$ the map $g^\sharp$ sends the fiber of ρ_K over any $p \in K$ to the fiber over $g^\sharp(p)$ of a locally trivial smooth fibration $\rho'_{g^\sharp(K)} : \Omega' \to g^\sharp(K)$ that we have previously fixed, i.e.

$$\rho'_{g^\sharp(K)} \circ g^\sharp = g^\sharp \circ \rho_K\,.$$

We will successively construct realizations $(\psi_j, \tilde{\psi}_j)$ of the monodromy conjugacy $(\tilde{g}_*, h)$, where ψ_j is a homeomorphism from a neighborhood of Z_j to a neighborhood of Z'_j, that extends ψ_{j-1}. Moreover, we will have

$$\rho'_{g^\sharp(K)} \circ \psi_j = \psi_j \circ \rho_K \tag{1.20}$$

on the boundary of each elementary big piece K intersecting Z_j.

The proof ends by applying Remark 1.5.14 to the last homeomorphism $\Phi := \psi_k$; this one conjugates $\mathcal{F}^\sharp$ to $\mathcal{F}'^\sharp$ on neighborhoods of $\mathcal{D}$ and $\mathcal{D}'$ and the holomorphic transversality property and the holomorphy property at the non-nodal singular points will result from the construction.

Assume that we have already constructed ψ_{j-1} and $\tilde{\psi}_{j-1}$. Let us fix a point $p_j \in K_j \cap Z_{j-1}$ and denote by K the big piece containing p_j. Consider the transverse discs

$$\Sigma_j := \rho_K^{-1}(p_j)\,, \qquad \Sigma'_j := \rho'^{-1}_{g^\sharp(K)}(p'_j) = g^\sharp(\Sigma_j)\,, \qquad p'_j := g^\sharp(p_j)\,,$$

and the liftings by their corresponding universal covering maps

$$\tilde{\Sigma}_j := q^{-1}(E_{\mathcal{F}}(\Sigma_j))\,, \quad \tilde{\Sigma}'_j := q'^{-1}(E_{\mathcal{F}'}(\Sigma'_j))\,.$$

The key of the induction process consists in applying the following Extension Lemma taking for ϕ the restriction of ψ_{j-1} to Σ_j and for $\tilde{\phi}_j$ the restriction of $\tilde{\psi}_{j-1}$ to $\tilde{\Sigma}_j$.

Lemma 1.5.19 ([21, Lemma 8.3.2]) *Let*

$$\phi : (\Sigma_j, p_j) \to (\Sigma'_j, p'_j)\,, \quad \tilde{\phi} : (\tilde{\Sigma}_j, \infty) \longrightarrow (\tilde{\Sigma}'_j, \infty)\,,$$

be a biholomorphism germ and a $\underleftarrow{\mathbb{C}\text{-Man}}$ *isomorphism defining a realization of* $(\tilde{g}_*, h)$ *on the germs* $(\Sigma_j, S^\sharp_{\mathcal{F}})$ *and* $(\Sigma'_j, S^\sharp_{\mathcal{F}'})$. *If* $\tilde{\phi}$ *and* $\tilde{g}$ *induce the same canonical morphism*

$$\tilde{\phi}_\star = \tilde{g}_\star : \pi_0(\tilde{\Sigma}_j) \to \pi_0(\tilde{\Sigma}'_j)\,,$$

then there exists a homeomorphism Φ *that extends* ϕ *along a neighborhood* Ω_j *of* K_j *and a lifting* $\tilde{\Phi}$ *of* Φ *through* $q \circ E_{\mathcal{F}}^{-1}$ *and* $q' \circ E_{\mathcal{F}'}^{-1}$, *that extends* $\tilde{\phi}$ *such that*

1. $(\Phi, \tilde{\Phi})$ *is a realization of* $(\tilde{g}_*, h)$ *on the germs* $(\Omega_j, S^\sharp_{\mathcal{F}})$ *and* $(\Phi(\Omega_j), S^\sharp_{\mathcal{F}'})$,
2. *on a neighborhood of each connected component* $\mathscr{C}$ *of the boundary of* K_j *the equality* $\rho' \circ \Phi = \Phi \circ \rho$ *is satisfied, where* ρ *(resp.* ρ'*) is the disc fibration over the big piece containing* $\mathscr{C}$ *(resp.* $\Phi(\mathscr{C})$*),*
3. $\tilde{\Phi}_{|\tilde{\Sigma}_{j+1}}$ *and* $\tilde{g}$ *induce the same canonical morphism from* $\pi_0(\tilde{\Sigma}_{j+1})$ *to* $\pi_0(\tilde{\Sigma}'_{j+1})$.

Property 2. above implies the uniqueness of the extension of ϕ on a neighborhood Ω of $Z_{j-1} \cap K_j$. Hence according to relation (1.20) with the index $j-1$, Φ and ψ_{j-1} coincide on Ω, and can be glued to define the homeomorphism ψ_j.

The proof of Lemma 1.5.19, which is technical, is based on the fact that by Theorem 1.5.12 the conjugacy of monodromies realized by ϕ implies that ϕ also conjugates the holonomies of $\mathcal{F}^\sharp$ and $\mathcal{F}'^\sharp$ along the whole piece K_j when K_j is big and along the irreducible component of K_j containing p_j when K_j is small. The extension Φ as a conjugacy of $\mathcal{F}^\sharp$ to $\mathcal{F}'^\sharp$ along K_j is then given by a classical theorem on regular foliations when K_j is big. When K_j is small, using Camacho-Sad indices equalities (1.19) the extension is given by a classical theorem on saddle singularities [27]. □

1.6 Topological Invariance of Camacho-Sad indices

1.6.1 Camacho-Sad index

We recall that at a singular point s of a foliation $\mathcal{F}$ on an invariant regular curve C, the Camacho-Sad index $\mathrm{CS}(\mathcal{F}, S, s)$ gives information about the winding of the leaves of $\mathcal{F}$ around C in a neighborhood of s. More precisely, if $\mathcal{F}$ is defined by the 1-form $vA(u,v)du + B(u,v)dv$ and $C = \{v = 0\} \ni (0,0) = s$, then

$$\mathrm{CS}(\mathcal{F}, C, s) := \mathrm{Res}_{u=0}\left(-\frac{B(u,0)}{A(u,0)}\right) = \lim_{z\to 0}\frac{1}{2i\pi}\int_{\Gamma_z}\frac{dv}{v}\,, \tag{1.21}$$

where $\Gamma_z = (\gamma(t), \mu_z(t))$ is a path obtained by lifting into the leaves the loop $\gamma(t) = (e^{2i\pi t}, 0)$ inside C with $\mu_z(0) = z$.

We will determine a large class of foliation germs $\mathcal{F}$ for which the Camacho-Sad indices along the exceptional divisor $\mathcal{E}_{\mathcal{F}}$, at the singular points of $\mathcal{F}^\sharp$, **are topological invariants** in the sense that if ϕ is a topological conjugacy from $\mathcal{F}$ to $\mathcal{G}$ then we have the equality

$$\mathrm{CS}(\mathcal{F}^\sharp, D, s) = \mathrm{CS}(\mathcal{G}^\sharp, \mathsf{A}_\phi(D), \mathsf{A}_\phi(s)) \tag{1.22}$$

for any vertex D adjacent to an edge s in $\mathsf{A}_{\mathcal{F}}$. Here the vertices and the edges of the dual trees $\mathsf{A}_{\mathcal{F}}$ and $\mathsf{A}_{\mathcal{G}}$ are identified respectively to the irreducible components and the singular points of the total transforms $S^\sharp_{\mathcal{F}}$ and $S^\sharp_{\mathcal{G}}$ and $\mathsf{A}_\phi : \mathsf{A}_{\mathcal{F}} \to \mathsf{A}_{\mathcal{G}}$ is the graph morphism introduced in Proposition 1.2.2.

We will further introduce two conditions (NFC) and (TR) on $\mathcal{F}$ making it possible to obtain the topological invariance of Camacho-Sad indices. But first we will examine the possible obstructions to this.

1.6.2 Different types of dynamical components

A straightforward induction process [21, §7.3] based on the following Camacho-Sad Index Formula [7, Appendix] for any invariant irreducible component D of $\mathcal{E}_{\mathcal{F}}$

$$\sum_{s\in \mathrm{Sing}(\mathcal{F}^\sharp)\cap D} \mathrm{CS}(\mathcal{F}^\sharp, D, s) = D\cdot D \tag{1.23}$$

gives that:

the equality (1.22) holds for any D and s in $\mathsf{A}_{\mathcal{F}}$ if it is satisfied when D is the strict transform of any isolated separatrix.

For a fixed isolated separatrix C we will examine different cases according to the nature of the dynamical component $\mathcal{D}$ of $S^{\sharp}_{\mathcal{F}}$ containing the strict transform $\mathscr{C}$ of C. Let us notice that Theorem 1.2.8, giving the existence of a non-nodal isolated separatrix meeting the dynamical component $\mathcal{D}$, does not ensure that there is an irreducible component in $\mathcal{D}$ with at least 3 singular points of $\mathcal{F}^{\sharp}$. However using this theorem and Index Formula (1.23) we can see (cf. [23, Proof of Lemma 11.6]) that there are only five possibilities for $\mathcal{D}$ and $\mathscr{C}$:

1. $\mathscr{C}$ is a nodal separatrix and $\mathcal{D}$ is reduced to $\mathscr{C}$;
2. The non-empty intersection $\mathcal{D}' := \mathcal{D} \cap \mathcal{E}_{\mathcal{F}}$ is a union $\mathcal{D}' = D_1 \cup \cdots \cup D_\ell$ of invariant irreducible components of $\mathcal{E}_{\mathcal{F}}$, where D_i meets D_{i+1} at a (non-nodal) single point s_i when $\ell > 1$, and $D_i \cap D_j$ is empty if $|i-j| \neq 0, 1$; moreover $\mathscr{C}$ meets D_1 at a non-nodal singular point s and one of the following configurations holds:
   (a) $\mathcal{D} = \mathscr{C} \cup \mathcal{D}'$, when $\ell > 1$ $\mathrm{Sing}(\mathcal{F}^{\sharp}) \cap \mathcal{D}$ is equal to $\{s, s_1, \ldots, s_{\ell-1}\}$, or to $\{s\}$ when $\ell = 1$, cf. Figure 1.8;
   (b) $\mathcal{D} = \mathscr{C} \cup \mathcal{D}'$, $\mathrm{Sing}(\mathcal{F}^{\sharp}) \cap \mathcal{D} = \{s, s_1, \ldots, , s_\ell\}$ and $s_\ell \in D_\ell \setminus \{s_{\ell-1}\}$ is a nodal singularity of $\mathcal{F}^{\sharp}$, cf. Figure 1.9;
   (c) $\mathcal{D} = \mathscr{C} \cup \mathcal{D}' \cup \mathscr{C}'$, $\mathscr{C}'$ is an isolated non-nodal separatrix meeting D_ℓ at singular points s', and $\mathrm{Sing}(\mathcal{F}^{\sharp}) \cap \mathcal{D}$ is equal to $\{s, s_1, \ldots, s_{\ell-1}, s'\}$ when $\ell > 1$ and to $\{s, s'\}$ when $\ell = 1$, cf. Figure 1.10.

   In all these cases any irreducible component of $\mathcal{E}_{\mathcal{F}}$ meeting $\mathcal{D}$ is either contained in $\mathcal{D}$ or dicritical.
3. There is an irreducible component of $\mathcal{D}$ containing at least 3 singular points of $\mathcal{F}^{\sharp}$.

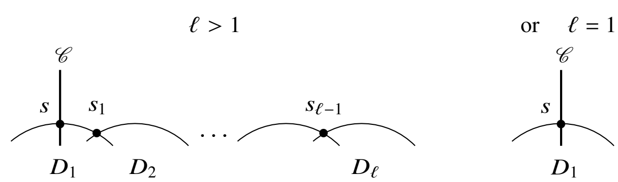


Fig. 1.8 Situation (2a): there is no nodal singular point on $\mathcal{D}$ and some dicritical component must meet $\mathcal{D}$.

We say that the dynamical componant $\mathcal{D}$ is **small** in cases 1 and 2 and that it is **big** in case 3. We will now go through each case.

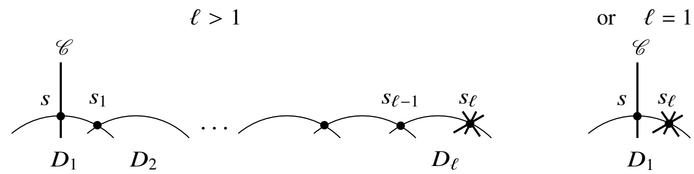


Fig. 1.9 Situation (2b): s_ℓ is the unique nodal singular point on $\mathcal{D}$ and some dicritical components may meet $\mathcal{D}$.

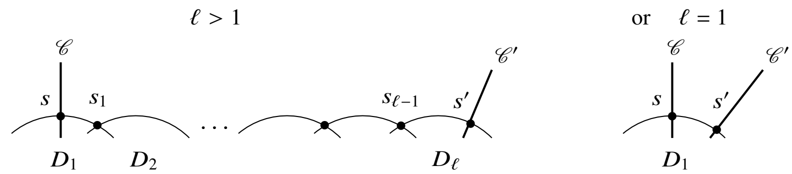


Fig. 1.10 Situation (2c): there is no nodal singular point on $\mathcal{D}$ and some dicritical component must meet $\mathcal{D}$.

1.6.3 Small dynamical components

We denote again by $\mathscr{C}$ the strict transform of an isolated separatrix of $\mathcal{F}$ and by $\phi : \mathcal{F} \to \mathcal{G}$ a topological conjugacy between $\mathcal{F}$ and $\mathcal{G}$.

In the case 1 the equality

$$\mathsf{CS}(\mathcal{F}^\sharp, \mathscr{C}, s) = \mathsf{CS}(\mathcal{G}^\sharp, \mathsf{A}_\phi(\mathscr{C}), \mathsf{A}_\phi(s))$$

is proven by R. Rosas under weak hypothesis [38, Proposition 13], see also [22, Theorem 1.12].

Index Formula (1.23) implies that:

- in case (2a) the Camacho-Sad index $\mathrm{CS}(\mathcal{F}^\sharp, \mathscr{C}, s)$ is a rational number that can be expressed as a continued fraction in terms of the self-intersections $D_i \cdot D_i$ of the irreducible components of $\mathcal{D} \cap \mathcal{E}_\mathcal{F}$; as these ones are invariant by topological conjugacies, cf. [46] or [20], the equality of Camacho-Sad indices (1.22) is satisfied;
- in case (2b) we have $\mathrm{CS}(\mathcal{F}^\sharp, D_\ell, s_\ell) \in \mathbb{R} \setminus \mathbb{Q}$ and consequently $\mathrm{CS}(\mathcal{F}^\sharp, \mathscr{C}, s) \in \mathbb{R} \setminus \mathbb{Q}$; in fact s must be a linearizable saddle of $\mathcal{F}^\sharp$ (recall that s is not a node). Then the proof of [22, Theorem 1.12] gives the equality (1.22).

We observe that in case (2c) any reduced singularity may appear at the point s and for this reason we will say that $\mathcal{D}$ is a **free (dynamical) component**. All the free components of **Poincaré type**, i.e. with $\mathrm{CS}(\mathcal{F}^\sharp, \mathscr{C}, s) \in \mathbb{C} \setminus \mathbb{R}$, are topologically conjugated. Indeed, this claim uses the following fact:

Lemma 1.6.1 *Let (λ_1, λ_2) and (μ_1, μ_2) be two pairs of $\mathbb{R}$-independent complex numbers and $g : \mathbb{C} \to \mathbb{C}$ be a $\mathbb{R}$-linear map such that $g(i\lambda_j) := i\mu_j$. Then the maps $\psi_j : \mathbb{C} \to \mathbb{C}$, $j = 1, 2$, defined by*

$$\begin{cases} \psi_j(z) := \exp\left(\mu_j^{-1} g(\lambda_j \log z)\right) & \text{if } z \neq 0\,, \\ \psi_j(0) := 0\,, \end{cases}$$

are well defined homeomorphisms and the map

$$\Psi : \mathbb{C}^2 \to \mathbb{C}^2\,, \quad \Psi(z_1, z_2) = \big(\psi_1(z_1), \psi_2(z_2)\big)$$

is a homeomorphism that conjugates the foliation defined by the meromorphic form $\lambda_1 \frac{dz_1}{z_1} + \lambda_2 \frac{dz_2}{z_2}$ to that defined by $\mu_1 \frac{dz_1}{z_1} + \mu_2 \frac{dz_2}{z_2}$.

Proof The map

$$(\mu_1 \log z_1 + \mu_2 \log z_2) \circ \Psi = g(\lambda_1 \log z_1 + \lambda_2 \log z_2)$$

is constant along the leaves of $\lambda_1 \frac{dz_1}{z_1} + \lambda_2 \frac{dz_2}{z_2}$. □

Thus, in case (2c) when $\mathcal{F}^\sharp$ is of Poincaré type at s, then it is also of Poincaré type at each singular point of the free component $\mathcal{D}$. Using similar methods as those developed by E. Paul in [34] one can see that the local conjugacies given by Lemma 1.6.1 at the singular points of the free component can be glued into a global C^0-conjugacy along the whole free component. Finally, for two topologically conjugated foliations whose reductions contain a Poincaré type free component, the equality (1.22) may not be satisfied at the singular points of the free component. This phenomenon forces us to exclude this case (2c) with the following assumption that there is no free dynamical component:

(NFC) *There is no dynamical component of $S^\sharp_{\mathcal{F}}$ without nodal singular points of $\mathcal{F}^\sharp$ whose compact irreducible components contain exactly two singular points of $\mathcal{F}^\sharp$.*

Notice that if $\phi : \mathcal{F} \to \mathcal{G}$ is a topological conjugacy between generalized curves and $\mathcal{F}$ satisfies (NFC) then $\mathcal{G}$ also satisfies (NFC) thanks to Corollary 1.2.12.

1.6.4 Big dynamical components

To obtain equality (1.22) in case 3 we will ask $\mathcal{F}$ to fulfill the following **transverse rigidity** assumption:

(TR) *If a dynamical component of $S^\sharp_{\mathcal{F}}$ contains an irreducible component with at least 3 singular points of $\mathcal{F}^\sharp$, it also contains an irreducible component whose holonomy group is topologically rigid, for instance unsolvable.*

We recall that a group G of germs of $\mathrm{Diff}(\mathbb{C},0)$ is **topologically rigid** if any orientation preserving homeomorphism germ that conjugates this group with another subgroup of $\mathrm{Diff}(\mathbb{C},0)$ is necessarily holomorphic. D. Cerveau and P. Sad showed that non abelian subgroups with generic linear part are topologically rigid [8] and by classical results of Y. Ilyashenko, A.A. Shcherbakov and I. Nakai [13, 41, 31] any unsolvable subgroup is topologically rigid. Condition (TR) above is Krull-generic [17]. We recall that a property $\mathscr{P}$ on the space Ω of germs of holomorphic differential 1-forms on $(\mathbb{C}^2,0)$ is **Krull-generic** if it is open dense for the Krull topology on Ω, i.e.

(a) for any $\omega\in\Omega$ and for any $p\in\mathbb{N}$ there is $\eta\in\Omega$ satisfying $\mathscr{P}$ with same p-jet at 0 as ω,
(b) if ω satisfies $\mathscr{P}$ then there is $q\in\mathbb{N}$ such that every $\xi\in\Omega$ with same q-jet as ω also satisfies $\mathscr{P}$.

The interest of Condition (TR) is that we can apply the main theorem of J. Rebelo in [37], whose proof remains valid for the extended statement below:

Theorem 1.6.2 (Transverse Rigidity Theorem) *Let $\mathcal{F}$ be a generalized curve and let $\mathcal{D}$ be a dynamical component of $S^{\sharp}_{\mathcal{F}}$ containing an irreducible component with topologically rigid holonomy group. Then $\mathcal{F}$ is **transversally rigid along** $\mathcal{D}$ in the following sense: for any homeomorphism $\phi:(\mathbb{C}^2,0)\to(\mathbb{C}^2,0)$ that conjugates $\mathcal{F}$ to another germ of foliation preserving the orientations of the ambient space and of the leaves, there is an open neighbourhood W of $\mathcal{D}\setminus\mathcal{N}$ such that ϕ is transversally holomorphic at each point of $E_{\mathcal{F}}(W)\setminus\{0\}$, where $\mathcal{N}\subset\mathcal{E}_{\mathcal{F}}$ is the set of nodal singular points of $\mathcal{F}^{\sharp}$.*

We recall that a germ of homeomorphism ϕ which conjugates two foliations $\mathcal{F}$ and $\mathcal{G}=\phi(\mathcal{F})$, is **transversally holomorphic at a point** p, if for any germs of submersion first integrals $u:(\mathbb{C}^2,p)\to(\mathbb{C},0)$ and $v:(\mathbb{C}^2,\phi(p))\to(\mathbb{C},0)$ there exists a germ of biholomorphism $\phi^{\flat}:(\mathbb{C},0)\to(\mathbb{C},0)$ such that $\phi^{\flat}\circ u=v\circ\phi$.

Remark 1.6.3 Similarly to Remark 1.5.18, any C^0-conjugacy Φ between $\mathcal{F}$ and $\mathcal{F}'$ which is transversally holomorphic outside a union of separators defines an excellent conjugacy between the monodromies of $\mathcal{F}$ and $\mathcal{F}'$ for which $(\Phi,\tilde{\Phi})$ is a realization on neighborhoods of the origin. □

Notice that for a C^0-conjugacy defined only on a neighborhood of a punctured separatrix $C\setminus\{0\}$, transverse holomorphy property is not suficient to obtain the invariance of the Camacho-Sad index along the strict transform of C. A trivial counter-example with C regular is given by performing the blowing-up E with center $\{0\}$ and by taking for $\mathcal{G}$ the germ of $E^{-1}\mathcal{F}$ at the meeting point p of the strict transform C' of C by E and the exceptional divisor $E^{-1}(0)$. Restricted to a neighborhood V of $C'\setminus\{p\}$, E is a biholomorphism that conjugates $\mathcal{G}_{|V}$ to $\mathcal{F}_{|E(V)}$. However, taking $\phi=(E_{|V})^{-1}$ we have

$$\mathrm{CS}(\mathcal{F},C,0)=\mathrm{CS}(\phi(\mathcal{F}),C',p)+1\,,$$

see [7, Proposition 2.1]. We thus must impose as additional hypothesis the compatibility of the conjugacy with the peripheral structure of C in the fundamental group of the complement of the $\mathcal{F}$-appropriate curve S.

1.6.5 Peripheral structure and Index Invariance Theorem

We fix local coordinates (u, v) at the meeting point s of the strict transform $\mathscr{C}$ of C with the exceptional divisor $\mathcal{E}_{\mathcal{F}}$, such that near s the equation $v = 0$ defines $\mathscr{C}$ and $u = 0$ defines $\mathcal{E}_{\mathcal{F}}$; we consider the loop m, resp. p, with common origin $a := (\varepsilon_1, \varepsilon_2)$, defined by $u(t) = \varepsilon_1$, $v(t) = \varepsilon_2 e^{2\pi i t}$, resp. by $u(t) = \varepsilon_1 e^{2\pi i t}$, $v(t) = \varepsilon_2$, $t \in [0, 1]$; we choose ε_1 and ε_2 small enough so that the loops $E_{\mathcal{F}} \circ m$ and $E_{\mathcal{F}} \circ p$ are contained in a Milnor ball B for the foliation $\mathcal{F}$; let us fix $b \in B \setminus S$ and a path α in $B \setminus S$ joining a to b. We call **peripheral structure of** C **in** $\pi_1(B \setminus S, b)$ the data of the two classes $\mathbf{m}, \mathbf{p} \in \pi_1(B \setminus S, b)$, respectively called **meridian** and **parallel**, defined by the path

$$\alpha^{-1} \vee (E_{\mathcal{F}} \circ m) \vee \alpha \quad \text{and} \quad \alpha^{-1} \vee (E_{\mathcal{F}} \circ p) \vee \alpha\,.$$

A consequence of the fact [12] that the subgroup $\langle \mathbf{m}, \mathbf{p} \rangle \subset \pi_1(B \setminus S, b)$ is equal to its normalizer in $\pi_1(B \setminus S, b)$, is that $\mathbf{m}$ and $\mathbf{p}$ do not depend on the choice of the path α, see [21, Corollary 6.1.4].

Theorem 1.6.4 *Let ϕ be a homeomorphism defined on an open tubular neighborhood V of $C \setminus \{0\}$ in $\mathbb{C}^2$ that conjugates $\mathcal{F}_{|V}$ to $\mathcal{G}_{|\phi(V)}$. If ϕ is transversally holomorphic and sends respectively the meridian and the parallel of the peripheral structure of C to the meridian and the parallel of $\phi(C)$, then the Camacho-Sad index of $\mathcal{F}^\sharp$ along the strict transform of C by $E_{\mathcal{F}}$ is equal to that of $\mathcal{G}^\sharp$ along the strict transform of $\phi(C)$ by $E_{\mathcal{G}}$.*

Proof Let (u, v) (resp. (u', v')) be local coordinates centered at the intersection point s (resp. s') of the strict transform $\mathscr{C} = \{v = 0\}$ (resp. $\mathscr{C}' = \{v' = 0\}$) of C (resp. $\phi(C)$) with the exceptional divisor $\mathcal{E}_{\mathcal{F}} = \{u = 0\}$ (resp. $\mathcal{E}_{\mathcal{G}} = \{u' = 0\}$). By the transversal holomorphy of ϕ the holonomies of $\mathscr{C}$ and $\mathscr{C}'$ are analytically conjugated and consequently

$$\mathrm{CS}(\mathcal{F}^\sharp, \mathscr{C}, s) - \mathrm{CS}(\mathcal{G}^\sharp, \mathscr{C}', s') \in \mathbb{Z}. \tag{1.24}$$

Let Γ_n be the lift of the loop $t \mapsto (\varepsilon_1 e^{2i\pi n t}, 0)$ to a leaf of $\mathcal{F}^\sharp$. Formula (1.21) implies that

$$\mathrm{CS}(\mathcal{F}^\sharp, \mathscr{C}, s) = \lim_{n \to \infty} \frac{1}{2i\pi n} \int_{\Gamma_n} \frac{dv}{v}\,.$$

Let ξ_n be a path contained in $\{u = \varepsilon_1, 0 < |v| < 1/n\}$ joining the points $\Gamma_n(1)$ and $\Gamma_n(0)$ in such a way that the variation of the argument of $v \circ \xi_n$ is bounded, consequently

$$\lim_{n \to \infty} \mathrm{Re} \left(\frac{1}{2i\pi n} \int_{\xi_n} \frac{dv}{v} \right) = 0$$

and

$$\mathrm{Re}(\mathrm{CS}(\mathcal{F}^\sharp,\mathscr{C},s)) = \lim_{n\to\infty}\frac{I_n}{n}, \quad \text{where} \quad I_n := \frac{1}{2i\pi}\int_{\Gamma_n\vee\xi_n}\frac{dv}{v}\in\mathbb{Z}\,. \tag{1.25}$$

The homotopy class of $E_{\mathcal{F}}(\Gamma_n \vee \xi_n)$ can be decomposed as $I_n\mathbf{m}_n + n\mathbf{p}_n$ using the peripheral structure $\mathbf{m}_n, \mathbf{p}_n$ of C in $\pi_1(V\setminus C, b_n) \hookrightarrow \pi_1(B\setminus S, b_n)$, $b_n = E_{\mathcal{F}}(\Gamma_n(0))$. Up to isotopy we can assume that $\phi^\sharp \circ u = u' \circ \phi^\sharp$ in a neighbourhood of the circle $\{|u| = \varepsilon_1, v = 0\}$, where $\phi^\sharp = E_{\mathcal{G}}^{-1} \circ \phi \circ E_{\mathcal{F}}$. As the variation of the argument of $v' \circ \phi^\sharp \circ \xi_n$ is bounded, because ϕ is transversely holomorphic, we have

$$\mathrm{Re}(\mathrm{CS}(\mathcal{G}^\sharp,\mathscr{C}',s')) = \lim_{n\to\infty}\frac{I'_n}{n}, \quad \text{where} \quad I'_n := \frac{1}{2i\pi}\int_{\phi^\sharp(\Gamma_n\vee\xi_n)}\frac{dv'}{v'}\in\mathbb{Z}\,, \tag{1.26}$$

and the homotopy class of $\phi(E_{\mathcal{F}}(\Gamma_n \vee \xi_n))$ can be decomposed as $I'_n\mathbf{m}'_n + n\mathbf{p}'_n$, where $\mathbf{m}'_n, \mathbf{p}'_n$ is the peripheral structure of C' in $\pi_1(\phi(V\setminus C), \phi(b_n)) \hookrightarrow \pi_1(B'\setminus S', \phi(b_n))$. By hypothesis the group morphism

$$\phi_* : \pi_1(V \setminus C, b_n) \to \pi_1(\phi(V \setminus C), \phi(b_n))$$

sends $\mathbf{m}_n$ into $\mathbf{m}'_n$ and $\mathbf{p}_n$ into $\mathbf{p}'_n$. Since

$$\phi_*(I_n\mathbf{m}_n + n\mathbf{p}_n) = I_n\mathbf{m}'_n + n\mathbf{p}'_n\,,$$

we obtain that $I_n = I'_n$. Hence relations (1.24), (1.25) and (1.26) give the equality $\mathrm{CS}(\mathcal{F}^\sharp,\mathscr{C},s) = \mathrm{CS}(\mathcal{G}^\sharp,\mathscr{C}',s')$. □

In addition global homeomorphisms always fulfill the peripheral structure assumption:

Proposition 1.6.5 ([21, Theorem 6.2.1]) *Let us consider Milnor balls B and B' for germs of curves S and S' at the origin of $\mathbb{C}^2$ and $b \in B \setminus S$. For any homeomorphism $\phi : B \to \phi(B) \subset B'$ such that $\phi(S \cap B) = S' \cap \phi(B)$, the map $\phi_* : \pi_1(B \setminus S, b) \to \pi_1(B' \setminus S', \phi(b))$ sends respectively the meridian and the parallel of the peripheral structure in $\pi_1(B \setminus S, b)$ of any irreducible component C of S to the meridian and the parallel of the peripheral structure of $\phi(C)$ in $\pi_1(B' \setminus S', b)$.*

Theorem 1.6.4, Proposition 1.6.5 and the previous analysis of small dynamical components give us:

Theorem 1.6.6 (Index Invariance Theorem) *For generalized curves satisfying assumptions (NFC) and (TR), the Camacho-Sad indices after reduction of singularities are topological invariants in the sense that equalities (1.22) are satisfied for any C^0-conjugacy $\phi : \mathcal{F} \to \mathcal{G}$.*

The hypothesis of this theorem are minimal. Condition (NFC) is necessary as we have already seen in §1.6.3. Counter-examples not fulfilling Condition (TR) are given by families of logarithmic foliations, whose topological classification is obtained by E. Paul:

Theorem 1.6.7 ([35, Théorème 3.5]) *Let ω_t be a family of germs of logarithmic forms*

$$f_1 \cdots f_p \sum_{j=1}^{p} \lambda_{j,t} \frac{df_j}{f_j}, \quad t \in (\mathbb{C}, 0),$$

on $(\mathbb{C}^2, 0)$ whose residues $\lambda_{j,t} \in \mathbb{C}$ are without relation with positive integer coefficients. If there is a family germ $\{g_t, t \in (\mathbb{C}, 0)\}$ of $\mathrm{GL}_2^+(\mathbb{R})$ such that each map g_t sends $2i\pi\lambda_{j,0}$ into $2i\pi\lambda_{j,t}$, then the family germ of foliations $\{\omega_t, t \in (\mathbb{C}, 0)\}$ is topologically trivial.

1.7 Excellence Theorem and topological moduli space

1.7.1 Excellence Theorem

On the set of foliation germs satisfying (NFC) and (TR) stated in §1.6.3 and §1.6.4, the equivalence relations of C^0-conjugacy and C^{ex}-conjugacy coincide:

Theorem 1.7.1 (Excellence Theorem) *Let $\mathcal{F}$ and $\mathcal{G}$ be two germs of foliations at $0 \in \mathbb{C}^2$ that are generalized curves fulfilling (NFC) and (TR) properties. Then the following assertions are equivalent:*

1. *there exists a homeomorphism germ $\phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ that conjugates $\mathcal{F}$ to $\mathcal{G}$;*
2. *there exists a homeomorphism germ $\Phi : (M_{\mathcal{F}}, \mathcal{E}_{\mathcal{F}}) \to (M_{\mathcal{G}}, \mathcal{E}_{\mathcal{G}})$ that conjugates $\mathcal{F}^\sharp$ to $\mathcal{G}^\sharp$; moreover Φ is holomorphic at each non-nodal singular point of $\mathcal{F}^\sharp$ and is transversely holomorphic at each point of $\mathcal{E}_{\mathcal{F}}$ that is regular for $\mathcal{F}^\sharp$ and not contained in a dicritical component.*

Proof It suffices to use Index Invariance Theorem 1.6.6, Classification Theorem 1.5.16 and Transverse Rigidity Theorem 1.6.2 as indicated in Figure 1.11.

1.7.2 Classification Problem: complete families and moduli space

Excellence Theorem reduces the problem of topological classification of germs of foliations at the origin of $\mathbb{C}^2$ to the problem of classification under excellent germs of homeomorphisms. This one can be formulated as follows.

Let us call **semi-local invariants** of a foliation germ $\mathcal{F}$ the data of the topology of its separatrices curve, the Camacho-Sad indices of $\mathcal{F}^\sharp$ and the holonomy representation morphisms of the invariant irreducible components of $\mathcal{E}_{\mathcal{F}}$. All these data are

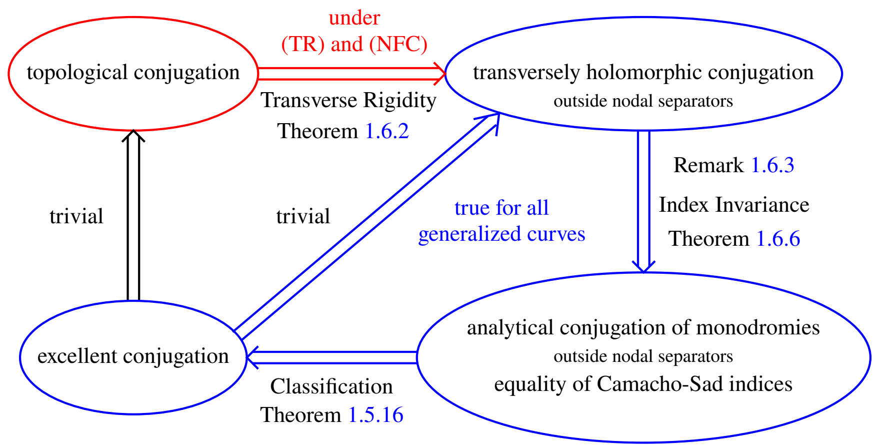


Fig. 1.11 Strategy of the proof of Excellence Theorem 1.7.1.

$\mathcal{C}^{\mathrm{ex}}$-invariants and therefore topological invariants under the hypothesis of Excellence Theorem 1.7.1. Let us denote by $\mathrm{SL}(\mathcal{F})$ the set of all foliation germs with same semi-local invariants as $\mathcal{F}$. The goal of $\mathcal{C}^0$-classification is to construct a natural set of foliation germs whose intersection with the topological class of any foliation in $\mathrm{SL}(\mathcal{F})$ is non-empty and totally discontinuous. For a large class of foliation germs a complete answer to this problem is given in the papers [23, 24, 25] that lead to a family of foliations in $\mathrm{SL}(\mathcal{F})$ depending holomorphically on a parameter that varies in a complex manifold of minimal dimension. This family has natural properties of factorization of any other holomorphic family in $\mathrm{SL}(\mathcal{F})$; moreover the redundancy of the topological types in this family is minimal and can be precisely described.

In the spirit of Teichmüller theory, it is convenient for our moduli problem to endow each foliation germ with a marking. For this, let us fix a foliation germ $\mathcal{F}_0$ that is a generalized curve and let us simply denote by $\mathcal{E}^\diamond := (\mathcal{E}, \Sigma)$ the pair $(\mathcal{E}_{\mathcal{F}_0}, \mathrm{Sing}(\mathcal{F}_0^\sharp))$. A **marked by** $\mathcal{E}^\diamond$ **foliation germ** is a pair $(\mathcal{F}, f)$ where $\mathcal{F}$ is a foliation germ at $(\mathbb{C}^2, 0)$ and f is a homeomorphism from $\mathcal{E}$ into $\mathcal{E}_{\mathcal{F}}$ such that

$$f(\Sigma) = \mathrm{Sing}(\mathcal{F}^\sharp) \quad \text{and} \quad f(D) \cdot f(D') = D \cdot D'$$

for any pair (D, D') of irreducible components of $\mathcal{E}$. Two $\mathcal{E}^\diamond$-markings f and g of a same foliation $\mathcal{F}$ are **equivalent** if $g^{-1} \circ f$ is isotopic to the identity map, by an isotopy fixing each point of Σ. Two marked by $\mathcal{E}^\diamond$ foliation germs $(\mathcal{F}, f)$ and $(\mathcal{G}, g)$ are $\mathcal{C}^{\mathrm{ex}}$**-conjugated** if there exists a $\mathcal{C}^{\mathrm{ex}}$-conjugacy germ $\Phi : (M_{\mathcal{F}}, \mathcal{E}_{\mathcal{F}}) \to (M_{\mathcal{G}}, \mathcal{E}_{\mathcal{G}})$ such that g and $\Phi \circ f$ are equivalent $\mathcal{E}^\diamond$-markings of $\mathcal{G}$.

Definition 1.7.2 We say that two marked by $\mathcal{E}^\diamond$ foliation germs $(\mathcal{F}, f)$ and $(\mathcal{G}, g)$ **have same semi-local type** if

1. for any irreducible component D of $\mathcal{E}$ and any $s \in \Sigma \cap D$, we have the equality of Camacho-Sad indices $\mathrm{CS}(\mathcal{G}^\sharp, g(D), g(s)) = \mathrm{CS}(\mathcal{F}^\sharp, f(D), f(s))$
2. for any irreducible component D of $\mathcal{E}$ with $D \cap \Sigma \neq \emptyset$, the compositions with the group morphisms induced by the markings

$$H_D^{\mathcal{F}} \circ f_* \text{ and } H_D^{\mathcal{G}} \circ g_* \ : \ \pi_1(D \setminus \Sigma, \cdot) \longrightarrow \mathrm{Diff}(\mathbb{C}, 0)$$

of the holonomy morphisms $H^{\mathcal{F}}_{f(D)}$ and $H^{\mathcal{G}}_{g(D)}$ of the foliations $\mathcal{F}^\sharp$ and $\mathcal{G}^\sharp$ along $f(D)$ and $g(D)$, are equal up to composition by an inner automorphism of $\mathrm{Diff}(\mathbb{C}, 0)$.

To obtain a "parametrization" by a finite dimensional manifold of the set of C^{ex}-conjugacy classes of marked foliation germs with fixed semi-local type, we need an additional hypothesis called **finite type**. This one was first introduced in [23] as a combinatorial property of an appropriate orientation of the dual graph $\mathsf{A}_{\mathcal{F}}$ and then reinterpreted in [23, 24] as the finiteness dimension $\tau(\mathcal{F})$ of a naturally associated cohomological space. This assumption is not very restrictive because in the set of 1-differential forms defining generalized curves, the subset of finite type foliations contains an open dense set for the Krull topology, see [29, Theorem 6.2.1] and [23, Remark 2.5].

Theorem 1.7.3 ([25, Theorem 3.7]) *Let $(\mathcal{F}, f)$ be a marked finite type foliation germ which is a generalized curve. Then there exists an equisingular marked family of foliation germs*

$$\mathcal{F}_{\mathbb{C}^\tau \times \mathbf{D}} := \left(\mathcal{F}_{z,d}, f_{z,d}\right)_{(z,d) \in \mathbb{C}^\tau \times \mathbf{D}} \tag{1.27}$$

such that

1. *$\mathbf{D}$ is a quotient of a finite product of totally disconnected subgroups of the circle $\mathbb{U}(1)$ and $\tau = \tau(\mathcal{F})$,*
2. *for any marked foliation germ $(\mathcal{G}, g)$ with same semi-local type as $(\mathcal{F}, f)$, there exists $(z, d) \in \mathbb{C}^\tau \times \mathbf{D}$ such that $(\mathcal{G}, g)$ is C^{ex}-conjugated to $(\mathcal{F}_{z,d}, f_{z,d})$,*
3. *if $(\mathcal{G}_t, g_t)_{t \in P}$ is an equisingular marked family of foliation germs with parameter space a connected manifold P satisfying $H^1(P, \mathbb{Z}) = 0$, then for any $t_0 \in P$ and $(z_0, d_0) \in \mathbb{C}^\tau \times \mathbf{D}$ such that $(\mathcal{G}_{t_0}, g_{t_0})$ is C^{ex}-conjugated to $\mathcal{F}_{z_0,d_0}$, there exists a unique holomorphic map $\lambda : P \to \mathbb{C}^\tau \times \mathbf{D}$ with $\lambda(t_0) = (z_0, d_0)$ such that for any $t \in P$ the marked foliation germs $(\mathcal{G}_t, g_t)$ and $(\mathcal{F}_{\lambda(t)}, f_{\lambda(t)})$ are C^{ex}-conjugated,*
4. *in a neighborhood $W_{t'}$ of any point $t' \in P$, the C^{ex}-conjugacies between the marked foliation germs $(\mathcal{G}_t, g_t)$ and $(\mathcal{F}_{\lambda(t)}, f_{\lambda(t)})$ in assertion 3 can be realized by excellent homeomorphisms Φ_t depending continuously on $t \in W_{t'}$.*

The notion of **marked family of foliation germs**, introduced in [25, §2.2], is a compatibility property of the markings with the local topological triviality in the parameter of the whole exceptional divisor. That of **equisingularity** is precisely defined in [24, §3]; basically it means that the family is locally equireducible and all the foliation germs in the family have same semi-local type. Assertion 4 in Theorem 1.7.3 follows from [25, Theorem 4.4].

We finally describe the redundancy of the C^{ex}-classes of the foliations in the family $\mathcal{F}_{\mathbb{C}^\tau\times\mathbf{D}}$. Let us define the C^{ex}**-moduli space of** $(\mathcal{F}, f)$ as the set

$$\mathrm{Mod}(\mathcal{F}, f) := \{\ [\mathcal{G}, g] \mid (\mathcal{G}, g) \text{ and } (\mathcal{F}, f) \text{ have same semi-local type}\}$$

of C^{ex}-conjugacy classes $[\mathcal{G}, g]$ of marked by $\mathcal{E}^\diamond$ foliations $(\mathcal{G}, g)$. To any equisingular marked family $\mathcal{G}_P := (\mathcal{G}_t, g_t)_{t\in P}$ of foliation germs with same semi-local type as $(\mathcal{F}, f)$ we associate its **moduli map**

$$\mathrm{mod}_{\mathcal{G}_P} : P \longrightarrow \mathrm{Mod}(\mathcal{F}, f)\,, \qquad t \mapsto [\mathcal{G}_t, g_t]\,.$$

Theorem 1.7.4 ([23, Theorem D]) *Let* $(\mathcal{F}, f)$ *be a marked finite type foliation germ which is a generalized curve. There is a natural abelian group structure on* $\mathrm{Mod}(\mathcal{F}, f)$ *given by an exact sequence*

$$0 \to \mathbb{Z}^p \xrightarrow{\alpha} \mathbb{C}^\tau \xrightarrow{\Lambda} \mathrm{Mod}(\mathcal{F}, f) \xrightarrow{\beta} \mathbf{D} \to 0$$

and for any section ζ *of* β *there exists an equisingular marked family* $\mathcal{F}_{\mathbb{C}^\tau\times\mathbf{D}}$ *of foliation germs with same semi-local type as* $(\mathcal{F}, f)$ *such that*

1. *all assertions in Theorem 1.7.3 are satisfied,*
2. *the moduli map* $\mathrm{mod}_{\mathcal{F}_{\mathbb{C}^\tau\times\mathbf{D}}}$ *is surjective and for any* $(z, d) \in \mathbb{C}^\tau \times \mathbf{D}$ *we have*

$$\mathrm{mod}_{\mathcal{F}_{\mathbb{C}^\tau\times\mathbf{D}}}(z, d) = \Lambda(z) \cdot \zeta(d)$$

where $\cdot$ *is the group operation in* $\mathrm{Mod}(\mathcal{F}, f)$.

Notice that the surjectivity of $\mathrm{mod}_{\mathcal{F}_{\mathbb{C}^\tau\times\mathbf{D}}}$ is equivalent to property 2 in Theorem 1.7.3.

A direct consequence of Theorem 1.7.4 is the explicit description of the redundancy of the C^{ex}-classes in the family (1.27):

Corollary 1.7.5 *Two marked foliation germs* $(\mathcal{F}_{z,d}, f_{z,d})$ *and* $(\mathcal{F}_{z',d'}, f_{z',d'})$ *in the marked family* $\mathcal{F}_{\mathbb{C}^\tau\times\mathbf{D}}$ *are* C^{ex}*-conjugated if and only if* $d = d'$ *and there exists* $N \in \mathbb{Z}^p$ *such that* $z' = z + \alpha(N)$.

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