CARLESON'S ε^2 CONJECTURE IN HIGHER DIMENSIONS AND FABER-KRAHN INEQUALITIES

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Abstract. In this paper we survey the proof of Carleson's ε^2 conjecture in the plane and its extension to higher dimensions. We also describe its connections with rectifiability and the so-called Faber-Krahn inequalities for the first eigenvalue of the Laplacian.

1. Introduction

One of the main objectives of geometric measure theory is the characterization of n-rectifiable sets. Recall that a set $E \subset \mathbb{R}^d$ is n-rectifiable if there are Lipschitz maps $f_i : \mathbb{R}^n \to \mathbb{R}^d$, $i \in \mathbb{N}$, such that

$$\mathcal{H}^n\Big(E \setminus \textstyle\bigcup_{i=1}^{\infty} f_i(\mathbb{R}^n)\Big) = 0. \tag{1.1}$$

We do not require n-rectifiable sets to have finite Hausdorff measure $\mathcal{H}^n$. Instead, (1.1) ensures that $\mathcal{H}^n(E)$ is σ-finite. Some well known classical characterizations of n-rectifiable sets, mainly due to Besicovitch, Federer, Marstrand, Mattila, and Preiss, are in terms of existence of tangents, densities, and the behavior of orthogonal projections.

In the 1990's, there appeared a need to develop a quantitative theory of rectifiability because of the possible applications to the Painlevé problem about removable singularities for bounded holomorphic functions and also because of the wish to understand the L^2 boundedness of singular integral operators on suitable rectifiable sets. This led to study the connection between rectifiability and the boundedness of different square functions involving different coefficients encoding geometric information. One of these square functions is the so-called Carleson's ε^2-square function.

Let Ω_1 be a Jordan domain in $\mathbb{R}^2$, and set $\Gamma = \partial\Omega_1$ and $\Omega_2 = \mathbb{R}^2 \setminus \overline{\Omega_1}$. For $x \in \mathbb{R}^2$ and $r > 0$, denote by $I_1(x,r)$ and $I_2(x,r)$ the longest open arcs of the circumference $\partial B(x,r)$ contained in Ω_1 and Ω_2, respectively (they may be empty). Then one defines

$$\varepsilon(x,r) = \frac{1}{r} \max\big(\big|\pi r - \mathcal{H}^1(I_1(x,r))\big|,\, \big|\pi r - \mathcal{H}^1(I_2(x,r))\big|\big). \tag{1.2}$$

The Carleson ε^2-square function is given by

$$\mathcal{E}(x)^2 := \int_0^1 \varepsilon(x,r)^2 \,\frac{dr}{r}. \tag{1.3}$$

Carleson's conjecture, now a theorem, asserts the following.

X.T is supported by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement 101018680). X.T. is also partially supported by MICINN (Spain) under the grant PID2020-114167GB-I00, the María de Maeztu Program for units of excellence (Spain) (CEX2020-001084-M), and 2021-SGR-00071 (Catalonia).

Theorem 1.1. *Let $\Omega_1 \subset \mathbb{R}^2$ be a Jordan domain, $\Omega_2 = \mathbb{R}^2 \setminus \overline{\Omega_1}$, and $\Gamma = \partial\Omega_1$. Let $\mathcal{E}$ be the associated square function defined in (1.3). Then the set of tangent points for Ω_1 coincides with the subset of those points $x \in \Gamma$ such that $\mathcal{E}(x) < \infty$, up to a set of zero measure $\mathcal{H}^1$. In particular, the set $G = \{x \in \Gamma : \mathcal{E}(x) < \infty\}$ is 1-rectifiable.*

The fact that $\mathcal{E}(x) < \infty$ for $\mathcal{H}^1$-a.e. tangent point in a Jordan curve was proved by Bishop in [Bi1] (see also [BCGJ]). The most difficult implication of Theorem 1.1, i.e, the fact that the set G is 1-rectifiable and tangents to Γ exist for $\mathcal{H}^1$-a.e. $x \in G$, was proved more recently by Ben Jaye, Michele Villa, and the author of this paper [JTV].

Regarding the notion of tangent for a domain, it is appropriate to consider a somewhat more general notion involving two disjoint domains. For a point $x \in \mathbb{R}^{n+1}$, a unit vector u, and an aperture parameter $a \in (0,1)$ we consider the two sided cone with axis in the direction of u defined by

$$X_a(x,u) = \big\{y \in \mathbb{R}^{n+1} : |(y-x)\cdot u| > a|y-x|\big\}.$$

Given disjoint open sets $\Omega_1, \Omega_2 \subset \mathbb{R}^{n+1}$ and $x \in \partial\Omega_1 \cap \partial\Omega_2$, we say that x is a tangent point for the pair Ω_1, Ω_2 if $x \in \partial\Omega_1 \cap \partial\Omega_2$ and there exists a unit vector u such that, for all $a \in (0,1)$, there exists some $r > 0$ such that

$$(\partial\Omega_1 \cup \partial\Omega_2) \cap X_a(x,u) \cap B(x,r) = \varnothing,$$

and moreover, one component of $X_a(x,u) \cap B(x,r)$ is contained in Ω_1 and the other in Ω_2. The hyperplane L orthogonal to u through x is called a tangent hyperplane at x. In case that $\Omega_2 = \mathbb{R}^{n+1} \setminus \overline{\Omega_1}$, we say that x is a tangent point for Ω_1.

Recently, in [FTV2], Ian Fleschler, Michele Villa, and the author have proven a higher dimensional version of Carleson's conjecture. Here we will review this result and the main ideas of the proof. We will also see the connections of Carleson's conjecture with Jones' traveling salesman theorem, the Alt-Caffarelli-Friedman formula, and quantitative Faber-Krahn inequalities which motivate this extension.

2. Carleson's conjecture and Jones' traveling salesman theorem

In the pioneering work [Jo], inspired in part by the multi-scale Littlewood-Paley techniques to characterize the regularity of functions in harmonic analysis, Peter Jones proved a celebrated traveling salesman theorem which quantifies the length of the shortest curve that contains a given set in the plane in terms of some β_∞ coefficients associated with the set. Jones' result has been very influential and has been the starting point of what is known now as the theory of quantitative rectifiability.

To state Jones' theorem we need to introduce some notation. Given a cube $Q \subset \mathbb{R}^d$, we denote

$$\beta_{\infty,E}(Q) = \inf_L \left\{ \sup_{y \in E\cap Q} \frac{\operatorname{dist}(y,L)}{\ell(Q)} \right\},$$

where the infimum is taken over all the lines $L \subset \mathbb{R}^d$ and $\ell(Q)$ is the side length of Q. So $2\ell(Q)\,\beta_{\infty,E}(Q)$ is the width of the thinnest strip that contains $E \cap Q$. The coefficient $\beta_{\infty,E}(Q)$ is scale invariant and it measures how close $E \cap Q$ is to some line.

Theorem 2.1 ([Jo])**.** *A subset $E \subset \mathbb{R}^d$ is contained in a curve with finite length if and only if*

$$\sum_{Q \in \mathcal{D}} \beta_{\infty,E}(3Q)^2 \ell(Q) < \infty, \tag{2.1}$$

where $\mathcal{D}$ is the family of all dyadic cubes in $\mathbb{R}^d$ and $3Q$ stands for the cube concentric with Q with triple side length. Further the length of the shortest curve Γ containing E satisfies

$$\mathcal{H}^1(\Gamma) \approx \operatorname{diam}(E) + \sum_{Q\in\mathcal{D}} \beta_{\infty,E}(3Q)^2 \ell(Q), \tag{2.2}$$

where the implicit constant is an absolute constant depending only on d.

The notation $A \approx B$ means that there exists an absolute constant $C > 0$, perhaps depending on the ambient dimension, such that $C^{-1}A \leq B \leq CA$. Theorem 2.1 was proved in the planar case $d = 2$ by Jones [Jo], while the extension to subsets of $\mathbb{R}^d$ with $d \geq 3$ (more precisely of the fact that (2.2) holds for the shortest curve Γ containing E) is due to Okikiolu [Ok].

Next we will announce a theorem of Bishop and Jones from 1994 that characterizes the existence of tangents to a Jordan curve in terms of the coefficients β_∞. To this end, it is convenient to change the cubes Q in the definition of the β_∞ coefficients by balls $B(x, r)$. So, given $E \subset \mathbb{R}^d$ and a ball B, we denote

$$\beta_{\infty,E}(B) = \inf_L \left\{ \sup_{y\in E\cap B} \frac{\operatorname{dist}(y, L)}{r(B)} \right\},$$

where the infimum is taken over all the lines $L \subset \mathbb{R}^d$ and $r(B)$ denotes radius of B. We will also write $\beta_{\infty,E}(x, r)$ instead of $\beta_{\infty,E}(B(x, r))$. The aforementioned theorem of Bishop and Jones asserts the following:

Theorem 2.2 ([BJ]). *Let $\Omega \subset \mathbb{R}^2$ be a Jordan domain and let $\Gamma = \partial\Omega$. Then, up to a set o null measure $\mathcal{H}^1$,*

$$\int_0^1 \beta_{\infty,\Gamma}(x, r)^2 \frac{dr}{r} < \infty \quad \text{at } x \in \Gamma \qquad \Leftrightarrow \qquad \Omega \text{ has a tangent at } x. \tag{2.3}$$

The proof of this theorem relies heavily on Jones' Theorem 2.1. Remark the function

$$J(x)^2 := \int_0^1 \beta_{\infty,\Gamma}(x, r)^2 \frac{dr}{r}$$

is called Jones' square function.

Using Theorem 2.2, we can now prove the "easy" implication in Theorem 1.1.

Proof of the finiteness of Carleson's square function at $\mathcal{H}^1$-a.e. tangent point. It is enough to show that, for any $x \in \Gamma$ and $r > 0$ small enough

$$\varepsilon(x, r) \lesssim \beta_{\infty,\Gamma}(x, r).$$

This follows by elementary geometric arguments. Indeed, let y and z be the extremes of the arc $I_1(x, r)$. Denote $\theta_i(x, r)$ the angles subtended by the arcs $I_i(x, r)$. See Figure 1. It is clear that the triangle with vertices x, y, z is contained in the thinnest strip containing $\Gamma \cap B(x, r)$ (we suppose that $B(x, r)$ is closed). Then, $\beta_{\infty,\{x,y,z\}}(x, r) \leq \beta_{\infty,\Gamma}(x, r)$, or equivalently, the height of the triangle from the vertex x till the side yz must be smaller than the width of that strip. That is, $r \cos \frac{\theta_1(x,r)}{2} \leq 2r\, \beta_{\infty,\Gamma}(x, r)$. Thus,

$$|\pi - \theta_1(x, r)| \approx \left|\sin \frac{\pi - \theta_1(x, r)}{2}\right| = \left|\cos \frac{\theta_1(x, r)}{2}\right| \leq 2\, \beta_{\infty,\Gamma}(x, r).$$

Obviously, the same estimate holds interchanging $\theta_1(x, r)$ by $\theta_2(x, r)$, and so it follows that $\varepsilon(x, r) \leq c\, \beta_{\infty,\Gamma}(x, r)$.

□

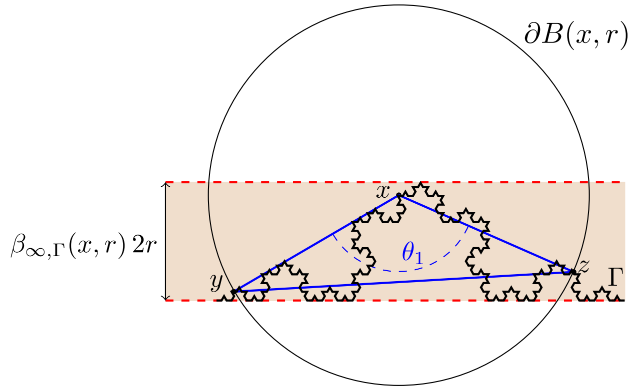

FIGURE 1. Comparison between $\beta_{\infty,\Gamma}(x,r)$ and $\varepsilon(x,r)$.

The results of Bishop and Jones have been extended to higher dimensions in different ways. In particular, David and Semmes, in the 1990's, introduced and studied the notion of uniform n-rectifiability for n-Ahlfors regular sets. A set $E \subset \mathbb{R}^d$ is n-Ahlfors regular if

$$\mathcal{H}^n(E \cap B(x,r)) \approx r^n \quad \text{for all } x \in E,\ 0 < r \leq \operatorname{diam}(E).$$

David and Semmes [DS1] proved that uniform n-rectifiable sets can be characterized in terms of some coefficients $\beta_{p,E}$ which we proceed to define. Given an $\mathcal{H}^n$-measurable set $E \subset \mathbb{R}^d$, $1 \leq p < \infty$, $x \in \mathbb{R}^d$, and $r > 0$, we denote

$$\beta_{p,E}(x,r) = \inf_L \left(\frac{1}{r^n} \int_{E \cap B(x,r)} \left(\frac{\operatorname{dist}(y,L)}{r} \right)^p d\mathcal{H}^n(y) \right)^{1/p},$$

where the infimum is taken over all the n-planes $L \subset \mathbb{R}^d$.

To describe the results of David and Semmes on uniform rectifiability would lead us too far in this paper. See the monographs [DS1], [DS2]. Instead, we just recall the following result, which characterizes n-rectifiable sets in terms of the finiteness of a square function involving the β_2 coefficients. It is worth comparing the rectifiability criterion below with the ones appearing in Theorems 1.1 and 2.2.

Theorem 2.3. *Let $E \subset \mathbb{R}^n$ be $\mathcal{H}^n$-measurable and such that $\mathcal{H}^n(E) < \infty$. Then E is n-rectifiable if and only if*

$$\int_0^1 \beta_{2,E}(x,r)^2 \frac{dr}{r} < \infty \qquad \text{for } \mathcal{H}^n\text{-a.e. } x \in E. \tag{2.4}$$

The fact that n-rectifiable sets satisfy (2.4) was proven in [To], while the converse implication is by Azzam and the author of this paper [AT]. Remark that the connection between rectifiability and the β_2 coefficients has been used to study the singular set for harmonic maps by Naber and Valtorta [NV]. Further, their techniques has been extended to the study of other related free boundary problems.

3. Carleson's conjecture in higher dimensions. Getting rectifiability

One of the difficulties in trying to extend Carleson's conjecture to higher dimensions is to guess which could be such natural extension and which coefficients one could use. Indeed, in the plane Carleson's conjecture involves Jordan domains, and moreover the arguments in [JTV] make an extensive use of the connectivity of the boundary of such domains. In particular, the connectivity implies the lower content regularity of $\partial\Omega_1$ when Ω_1 is Jordan domain. That is, it holds

$$\mathcal{H}^1_\infty(B(x,r)\cap\partial\Omega_1)\gtrsim r \quad \text{for all } x\in\partial\Omega_1,\ 0<r\leq \operatorname{diam}(\Omega_1).$$

Here $\mathcal{H}^s_\infty$ denotes the s-dimensional Hausdorff content, defined by $\mathcal{H}^s_\infty(E)=\inf\{\sum_i \operatorname{diam}(A_i)^s : E\subset\bigcup_i A_i\}$ for any set E in the Euclidean space. In $\mathbb{R}^{n+1}$, it is well known that the connectivity of the boundary of a domain is a rather weak assumption and no quantitative information about the n-dimensional content $\mathcal{H}^n_\infty(B(x,r)\cap\partial\Omega_1)$ can be obtained from this.

Next we will define the coefficients $\varepsilon_n(x,r)$ introduced in [FTV2]. For $x\in\mathbb{R}^{n+1}$, $r>0$ and an affine half-space H such that $x\in\partial H$, denote

$$S^1_H(x,r)=S(x,r)\cap H,\qquad S^2_H(x,r):=S(x,r)\cap(\mathbb{R}^{n+1}\setminus\overline{H}), \tag{3.1}$$

where $S(x,r)=\partial B(x,r)$. Given two disjoint Borel sets $\Omega_1,\Omega_2\subset\mathbb{R}^{n+1}$, we put

$$\varepsilon_n(x,r):=\frac{1}{r^n}\inf_H \mathcal{H}^n\left((S^1_H(x,r)\setminus\Omega_1)\cup(S^2_H(x,r)\setminus\Omega_2)\right). \tag{3.2}$$

It is clear that if Ω_1 and Ω_2 are complementary (open) half-spaces, then $\varepsilon_n(x,r)=0$ for any $x\in\partial\Omega_1=\partial\Omega_2$ and $r>0$. Note that in the plane $\varepsilon_1(x,r)\lesssim\varepsilon(x,r)$, but the opposite inequality fails, in general. We write

$$\mathcal{E}_n(x)^2:=\int_0^1\varepsilon_n(x,r)^2\,\frac{dr}{r}. \tag{3.3}$$

One of the main results from [FTV2] is the following:

Theorem 3.1. *For $n\geq 1$ let $\Omega_1,\Omega_2\subset\mathbb{R}^{n+1}$ be two disjoint Borel subsets. Then the set $\{x\in\mathbb{R}^{n+1}:\mathcal{E}_n(x)<\infty\}$ is n-rectifiable.*

Remark that this theorem is valid for arbitrary disjoint Borel sets Ω_i (not necessarily open) and it is new even in the plane. So the finiteness of the square function $\mathcal{E}_n$ (smaller than $\mathcal{E}$ in the case $n=1$) on some set $E\subset\mathbb{R}^{n+1}\setminus\Omega_1\cup\Omega_2$ implies the n-rectifiability of E and so the fact that E has σ-finite measure $\mathcal{H}^n$. In turn, the n-rectifiability of E implies the existence of approximate tangents of E at $\mathcal{H}^n$-a.e. $x\in E$ (in case that $\mathcal{H}^n(E)<\infty$). So at first sight, this result may look stronger that Theorem 1.1. However, Theorem 3.1 does not ensure the existence of "true" tangents for the pair of sets Ω_1, Ω_2 (even if they are open). So the assumptions in the this theorem are weaker than the ones in Theorem 1.1, but the conclusion is also weaker.

So to guarantee the existence of tangents for Ω_1,Ω_2 we need stronger assumptions, and probably other coefficients than the ε_n's. Indeed, notice that $\varepsilon_n(x,r)$ does not detect sets of dimension smaller than n, nor purely n-unrectifiable subsets. For example, let Ω_2 be the lower (open) half space $\mathbb{R}^{n+1}_-$, and let Ω_1 be $\mathbb{R}^{n+1}_+$ minus a union of countably many compact sets which accumulate on the hyperplane $E:=\{x_{n+1}=0\}$ and either have dimension smaller than n or are purely n-unrectifiable. Then it is clear that $\mathcal{E}_n(x)=0$ in E, but one may construct Ω_1 so that there are no tangents for the pair Ω_1, Ω_2, at any $x\in E$.

The proof of Theorem 3.1 combines compactness arguments and a stopping time construction inspired by techniques developed previously by David and Semmes, and extended to the non-doubling setting by Léger. The arguments are more elaborated and difficult than the ones from the planar case in [JTV].

4. The Alt-Caffarelli-Friedman monotonicity formula and the Friedland-Hayman inequality

Given a bounded open set V in a Riemannian manifold $\mathbb{M}^n$ (such as $\mathbb{R}^n$ or $\mathbb{S}^n$), we say that $u \in W_0^{1,2}(V)$ is a Dirichlet eigenfunction of V for the Laplace-Beltrami operator $\Delta_{\mathbb{M}^n}$ if $u \not\equiv 0$ and

$$-\Delta_{\mathbb{M}^n} u = \lambda\, u,$$

for some $\lambda \in \mathbb{R} \setminus \{0\}$. The number λ is the eigenvalue associated with u. It is well known that all the eigenvalues of the Laplace-Beltrami operator are positive and the smallest one, i.e., the first eigenvalue λ_V, satisfies

$$\lambda_V = \inf_{u \in W_0^{1,2}(V)} \frac{\int_V |\nabla u|^2\, dx}{\int_V |u|^2\, dx}. \tag{4.1}$$

Further the infimum is attained by an eigenfunction u which does not change sign, and so which can be assumed to be non-negative. Also, from (4.1) we infer that, if that $U, V \subset \mathbb{M}^n$ are open, then

$$U \subset V \quad \Rightarrow \quad \lambda_U \geq \lambda_V. \tag{4.2}$$

In the case $\mathbb{M}^n = \mathbb{S}^n$, one defines the characteristic constant of V as the positive number α_V such that $\lambda_V = \alpha_V(n - 1 + \alpha_V)$.

The Alt-Caffarelli-Friedman (ACF) monotonicity formula is an important inequality which plays an essential role in many free boundary problems. It asserts the following:

Theorem 4.1 (Alt, Caffarelli, Friedman). *Let $x \in \mathbb{R}^{n+1}$ and $R > 0$. Let $u_1, u_2 \in W^{1,2}(B(x,R)) \cap C(B(x,R))$ be nonnegative subharmonic functions such that $u_1(x) = u_2(x) = 0$ and $u_1 \cdot u_2 \equiv 0$. Set*

$$J(x,r) = \left(\frac{1}{r^2}\int_{B(x,r)} \frac{|\nabla u_1(y)|^2}{|y-x|^{n-1}} dy\right) \cdot \left(\frac{1}{r^2}\int_{B(x,r)} \frac{|\nabla u_2(y)|^2}{|y-x|^{n-1}} dy\right) \tag{4.3}$$

Then $J(x,r)$ is an absolutely continuous function of $r \in (0,R)$ and

$$\frac{\partial_r J(x,r)}{J(x,r)} \geq \frac{2}{r}\big(\alpha_1 + \alpha_2 - 2\big). \tag{4.4}$$

where α_i, for $i = 1,2$, are the characteristic constants of the open subsets $V_i \subset \mathbb{S}^n$ given by

$$V_i = \big\{ r^{-1}(y - x) : y \in \partial B(x,r),\, u_i(y) > 0 \big\}.$$

Further, for $r \in (0, R/2)$, we have

$$\frac{1}{r^2}\int_{B(x,r)} \frac{|\nabla u_i(y)|^2}{|y-x|^{n-1}} dy \lesssim \frac{1}{r^{n+1}} \|\nabla u_i\|^2_{L^2(B(x,2r))}. \tag{4.5}$$

The Friedland-Hayman [FH] inequality asserts that, for any two disjoint open subsets $V_1, V_2 \subset \mathbb{S}^n$ and $\alpha_i \equiv \alpha_{V_i}$, we have

$$\alpha_1 + \alpha_2 - 2 \geq 0,$$

so that $J(x,r)$ is non-decreasing on r, by (4.4). In fact, more is known: by Sperner's inequality [Sp], among all the open subsets with a fixed measure $\mathcal{H}^n$ in $\mathbb{S}^n$, the one that minimizes the characteristic constant is a spherical ball with the same measure $\mathcal{H}^n$. That is to say, given $V \subset \mathbb{S}^n$, if Δ is a spherical ball such that $\mathcal{H}^n(\Delta) = \mathcal{H}^n(V)$ and $\bar{\alpha}_V$ denotes its characteristic constant, then

$$\alpha_V \geq \bar{\alpha}_V.$$

Further, for $V_1, V_2 \subset \mathbb{S}^n$, if one of the sets V_i differs from a hemisphere by a surface measure h, that is,

$$\left|\mathcal{H}^n(V_i) - \frac{1}{2}\mathcal{H}^n(\mathbb{S}^n)\right| \geq h$$

either for $i = 1$ or $i = 2$, then

$$\alpha_1 + \alpha_2 - 2 \geq c\, h^2$$

for some fixed $c > 0$.

In [AKN2], Allen, Kriventsov and Neumayer have shown a very interesting connection between the $\varepsilon(x,r)$ coefficient of Carleson in the plane and the Friedland-Hayman inequality, which we proceed to describe. Consider two disjoint open sets $V_1, V_2 \subset \mathbb{S}^1$ and let $\alpha_i = \alpha_{V_i}$ be the characteristic constant of V_i. Analogously, set $\lambda_i = \lambda_{V_i}$. In [AKN2] it is remarked that

$$\varepsilon(0,1)^2 \lesssim \alpha_1 + \alpha_2 - 2, \tag{4.6}$$

where $\varepsilon(0,1)$ is defined as in (1.2), with $\Omega_i \cap \mathbb{S}^1$ replaced by V_i. Indeed, V_i is the union of a finite or countable collection of disjoint open arcs $\{J^i_j\}_j$, and from the definition of Dirichlet eigenvalues it follows that the family of the Dirichlet eigenvalues of V_i coincides with the union of the Dirichlet eigenvalues of all the intervals J^i_j. Then, from (4.2) we infer that the first eigenvalue λ_i of V_i equals the first eigenvalue of the largest interval I_i from the family $\{J^i_j\}_j$. That is, $\lambda_{I_i} = \lambda_i$ and $\alpha_{I_i} = \alpha_i$, for $I_i := I_i(0,1)$ as in (1.2).

Let $\gamma_i = \mathcal{H}^1(I_i)/(2\pi)$. Since the first eigenfunction for I_i is the function $u_i(\theta) = \sin((2\gamma_i)^{-1}\theta)$ (modulo a translation in the torus), we have $\alpha_i = (\lambda_i)^{1/2} = (2\gamma_i)^{-1}$. Suppose, for example, that $\varepsilon(0,1) = \left|\pi - \mathcal{H}^1(I_1)\right|$. Let $\tilde{\alpha}_2$ the characteristic of $\mathbb{S}^1 \setminus \overline{I_1}$. Since $I_2 \subset \mathbb{S}^1 \setminus \overline{I_1}$, we have $\alpha_2 \geq \tilde{\alpha}_2$. Thus,

$$\alpha_1 + \alpha_2 - 2 \geq \alpha_1 + \tilde{\alpha}_2 - 2 = \frac{1}{2\gamma_1} + \frac{1}{2(1-\gamma_1)} - 2 = \frac{1 - 4\gamma_1(1-\gamma_1)}{2\gamma_1(1-\gamma_1)} = \frac{2(\frac{1}{2} - \gamma_1)^2}{\gamma_1(1-\gamma_1)} \approx \frac{\varepsilon(0,1)^2}{\gamma_1(1-\gamma_1)},$$

which completes the proof of (4.6), since $\gamma_1 \in (0,1)$. Further, in case that $I_1(0,1)$ and $I_2(0,1)$ are complementary arcs, arguing as above, one can deduce

$$\min\big(1, \alpha_1 + \alpha_2 - 2\big) \approx \varepsilon(0,1)^2.$$

See also [Bi2] for a very related discussion.

In higher dimensions a similar estimate holds:

Theorem 4.2 ([FTV1]). *Let $V_1, V_2 \subset \mathbb{S}^n$ be disjoint relatively open sets and let $\varepsilon_n(0,1)$ be defined as in (3.2), with Ω_i replaced by V_i. Let $\alpha_i = \alpha_{V_i}$ for $i = 1, 2$. Then*

$$\varepsilon_n(0,1)^2 \lesssim \alpha_1 + \alpha_2 - 2.$$

The proof of this result is more difficult than the one for (4.6). It requires the application of a quantitative Faber-Krahn inequality. The arguments will be explained in the next section.

5. Quantitative Faber-Krahn inequalities

Given a bounded open set V in a Riemannian manifold, we denote by λ_V the first Dirichlet eigenvalue of V and by u_V the associated non-negative eigenfunction, normalized so that $\|u_V\|_{L^2(V)} = 1$.

The classical Faber-Krahn inequality asserts that among all bounded open sets with a fixed volume, a ball minimizes the first eigenvalue. Following many previous works , Brasco, De Philippis, and Velichkov proved in [BDV] the following quantitative version of the Faber-Krahn inequality.

Theorem 5.1. *For $n \geq 2$, let $V \subset \mathbb{R}^n$ be a bounded open set with $\mathcal{H}^n(V) = 1$. Then*

$$\lambda_V - \lambda_B \geq c \inf_B \mathcal{H}^n(V \triangle B)^2, \tag{5.1}$$

where c is a positive absolute constant and the infimum is taken over all balls B with $\mathcal{H}^n(B) = 1$.

The inequality above is sharp in the sense that the power 2 on the right hand side cannot be lowered.

The classical Faber-Krahn inequality also holds for subdomains of the sphere $\mathbb{S}^n$ or the hyperbolic space $\mathbb{H}^n$. In this context one should consider geodesic balls. That is, for open subsets V of $\mathbb{S}^n$ or $\mathbb{H}^n$ with a given volume, the minimal value of λ_V is attained again by a geodesic ball among all open bounded open sets of the same volume. Recently Allen, Kriventsov, and Neumayer [AKN1] have obtained the following quantitative form.

Theorem 5.2. *For $n \geq 2$, let $\mathbb{M}^n$ be either $\mathbb{R}^n$, $\mathbb{S}^n$, or $\mathbb{H}^n$, and let $\beta > 0$. Let V be a relatively open subset of $\mathbb{M}^n$ and let B be a geodesic ball in $\mathbb{M}^n$ such that $\mathcal{H}^n(B) = \mathcal{H}^n(V)$. In the case $\mathbb{M}^n = \mathbb{S}^n$, suppose also that $\beta \leq \mathcal{H}^n(V) \leq \mathcal{H}^n(\mathbb{S}^n) - \beta$, and in the case $\mathbb{M}^n = \mathbb{R}^n$ or $\mathbb{M}^n = \mathbb{H}^n$ just that $\mathcal{H}^n(V) \leq \beta$. Denote by λ_V and λ_B the respective first Dirichlet eigenvalues of $-\Delta_{\mathbb{M}^n}$ in V and B, and let u_V and u_B be the corresponding eigenfunctions normalized so that they are positive and $\|u_V\|_{L^2(\mathbb{M}^n)} = \|u_B\|_{L^2(\mathbb{M}^n)} = 1$. Then*

$$\lambda_V - \lambda_B \geq c(\beta) \inf_{x \in \mathbb{M}^n} \Big(\mathcal{H}^n(V \triangle B_x)^2 + \int_{\mathbb{M}^n} |u_V - u_{B_x}|^2 \, d\mathcal{H}^n \Big), \tag{5.2}$$

where $c(\beta) > 0$ and B_x denotes a ball centered at x with the same $\mathcal{H}^n$ measure as B. In the case $\mathbb{M}^n = \mathbb{S}^n$, (5.2) also holds with the infimum over $\mathbb{S}^n$ replaced by the evaluation at x equal to $\mathbb{S}^n$-barycenter of V (possibly with a different constant $c(\beta)$).

The assumption involving the parameter β is necessary to prevent the domain from being too big and, in the case $\mathbb{M}^n = \mathbb{S}^n$, also too small. The $\mathbb{S}^n$-barycenter of $V \subset \mathbb{S}^n$ is defined by $x_V/|x_V|$, with x_V as above. So this belongs to $\mathbb{S}^n$ and it is defined only when $x_V \neq 0$. Notice the presence of the additional term $\int_{\mathbb{M}^n} |u_V - u_{B_x}|^2 \, d\mathcal{H}^n$ in the inequality (5.2) when compared to (5.1). In this term we assume that the functions u_V and u_{B_x} vanish outside of V and B_x respectively.

Next we show how Theorem 4.2 can be deduced from the estimate (5.2) (with x equal to $\mathbb{S}^n$-barycenter of V in place of the the infimum over $\mathbb{S}^n$).

Proof of Theorem 4.2. Without loss of generality, we assume that $\mathcal{H}^n(V_1) \leq \mathcal{H}^n(V_2)$. Let $\bar{\alpha}_i$ be the characteristic of the spherical ball $B_i \subset \mathbb{S}^n$ with the same $\mathcal{H}^n$ measure as V_i. The Friedland-Hayman inequality ensures that $\alpha_1 + \alpha_2 - 2 \geq 0$. Then we write

$$\alpha_1 + \alpha_2 - 2 = (\alpha_1 - \bar{\alpha}_1) + (\alpha_2 - \bar{\alpha}_2) + (\bar{\alpha}_1 + \bar{\alpha}_2 - 2) \geq 0. \tag{5.3}$$

Recall that Sperner's inequality asserts that, among all the open subsets with a fixed measure $\mathcal{H}^n$ on $\mathbb{S}^n$, the one that minimizes the characteristic constant is a spherical ball with that measure $\mathcal{H}^n$. Hence,

$$\alpha_i \geq \bar{\alpha}_i.$$

So the three summands on the right hand side of (5.3) are non-negative. Further, if one of the balls B_i differs from a hemisphere by a surface measure h_0, that is,

$$h_0 = \max_i \left| \mathcal{H}^n(B_i) - \frac{1}{2}\mathcal{H}^n(\mathbb{S}^n) \right|,$$

then

$$\bar{\alpha}_1 + \bar{\alpha}_2 - 2 \geq c\, h_0^2. \tag{5.4}$$

See Corollary 12.4 from [CS], for example. So to prove the theorem we can assume that, for $i = 1, 2$,

$$\left| \mathcal{H}^n(B_i) - \frac{1}{2}\mathcal{H}^n(\mathbb{S}^n) \right| \leq \frac{1}{100}\, \mathcal{H}^n(\mathbb{S}^n), \tag{5.5}$$

because otherwise

$$\alpha_1 + \alpha_2 - 2 \gtrsim 1$$

and the statement in the theorem is trivial. Observe that (5.5) implies that

$$\beta \leq \mathcal{H}^n(V_i) = \mathcal{H}^n(B_i) \leq \mathcal{H}^n(\mathbb{S}^n) - \beta \tag{5.6}$$

for a suitable absolute constant β. From this estimate it follows that $\bar{\alpha}_i \approx 1$ for $i = 1, 2$. So we can assume that $|\alpha_i - \bar{\alpha}_i| \leq \frac{1}{2}\,\bar{\alpha}_i$ because otherwise the theorem follows trivially from (5.3). So $\alpha_i \approx \bar{\alpha}_i \approx 1$. Then from the identity $\lambda_i = \alpha_i(\alpha_i + n - 1)$ (where $\lambda_i \equiv \lambda_{V_i}$) it follows immediately that

$$\alpha_i - \bar{\alpha}_i \approx \lambda_i - \lambda_{B_i}. \tag{5.7}$$

For $i = 1, 2$, let $x_i \in \mathbb{S}^n$ be the barycenter of V_i. We assume that the barycenter exists because otherwise this means that $\int_{V_i} y\, d\mathcal{H}^n(y) = 0$, while $\left|\int_{B_i} y\, d\mathcal{H}^n(y)\right| \gtrsim 1$ because of (5.6) (independently of the choice of its center in $\mathbb{S}^n$). Hence, by the Allen-Kriventsov-Neumayer theorem, using (5.6) and (5.7), we would obtain

$$\alpha_i - \bar{\alpha}_i \approx \lambda_i - \lambda_{B_i} \gtrsim \mathcal{H}^n(V_i \triangle B_i)^2 \geq \left| \int_{V_i} y\, d\mathcal{H}^n(y) - \int_{B_i} y\, d\mathcal{H}^n(y) \right|^2 \gtrsim 1,$$

which would yield the conclusion of the theorem. The same argument shows that, in fact, we have

$$\left| \int_{V_i} y\, d\mathcal{H}^n(y) \right| \approx 1.$$

From now on we assume that the spherical balls B_i, $i = 1, 2$, are centered in the barycenters x_i of the V_i's. In the case when $\mathcal{H}^n(V_i) + \mathcal{H}^n(V_2) = \mathcal{H}^n(\mathbb{S}^n)$ it follows easily that the barycenters of V_1 and V_2 are opposite points in $\mathbb{S}^n$, and B_1 and B_2 are complementary balls in $\mathbb{S}^n$. When the preceding condition does not hold, we need to be a little more careful.

Let

$$\theta_0 = \mathcal{H}^n(\mathbb{S}^n) - \mathcal{H}^n(V_1) - \mathcal{H}^n(V_2),$$

and suppose that $\theta_0 > 0$. For $i = 1, 2$, let

$$y_i = \frac{1}{\mathcal{H}^n(V_i)} \int_{V_i} y \, d\mathcal{H}^n(y),$$

so that $x_i = y_i/|y_i|$. Also, let

$$y_2^* = \frac{1}{\mathcal{H}^n(\mathbb{S}^n \setminus V_1)} \int_{\mathbb{S}^n \setminus V_1} y \, d\mathcal{H}^n(y).$$

Notice that $y_2^* = y_2$ if $\theta_0 = 0$ and that

$$\mathcal{H}^n(V_1)\, y_1 + \mathcal{H}^n(\mathbb{S}^n \setminus V_1)\, y_2^* = \int_{\mathbb{S}^n} y \, d\mathcal{H}^n(y) = 0.$$

So the barycenter x_1 and the point $x_2^* = y_2^*/|y_2^*|$ are antipodal points in $\mathbb{S}^n$. We also have

$$\begin{aligned} |y_2 - y_2^*| \leq & \left| \frac{1}{\mathcal{H}^n(V_2)} - \frac{1}{\mathcal{H}^n(\mathbb{S}^n \setminus V_1)} \right| \int_{\mathbb{S}^n \setminus V_1} |y| \, d\mathcal{H}^n(y) \\ & + \frac{1}{\mathcal{H}^n(\mathbb{S}^n \setminus V_1)} \left| \int_{V_2} y \, d\mathcal{H}^n(y) - \int_{\mathbb{S}^n \setminus V_1} y \, d\mathcal{H}^n(y) \right| \lesssim \theta_0 + \theta_0 \approx \theta_0. \end{aligned}$$

Also, let B_2^* be a spherical ball centered in x_2^* with measure $\mathcal{H}^n(B_2^*) = \mathcal{H}^n(\mathbb{S}^n \setminus V_1)$. Since

$$|\mathrm{rad}_{\mathbb{S}^n}(B_2) - \mathrm{rad}_{\mathbb{S}^n}(B_2^*)| \approx |\mathcal{H}^n(B_2) - \mathcal{H}^n(B_2^*)| \lesssim \theta_0,$$

using also (5.6), we infer that

$$\mathrm{dist}_H(\partial_{\mathbb{S}^n} B_2, \partial_{\mathbb{S}^n} B_1) = \mathrm{dist}_H(\partial_{\mathbb{S}^n} B_2, \partial_{\mathbb{S}^n} B_2^*) \lesssim \theta_0, \tag{5.8}$$

where dist_H denotes the Hausdorff distance. On the other hand, from the definition of θ_0, it follows that

$$h_0 = \max_i \left| \mathcal{H}^n(V_i) - \frac{1}{2} \mathcal{H}^n(\mathbb{S}^n) \right| \geq \frac{1}{2} \theta_0.$$

Then, by (5.4),

$$\bar{\alpha}_1 + \bar{\alpha}_2 - 2 \gtrsim \theta_0^2 \gtrsim \mathrm{dist}_H(\partial_{\mathbb{S}^n} B_2, \partial_{\mathbb{S}^n} B_1)^2. \tag{5.9}$$

To estimate $\varepsilon_n(0, 1)$ we denote by L_1 be the n-plane that contains $\partial_{\mathbb{S}^n} B_1$, and we let L_0 be the n-plane through the origin parallel to L_1. Then we choose H to be the open half-space that contains B_1 and whose boundary is L_0 (notice that B_1 is contained in a hemisphere because of the assumption $\mathcal{H}^n(V_1) \leq \mathcal{H}^n(V_2)$). We denote $S_1 = \mathbb{S}^n \cap H$, $S_2 = \mathbb{S}^n \setminus \overline{H}$, and then we write

$$\varepsilon_n(0, 1) \leq \mathcal{H}^n(S_1 \setminus V_1) + \mathcal{H}^n(S_2 \setminus V_2) \leq \sum_{i=1}^{2} \mathcal{H}^n(B_i \setminus V_i) + \sum_{i=1}^{2} \mathcal{H}^n(S_i \triangle B_i).. \tag{5.10}$$

By the theorem of Allen-Kriventsov-Neumayer, recalling the assumption (5.5),

$$\mathcal{H}^n(B_i \setminus V_i)^2 \lesssim \alpha_i - \bar{\alpha}_i. \tag{5.11}$$

To deal with the term $\mathcal{H}^n(S_i \triangle B_i)$ we will estimate $\mathrm{dist}_H(\partial_{\mathbb{S}^n} B_i, \mathbb{S}^n \cap L_0)$ for $i = 1, 2$. In the case $i = 1$, since L_0 is parallel to L_1 and L_0 splits $\mathbb{S}^n$ in two hemispheres, we have

$$\mathrm{dist}_H(\partial_{\mathbb{S}^n} B_1, \mathbb{S}^n \cap L_0) \lesssim |\mathcal{H}^n(B_1) - \tfrac{1}{2} \mathcal{H}^n(\mathbb{S}^n)| \leq h_0. \tag{5.12}$$

Also, for $i = 2$,

$$\mathrm{dist}_H(\partial_{\mathbb{S}^n} B_2, \mathbb{S}^n \cap L_0) \leq \mathrm{dist}_H(\partial_{\mathbb{S}^n} B_2, \partial_{\mathbb{S}^n} B_2^*) + \mathrm{dist}_H(\partial_{\mathbb{S}^n} B_2^*, \mathbb{S}^n \cap L_0).$$

Since $\partial_{\mathbb{S}^n} B_2^* = \partial_{\mathbb{S}^n} B_1$, from (5.8) and (5.12) we get

$$\mathrm{dist}_H(\partial_{\mathbb{S}^n} B_2, \mathbb{S}^n \cap L_0) \lesssim \theta_0 + \mathrm{dist}_H(\partial_{\mathbb{S}^n} B_1, \mathbb{S}^n \cap L_0) \lesssim \theta_0 + h_0 \lesssim h_0.$$

So, for a suitable $C_2 > 0$,

$$\mathrm{dist}_H(\partial_{\mathbb{S}^n} B_i, \mathbb{S}^n \cap L_0) \leq C_2\, h_0 \quad \text{for both } i = 1, 2. \tag{5.13}$$

By (5.10), (5.11), and the preceding estimate, we get

$$\varepsilon_n(0,1)^2 \lesssim \sum_{i=1}^{2} (\alpha_i - \bar{\alpha}_i) + h_0^2 \lesssim \sum_{i=1}^{2} (\alpha_i - \bar{\alpha}_i) + (\bar{\alpha}_1 + \bar{\alpha}_2 - 2) = \alpha_1 + \alpha_2 - 2.$$

□

Given two disjoint open sets $\Omega_1, \Omega_2 \subset \mathbb{R}^{n+1}$ and $x \in \mathbb{R}^{n+1}$, $r > 0$, we consider the sets $V_i(x,r) = \{r^{-1}(x - y) : y \in S(x,r) \cap \Omega^i\}$ and we denote

$$\alpha_i(x,r) = \alpha_{V_i(x,r)}. \tag{5.14}$$

By Theorem 4.2,

$$\varepsilon_n(x,r)^2 \lesssim \min(1, \alpha_1(x,r) + \alpha_2(x,r) - 2). \tag{5.15}$$

Recall that the same estimate holds in the planar case with $\varepsilon_n(x,r)$ replaced by $\varepsilon(x,r)$, and in fact "typically" we have $\varepsilon(x,r)^2 \approx \min(1, \alpha_1(x,r) + \alpha_2(x,r) - 2)$, as we mentioned above. In view of these facts, it is natural to wonder if a version of Carleson's conjecture can hold in higher dimensions in terms of the square function

$$\mathcal{A}(x)^2 := \int_0^1 \min(1, \alpha_1(x,r) + \alpha_2(x,r) - 2)\, \frac{dr}{r}. \tag{5.16}$$

The coefficient $a(x,r) := \min(1, \alpha_1(x,r) + \alpha_2(x,r) - 2)$ is a much stronger quantity than what (5.15) suggests. Indeed, the coefficient $a(x,r)$ also detects sets of dimension $s \in (n-2, n)$ in $S(x,r)$. To be more precise, we need some additional notation. For Ω_1, Ω_2 as above and for $x \in \mathbb{R}^{n+1}$, $r > 0$, we take a half-space H such that $x \in \partial H$ and we consider the associated half spheres $S_H^1(x,r)$, $S_H^2(x,r)$ introduced in (3.1). For a given $b \in (0,1)$, $s \in (n-2, n)$, $c_0 > 0$, and $i = 1, 2$, we consider the following subsets of "thick" points from $S(x,r)$:

$$T^i_{s,b,c_0}(x,r,H) := \{y \in S^i_H(x,r) \backslash \Omega_i : \mathcal{H}^s_\infty(B(y, b\, \mathrm{dist}_{S(x,r)}(y,H)) \cap S(x,r) \backslash \Omega_i) \geq c_0\, \mathrm{dist}_{S(x,r)}(y,H)^s\},$$

where $\mathrm{dist}_{S(x,r)}$ denotes the geodesic distance in $S(x,r)$, and we put

$$V_{s,b,c_0}(x,r,H) := T^1_{s,b,c_0}(x,r,H) \cup T^2_{s,b,c_0}(x,r,H).$$

We define

$$\varepsilon_s(x,r) = \inf_H \frac{1}{r^s} \int_{V_{s,b,c_0}(x,r,H)} \left(\frac{\mathrm{dist}(y,H)}{r}\right)^{n-s} d\mathcal{H}^s_\infty(y) \tag{5.17}$$

where the infimum is taken over all affine half-spaces $H \subset \mathbb{R}^{n+1}$ such that $x \in \partial H$. Remark that the integral of a function $f : \mathbb{R}^{n+1} \to [0, \infty)$ with respect to the Hausdorff content $\mathcal{H}^s_\infty$ is given by

$$\int_{\mathbb{R}^{n+1}} f\, d\mathcal{H}^s_\infty = \int_0^\infty \mathcal{H}^s_\infty(\{x \in \mathbb{R}^{n+1} : f(t) > t\})\, dt.$$

In [FTV1] the following has been proven:

Theorem 5.3. *Let $n \geq 1$ and $0 < s < n$. Let Ω_i and $\varepsilon_s(x,r)$ be as above. Then for any $s \in (n-2, n)$, $c_0 > 0$ and $b \in (0,1)$, we have*

$$\varepsilon_s(x,r)^2 \lesssim_{s,c_0,b} a(x,r). \tag{5.18}$$

The proof of this estimate in [FTV1] follows the same lines of the proof of Theorem 4.2. The main difference is that, instead of the estimate (5.2), it uses new quantitative Faber-Krahn inequalities involving s-dimensional Hausdorff contents and capacities, which are also proved in [FTV1]. In the case $n \geq 3$, one of these inequalities, which is sharp up to a constant factor, reads as follows:

Theorem 5.4. *Given $n \geq 3$, let $\mathbb{M}^n$ be either $\mathbb{R}^n$ or $\mathbb{S}^n$, and let $\beta > 0$, $a \in (0,1)$. Let Ω be a relatively open subset of $\mathbb{M}^n$ and let B be a geodesic ball in $\mathbb{M}^n$ such that such that $\mathcal{H}^n(B) = \mathcal{H}^n(\Omega)$ with radius r_B. In the case $\mathbb{M}^n = \mathbb{S}^n$, suppose also that $\beta \leq \mathcal{H}^n(\Omega) \leq \mathcal{H}^n(\mathbb{S}^n) - \beta$, and in the case $\mathbb{M}^n = \mathbb{R}^n$ just that $\mathcal{H}^n(\Omega) \leq \beta$. Denote by λ_Ω and λ_B the first Dirichlet eigenvalues of $-\Delta_{\mathbb{M}^n}$ in Ω and B, respectively. Then, there is some constant $C(a,\beta) > 0$ such that*

$$\lambda_\Omega - \lambda_B \geq C(a,\beta) \inf_{x \in \mathbb{M}^n} \left(\sup_{t \in (0,1)} \fint_{\partial_{\mathbb{M}^n}((1-t)B_x)} \frac{\mathrm{Cap}_{n-2}(B_{\mathbb{M}^n}(y, atr_B) \setminus \Omega)}{(t\, r_B)^{n-3}} \, d\mathcal{H}^{n-1}(y) \right)^2, \tag{5.19}$$

where B_x denotes a ball centered in x with radius r_B. In the case $\mathbb{M}^n = \mathbb{S}^n$, (5.19) also holds with the infimum over $\mathbb{M}^n$ replaced by the evaluation at x equal to $\mathbb{S}^n$-barycenter of V (possibly with a different constant $C(a,\beta)$).

In (5.19), $\partial_{\mathbb{M}^n}(E)$ stands for the relative boundary of a set $E \subset \mathbb{M}^n$ and $B_{\mathbb{M}^n}(x,r)$ is a ball in $\mathbb{M}^n$ with center x and (geodesic) radius r. Also, for any $s > 0$ and $E \subset \mathbb{M}^n \subset \mathbb{R}^{n+1}$, we consider the capacity

$$\mathrm{Cap}_s(E) = \sup\left\{\mu(E) : \mu \in M_+(E),\ \|U_s\mu\|_{\infty,E} \leq 1\right\},$$

where $M_+(E)$ is the family of all Radon measures supported in E and $U_s\mu(x) = \int \frac{1}{|x-y|^s}\, d\mu(y)$.

6. Carleson's conjecture in higher dimensions. Existence of tangents

The discussion in the previous section suggests that the square function $\mathcal{A}$ defined in (5.16) is a good one for a possible extension of Carleson's ε^2 conjecture to higher dimensions. This is confirmed by the next result from [FTV2].

Theorem 6.1. *For $n \geq 1$ let $\Omega_1, \Omega_2 \subset \mathbb{R}^{n+1}$ be two disjoint open subsets. Suppose that $\Omega_1 \cup \Omega_2$ satisfies the capacity density condition (CDC). Then, up to a set of zero $\mathcal{H}^n$ measure,*

$$\mathcal{A}(x) < \infty \quad \text{if and only if } x \text{ is a tangent point of the pair } \Omega_1, \Omega_2.$$

The CDC for $\Omega_1 \cup \Omega_2$ is a thickness condition of the complement of $\Omega_1 \cup \Omega_2$ which is equivalent to the existence of some $s \in (n-1, n+1]$ and some $r_0, c > 0$ such that

$$\mathcal{H}^s_\infty(B(x,r) \setminus (\Omega_1 \cup \Omega_2)) \geq c\, r^s \quad \text{for all } x \in \partial(\Omega_1 \cup \Omega_2) \text{ and all } r \in (0, r_0).$$

For example, if Ω_1 is a Jordan domain and $\Omega_2 = \mathbb{R}^2 \setminus \overline{\Omega_1}$, then the CDC holds $\Omega_1 \cup \Omega_2$ (with $s = 1$ in the condition above). So Theorem 6.1 implies Theorem 1.1 in the case $n = 1$, and in fact it can be applied to more general open sets.

The fact that the square function $\mathcal{A}$ is finite at tangent points of the pair Ω_1, Ω_2 follows by a rather easy application of the ACF monotonicity formula. Next we show the detailed arguments, taken from [FTV1].

Proof of the finiteness of $\mathcal{A}(x)$ at $\mathcal{H}^n$-a.e. tangent point $x \in \mathbb{R}^{n+1}$. Let $\Omega_1, \Omega_2 \subset \mathbb{R}^{n+1}$ be as in Theorem 6.1. Let Ω_1^j, Ω_2^k, for $j, k \geq 1$, be the respective connected components of Ω_1 and Ω_2. Notice that any tangent point for the pair Ω_1, Ω_2 is a tangent point for a pair of components Ω_1^j, Ω_2^k. So it suffices to show that for every $j, k \geq 1$, it holds $\mathcal{A}(x) < \infty$ at every tangent point x of the pair Ω_1^j, Ω_2^k.

Denote by g_1^j, g_2^k the respective Green functions of the domains Ω_1^j, Ω_2^k. Let $p_1 \in \Omega_1^j$, $p_2 \in \Omega_2^k$, and consider the functions

$$u_1(y) = g_1^j(y, p_1), \qquad u_2(y) = g_2^k(y, p_2).$$

We extend u_1 and u_2 by 0 respectively in $(\Omega_1^j)^c$ and $(\Omega_2^k)^c$ and abusing notation we still denote by u_1, u_2 such extensions. The CDC condition of $\Omega_1 \cup \Omega_2$ ensures the Wiener regularity of Ω_i and thus the continuity of u_1 and u_2 away from the poles p_1, p_2.

Let $d = \frac{1}{6} \min_i \operatorname{dist}(p_i, \partial\Omega_1^j \cup \partial\Omega_2^k)$. For all $x \in \mathbb{R}^{n+1} \setminus (\Omega_1^j \cup \Omega_2^k)$ and all $r \in (0, d)$, by the ACF monotonicity formula, we have

$$\frac{\partial_r J(x,r)}{J(x,r)} \geq \frac{2}{r}\big(\alpha_1(x,r) + \alpha_2(x,r) - 2\big),$$

with $J(x,r)$ and $\alpha_i(x,r) = \alpha_i$ as in (4.3), (4.4), and (5.14). Integrating on r, for any $\rho \in (0,d)$ we derive

$$\int_\rho^d \frac{\alpha_1(x,r) + \alpha_2(x,r) - 2}{r}\, dr \leq \log \frac{J(x,d)}{J(x,\rho)}.$$

Thus,

$$\int_0^d \frac{\alpha_1(x,r) + \alpha_2(x,r) - 2}{r}\, dr \leq \log \frac{J(x,d)}{\inf_{0<\rho\leq d} J(x,\rho)}.$$

Hence, in order to show that $\mathcal{A}(x) < \infty$ for a given $x \in \partial\Omega_1^j \cap \partial\Omega_2^k$, it suffices to show that

$$\frac{J(x,d)}{\inf_{0<\rho\leq d} J(x,\rho)} < \infty.$$

Notice first that, by (4.5),

$$J(x,d) \lesssim \fint_{B(x,2d)} |\nabla u_1|^2\, dy \;\cdot\; \fint_{B(x,2d)} |\nabla u_2|^2\, dy.$$

By the Caccioppoli inequality and the subharmonicity of u_i, for each i,

$$\fint_{B(x,2d)} |\nabla u_i|^2\, dy \lesssim \frac{1}{r^2} \fint_{B(x,3d)} |u_i|^2\, dy \lesssim \frac{1}{r^2}\bigg(\fint_{B(x,4d)} u_i\, dy\bigg)^2.$$

So the continuity of u_i implies that $J(x,d) < \infty$.

To estimate $J(x,\rho)$ from below, let $\varphi_{x,\rho}$ be a C^∞ bump function such that $\chi_{B(x,\rho/2)} \leq \varphi_{x,\rho} \leq \chi_{B(x,\rho)}$, with $\|\nabla\varphi_{x,\rho}\|_\infty \lesssim \rho^{-1}$. Then, denoting by $\omega_i^{p_i}$ the harmonic measure for Ω_i with pole at

p_i, by the properties of the Green function, it holds

$$\omega_i^{p_i}(B(x,\rho/2)) \leq \int \varphi_{x,\rho}\, d\omega^{p_i} = -\int \nabla\varphi_{x,\rho}\, \nabla u_i\, dy \leq \|\nabla\varphi_{x,\rho}\|_2\, \|\nabla u_i\|_{2,B(x,\rho)}$$
$$\lesssim \rho^{(n-1)/2}\bigg(\int_{B(x,\rho)} \frac{\rho^{n-1}|\nabla u_i|^2}{|x-y|^{n-1}}\, dy\bigg)^{1/2} = \rho^n\bigg(\frac{1}{\rho^2}\int_{B(x,\rho)} \frac{|\nabla u_i|^2}{|x-y|^{n-1}}\, dy\bigg)^{1/2}.$$

Therefore,

$$J(x,\rho)^{1/2} \gtrsim \frac{\omega_1^{p_1}(B(x,\rho/2))}{\rho^n} \cdot \frac{\omega_2^{p_2}(B(x,\rho/2))}{\rho^n}.$$

Hence, it suffices to show that

$$\liminf_{\rho\to 0} \frac{\omega_i^{p_i}(B(x,\rho/2))}{\rho^n} > 0 \quad \text{for } \mathcal{H}^n\text{-a.e. tangent point } x \in \partial\Omega_1^j \cap \partial\Omega_2^k, \tag{6.1}$$

for $i = 1, 2$. To this end, consider a subset E of the tangent points for the pair Ω_1^j, Ω_2^k such that $\mathcal{H}^n(E) < \infty$. Since the set of tangent points is n-rectifiable, we have

$$\lim_{\rho\to 0} \frac{\mathcal{H}^n(B(x,\rho)\cap E)}{(2\rho)^n} = 1 \quad \text{for } \mathcal{H}^n\text{-a.e. } x \in E. \tag{6.2}$$

By standard arguments, using that the tangent points for the pair Ω_1^j, Ω_2^k are cone points for Ω_1^j and Ω_2^k, it follows that $\mathcal{H}^n|_E$ is absolutely continuous with respect to $\omega^{p_i}|_E$ for $i = 1, 2$ (see [AAM, Theorem III], for example[1]). Then, by the Lebesgue-Radon-Nykodim differentiation theorem, we have

$$\lim_{\rho\to 0} \frac{\mathcal{H}^n(B(x,\rho)\cap E)}{\omega_i^{p_i}(B(x,\rho)\cap E)} < \infty \quad \text{for } \omega^{p_i}\text{-a.e. } x \in E.$$

Since null sets for $\omega_i^{p_i}$ are also null sets for $\mathcal{H}^n|_E$, we infer that

$$\lim_{\rho\to 0} \frac{\omega_i^{p_i}(B(x,\rho)\cap E)}{\mathcal{H}^n(B(x,\rho)\cap E)} > 0 \quad \text{for } \mathcal{H}^n\text{-a.e. } x \in E. \tag{6.3}$$

Multiplying (6.2) and (6.3), we deduce (6.1). This completes the proof of the fact that $\mathcal{A}(x) < \infty$ at $\mathcal{H}^n$-a.e. tangent point of the pair Ω_1, Ω_2. □

Remark that the first proof by Bishop of the fact that $\mathcal{E}(x) < \infty$ at $\mathcal{H}^1$-a.e. tangent point for a Jordan domain is not the one shown in Section 2 relying on Theorem 2.2. Instead, the original proof by Bishop is close in spirit to the one above showing the finiteness of $\mathcal{A}(x)$ at $\mathcal{H}^n$-a.e. tangent point for Ω_1, Ω_2. In place of the Alt-Caffarelli monotonicity formula, Bishop's proof uses an estimate for harmonic measure due to Beurling. See [BCGJ].

The proof of the converse implication in Theorem 6.1 is more complicated. The strategy is as follows. First notice that, by Theorem 4.2, $\mathcal{E}_n(x) \lesssim \mathcal{A}(x)$, and then by Theorem 3.1 the set of points $G_0 := \{x \in \mathbb{R}^{n+1} : \mathcal{A}(x) < \infty\}$ is n-rectifiable. So it remains to show that approximate tangents for this set are also "true" tangents of $\partial\Omega_1$ and $\partial\Omega_2$. To this end, we prove the following slicing result, which may have some independent interest:

[1]Actually, in Theorem III from [AAM] it is assumed that $\partial\Omega$ is lower n-content regular in order to prove the mutual absolute continuity of $\mathcal{H}^n|_E$ and $\omega^{p_i}|_E$. However, a quick inspection of the arguments shows that for the absolute continuity $\mathcal{H}^n|_E \ll \omega_i^{p_i}|_E$ one only needs Ω_i to be Wiener regular.

Proposition 6.2. *Let $B(0,r_0) \subset \mathbb{R}^{n+1}$ and let Γ be a Lipschitz graph through the origin with slope at most τ. Let $B \subset B(0,r_0)$ be a ball with $\mathrm{rad}(B) \leq \frac{1}{10} r_0$ such that $\mathrm{dist}(B,\Gamma) \geq 100\tau r_0$. Let $K \subset B$ and $G \subset \Gamma$ both be compact sets. Then, for any $s > 1$,*

$$\mathrm{Cap}_s(K)\,\frac{\mathcal{H}^n(G)^2}{r_0^n} \leq C(\tau) \int_G \int_0^\infty \mathrm{Cap}_{s-1}(K \cap S(z,r))\,dr\,d\mathcal{H}^n(z).$$

To apply this proposition, we split G_0 into a countable collection of subsets G_j contained in Lipschitz graphs Γ_j with small slope and a set of zero measure $\mathcal{H}^n$. By the CDC assumption, Proposition 6.2 applied to $\Gamma = \Gamma_j$, $G = G_j$, $x \in \Gamma_j$, and a ball $B(x,r_0)$, ensures that if there is a big piece of $(\partial\Omega_1 \cap \partial\Omega_2) \cap B(x,r_0)$ far away from Γ_j, then, for some $s > n-1$, $\mathrm{Cap}_{s-1}((\partial\Omega_1 \cup \partial\Omega_2) \cap S(x,r))$ is large for many $x \in G_j$ and many radii $r \approx r_0$. In turn, this implies that $\varepsilon_s(x,r)$ is large for these points x and radii r. So by Theorem 5.3 the coefficients $a(x,r)$ are large too. Integrating on r, if follows that $\mathcal{A}(x)$ is large for many points $x \in G_j$, which leads to a contradiction if r_0 is small enough.

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