ICMI Study 26
ADVANCES IN GEOMETRY EDUCATION
REIMS (France), 23-26 April 2024

DEFINITIONS OF PROSPECTIVE PRIMARY TEACHERS CONCERNING THE AREA OF 2D FIGURES

Sofía Caviedes[1], Genaro de Gamboa[2], Edelmira Badillo[2], Luis Pino Fan[1]
[1] Universidad de Los Lagos, [2] Universitat Autònoma de Barcelona

The aim of the present study is to characterize the personal definitions about the area of 2D figures of a group of prospective primary teachers. For this purpose, tools of the Ontosemiotic Approach to Mathematical Knowledge and Instruction are used. A content analysis is carried out to identify the emergence of different objects and processes in the definitions of prospective primary teachers. The results show that, for the most part, personal definitions of prospective primary teachers match institutionalized definitions of area as an object of reference. Likewise, the results show that the structure of the definitions provided is given by different objects and processes that allow describing the way in which prospective primary teachers use the different area partial meanings in their definitions. Finally, the results show tendencies of prospective primary teachers in the use of geometric, numerical, and algebraic representations to define area.

INTRODUCTION

Definitions of geometric concepts are a fundamental component of teachers who teach mathematics (Ball et al., 2008; Zazkis & Leikin, 2008), as they reveal a series of logical relationships, for example, between propositions, didactic learning sequences, mathematical connections, or mathematical communication (Leikin & Winicki-Landman, 2001). The variety of ideas about the structure of a mathematical definition and the associated challenges, has led researchers to suggest that definitions should be explicitly addressed as part of initial and continuing teacher education (Leikin & Winicki-Landman, 2001; Zazkis & Leikin, 2008). This is because teachers play a determining role in the definitions used in the classroom and in the role given to these definitions in mathematical activity. However, research that addresses the knowledge that teachers have about mathematical definitions is scarce, pointing mostly to the difficulties associated with this process (Leikin & Winicki-Landman, 2001). Particularly, some studies show the difficulties of prospective and in-service teachers in defining geometric concepts, as quadrilaterals or polygons (e.g., Avcu, 2022; Fujita & Jones, 2007; Leikin & Zazkis, 2010; Miller, 2018). In explicit relation to the area of 2D figures, studies are more scarce and suggest that prospective teachers tend to define area as "length x width" (Livy et al., 2012), coinciding with students' definitions (e.g., Zacharos, 2006). Understanding the personal definitions of prospective primary teachers is a crucial step before designing targeted interventions, as it provides insights into the specific areas of difficulty and misconceptions, and therefore, allows to address the root causes of the difficulties outlined above. In this context, the aim of this paper is to characterize the personal definitions about the area of 2D figures of a group of prospective primary teachers (PPT). Personal definitions are understood as those interpretations made by PPTs of a formal definition of the area in the sense of Tall and Viner (1981). For this purpose, we take into consideration tools from the Ontosemiotic Approach to Mathematical Knowledge and Instruction, which we describe in the following section.

Lowrie, T., Gutiérrez, A., & Emprin, F. (Eds.), *Proceedings of the 26th ICMI Study Conference (Advances in Geometry Education)* (pp. 167-174). ICMI.

Definitions in the teaching of geometry

According to Tall and Vinner (1981) and Vinner (1983) the definition of a concept corresponds to a verbal statement used to specify such concept, explaining it precisely and in a non-circular way. The latter implies that the term to be defined should not be used in the definition (Zaslavsky & Shir, 2005). Tall and Vinner (1981) make a classification of concept definitions, identifying a formal definition of the concept and a personal definition of the concept. The formal definition of the concept corresponds to that which is accepted by the mathematical community in general (Bingölbali & Monaghan, 2008; Moore, 1994). To understand this type of definition, PPTs must generate their own interpretations of the definition (Viholainen, 2008), which may change from one person to another and from one context to another (Pinto & Tall, 2002). In this context of change, Tall and Vinner (1981) point out that personal definitions correspond to a "discursive description that the learner uses for his own explanation of his evoked concept image" (p. 152), this evoked image being the one that is activated at a given moment. The authors point out that the personal definition of a concept may be equivalent to the formal definition, or it may disagree and, consequently, be incomplete or erroneous (Tall & Vinner, 1981). Thus, both definition itself and concept image come into play in the process of defining, and the understanding of a geometric concept is determined by the link between the concept image and the concept definition (Gutiérrez & Jaime, 2012). The concept image is often in conflict with the formal definition of a concept (Tall & Vinner, 1981). For example, even if a student has been taught, and is able to give a formal definition of a parallelogram as a quadrilateral with two pairs of parallel opposite sides, he or she may not consider rectangles, squares, and rhombuses as parallelograms, because the concept image they have of these is that not all angles ("amplitude") or sides ("length") can be equal. In the negotiation of this type of conflicts, the role of definitions can be decisive (Vinner, 1991), since they allow establishing relationships between the conditions (e.g., parallelism) that geometric figures must meet to be classified as parallelograms.

In line with the above, this paper delves into personal definitions that PPTs have about the area of 2D figures. To do so, we use some of the tools of the Ontosemiotic Approach which attributes an essential role to language and categorizes types of objects that emerge in mathematical activity.

Ontosemiotic Approach (OSA)

Godino et al. (2007) define mathematical practice as any action or manifestation performed by someone to solve mathematical problems and communicate the solution to others. The set of such practices is referred to as a system of practices, composed by operative facet (e.g., problems) and discursive facet (e.g., arguments), which are mutually interrelated (Wilhemi et al., 2007). Since defining mathematical objects involves "more than anything else the conflict between the structure of mathematics, as conceived by professional mathematicians, and the cognitive processes of concept acquisition" (Vinner, 1991, p.65), mathematical definitions represent a discursive component of mathematical system of practices (e.g., the formalized language used in defining mathematical concepts), which comes after the initial practical engagement or operative practices (Wilhemi et al., 2007). In this sense, language is the articulating axis of the operative-discursive facets, allowing the communication of procedures and meanings involved in mathematical activity (Wilhemi et al., 2007). Operational and discursive practices can be attributed to an individual or shared within an institution (Godino et al., 2007). In the first case, reference is made to the personal meaning of the object. In the second case, reference is made to the corresponding institutional meaning. The personal meaning of the object is conditioned by its relation/interpretation of the institutional or formal meaning of such object (Tall & Vinner, 1981). A personal object is then an emergent of the system of personal practices linked to a given problem situation. The description and analysis of mathematical activity is conditioned by six primary objects (Godino et al., 2007): linguistic elements; problem-situations; concepts; properties/propositions; procedures; arguments. The primary objects detailed in the OSA emerge from systems of practices which underlines their complexity and the need to articulate them. The later can be done through epistemic configurations (networks of institutional objects) or cognitive configurations (networks of personal objects including cognitive constructs such as conceptual images). Moreover, the relationships that can be established between primary objects take place through different processes, such as definition, which can be considered from different dual facets giving rise to other fundamental epistemic/cognitive processes in mathematical activity (Godino et al., 2007): materialization-idealization (ostensive-non-ostensive), particularization-generalization (intensive-extensive), decomposition-reification (unitary-systemic) and representation-meaning (expression-content).

From the above-mentioned background, in this paper definition is understood as a process that involves the cognitive configuration of PPTs and, thus, their personal meanings. Hence, it is accepted that the personal definition of area constitutes the personal meaning of such an object from the institutionalization-personalization duality (Godino et al., 2007).

METHOD

The study is situated in an interpretive paradigm and follows a qualitative approach (Cohen et al., 2007). A content analysis involving deductive coding is performed, being the unit of analysis the written protocols of PPTs. An epistemic configuration of area (Caviedes et al., 2021) is used to identify how the cognitive configuration that is activated in the definitions of PPTs matches up the institutional definition of area (in the sense of Tall and Vinner, 1981). The institutionalized definition of area is based on three partial meanings presented in the epistemic configuration mentioned above, which can be summarized as follows: *two-dimensional surface that can be covered with units of measurement and/or calculated by means of formulas*. Deductive coding makes it possible to identify the primary objects that account for the cognitive configuration of PPTs, and thus the personal meanings associated with area and the processes in their dual facets. The process of defining is then characterized by the gradual, systematic, and progressive emergence of different objects, processes, and meanings (Godino et al., 2007).

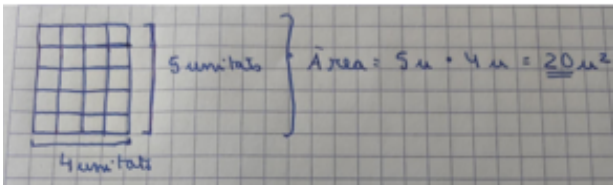
A semi-structured open-ended questionnaire was designed to be solved individually and in writing by a group of 70 PPT in the third year of the Primary Education degree, in the 2020-21 academic year at a public University in Spain. The questionnaire was part of an evaluated activity, had a total of 8 tasks and sought to problematize mathematical and didactic knowledge about area of PPTs (see Caviedes et al., 2023a, 2023b). Although PPTs had worked on activities related to area (e.g., use of different contexts and procedures to solve tasks, analysis of students' responses), they had not worked on its definition. The questionnaire was applied by the professor in charge of the subject, in online format due to the COVID-19 health contingency, and the PPTs had one week to answer it. Due to the objective of this paper, evidence of the written protocols corresponding to Task 7 is presented, posing the following problem situation: *If you had to introduce the geometric content of area in 5th grade, how would you define area?* It's worth mentioning that PPTs were told to focus on *how* to define.

Analysis

The analysis is carried out with the support of MAXQDAplus software. A system of a priori categories is established, given by the epistemic configuration presented in Caviedes et al. (2021) and by the processes of the OSA. The definitions provided by the PPTs allow inferring the emergence of primary objects associated with three partial meanings of area (Caviedes et al., 2021): (1) Sp1-area as space delimited by a closed line; (2) Sp2-area as two-dimensional units covering a surface; (3) Sp3-area as a product of two linear dimensions. The emergence of partial meanings allows us to identify the way in which personal definitions of PPTs enable the emergence of processes of representation/meaning and idealization/materialization. The first is associated with the expression-content duality, where a given representation informs about the structure of that object (Font & Rubio, 2017). The second process is associated with the ostensive-non-ostensive duality, since mathematical objects are in general not perceptible, but are used in mathematical practices through their associated ostensive (e.g., notations, graphs, etc.) (Font & Rubio, 2017). We identify 3 types of definitions: (1) definitions that involve two dual processes, (2) definitions that involve one dual process; (3) definitions that do not involve dual processes.

The definition of PPT 55 (Figure 1) allows inferring the emergence of a representation/meaning process, since PPT 55 attributes a content (area) to a rectangular surface, granting multiplicative meaning to the structure of rows and columns of the area model.

PPT 55: *First, I would define it as the amount of space or surface area covered by a flat (two-dimensional) figure. The area is measured in square units of a fixed size, such as square centimeters, square inches, square miles, etc. Thus, if we delimit on the plan any figure, to find the area we count how many squares of a certain size will cover the region inside the polygon. So, an example would be the following, where we find a square formed by the following units (5x4):*



In this case, you can count the squares and get 20, so the area is 20 square units. However, this method can be ineffective if the rectangle is of larger dimensions or the units are smaller... in this case you can use multiplication, 5 x 4, since there are 4 rows of 5 squares.

Figure 1: Definition of PPT 55

The emergence of the concept of the square unit in the two-dimensional treatment of area (which can be expressed as the product of two lengths) is inferred, as PPT 55 mentions, superficially, that any figure (polygonal) can be measured using squares. Similarly, a process of idealization/materialization is observed, since PPT 55 evokes the properties of the units of measurement, which materializes the idea of area as a space that can be measured in square units and, subsequently, the algorithm that represents the area formula. In this sense, the idealization/materialization process allows giving a material sense to the area formula (the squares that cover the figure and the structure in rows and columns). From this process we infer the emergence of the concept of spatial structuring and the procedure of obtaining the area in an additive way, by means of graphic and symbolic representations.

Thus, a personal meaning associated with Sp1, area as space delimited by a closed line; Sp2, area as the number of two-dimensional units that cover a surface; and Sp3, area as the product of two linear dimensions, is inferred.

Moreover, we infer the emergence of two concepts, the surface as an attribute that allows objects to be measured, which has length and width (two-dimensionality), and the square units of measurement, as a region of the plane delimited by a square and as a fixed quantity of a physical magnitude.

The definition of PPT 65 (Figure 2) allows inferring the emergence of a representation/meaning process. PPT 65 attributes a content (area) to a polygonal figure, giving meaning to the property of being a bounded/enclosed space with a surface extension that can be calculated. To exemplify this, PPT 65 makes use of a counterexample (non-delimited/enclosed surface). It is possible to infer that the personal definition of PPT 65 reduces the area to the surface of polygons, without contemplating, for example, concave or irregular surfaces. In this way, a bias that relates area to conventional geometric representations is evident. In addition, it is possible to infer a personal meaning associated with Sp1, area as space delimited by a closed line, through the emergence of the concept of surface as a quality that allows objects to be measured, which has length and width (two-dimensionality).

PPT 65*: The area is a surface that we find within limits, therefore, we could say that it is the surface of a closed space. Moreover, this surface is flat (two-dimensional). We can calculate the area of polygons because they are a closed space, for example:*

But we CANNOT calculate the area of surfaces that are not bounded, for example:

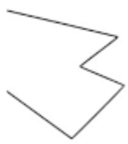


Figure 2: Definition of PPT 65

The definition of PPT 56 (Figure 3) allows us to exemplify those definitions that do not show evidence of dual processes. It is possible to infer a personal meaning associated with Sp1, area as space delimited by a closed line and, likewise, the emergence of the concept of surface as a quality of two-dimensional objects. However, it is not possible to infer the emergence of dual processes, since the definition of PPT 56 does not make explicit the notion of closed or internal space of a two-dimensional figure or object, that is, there is no reference to the two-dimensionality of the area (beyond the use of the concept of surface), since a "thing" can have one, two or three dimensions.

PPT 56: *To introduce the concept of area in the fifth-grade classroom, I would define it as follows: area is the surface occupied by something.*

Figure 3: Definition of PPT 56

RESULTS

Table 1 presents the two dual processes derived from the personal definitions of PPTs.

Emerging processes	Descriptor (Personal meaning)	Partial meanings	Frequency
Idealization-materialization	(1) The emergence of the iteration procedure of units of measurement (standard and non-standard), and their respective properties, materializes the idea of area as a space that can be measured in square units.	Sp1, Sp2 Sp1, Sp2, Sp3	1** 2**
	(2) The emergence of the product formula procedure, through the rectangular model of multiplication, materializes the rectangle formula by structuring rows and columns.	Sp2, Sp3	1**
Representation-meaning	(1) A content is attributed to the square polygonal surface, giving meaning to the property of being a delimited/closed space that can be decomposed into other figures.	Sp1	1*
	(2) A content is attributed to a polygonal figure, giving meaning to the property of being a delimited/enclosed space with a surface area that can be measured.	Sp1 Sp3 Sp1, Sp2 Sp1, Sp3 Sp1, Sp2, Sp3	26* 2* 6* 17* 1**
	(3) A content is attributed to the rectangular surface, giving multiplicative meaning to the row and column structure of the area model.	Sp1, Sp2, Sp3	3**

Table 1: Meanings and processes emerging from the definitions of PPTs (N=60). [**indicates the number of PPTs mobilizing two dual processes in their personal definitions; *indicates the number of PPTs mobilizing one dual process in their personal definitions]

Each process is linked to one or more partial meanings of area (which constitute the institutional definition), and thus to specific primary objects which can be seen in detail in Caviedes et al. (2021).Table 1 shows that a majority of PPTs use the second representation/meaning process (attributing a content to a polygonal figure, giving meaning to the property of being a delimited/enclosed space with a measurable surface extension) based on Sp1. On the contrary, a minority of PPTs use the other processes of representation/ meaning, or the processes of idealization/materialization, which would allow them to make more complete personal definitions through the joint mobilization of the three partial meanings of the area. In addition, Table 1 also shows the relationship between meanings and processes. It is observed that the idealization/materialization processes occur only when two or three partial meanings of the area are connected. In the case of the representation/ meaning processes, it is observed that the first one (attributing a content to a square polygonal surface by decomposing it into other figures) occurs when mobilizing only Sp1. The second representation/ meaning process (attributing a content to a polygonal surface by measuring) occurs both by mobilizing the different partial meanings and by mobilizing only Sp1 and Sp3. The third representation/ meaning process (attributing a content to a rectangular polygonal surface by structuring rows and columns) occurs when Sp1, Sp2 and Sp3 are mobilized together.

The definitions of PPTs present a structure that differs in the emergence of the mathematical primary objects corresponding to the three partial meanings of the area, and in the processes that these meanings make possible. This way, the personal meaning evidenced, and therefore the concept image, can be equivalent to the institutionalized meaning if Sp1, Sp2 and Sp3 are mobilized in the process of defining. The joint mobilization of these meanings would account for robust personal definitions. Isolated mobilization of partial meanings would account for incomplete personal definitions.

CONCLUSIONS

The aim of this study was to characterize the personal definitions of the area of 2D figures of a group of PPT. The use of the configuration of objects and processes as a theoretical/analytical tool makes explicit the complexity underlying the process of defining (De Villiers et al., 2009), showing that this process involves describing objects and concepts, identifying properties of mathematical objects (Zaslavsky & Shir, 2005), communicating mathematical ideas, and coordinating different representations (Sinclair et al., 2012). The mobilization of different partial meanings makes it possible the emergence of certain processes that strengthen and complexify the definitions of PPTs. Hence, primary objects and processes account for the concept image that underlies personal definitions of PPTs (Tall & Vinner, 1981). Moreover, it is evidenced that PPTs concept image about the area of 2D figures can be put in correspondence with three partial meanings of the area (Caviedes et al., 2021), whose primary objects vary according to the intra-mathematical context in which they are framed. Therefore, this study considers the complexity of the process of defining (in terms of objects and processes) as a differentiating aspect of it and, at the same time, unifying, in the sense that it allows characterizing robust definitions through partial meanings of reference. In this sense, we agree that learning the process of defining should be an essential part of the learning process of mathematics in teacher training (Zaslavsky et al., 2003), since personal definition can have an impact on how they interpret the curriculum, what activities to propose and how to interpret students' mathematical practices. However, studies involving the design of teaching experiments based on these results are needed to strengthen the definitions and address the respective difficulties of PPTs.

Acknowledgements

Study funded by ANID-Chile Strengthening of doctoral programmes nº 2022-86220016. PID2019-104964GB-I00 and GIPEAM, 2021 SGR 00159-AGAUR.

References

Avcu, R. (2022). Pre-service middle school mathematics teachers' personal concept definitions of special quadrilaterals. *Mathematics Education Research Journal,* 1-46.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389–407. https://doi.org/10.1177/0022487108324554

Bingölbali, E., & Monaghan, J. (2008). Concept image revisited. *Educational Studies in Mathematics, 68*(1), 19–35. https://doi.org/10.1007/s10649-007-9112-2

Caviedes, S., de Gamboa, G., & Badillo, E. (2021). Mathematical objects that configure the partial area meanings mobilized in task-solving. *International Journal of Mathematical Education in Science and Technology*, *54*(6), 1092-1111. https://doi.org/10.1080/0020739X.2021.1991019

Caviedes, S., de Gamboa, G., & Badillo, E. (2023a). Preservice teachers' knowledge mobilized in solving area tasks. *Journal on Mathematics Education, 14*(1), 35–54. https://doi.org/10.22342/jme.v14i1.pp35-54

Caviedes Barrera, S., de Gamboa Rojas, G., & Badillo Jiménez, E. (2023b). Mathematical and didactic knowledge of preservice primary teachers about the area of 2d figures. *Avances de Investigación en Educación Matemática*, (24), 1-20. https://doi.org/ 10.35763/aiem24.4076

De Villiers, M., Govender, R., & Patterson, N. (2009). Defining in geometry. En T. V. Craine, y R. Rubenstein (Eds.). *Understanding geometry for a changing world* (pp.189–203). National Council of Teachers of Mathematics.

Font, V., & Rubio, N. V. (2017). Procesos matemáticos en el enfoque ontosemiótico. En J. M. Contreras, P. Arteaga, G. R. Cañadas, M. M. Gea, B. Giacomone y M. M. López-Martín (Eds.), *Actas del Segundo Congreso International Virtual sobre el Enfoque Ontosemiótico del Conocimiento y la Instrucción Matemáticos*, (pp.1–27).

Godino, J. D., Batanero, C., & Font, V. (2007). The onto-semiotic approach to research in mathematics education. *Zdm, 39*, 127-135. https://doi.org/10.1007/s11858-006-0004-1

Gutiérrez, Á., & Jaime, A. (2012). Reflections on the Teaching of Geometry in Primary and Secondary Schools. *Tecné, Episteme y Didaxis: TED, 32*, 55-70.

Leikin, R., & Winicki-Landman, G. (2001). Defining as a vehicle for professional development of secondary school mathematics teachers. *Mathematics Teacher Education and Development, 3*, 62–73.

Livy, S., Muir, T., & Maher, N. (2012). How do they measure up? Primary pre-service teachers' mathematical knowledge of area and perimeter. *Mathematics Teacher Education and Development, 14*(2), 91-112.

Miller, S. M. (2018). An analysis of the form and content of quadrilateral definitions composed by novice pre-service teachers. *The Journal of Mathematical Behavior, 50*, 142–154.

Moore, R. C. (1994). Making the transition to formal proof. *Educational Studies in Mathematics, 27*(3), 249–266. https://doi.org/10.1007/BF01273731

Pinto, M., & Tall, D. (2002). Building formal mathematics on visual imagery: A case study and a theory. *For the Learning of Mathematics, 22*(1), 2–10.

Sinclair, N., Pimm, D., Skelin, M., & Zbiek, R. M. (2012). *Developing essential understanding of geometry for teaching mathematics in grades 6–8.* National Council of Teachers of Mathematics.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics, 12*(2), 151–169.

Viholainen, A. (2008). Incoherence of a concept image and erroneous conclusions in the case of differentiability. *The Mathematics Enthusiast, 5*(2), 231–248. https://doi.org/10.54870/1551-3440.1104

Vinner, S. (1983). Concept definition, concept image and the notion of function. International *Journal of Mathematical Education in Science and Technology, 14*(3), 293–305.

Wilhelmi, M. R., Godino, J. D., & Lacasta, E. (2007). Didactic effectiveness of mathematical definitions: The case of the absolute value. *International Electronic Journal of Mathematics Education, 2*(2), 72–90.

Zacharos, K. (2006). Prevailing educational practices for area measurement and students' failure in measuring areas. *The Journal of Mathematical Behavior*, *25*(3), 224-239.

Zaslavsky, O., & Shir, K. (2005). Students' conceptions of a mathematical definition. *Journal for Research in Mathematics Education, 36*(4), 317–346. https://doi.org/10.2307/30035043

Zazkis, R., & Leikin, R. (2008). Exemplifying definitions: A case of a square. *Educational Studies in Mathematics, 69*, 131–148. https://doi.org/10.1007/s10649-008-9131-7