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Dear Professor Kamthan,

I have receivad your letters of the May 1 and of the June 8 enclosing the revised versions of yours papers "FK-space of entire Dirichlet functions" and "Growth of a Moromorphic Function".

In the actual form the papers will be published in the Collectanea Math. However in the paper"FK-space....." I shall put the condition

$$(1) \qquad \overline{\lim} \frac{\log n}{\lambda_n} = \Lambda < \infty$$

because without (1) the condition

$$\frac{\log |a_n|}{\lambda_n} \to -\infty$$

is not sufficient in order to affirm that $\sum a_n e^{\lambda_n s}$ is convergent in every point of the complex plane. Moreover without (1) in the p.3 line 2 the condition $k > x_i$ is not sufficient in order to state that

$$\sum e^{(x_i - k)\lambda_n} < \infty$$

On the other hand in the proof of (1.3) in p. 5 line 10-12 without (1) from

$$\lim \frac{\log | a_j|}{\lambda_j} = -\infty$$

it does not necessarily follow that

$$f(s) = \sum a_j e^{s\lambda_j}$$

is an entire Dirichlet function.

In the proof of theorem 2 the condition (1) is also necessary

Regarding your question the answer is affirmative, and the proof is the following:As you suppose that f(s) is represented by a Dirichlet serues absolutely convergent in every point of complex plane, it is evident that $f(\sigma + it)$ for every σ will be an almost periodic function of t.

Then it is known that

$$\frac{1}{2T}\int_{-T}^{T} |f(\sigma + ia + it)|^{\delta} dt$$

tends to $A_\delta(\sigma)$ uniformly relative to a.

Now we can prove that if $\sigma_1 < \sigma < \sigma_2$ we have

$$A_\delta(\sigma) \leq \max\left[A_\delta(\sigma_1),\ A_\delta(\sigma_2)\right] \tag{2}$$

In fact if on the contrary

$$A_\delta(\sigma) > \max\left[A_\delta(\sigma_1),\ A_\delta(\sigma_2)\right] \tag{3}$$

According to the result stated above it would follow that there exists a T_0 such that

$$\frac{1}{2T_0}\int_{-T_0}^{T_0} |f(\sigma + it)|^{\delta} dt > \tag{4}$$

$$> \max\left[\mathrm{Sup}\left(\frac{1}{2T_0}\int_{-T_0}^{T_0} |f(\sigma_1 + ia + it)|^{\delta} dt\right),\right.$$

$$\left.\mathrm{Sup}\left(\frac{1}{2T_0}\int_{-T_0}^{T_0} |f(\sigma_2 + ia + it)|^{\delta} dt\right)\right]$$

but following a similar method to Polya and Szego (Anfgaben und

Lehrsatze ans der Analysis 1º Band, III Abschm., Losingen 310) we can prove that (4) and therefore (3) are not true, hence we have proved (2).

Finally as $e^{\alpha s/\delta} f(s)$ is also represented by a Dirichlet series absolutely convergent for every s we have

$$e^{\alpha\sigma} A_\delta(\sigma) \leq \max\left[e^{\alpha\sigma_1} A(\sigma_1),\ e^{\alpha\sigma_2} A_\delta(\sigma_2) \right]$$

And ~~we~~ if we suppose α such that

$$e^{\alpha\sigma_1} A_\delta(\sigma_1) = e^{\alpha\sigma_2} A(\sigma_2)$$

i.e.

$$\alpha = \frac{\log A_\delta(\sigma_2) - \log A_\delta(\sigma_1)}{\sigma_1 - \sigma_2}$$

we shall have

$$\log A_\delta(\sigma) \leqslant \frac{\sigma - \sigma_1}{\sigma_2 - \sigma_1} \log A_\delta(\sigma_2) + \frac{\sigma_2 - \sigma}{\sigma_2 - \sigma_1} \log A_\delta(\sigma_1)$$

This is the result of which you wish to know the proof.

With kindest regards

Your sincerely

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