

Second quarterly report of the Contract

N62558-4484 NR 043-266 (1 October - 31

December 1965

=====

Before finishing the first part of this research I shall state a result of a different kind but which can be proved by almost the same method as the previous theorems. But to state it I have to give some definitions

Let

$$(1) \quad f(z) = \sum_{n=0}^{\infty} b_n z^n$$

be a convergent series in the circle $|z| < 1$. If

$$\limsup_{r \rightarrow 1} \frac{\log_2 M(r, f)}{-\log(1-r)} > 1$$

there exists a function $V(r)$ such that

$$\limsup_{r \rightarrow 1} \frac{\log_2 M(r, f)}{\log V(r)} = 1$$

and moreover such that if we write

$$V_1(-\log(1-r)) = V(r)$$

the function $V_1(R)$ will have very similar properties to those of $W(r)$

(second report of the contract N62558-4161).

Therefore, given a function $V(r)$ with the just mentioned properties we can consider the space $E_{V,t}$ formed by all the functions (1) holomorphic ^{on} $|z| < 1$ and such that

$$\limsup_{r \rightarrow 1} \frac{\log_2 M(r, f)}{\log V(r)} = 1$$

and

$$\log_2 M(r, \mathcal{F}) \leq (1+t(r)) \log V(r)$$

where $\mathcal{F}(r) = \sum_{n=0}^{\infty} |b_n| r^n$ and where $t(r) > 0$ and $t(r) \rightarrow 0$ as $r \rightarrow 1$. Moreover I suppose that the topology of $E_{V,t}$ is the compact open topology, i.e., the topology of the uniform convergence on the compact sets. Then I have proved that $E_{V,t}$ is of second category.

Let H be the set of all those elements of the space $E_{V,t}$ which can be continued analytically outside $|z| \leq 1$, and let H^* be the complement of H with respect to $E_{V,t}$, then we can state the following theorem.

THEOREM. XVIII.- $H \subset \Delta$, where Δ is a F_σ -set of first category and $H^* \supset \Delta'$, where Δ' is a dense G_δ -set of second category.

This result reminds us of some theorems of Polya [1], but Polya's results use another space and another topology.

In this quarter I have begun to work on the second part of the research.

I shall consider again an entire function $F(z) = \sum_{n=0}^{\infty} a_n z^n$ of infinite order and let $W(r)$ be a function with the properties defined in the second report of the contract N62558-4161, and such that

$$(2) \quad \limsup_{r \rightarrow \infty} \frac{\log_2 M(r, F)}{\log W(r)} = 1.$$

It is very easy to prove that

$$\limsup_{r \rightarrow \infty} \frac{\log_2 M(r, \mathcal{F})}{\log W(r)} = 1$$

where $\mathcal{F}(r) = \sum_0^\infty |a_n| r^n$.

Hence there exists a function $t(r)$ ($t(r) > 0$ and $t(r) \rightarrow 0$ as $r \rightarrow \infty$) such that the family S of the functions

$$\sum_0^\infty t_n z^n$$

verifies $S \subset S_{w,t}$, where $S_{w,t}$ is the space defined in the fourth report of the contract N62558-4161.

Littlewood and Offord suppose that $F(z)$ is of finite order and with the aid of Rademacher's functions obtain a one-to-one correspondence between the set $\{\sigma \mid 0 < \sigma < 1, \sigma \neq k2^{-n}, \text{ where } k \text{ and } n \text{ are any natural numbers}\}$ and a subset of S .

We suppose that $F(z)$ is of infinite order and I shall consider a slightly different correspondence:

Let β_0 be the interval $0 \leq \sigma \leq 1$ and we consider the set β_1 formed of all the elements of β_0 such that

$$\sigma = (2k + 1)2^{-n}$$

where k is any non-negative entire and n is any natural number. Then I represent by β_2 the complement of β_1 with respect to β_0 . With these notations it follows that for every $\sigma \in \beta_2$ the function

$$F(z, \sigma) = \sum_0^\infty r_n(\sigma) a_n z^n \in S \quad (\text{for } \sigma \neq 0, 1)$$

$$F(z, 0) = \sum_0^{\infty} r_n(0+0) a_n z^n \in S,$$

$$F(z, 1) = \sum_0^{\infty} r_n(1-0) a_n z^n \in S$$

where $\{r_n(\sigma)\}$ are Rademacher's functions.

On the other hand if $\sigma \in \beta_1$ we shall have

$$F(z, \sigma + 0) = \sum_0^{\infty} r_n(\sigma + 0) a_n z^n \in S$$

$$F(z, \sigma - 0) = \sum_0^{\infty} r_n(\sigma - 0) a_n z^n \in S$$

and

$$F(z, \sigma + 0) \neq F(z, \sigma - 0).$$

Now I write

$$\beta_1' = \bigcup_{\sigma \in \beta_1} (\sigma + 0),$$

$$\beta_1'' = \bigcup_{\sigma \in \beta_1} (\sigma - 0)$$

and

$$\beta = \beta_1 \cup \beta_1' \cup \beta_1''$$

Evidently β can be put into one-to-one correspondence with the interval $0 \leq \sigma \leq 1$ in which instead of the points of the form

$$(3) \quad \sigma = (2k + 1) 2^{-n}$$

we consider $\sigma + 0$ and $\sigma - 0$, i.e. the point of the form (3) is counted twice. Hence if for $\sigma \in \beta_1$ I write $\sigma + 0 > \sigma - 0$ I shall have stated a total order for β .

If we suppose β with the order topology and S with the topologyrelativization of that of $S_{w,t}$ to S it is easy to state.

THEOREM XLX.- The spaces β and S are homeomorph and $g: \sigma \rightarrow F(z, \sigma)$ is a homeomorphism.

On the other hand if F is any entire function which verifies (2) and if $\mu(r)$ and $\nu(r)$ are defined by

$$\mu(r) = |a_{\nu(r)}| r^{\nu(r)} = \max_{n \geq 0} (|a_n| r^n)$$

then we can state:

LEMMA 1.- If $k > (W(r))^{1+o(1)}$ and $q < k(\log k)^{1-\delta}$ then

$$\sum_{n=1}^{\infty} n^q |a_n| r^n \leq (1 + \alpha(r))^{\nu(r) - k/2} \mu(r)$$

where $R = r(1 + \alpha(r))$ and $\alpha(r) = \left(\frac{\log r}{\log W(r)} \right)^{o(1)}$

COROLLARY OF LEMMA 1.-

$$\mu(r) \leq F(r) \leq \mu(r) (W(r))^{1+o(1)}$$

The lemma and the corollary are a translation to infinite order of two results of Littlewood and Offord [2, lemma 1 and corollary] However I have used the corollary independently in some of my previous papers.

REFERENCES

=====

- 1.- Polya, G.- Ueber die Potenzreihen deren Konvergenzkreis natürliche Grenze ist (Acta Math. t. 41, pag. 99).
- 2.- Littlewood, J.E. and Offord, A.C. - On the distribution of zeros and a-values of a random integral function (II) (Annals of Math. vol. 49, 1948).

Barcelona 31 December 1965

F. Sunyer Balaguer
 Angel Guimera 66 pral. 2ª
 Barcelona - 17 - Spain