On the asymptotic paths of entire functions represented by Dirichlet series

1.- Introduction. - Let

$$(1) \qquad \sum a_n e^{\lambda_n s} \qquad (\lambda_n < \lambda_{n+1},\ \lim \lambda_n = \infty)$$

be a Dirichlet series absolutely convergent ~~for~~ at every point s and ~~[illegible]~~ let $F(s)$ be the entire function represented by (1). Suppose that (2) $s(u) = \sigma(u) + it(u)$ is a continuous function of the real variable u ($0 \leq u < \infty$) such that $\lim_{u \to \infty} s(u) = \infty$. According to the theory of the almost periodic functions ~~if for every u we have $|\sigma(u)| < k$, where k is a finite constant the path (2) cannot be an asymptotic path of $F(s)$.~~ Only if $|\sigma(u)| \to \infty$ ~~[illegible]~~ as $u \to \infty$ the path (2) ~~[illegible]~~ can be ~~[illegible]~~ asymptotic. It is evident that if $\sigma(u) \to -\infty$ then $F(s) \to a_0$ but these asymptotic paths are not interesting.

On the contrary ~~[illegible] theorem on the number of [illegible]~~ when $\sigma(u) \to +\infty$ and $F(s(u)) \to c$ ($c =$ a finite constant) we have

⊗ When I say that two asymptotic paths are distinct I shall suppose that between these paths F(S) is not bounded

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~~(1) Instead of this definition we can give the following: Two asymptotic paths are distinct if between them F(S) is not bounded. With this definition the results would be more general but the proof more difficult~~

an interesting asymptotic path. In 2 I give a theorem on the number of ~~distinct~~ these paths which are contained in a given strip and ~~the~~ are ~~distinct~~ distinct ~~the values of~~ ~~e~~ ~~the~~ The proof is obtained by using the interesting method used by Macintyre [ ] in the proof of the ~~theorem of~~ Denjoy-Carleman-~~Ahlfors~~ Ahlfors theorem.

In 3 I obtain a translation to a class of Dirichlet series of a classical theorem of Wiman for the Taylor series.

2. - again I consider ~~again~~ a continuous ~~curve~~ curve (not necessarily an asymptotic path)

$$(\alpha) \qquad s(u) = \sigma(u) + i t(u)$$

such that $s(u) \to \infty$ as $u \to +\infty$ and now I suppose that if $u_1 < u_2$ then $\sigma(u_1) < \sigma(u_2)$ and that $\sigma(0) = 0$. Then I define the strip

$$S = \{ s = \sigma(u) + it : 0 \le u < +\infty, t(u) \le t \le t(u) + A \}$$

~~I shall say that two asymptotic paths are distinct if ... suppose that the corresponding values of ... are distinct values.~~

~~With these definitions we can state the following theorem~~

On the other hand I write

$$M(\sigma, F) = \sup_{-\infty < t < +\infty} |F(\sigma + it)|$$

and

$$M(\sigma, F, \mathcal{S}) = \sup_{\substack{-\infty < t < +\infty \\ s \in \mathcal{S}}} |F(\sigma + it)|.$$

With these definitions we can state the following theorem:

Theorem I. – If

$$F(s) = \sum a_n e^{\lambda_n s}$$

where $\sum a_n e^{\lambda_n s}$ is absolutely convergent at every point s and if $F(s)$ has n distinct asymptotic paths in $\mathcal{S}$, Then

$$\liminf \frac{\log M(\sigma, F)}{e^{\pi(n-1)\sigma/A}} \geq \liminf \frac{\log M(\sigma, F, \mathcal{S})}{e^{\pi(n-1)\sigma/A}} > 0$$

Proof. – Evidently without loss of generality we can suppose that the n asymptotic paths l_k $(k = 1, 2, \dots, n)$ do not intersect.

⊛ this mapping will be represented by
$z = \Psi(s)$ and it is conformal except on the
cuts.

Now consider the part S_Φ of S for which $\sigma \le \Phi$. Then ~~point of l_k for which $\sigma = 0$ and B_k is the first point~~ by a method used by Macintyre [ ] we can obtain the map S_Φ cut

~~Lemma. If L is the lower bound of the length in the s plane~~

by the n curves l_k on the rectangle

(3) $\quad 0 \le x \le \Phi' \quad |y| \le A/2$ of the plane $z = x+iy$

cut along n parallels to the axis of the x. Then again following ~~again~~ Macintyre we can prove

Lemma 1. – If L is the lower bound of the length in the s plane of all curves ~~belonging~~ to S_Φ and joining a point of the $\sigma = 0$ to a point of $\sigma = \Phi$ but not intersecting any curve L_k, then Φ' verifies the inequality

These rectangles will be ~~designed~~ denoted by Δ_k $(k = 1, 2, \ldots n-1)$

$\Phi' \geq L^2/\Phi$

It is evident that ~~there are n~~ ~~rectangles~~ the rectangle (S) is formed by at most n+1 rectangles of which n−1 are bounded by the n parallels corresponding to the l_k $(k=1, 2, \ldots n)$. Therefore at least one of these rectangles has a width not greater than $A/(n-1)$; and I denote it by $\Delta_{\Phi'}$.

~~Since $F(S)$ is bounded on l_k $(k=1, \ldots n)$ and also for $\sigma=0$ we can suppose without loss of generality that on l_k and for $\sigma=0$ we have $|F(S)|=1$~~

Under the hypothesis that we suppose it is possible to prove that there exists a value $\theta>0$ such that if $\Phi'>\theta$ we can determine a x_0 such that for every k

$$\sup_{x=x_0,\, z\in\Delta_k} |F(\Psi^{-1}(z))|>1$$

where Ψ^{-1} is the inverse function of Ψ

Therefore according to a precision of a theorem of Lindelöf we have

$$\liminf_{\Phi'\to\infty} \frac{\log M(\Phi', F(\Psi^{-1}), \Delta_{\Phi'})}{e^{\pi(n+1)(\Phi'-x_0)/A}} > 0$$

Since ~~following the lemma~~ $L \geq \Phi$, following the lemma, $\Phi' \geq \Phi$ and as

$$M(\Phi, F, \mathcal{S}) \geq M(\Phi', F(\Psi^{-1}), \Delta_{\Phi'})$$

it follows ~~the~~ theorem I

3. - Let $F(s)$ be an entire function represented by a Dirichlet series $\sum a_n e^{\lambda_n s}$ where the sequence $\{\lambda_n\}$ has an upper density D and is such that $\inf(\lambda_{n+1} - \lambda_n) > h$. ~~Then following a result~~ Moreover I suppose that we have defined the function $\rho(\sigma)$ such that

$$\lim \rho(\sigma) = \rho \qquad \rho'(\sigma)\sigma \to 0$$

$$\log M(\sigma, F) \leq e^{\rho(\sigma)\sigma}$$

where ρ is the Ritt's order of $F(s)$

On the other hand following a result of Mandelbrojt [ ] there exists a sequence $\{\sigma_n\}$ such that for every

$$s = \sigma_n + it$$

there exists a point s' such that

$$|s' - s| < \pi D + o(1)$$

$$\log|F(s')| > e^{\rho(\sigma_n)\sigma_n - \rho d}(1-o(1))$$

where $d = D(7 - 3\log(hD))$,

~~If I has an asymptotic path in which σ → +∞ using a result of Milloux~~

Now ~~it is necessary~~ I ~~have~~ need ~~of~~ a result of Milloux, i.e.,

Lemma 2. - Let $F(s)$ be a holomorphic function in $|s| \leq R$ such that

$$\log|F(s)| \leq M$$

and if ~~...~~ on a path ~~...~~ joining $s = 0$ with a point of $|s| = R$ the function is bounded by

$$\log|F(s)| \leq m \qquad m < M$$

then for $|s| \leq r < R$

$$\log|F(s)| < M - (M - m)\frac{2}{\pi}\arcsin\frac{R-r}{R+r}.$$

If $F(s)$ has an asymptotic path in which $\sigma \to +\infty$ we denote by s_n a point such that s_n belongs to the asymptotic path and $s_n = \sigma_n + it_n$ using ~~the~~ lemma 2 and the properties of $\rho(\sigma)$

× Then for the same value of R and for $\rho < \rho_0$ we shall have $1 - \frac{2}{\pi} \arcsin \frac{R - \pi D}{R + \pi D} < e^{\rho d - \rho R}$
and hence

we can prove that for every $R>\pi D$ we have

$$1-\frac{2}{\pi}\arcsin\frac{R-\pi D}{R+\pi D} > e^{-\rho d-\rho R}$$

Therefore if ρ_0 is a function of D and d such that there exists a value of $R>\pi D$ which verifies

$$1-\frac{2}{\pi}\arcsin\frac{R-\pi D}{R+\pi D}=e^{-\rho_0 d-\rho_0 R}$$

we have proved the following:

Theorem II. - If

$$F(s)=\sum a_n e^{\lambda_n s}$$

is a Dirichlet series ~~absolutely~~ convergent at every point s and if ρ_0 represents the function of D and d defined above where D is the upper density of $\{\lambda_n\}$ and

$$d=D(7-3\log(hD))$$

then if the Ritt's order ρ of the function $F(s)$ verifies $\rho<\rho_0$ then $F(s)$ has no asymptotic path such that $\sigma\to+\infty$.

This is the translation of

a classical theorem of Wiman to the Di-
richlet series