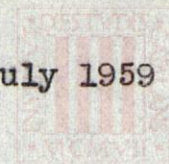


Barcelona 3 july 1959



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FERRAN SÚNYER I BALAGUER

Prof. R. P. Boas
Evanston

DDear Professor Boas,

Enclosed I have the pleasure to send you a paper to be published in the Proc. Am. Math. Soc. if you think it convenient. If you think more proper you may cut out the § 4 and communicate the contents of the same to Mr. Rahman for the case he would like to make personally the correction.

Thanking you in anticipation I remain

yours sincerely

Some remarks on Q.I. Rahman's paper "On entire
Functions defined by a Dirichlet series"

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F. Sunyer i Balaguer

1. The first part of the theorem 2 of Rahman [1] (1) can be improved and we may state the following result (2):

THEOREM A. If $h = \infty$, then the type T_S of $f(s)$ in each horizontal strip $S(\lambda a)$, with $a > 0$, satisfy $T_S = T$.

Proof. If $a > a' > 0$, evidently

$$M_S(\sigma) \geq M_{S'}(\sigma)$$

where $S = S(\lambda a)$ and $S' = S(\lambda a')$; and therefore

$$T_S \geq T_{S'}$$

From theorem 2 of Rahman it follows

$$T_S \geq T_{S'} \geq e^{-\lambda \rho a'} T$$

and hence, for if $a' \rightarrow 0$,

$$T_S \geq \lim T_{S'} \geq T$$

But obviously $T_S \leq T$, and therefore $T_S = T$.

2. The Theorem A is included in the following theorem

THEOREM B. As we suppose $h > 0$, in each horizontal strip $S(a)$, with $a > D$, the type T_S satisfy

$$T \geq T_S \geq e^{-\beta \rho} T$$

where $\beta = \pi D + D(7 - 3 \log(hD))$.

If $h = \infty$, we have $D = 0$ and therefore $\beta = 0$, and since from theorem B follows $T_S = T$, the theorem B include the theorem A, as we have said before.

Proof of theorem B. According to a result of Mandelbrojt [2, theorem a] for any $s_0 = \sigma_0 + it_0$, inside the circle

$$(1) \quad |s - s_0| \leq \pi D + \varepsilon,$$

where ε is a given positive quantity, arbitrarily small, but fixed, there will exist a point s' at which (δ)

$$(2) \quad \log |f(s')| > \log |a_n| - \lambda_n \sigma_0 - \log (\lambda_n \Lambda_n) - c_\varepsilon,$$

where c_ε is a constant which depends on ε ; and this inequality will hold for every value of n .

On the other hand, according to a result contained in the same paper of Mandelbrojt [2, p. 355] for values of n sufficiently large

$$(3) \quad -\log (\lambda_n \Lambda_n) - c_\varepsilon > -\lambda_n (D(7 - 3 \log(hD)) + \varepsilon).$$

Then evidently, if σ is smaller than a certain negative quantity, ~~the~~ the value of n which makes maximum the expression:

$$\log |a_n| - \lambda_n \sigma$$

will be one of those which satisfy (3). As a consequence of (2), we shall conclude that, if σ_0 is smaller than a negative quantity, in the circle (1) there will exist a point s' at which the following inequality will hold:

$$\log |f(s')| > \log \mu(\sigma_0 + d + \varepsilon).$$

where $d = D(7 - 3 \log(hD))$. Moreover, according to Sigimura [3, theorem 5], as D is finite,

$$\log \mu(\sigma) = (1 - o(1)) \log M(\sigma)$$

and consequently we shall have

$$\log |f(s')| > (1 - o(1)) \log M(\sigma_0 + d + \varepsilon)$$

Therefore if we write $s' = \sigma' + it'$, we shall have:

$$\log M_S(\sigma') > (1 - o(1)) \log M(\sigma_0 + d + \varepsilon)$$

And, since $\sigma \geq \sigma_0 - \pi D - \varepsilon$, and ε is arbitrary,

$$T_S \geq e^{-\beta p} T$$

On the other hand, the inequality

$$T \geq T_S$$

is evident.

3. On representing by Δ the maximum density of $\{\lambda_n\}$ introduced by Polya [4], we can state the following theorem.

THEOREM C. As we suppose $h > 0$, in each horizontal strip $S(\kappa a)$, with $a > \Delta$, the type ~~maximum~~ T_S satisfy $T_S = T$.

Since Δ can be $> D$, this theorem not contains the theorem B.

The theorem C is a corollary of my generalization of a result due to Polya [5, lema 2,3].

4. I believe that proofs of the theorem 1 and of the second part of the theorem 2 of Rahman are not completely correct ("); since σ_j^* can be

a discontinuous function of σ_j , and we can neither affirm that

$$\liminf_{\sigma_j \rightarrow -\infty} \frac{\log \log M_S(\sigma_j)}{-\sigma_j} = \lambda_S$$

nor that

$$\liminf_{\sigma_j \rightarrow -\infty} \frac{\log M_S(\sigma_j)}{-\rho \sigma_j} = \tau_S$$

and therefore we can neither affirm that $\lambda_S \geq \lambda$ nor that $\tau_S \geq e^{-\pi \rho a} \tau$

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2. S. Mandelbrojt, Sur une inégalité fondamentale, Ann. Ecole Norm. (3) vol.63 (1947) p.351.
3. K. Sigimura, Übertragung einiger Satze aus der Theorie der ganzen Funktionen auf Dirichletschen Reihen, Math. Zeitschrift vol.29 (1929) p. 264.
4. G. Polya, Untersuchungen über Lucken und Singularitate von Potenzreihen, Math. Zeitschrift vol.22 (1929) p.549.
5. F. Sunyer i Balaguer, Sobre la Distribución de los valores de una función entera representada por una serie de Dirichlet lagunar, Rev. de la Acad. de Ciencias de Zaragoza (2) vol 5 (1950) p. 25-73.

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FOOT-NOTES

(1). Numbers in brackets refer to the bibliography at the end of the paper.

(2). For notations see [1].

(3). When the Dirichlet series contains a constant term constant the definition of the $\Lambda(r)$ and, therefore, of the Λ_n^* given by Mandelbrojt must vary slightly.

(4). The results however might be exact; particularly if $\rho < \infty$ I think it likely they are exact, but the proofs of these results seems to be rather difficult.

$$\frac{\log \Lambda_n^{*(m)}}{\mu_n} \leq D_n^{(m)} + 2 D_n^{*(m)} (\sqrt{2} \mu_n) - 3 D_n^{(m)} \log \frac{h_n^{(m)} D_n^{(m)}}{e}$$

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$$D_n^{(m)} \log \frac{h_n^{(m)} D_n^{(m)}}{e} > \frac{(1+\varepsilon)^2}{1-\varepsilon} D_n^{*} \log \frac{h_n D_n^{*}}{e} + D_n^{(m)} \log (1+\varepsilon)^2$$

$$\frac{\log \Lambda_n^{*(m)}}{\mu_n} \leq (1+\varepsilon) 3 D_n^{*} + \frac{(1+\varepsilon)^2}{1-\varepsilon} [3 D_n^{*} - 3 D_n^{*} \log (h_n D_n^{*})]$$