ON THE ASYMPTOTIC PATHS OF ENTIRE FUNCTIONS

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In homage to Professor Macintyre

1.- INTRODUCTION.- Let

(1) $\sum a_n e^{\lambda_n s}$ $\qquad (\lambda_n < \lambda_{n+1}, \lim \lambda_n = \infty)$

be a Dirichlet series absolutely convergent at every point s and let F(s) be the entire function represented by (1). Suppose that

(2) $s(u) = \sigma(u) + it(u)$

is a continuous function of the real variable $u(\sigma \leq u < \infty)$ such that $\lim_{u \to \infty} s(u) = \infty$. According to the theory of the almost periodic functions only if $|\sigma(u)| \to \infty$ as $u \to \infty$ the path (2) can be asymptotic. It is evident that if $\sigma(u) \to -\infty$ then $F(s) \to a_0$ but these asymptotic paths are not interesting.

On the contrary when $\sigma(u) \to +\infty$ and $F(s(u)) \to c$ (c = a finite constant) we have an interesting asymptotic path. In 2 I give a theorem on the number of these paths which are contained in a given strip and are distinct; when I say that two asymptotic paths are distinct I shall suppose that between these paths F(s) is not bounded. The proof is obtained using the interesting method used by Macintyre [1] in the proof of the Denjoy-Carlemang-Ahlfors theorem.

In 3 I obtain a translation to a class of Dirichlet series of a classical theorem of Wiman for the Taylor series.

2.- I again consider a continuous curve (not necessarily an asymptotic path)

(a) $$s(u) = \sigma(u) + it(u)$$

such that $s(u) \to \infty$ as $u \to +\infty$ and now I suppose that if $u_1 < u_2$ then $\sigma(u_1) < \sigma(u_2)$ and that $\sigma(0) = 0$. Then I define the strip

$$S = \{ s = \sigma(u) + it : 0 \leq u < +\infty,\ t(u) \leq t \leq t(u) + A \}.$$

On the other hand I write

$$M(\sigma, F) = \sup_{-\infty < t < +\infty} |F(\sigma + it)|$$

and

$$M(\sigma, F, S) = \sup_{\substack{-\infty < t < +\infty \\ s \in S}} |F(\sigma + it)|.$$

With these definitions we can state the following theorem:

THEOREM I.- If

$$F(s) = \sum a_n e^{\lambda_n s}$$

where $\sum a_n e^{\lambda_n s}$ is absolutely convergent at every point s and if F(s) has n distinct asymptotic paths in S, then

$$\liminf \frac{\log M(\sigma, F)}{e^{\pi(n-1)\sigma/A}} \geq \liminf \frac{\log M(\sigma, F, S)}{e^{\pi(n-1)\sigma/A}} > 0$$

Proof.- Evidently without loss of generality we can suppose that the n asymptotics paths $\ell_k (k = 1, 2, \ldots, n)$ do not intersect.

Now consider the part S_{Φ} of S for which $\sigma \leq \Phi$ then by a method used by Macintyre [1] we can map S_{Φ} cut by the n curves ℓ_k on the rectangle

$$(3) \qquad 0 \leq x \leq \Phi' \quad , \qquad |y| \leq A/2 \qquad \text{of the plane} \quad z = x + iy$$

cut along n paralels to the axis of the x; this mapping will be represented by $z = \psi(s)$ and it is conformal except on the cuts. Then again following Macintyre we can prove.

LEMMA 1.- If L is the lower bound of the length in the s plane of all curves belonging to S_{Φ} and joining a point of the $\sigma = 0$ to a point of $\sigma = \Phi$ but not intersecting any curve ℓ_k, then Φ' verifies the inequality

$$\Phi' \geq L^2/\Phi \; .$$

It is evident that the rectangle (3) is formed by at most n + 1 rectangles of which n - 1 are bounded by the n paralels corresponding to the $\ell_k (k = 1, 2, \ldots, n)$. These rectangles will be denoted by $\Delta_k (k = 1, 2, \ldots, n - 1)$. Therefore at least one of these rectangles has a width not greater than $A/(n - 1)$; I denote it by $\Delta_{\Phi'}$.

Under the hypothesis that we suppose it is possible to prove that there exists a value $\Theta > 0$ such that if $\Phi' > \Theta$ we can determine a x_0 such that for every k

$$\sup_{x = x_0, \, z \in \Delta_k} |F(\psi^{-1}(z))| > 1$$

where ψ^{-1} is the inverse function of ψ .

Therefore according to a precision of a theorem of Lindelöf we have

$$\liminf_{\Phi'\to\infty} \frac{\log M(\Phi', F(\psi^{-1}), \Delta_{\Phi'})}{e^{\pi(n-1)(\Phi'-x_0)/A}} > 0 \, ,$$

Since $L \geq \Phi$, following the lemma 1, $\Phi' \geq \Phi$ and as

$$M(\Phi, F, S) \geq M(\Phi', F(\psi^{-1}), \Delta_{\Phi'})$$

it follows theorem I.

3.- Let $F(s)$ be an entire function represented by a Dirichlet series $\sum a_n e^{\lambda_n s}$ where the sequence $\{\lambda_n\}$ has an upper density D and is such that $\inf(\lambda_{n+1} - \lambda_n) > h$. Moreover I suppose that we have defined the function $\rho(\sigma)$ such that

$$\lim \rho(\sigma) = \rho \quad , \quad \rho'(\sigma)\sigma \to 0 \, ,$$

$$\log M(\sigma, F) \leq e^{\rho(\sigma)\sigma}$$

where ρ is the Ritt's order of $F(s)$.

On the other hand following a result of Mandelbrojt [2] there exists a sequence $\{\sigma_n\}$ such that for every

$$s = \sigma_n + it$$

there exists a point s' such that

$$|s' - s| < \pi D + O(1)$$

$$\log |F(s')| > e^{\rho(\sigma_n)\sigma_n - \rho d}(1-o(1))$$

where $d = D(7 - 3\log(hD))$.

Now I need a result of Milloux, i. e.,

LEMMA 2.- Let $f(s)$ be a holomorphic function in $|s| \leq R$ such that

$$\log |f(S)| \leq M$$

and if on a path joining $s = 0$ with a point of $|s| = R$ the function is bounded by

$$\log |f(s)| \leq m \qquad , \qquad m < M ,$$

then for $|s| \leq r < R$

$$\log |f(s)| < M - (M - m)\, \frac{2}{\pi}\, \text{arc} \sin \frac{R - r}{R + r}.$$

If $F(s)$ has an asymptotic path in which $\sigma \rightarrow +\infty$ we denote by s_n a point such that s_n belongs to the asymptotic path and $s_n = \sigma_n + it_n$ using lemma 2 and the properties of $\rho(\sigma)$ we can prove that for every $R > \pi D$ we have

$$1 - \frac{2}{\pi}\, \text{arc} \sin \frac{R - \pi D}{R + \pi D} \quad e^{-\rho d - \rho R}$$

Therefore if ρ_0 is a function of D and d such that there exists a value of $R > \pi D$ which verefies

$$1 - \frac{2}{\pi}\, \text{arc} \sin \frac{R - \pi D}{R + \pi D} = e^{-\rho_0 d - \rho_0 R}$$

then for the same value of R and for $\rho < \rho_0$ we shall have ~~XXXX~~ $1 - \frac{2}{\pi} \text{arc} \sin \frac{R - \pi D}{R + \pi D} < e^{\rho d - \rho R}$ and hence we have proved the following:

THEOREM II.- If

$$F(s) = \sum a_n e^{\lambda_n s}$$

is a Dirichlet series convergent at every point s and if ρ_o represents the function of D and d defined above where D is the upper density of $\{\lambda_n\}$ and

$$d = D(7 - 3 \log(hD))$$

then if the Ritt's order ρ of F(s) verifies $\rho < \rho_o$ the function F(s) has no asymptotic path such that $\sigma \to +\infty$.

This is the translation of a classical theorem of Wiman to the Dirichlet series.

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REFERENCES

1.- Macintyre, A.J.- On the asymptotic paths of integral functions of entire order (Jour. London Math. Soc. 1935)

2.- Mandelbrojt, S.- Séries adhérentes régularisation des suites applications (Col. Monographies sur la teorie des fonctions, Paris 1952).