Barcelona 24 december 1959

Prof. R. P. Boas
Evanston

Dear Professor Boas,

I received your kind letter of ~~the~~ 13 december 1959 together with a letter from Srivastava which ~~I return you~~. I send back to you

We can see that the proof of Srivastava is incorrect as follows:

According to Hardy and Riesz (The general theory of Dirichlet's series, Cambridge Tract nº 18 p.8) if the abscissa of absolute convergence of the series $\sum_1^\infty a_n e^{-\lambda_n s}$ <u>is positive</u>, it is given by:

$$\limsup_{n\to\infty} \frac{\log \sum_{k=1}^{n} |a_k|}{\lambda_n}$$

Therefore <u>only</u> if the abscissa of absolute convergence σ' of the series $f_1(s)=\sum_1^\infty a_{1,n} e^{s\lambda_{1,n}}$ <u>is negative</u>, we can ~~affirm that~~ state that

$$\sigma' = -\limsup_{n\to\infty} \frac{\log A_1(n)}{\lambda_{1,n}}$$

(never $\sigma' = \limsup_{n\to\infty} \frac{-\log A_1(n)}{\lambda_{1,n}}$). Hence from $\sigma'=+\infty$ it does not necessarily follows that

$$-\limsup_{n\to\infty} \frac{\log A_1(n)}{\lambda_{1,n}} = +\infty, \quad (1)$$

and in general we have

$$-\limsup_{n\to\infty} \frac{\log A_1(n)}{\lambda_{1,n}} = 0,$$

and only when $a_{1,n}=0$ for every n, we shall have (1). And equally for $f_2(s)$ and

$$-\limsup_{n\to\infty} \frac{\log A_2(n)}{\lambda_{2,n}}$$

On the other hand, beside the proof that I have suggested in my letter 23 october there exists the following: If

$$\sum_{\phi}^{\infty}|a_{1,n}|e^{\sigma\lambda_{1,n}} \quad \text{and} \quad \sum_{\phi}^{\infty}|a_{2,n}|e^{\sigma\lambda_{2,n}}$$

are convergent for every σ, since $\lambda_n \sim \lambda_{1,n} \sim \lambda_{2,n}$ we shall have

(2) $$\sum_{\phi}^{\infty}|a_{1,n}|e^{\sigma\lambda_n} < \infty$$

(3) $$\sum_{\phi}^{\infty}|a_{2,n}|e^{\sigma\lambda_n} < \infty$$

for every σ. It follows from (2) that $\lim_{n\to\infty}|a_{1,n}|=0$, and therefore from (3) it follows

$$\sum_{\phi}^{\infty}|a_{1,n}||a_{2,n}|e^{\sigma\lambda_n} < \infty$$

and, since $|a_n| \sim |a_{1,n}||a_{2,n}|$, we have

$$\sum_{\phi}^{\infty}|a_n|e^{\sigma\lambda_n} < \infty$$

for every σ.

I wish you a merry Christmas and a happy New Year.

sincerely yours