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In this report, we examine the noticing of a group of secondary-school mathematics teachers of aspects of the mathematical discourse during the collaborative discussion of a task on teaching angles. Our framework integrates aspects of the mathematical discourse in content-specific teaching that are responsive to students' common challenges in learning angles, as a focus of mathematics teachers' noticing. The data were collected during a one-day workshop. Using hybrid methods of deductive and inductive analysis, we examine what teachers identify, interpret, and decide regarding mathematical vocabulary, mathematical explanations, and graphical examples for teaching angles. A result is that the teachers in the group engaged in noticing mathematical-linguistic explanations with multimodal examples for teaching angles with meaning in dynamic contexts and in relation to non-metric properties.

Keywords: Collaborative discussion, mathematical discourse in teaching, teachers' noticing, teaching angles.

Introduction

In this report, we present a study initiated as part of a research and developmental project with secondary-school mathematics teachers on the mathematical discourse in content-specific teaching. One of the project strategies was the participation of the teachers and of the researchers as facilitators in one-day workshops to discuss professional tasks on aspects of the mathematical discourse in teaching. In Planas and Alfonso (2023), for example, the noticing of two groups of participant teachers was analysed, with a focus on mathematical-linguistic practices of naming and explaining, alongside other foci spontaneously introduced by the teachers. The design and implementation of the workshop in the current study include tasks with written prompts aimed at identifying, interpreting, and deciding on mathematical vocabulary, mathematical explanations, and graphical examples in fictional situations of teaching angles. A notion of supporting students' common challenges in content-specific learning by using aspects of the mathematical discourse in teaching is at the basis of the task design (for details, see Rave-Agudelo & Planas, 2022; 2023). Our study is therefore situated at the intersection of research on mathematics teachers' noticing (van Es & Sherin, 2002; Jacobs et al., 2010) and research on mathematical language and discourse in teaching (Morgan, 2013; Planas, 2021). The study of mathematics teachers' noticing is a relevant line of research, as it is a fundamental aspect of the teaching profession (König et al., 2022). It is also important to ensure that mathematical language and discourse have a place in teacher professional development programmes on mathematics teaching (Planas, 2021). We aimed at examining responses to the following research question: *What does a group of secondary-school mathematics teachers notice regarding aspects of the mathematical discourse when discussing tasks on teaching angles?*

Theoretical framework

Our framework integrates three components: aspects of the mathematical discourse in content-specific teaching, students' common challenges in content-specific mathematics learning, and mathematics teachers' noticing.

The presence of mathematical vocabulary, mathematical explanations, and graphical examples in teaching has the potential to communicate and facilitate the communication of mathematical meanings (Planas, 2021). In our study, these aspects are inspired by the *Mathematics Discourse in Instruction* (MDI) framework (Adler, 2021), particularly regarding the mathematical-linguistic practices of naming and explaining, and the mathematical-graphical practice of exemplifying. We understand *mathematical vocabulary* in relation to mathematical naming, as the set of words or phrases from the mathematical discourse used to name objects, symbols, properties, actions, procedures, and practices in mathematics (Adler, 2021). This vocabulary serves to answer questions such as: *What is this?* (Planas & Alfonso, 2023), *What is it called?* or *What are we talking about?* We understand *mathematical explanations* in relation to mathematical explaining, as sentences incorporating mathematical vocabulary that build mathematical meaning by establishing relationships between objects, symbols, properties, actions, procedures, and practices in mathematics to facilitate students' thinking. These explanations address questions such as: *What is said about something? How does it happen? How is it done?* or *Why is this?* (Planas & Alfonso, 2023). This understanding of mathematical explanation aligns with Leinhardt's (2001) notion of *instructional explanation*, in which the disciplinary nature of the mathematical content is linked to teaching practices that support students' thinking. We understand *graphical examples* in relation to mathematical exemplifying, as graphical representations (pictures, illustrations, photographs, diagrams, etc.) of mathematical examples. A mathematical example is a particular case within a broader equivalence class (idea, concept, technique, ... in mathematics) that supports reasoning and generalisation (Watson & Mason, 2002). A graphical representation is thus a visual mode of communication that conveys complex objects, structures, and processes holistically by integrating visual elements and interpretation rules (Winn, 1987).

A starting point for our investigation is the lack of consensus in defining the concept of angle and its implications for teaching and learning. For our study, common challenges in learning angles were identified in the literature on mathematics education (Rave-Agudelo & Planas, 2022). The studies selected highlighted two common challenges in learning angles that can occur simultaneously: the tendency to communicate meaning about angles in static contexts, without considering dynamic contexts (Mitchelmore & White, 2000), and the tendency to confuse the numerical value of angle amplitude with the angle itself, while overlooking non-metric properties (Tanguay & Venant, 2016).

Regarding mathematics teachers' noticing, van Es and Sherin (2002) conceptualise it by considering the identification of specific aspects of classroom interactions, the connection between these aspects and broad principles of teaching and learning, and the use of contextual knowledge to reason about these interactions. In this initial phase of the study, we adopt the conceptualisation proposed by Jacobs et al. (2010), which involves processes of identifying, interpreting, and deciding on relevant aspects of teaching and learning based on pre-established criteria. While other studies within the broader

project have shown that teachers notice additional aspects beyond those considered in the design of professional development workshops (Planas & Alfonso, 2023), the characterization and description of these processes constitute a crucial step in understanding the complexity of mathematics teachers' noticing. In order to study mathematics teachers' noticing focused on aspects of mathematical discourse in the teaching angles, we adapted Jacobs et al.'s (2010) processes as follows: (i) *Identifying* mathematical vocabulary, mathematical explanations and/or graphical examples of angles in teaching accounts; (ii) *Interpreting* the potential impact of the identified aspects of the mathematical discourse on supporting students with common challenges in their learning of angles; (iii) *Deciding* on proposals of mathematical explanations with mathematical vocabulary and/or graphical examples of angles with the potential to support the students' learning.

Methods

The empirical context of this study was a one-day workshop on aspects of the mathematical discourse in teaching angles. The first author was the teacher educator. Five secondary-school mathematics teachers participated in this workshop. Two participants had a degree in mathematics, and the other three had a degree in engineering. Their professional experience in teaching mathematics ranged from six months to 25 years. They all reported that they had not previously participated in development programmes on aspects of the mathematical discourse in teaching.

In Rave-Agudelo and Planas (2023), we report on the workshop design and the structure, following three stages. The first stage consisted of the presentation and group discussion of two introductory documents. The first document presented and illustrated two students' common challenges in learning angles. The second document illustrated mathematical vocabulary, mathematical explanations and graphical examples for teaching angles. The group discussion of these documents helped introduce the teachers to relevant aspects from the literature on teaching and learning angles. This group discussion also facilitated presentations of the teachers' professional experiences. The second stage involved the teachers' individual responses to two tasks related to teaching angles. Each task provided a fictional account of teaching with three prompting questions. These prompts were invitations to identify, interpret, and decide on mathematical vocabulary, mathematical explanations, and graphical examples to support students' learning. The third stage consisted of a group discussion of the tasks. The teacher educator started the discussion by asking for responses to each of the task prompts. The teachers participated by sharing their answers and focusing on aspects of the mathematical discourse in the tasks that they saw as related to the common challenges in learning. The teacher educator tried not to interrupt the discussion.

In this report, we consider the group discussion of the first task. This task presented a fictional situation in which a student asked the teacher for a mathematical explanation after reading a definition of angle and its classification in a mathematics textbook. In the textbook, the mathematical vocabulary and the mathematical explanations focused on communicating the meaning of angles in static contexts. The graphical examples focused on communicating the metric property. The student's utterance was as follows: '*If an angle is the portion of the plane between two straight lines, why do we take a different portion in the reflex angle? The outer one*'. The everyday meaning of the word

"between" differs from its mathematical meaning, particularly in the phrase "the plane between two straight lines". Figure 1 shows an English version of an extract from the task.

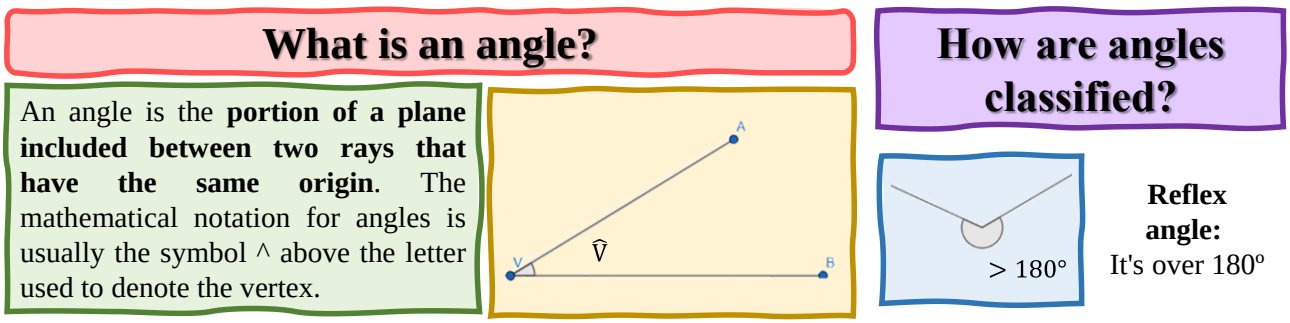

Figure 1: English version of an extract of the task

The prompts provided in the tasks were: (1) What mathematical vocabulary, mathematical explanations, and graphical examples are communicated to the student during teaching? (2) What effects might these aspects of the mathematical discourse have on the students' learning of the concept of angle? (3) What mathematical explanations with mathematical vocabulary and graphical examples, could serve in teaching that is responsive to this student's learning?

The workshop was audio-recorded and transcribed by the first author. The transcript of the group discussion for the first task was divided into three segments, each associated with one prompt. The analysis was then focused on examining evidence of identifying, interpreting, and deciding processes as constituent elements of mathematics teacher' noticing. In this process, we used hybrid methods of deductive and inductive coding analysis (Nowell et al., 2017). A preliminary analysis of the teachers' written responses formed the basis for creating initial codes. The turns in the group discussion were coded deductively, when possible, and this procedure led to the induction of additional codes. The coding was thematically oriented and included processes of noticing (identifying, interpreting, deciding), aspects of the mathematical discourse (mathematical vocabulary, mathematical explanations, graphical examples) and emergent related features. There were variations in the coding process across the task prompts for questions 2 and 3. For the second prompt, for example, the potential effect of the relationship to the students' common challenges in learning and to other aspects of the teachers' professional experience were included. For the third prompt, the potential response to the common challenges in learning and to the other aspects of teaching and learning were added to the created codes. Together with the teachers' turns in the group discussion, the episodes served as a primary unit of analysis. Episodes were created by selecting sets of turns where the teachers' noticing and the aspect of the mathematical discourse in focus were consistent, despite nuances and differences in the features discussed. Although the coding was overall guided by the framework, we remained interpretative and inductive in the analysis of the episodes. For the sake of rigour, we discussed the coding of the turns and the interpretation of the episodes. We present the episodes in an English version.

Analysis and findings

In this section, we present evidence of the participant teachers' noticing of aspects of the mathematical discourse in the teaching of angles.

Identifying aspects of the mathematical discourse in teaching angles

The teachers discussed responses to the first prompt concerning the identification of aspects of the mathematical discourse, but they did not address the identification of mathematical explanations. The extract below illustrates the identification of mathematical vocabulary and graphical examples of angles. In the first part of the discussion, the teacher educator repeated the first prompt, and the teachers identified mathematical vocabulary related to angles. In [2], Senú mentioned the Catalan words for "vertex" and "amplitude". In [3], Tule referred to "plane" and "ray", which Senú repeated in [4]. In [5], Chibcha mentioned "lines" and "origin". In [6], Senú noted that the teacher in the task spent time discussing the classification of angles. In the second part of the discussion, the teacher educator asked about the presence of mathematical explanations and graphical examples, and the teachers focused on graphical examples. In [8], Tule mentioned that there were no graphical examples, but in [9], Nutabe referred to the shapes of angles in his written and graphical response to the task. In [10], Katío referred to the measurements in the graphical examples related to the classification of the angles. In [11], Tule distinguished between graphical representations and graphical examples, associating graphical representations with abstract contexts and graphical examples with real-world contexts. The episode thus illustrates the identification of mathematical vocabulary and multimodal examples (both graphical and real-world context) related to angles.

1	Educator:	What mathematical vocabulary, mathematical explanations, and graphical examples are communicated to the student during teaching?
2	Senú:	She provides many definitions. She talks about the vertex, the angle, and the amplitude…
3	Tule:	About the plane, the ray.
4	Senú:	About the ray…
5	Chibcha:	Straight lines, origin.
6	Senú:	And then she spends a lot of time sorting the angles according to their measure, right?
7	Educator:	What do you think about mathematical explanations and graphical examples?
8	Tule:	There are not any graphical examples, are there?
9	Nutabe:	I have drawn the shape of angles.
10	Katío:	Including the measurements.
11	Tule:	Yes, I mean, these are graphical representations, but when I hear the word example, I think back to when I was a child. I suppose… I need to relate it to something familiar, something tangible that I can physically interact with, right? I mean, for me, those are more than examples; they are graphical representations. But when I think of examples, I picture things like the hands of the clock or the handle of a door...

Interpreting the potential impact of the identified aspects of the mathematical discourse

The teachers interpreted the effects of the identified aspects of the mathematical discourse in relation to common learning challenges, as well as their professional knowledge and teaching experience. In [18], the teacher educator asked for a response to the second prompt. In [19], Senú associated the identified aspects with the communication of meanings for angles in static contexts, without mentioning turns and movements. In [20] and [22], Tule pointed out that "turning" is a missing term that does not appear in the task. In [21], Chibcha, the most experienced teacher, mentioned that the mathematical explanations in the textbook are contradictory. She noticed that these explanations fail

to support the distinction between static contexts, where the angle is described as a region of the plane, and dynamic contexts, where the vertex is emphasised as the centre of rotation. This episode illustrates the interpretation of the potential impact of the mathematical discourse on learning, in relation to the introductory documents discussed in the workshop, to professional knowledge, and to teaching experience.

18	Educator:	Could these things be connected to the common challenges we have been working on? Could you explain how they are connected?
19	Senú:	Yes, sure. She teaches..., in other words, she teaches it in a totally static way, doesn't she? The angle is not presented as a turn; instead, an angle is this, and it is classified in this way, but they don't see that it can be associated with movement.
20	Tule:	The turn is absolutely absent; it does not appear anywhere.
21	Chibcha:	But, you see, it does... That is, in the first explanation, it is the portion of the plane, and in the second, the vertex is part of the angle. Also, she contradicts herself, because the vertex is the centre of the turn, not the ray, but the centre of the turn is important for the angle.
22	Tule:	The turn is absent at any moment.

Deciding on mathematical explanations with multimodal examples

The teachers decided on mathematical explanations with related examples using different modes of communication. The structure of the prompt and the dynamics of the workshop facilitated decision-making regarding these modal aspects of the mathematical discourse in an integrated manner. In [28], the teacher educator asked for an answer, that is, a decision that resolved the doubt raised by the student in the task. In [29], Senú read the explanation in his written answer and stated that moving a ray to one side is not the same as moving it to the other side, which implies a turn. In [30], the educator asked for a graphical example, and in [31], Senú made a hand gesture. In [32], the educator asked if he would present the example with hand gestures, to which Senú responded affirmatively in [33]. In [34], Nutabe presented an example previously discussed that had meaning in static contexts. In [35], Tule presented an example linked to the movement of a clock hand to explain the angle as a rotation and the direction of the rotation. In this episode, mathematical explanations and examples responded to common learning challenges by focusing on communicating dynamic contexts and non-metric properties. Furthermore, the way in which the teachers present an explanation with examples reflects some group decision-making. Although the teachers use words from the mathematical discourse when presenting mathematical explanations, this did not seem to be the focus of their discussion. They also made suggestions that were not related to learning challenges but to other aspects, such as emotions in [34]. Another relevant aspect was the use of examples in modes of communication other than graphical representations, such as hand gestures and manipulatives, suggesting that teachers considered visual representations as complementary to linguistic explanations.

28	Educator:	Can you provide a concrete answer for this student?
29	Senú:	What I wrote is that… I mean, when you go from one ray to the other ray, doing this is not the same as doing that; it is different, hence associated with a turn.
30	Educator:	And how would you present the graphical example?
31	Senú:	Like this, isn't it?
32	Educator:	With hands? With gestures?
33	Senú:	With hand gestures, yes.
34	Nutabe:	I liked the example of the pizza; I think they appeals to emotions, and...

35 Tule: In addition to the pizza, I used the example of the clock hands. I put one in a different colour, and we played as if we had a time machine. Let's move time forward or backward. Then they see that if you do it on one side, you can do it on the other side too. I am not sure.

Discussion and conclusions

The group discussion of the task demonstrates processes of noticing in collaboration amongst the teachers, focused on aspects of the mathematical discourse in the teaching angles. We suggest that structuring the task around prompts aimed at identifying, interpreting, and deciding facilitated the analysis of data and mediated the production of findings in our approach to understanding *What does a group of secondary-school mathematics teachers notice regarding aspects of the mathematical discourse when discussing tasks on teaching angles?* The teachers identified several aspects of the mathematical discourse in the fictional teaching account. The absence of mathematical explanations in the process of identification can be linked to the complexity of what is meant by mathematical explanation. However, it can also be attributed to the design of prompts that did not sufficiently encourage the teachers to identify explanations as an aspect to be discussed in the workshop. Instead, they may have been involved in other processes of noticing related to mathematics teaching and to mathematical discourse in teaching that we did not capture in our study. Although the processes of interpretation are not related to each of the aspects of the mathematical discourse prompted in the task, the teachers interpreted the potential effects of these discourse aspects on the common learning challenges presented to them, incorporating their professional knowledge and experiences. The group discussion of the introductory documents led to the recognition of common challenges in learning angles. The processes of decision-making included mathematical explanations and multimodal examples that are responsive to common challenges in learning angles. Finally, as stated in our framework, by considering mathematics teachers' noticing through identifying, interpreting, and deciding, we are only beginning to understand the complexity and learning richness of the noticing that can occur during a developmental workshop.

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