Second quarterly report of the Contract N62558-3079
NR 043-266. (1 October - 31 December 1961).

The notations and definitions of the first report will be adopted, but since now I suppose that it is posible that $\sum_2^\infty \frac{1}{\lambda_n}=\infty$, I will write besides

$$\Lambda(z)=\begin{cases} z\prod_2^\infty\left(1+\frac{z^2}{\lambda_n^2}\right) & \text{if } \lambda_1=0 \\ \prod_1^\infty\left(1+\frac{z^2}{\lambda_n^2}\right) & \text{if } \lambda_1>0 \end{cases}$$

$$L(R)=\int_0^\infty e^{-Rr}\Lambda(r)dr$$

(these definitions when $\lambda_1>0$ are of Mandelbrojt [1]). If the upper mean density $\bar{D}^*$ of the sequence $\{\lambda_n\}$ (1) is finite, the function $\Lambda(z)$ is entire and $L(R)$ is finite for $R>\pi\bar{D}^*$.

Definition of the adherence hypothesis $A(g(\sigma),p(\sigma),\{\lambda_n\})$.-

We say that the functions $g(\sigma)$ and $p(\sigma)$, and the sequence $\{\lambda_n\}$ satisfy the hypothesis $A(g(\sigma),p(\sigma),\{\lambda_n\})$ if there exists a continuous non-increasing function $h(\sigma)$, with $\lim h(\sigma)=\bar{D}^*$, where $\bar{D}^*$ is the upper mean density of $\{\lambda_n\}$, such that

$$\bar{D}^*<g, \qquad \log L(\pi h(\sigma))<p(\sigma)+M \qquad (M<\infty)$$

$$\int^\infty\left[p(\sigma)-\log L(\pi h(\sigma))\right]\exp\left(-\frac{1}{2}\int^\sigma\frac{dx}{g(x)-h(x)}\right)d\sigma=\infty$$

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(1) When $\lambda_1=0$ the definition of the mean density $\bar{D}^*$ given by Mandelbrojt [1] must vary slightly

Now we can prove

THEOREM III.- Suppesing the following conditions are satisfied

1° $\{\lambda_n\}$ is such that $0 \leq \lambda_n < \lambda_{n+1}$ and $\bar{D}^* < \infty$

2° $F(s)$ is a holomorphic function in Δ, and the linear combinations $\varphi(s) \in \Phi$ representa $F(s)$ in Δ with the logaritmic b-precision $p(\sigma)$. Where $b > 2\pi \bar{D}^*$ and Δ is the strip $\{\sigma > \sigma_0, |t| < \pi g(\sigma)\}$ with $g(\sigma) > \bar{D}^*$ and $\lim g(\sigma) > \bar{D}^*$.

3° the hypothesis $A(g(\sigma), p(\sigma), \{\lambda_n\})$ is satisfied

Then for any bounded domain $\chi \subset \Delta$ and such that the distance between the frontier of χ and the frontier of Δ is greater than $\pi \bar{D}^*$, we have $F(s) \in K(\{\lambda_n\}, \chi)$.

In [3] I have stated a similar result but the conclusion was $F(s) \in K(\{\lambda_n\} + \{-\lambda_n\}, \chi)$. A more acurate proof gives the theorem III.

Let H_x be the half-plane $\sigma > x$, then combining theorem III and many results of Schwartz and Bernstein [2, pag. 135-137] we obtain

THEOREM IV.- Then

1° $\{\lambda_n\}$ is such that $0 \leq \lambda_n < \lambda_{n+1}$, ~~and~~ $\bar{D}^* < \infty$. and $B < \infty$, where B is the maximum density of $\{\lambda_n\}$.

2° $\Delta = \{\sigma > \sigma_0, |t| < \pi g(\sigma)\}$, where $g(\sigma)$ is a continuous function of bounded variation in $\sigma_0 < \sigma < \infty$ such that $g(\sigma) > \bar{D}^*$, $\lim g(\sigma) > \bar{D}^*$ and $g(\sigma) > \bar{D}^* + B$ for $x < \sigma < x + \alpha$, where $\alpha > 2\pi D$, ~~and B is the maximum density of~~ $\cancel{\{\lambda_n\}}$.

Then $F(s) \in W(\Delta, \{\lambda_n\}, b, A)$ with $b > 2\pi \bar{D}^*$ if, and only if,

(i) $F(s)$ is holomorphic in $\Delta \cup H_{x_1}$, where x_1 is the lower real number such that $g(\sigma) > B$ for $x_1 < \sigma < x$.

(ii) There exists a Dirichlet series $\sum d_n e^{-\lambda_n s}$ and a sequence $\{n_k\}$ of natural numbers such that

$$\lim_{k\to\infty} S_{n_k}(s)=F(s)$$

uniformly in every domain

$$\sigma \geq x_1 + \varepsilon \qquad \left|\frac{t}{\sigma - x_1}\right| \leq C$$

for any $\varepsilon > 0$ and any $C > 0$, where

$$S_{n_k}(s)=\sum_{1}^{n_k} d_n e^{-\lambda_n s}$$

and where the sequence $\{n_k\}$ depends only on $\{\lambda_n\}$

In the next quarter I shall try to see how the results previously obtained change when Δ is not horizontal.

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