Barcelona, 14th february 1959

Prof. A.J. Macintyre
Cincinati

Dear Professor Macintyre,

In reply to the questions you asked me in your two kind letters of the 11th January I have the pleasure to inform you as follows:

(1) At present I should not be able to speak English, but as I translate it easily I think that if I strived hard for it I could before long give a course in this language.

(2) Imshould have no inconvenience at all in leaving Barcelona for one rear.

(3) My task in Barcelona is merely research work amd therefore of a thoroughly personal kind.

(4) The greatest difficulty is that the Spanish University Authorities are unable to accept an interchange of professors as they are in lack of funds for it.

(5) In view of this difficulty I have made no enquiries from the Consulate of U.S.A. on whether I should be able to get a visa, but I thinnk there would be no difficulty.

(6) If in spite of the foresaid you were interested in knowing my publications, age, etc. etc. I should be only too glad to send you the same.

(7) I have been known - among other ones - by the professors Mandelbrojt and Milloux, and I suppose it would not be difficult in having some statement from them.

Regarding the lecture you would like to develop, the result you want to show is quite interesting. Mereover, of this result it follows that: If D is a multiply-conected open domain such that $z=0\in D$ and $z=\infty\notin D$, and if

$$R=\min_{z\in c}|z|$$

where c is the component of the complement of D such thet $z=\infty\in c$, then there exists a $\lambda=\lambda(D)$ such thet any function

$$f(z)=\sum_{n=0}^{\infty}a_n z^n,$$

where $a_n=0$ for $n_k< n<\lambda n_k$ and where the radius of convergence

$$r = 1/\overline{\lim}\ |a_n|^{1/n}$$

satisfies the condition $r < R$, has a singular point in D.

Thanking you for the interest shown I beg to be

yours sincerely

Barcelona 31 march 1959

Prof. A. J. Macintyre
Cincinati

Dear Professor Macintyre,

I received your letter of the 7th inst. and thank you very much for your kind interest. Regarding the contents of the same I have the pleasure to inform you of the following:

(i) As I already told you I am suffering from a paralysis which disables me from walking so that I am compelled to ride in a small bath chair, the dimensions of which and the circumstances of its being collapsible allow me to go by car (railway, etc.) as well as to use a lift. I can not manage a staircase. Therefore I should like to live near the University where I had to lecture.

(ii) I am 47 years old (on a sheet enclosed I am informing you of my publications and any other details).

(iii) Your suggestion to send you a lecture recorded by magnetophone is unfortunally impracticable at present as my English pronunciation is not correct. Therefore if you could arrange for my visit to the United States I ought to know it 9 or 10 months before so as to be able to improve my pronunciation.

Reiterating my thanks for your kind interest. I remain

Yours sincerely

Barcelona 26 may 1959

Prof. A. J. Macintyre
Cincinati

Dear Professor Macintyre,

In my possession your kind letter of the 4 May and according to your indication I have the pleasure to submit you in the following the demonstration of the general case of your conjecture. The result I demonstrate may be enunciated as follows:

Let D be any open domain, and let f(x) be a regular function in D such that

$$f(z)=\sum c_n z^n$$

converge in $\{|z|<r\}\subset D$, and where $c_n=0$ if $n_k<n\leq\lambda n_k$. For any bounded closed domain D_1 such that $D_1\subset D$ it exist a number $\lambda_0(D,D_1)$ such that if $\lambda>\lambda_0(D,D_1)$, then there is overconvergence in D_1

Proof.- We shall denote by CD the complement of D, by CD_1 the complement of D_1 and by $\{h_i\}$ the components of CD_1. Evidently there exists only a finit number of h_i such that $h_i\cap CD\neq 0$.

Let h_1 be the component such that $\infty\in h_1$ and we supose that $h_2, h_3, \dots, h_n$ are all the boundeds components such that $h_i\cap CD\neq 0$

Then we write

$$D_2=D_1+\bigcup_{i=n+1}^{\infty} h_i$$

and it follows that D_2 is a bounded closed domain and that $D_2\subset D$. Evidently we may supose that the boundary of D_2 is composet by polygons

Obwiously there exists 2(n-1) polygonal arcs

$$L_1, L_2, \dots, L_{n-1}$$

$$L'_1, L'_2, \dots, L'_{n-1}$$

such that $L_k\cap L'_j=0$ and $D_2-\bigcup_1^{n-1}L_k$ and $D_2-\bigcup_1^{n-1}L'_k$ are simply connected.

On the other hand we shall denote by C(s,R) the open circle $|z-s|<R$ and by S(L,R) the strip

$$S(L,R)=\bigcup_{s\in L}C(s,R)$$

then for R sufficiently small

$$S(L_k,R) \cap S(L'_j,R)=0$$

Hence, for R sufficiently small we may define two closed simply connected domains

$$D_3=D_2-\bigcup_1^{n-1} S(L_k,R) \qquad D'_3=D_2-\bigcup_1^{n-1} S(L'_k,R)$$

such that $D_3+D'_3=D_2$

The proof of the theorem is now immediate. Evidently we may define a open bounded domain $\Delta=\Delta(D,D_3)$ simply connected and such that

$$D_3\subset\Delta \qquad \{|z|<r\}\subset\Delta \qquad \overline{\Delta}\subset D$$

where $\overline{\Delta}$ is the closure of Δ. Moreover it is known that if F(z) is regular in Δ and

$$M(\Delta)=\underset{z\in\Delta}{\text{l.u.b.}}|F(z)| \qquad M(d)=\underset{|z|<r/2}{\text{l.u.b.}}|F(z)| \qquad M(D_3)=\underset{z\in d_3}{\text{l.u.b.}}|F(z)|$$

we have

$$(1) \qquad M(D_3)\leq[M(\Delta)]^{\theta}[M(d)]^{1-\theta}$$

where θ dependonly of Δ, D_3 and d.

Obviusly if

$$s_{n_k}(z)=\sum_0^{n_k} c_n z^n$$

we have

$$\underset{|z|<r/2}{\text{l.u.b.}}|f(z)-s_{n_k}(z)|<H(1/2)^{\lambda n_k} \qquad \underset{z\in\Delta}{\text{l.u.b.}}|f(z)-s_{n_k}(z)|\leq H_1^{n_k}$$

where H and H_1 are two constansts that only depends of Δ. Therefore by (1)

$$\underset{z\in D_3}{\text{l.u.b.}}|f(z)-s_{n_k}(z)|\leq H^{1-\theta}\left[H_1^{\theta}(1/2)^{\lambda(1-\theta)}\right]^{n_k}$$

hence if $\lambda>\lambda_0(\Delta,D_3)$ there is overconvergence in D_3 And since, similarly, we may proof that if $\lambda>\lambda_0(\Delta',D'_3)$ there is overconvergence in D'_3, by

$$D_3+D'_3=D_2\supset D_1$$

it follow the theorem

Believe me to be

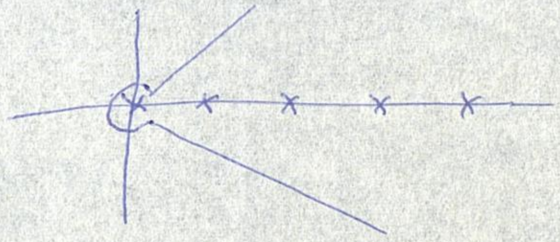
yours sincerely

F. Sunyer Balaguer

Lengths of gaps and region of overconvergence.

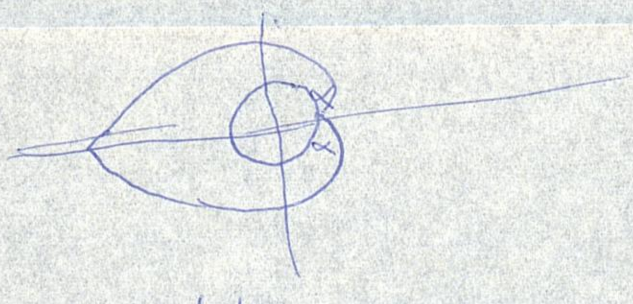
This report is based on the method which has proved so useful for related problems. If $f(z) = \sum_0^\infty c_n z^n$ is regular for $|z| < 1$ and for part but not the whole of $|z| = 1$ then the c_n can be interpolated by an entire function $G(z)$ with $G(n) = c_n$, and the growth properties of $G(z)$ are related to the region into which $f(z)$ may be analytically continued. In the converse direction the representation

$$f(-z) = \frac{1}{2i}\int \frac{G(\zeta)\, z^\zeta\, d\zeta}{\sin \pi\zeta}$$



is useful. Its validity is due to the fact that the integrand tends to zero along arcs $|\zeta| = n + \frac{1}{2}$ if $|z| < 1$ and the calculus of residues may be exploited to give the power series expansion. If however $G(\zeta)$ was sufficiently small along special sequences of arcs $|\zeta| = n + \frac{1}{2}$ the same procedure would give overconvergence. This conclusion could be anticipated for instance from the existence of groups of zeros of c_n of a greater density than is normally possible.

The situation that $G(\zeta)$ is regular and of minimum type in an angle $|\arg \zeta| \leq \alpha$ is associated with continuation of $f(z)$ into a region outside $|z| = 1$ bounded by logarithmic spirals



$f(z)$ will be regular for $|z| < 1$ and between two spirals meeting in $z = 1$ ~~making~~ where they make angles α with $|z| = 1$.

Diagram

If we further suppose $G(n) = 0$ for $n_k \leq n \leq (1+\Delta) n_k$ then $G(z)/\sin \pi z$ is regular in $n_k \leq |z| \leq n_k(1+\Delta)$ for $|\arg z| \leq \alpha$. On the radial parts of the boundary $G(z)/\sin \pi z$ is $O\{\exp[(-\pi \sin \alpha + \epsilon)|z|]\}$. From this we can infer that it is small in the interior. An exact harmonic majorant could be exhibited in terms of elliptic functions. Approximations using the two constants inequality are enough to establish an overconvergence region

Diagram

(i) whose boundary touches the spirals at z = 1 (in accordance with Bourion's theorem.

(ii) which approaches the known region of regularity as Δ tends to infinity.

Similar arguments are available if $G(n) = 0$ for a sufficiently dense set of integers in the gap. In this way Fabry's gap theorem and the gap theorem of Polya can both be exhibited as immediate corollaries of overconvergence theorem. The larger the arc ~~on~~ on which $f(z)$ is regular the smaller the proportion of zeros $G(n)$ needed to establish overconvergence. The length of the gap is more associated with radial extent of overconvergence

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A THEOREM ON OVERCONVERGENCE

F. SUNYER I BALAGUER

The conjecture announced by A. J. Macintyre [2; 3] is equivalent to the theorem stated and proved below.

THEOREM. *Let D be an open domain containing the origin and let $f(z)$ be a function regular in D with the expansion $f(z)=\sum_0^\infty c_nz^n$. Let D_1 be a bounded closed domain contained in D. Then there exists a positive number $\lambda_0=\lambda_0(D, D_1)$ such that if $c_n=D$ for a sequence of intervals $n_k \leqq n \leqq \lambda n_k$ with $\lambda > \lambda_0$, then the subsequence of partial sums $s_{n_k} = \sum_0^{n_k} c_nz^n$ converges uniformly to $f(z)$ in D_1.*

PROOF. Let CO and CD_1 denote the complements of D and D_1 respectively and let k_i, $i=1, 2, \cdots$ be the components of CD_1. The components can be considered as disjoint and there exists only one unbounded component. The one unbounded component will be denoted as h_1.

One can assert that there exists only a finite number of components k_i such that

$$h_i \cap c_D \neq \varnothing, \tag{1}$$

where $\varnothing$ is the empty set. This assertion is proved as follows. Assume that there exists an infinite number of components h_i, $i \geqq 2$, such that (1) is valid. A bounded sequence of points a_i can be formed where $a_i \in h_i \cap CO$, $i \geqq 2$. Every a_i is an element of CD and hence the distance d from D_1 is at least $\delta > 0$. The limit point a of the sequence then must be such that $d(a, D_1) \geqq \delta > 0$. Thus a is an element of CD_1 and all points z in $|z-a| < \delta$ must be in the same component.

Let the finite number of components be enumerated as h_i, $i=1, 2, \cdots, N$. Considering now

$$D_2 = D_1 + \bigcup_{i=N+1}^{\infty} h_i, \tag{2}$$

then D_2 is a bounded closed domain and $D_2 \subset D_1$. Since $\sum_{i=N+1}^{\infty} h_i$ is bounded and D_1 is bounded by hypothesis, D_2 is bounded. Also, since $h_i \cap CD = \varnothing$, $i \geqq N+1$, then $h_i \subset D$ and D_2 is contained in D. To prove that D_2 is closed note that its complement is $\sum_{i=1}^N h_i$ and is open.

Now $N-1$ polygonal arcs $L_1, L_2, \cdots, L_{N-1}$ can be chosen such that $D_2 - \sum_1^{N-1} L_k$ is simply connected. Also $N-1$ other polygon arcs $L_1', L_2', \cdots, L_{N-1}'$ can be so chosen that $L_k \cap L_j' = \varnothing$ and $D_2 - \sum_1^{N-1} L_k'$ is simply connected. Consider now the open circle $C(s, R)$ or $|z-s| < R$ and let $S(L, R) = \bigcup_{s \in L} C(s, R)$. Thus $S(L, R)$ is a strip enclosing the polygonal arc L. For R sufficiently small,

$$S(L_k, R) \cap S(L_j', R) = \varnothing. \tag{3}$$

Hence for R sufficiently small two closed simply connected domains can be defined, $D_3 = D_2 - \bigcup_1^{N-1} S(L_k, R)$ and $D_3 = D_2 - \bigcup_1^{N-1} S(L_j', R)$ such that $D_3 + D_3' = D_2$. This follows from

$$\begin{aligned} D_3 + D_3' &= D_2 - \left\{\bigcup_{k=1}^{N-1} S(L_k, R)\right\} \cap \left\{\bigcup_{j=1}^{N-1} S(L_j', R)\right\} \\ &= D_2 \end{aligned}$$

by (3).

The proof of the theorem follows. An open bounded simply connected domain $\Delta = \Delta(D, D_3)$ can now be defined such that $D_3 \subset \Delta$, $\{|z| < R\} \in \Delta$, $\bar{\Delta} \subset D$ where r is the radius of convergence of $f(z) = \sum_0^\infty c_nz^n$ and $\bar{\Delta}$ is the closure of Δ. From the Nevanlinna two-constant theorem, if $F(z)$ is regular in Δ

$$M(\Delta) = \operatorname*{l.u.b.}_{z \in \Delta} |F(z)|, \qquad M(d) = \operatorname*{l.u.b.}_{|z| < \tau/2} |F(z)|,$$

then [1]

$$M(D_3) = \operatorname*{l.u.b.}_{z \in D_3} |F(z)| \leqq \{M(\Delta)\}^\theta \{M(d)\}^{1-\theta} \tag{4}$$

where $\theta > 0$ depends on D_3 and Δ. Using the majorization of r_{n_k}, where $r_{n_k} = f(z) - s_{n_k}$, $s_{n_k} = \sum_{nk}^\infty c_nz^n$ we get $\text{l.u.b.}_{|z|<k/2} |r_{n_k}| < H(3/4)^{\lambda n_k}$ and $\text{l.u.b.} |r_{n_k}| < H_1^{h_k}$ where H and H_1 are two constants depending on Δ and $f(z)$. Thus by (4),

$$\operatorname*{l.u.b.}_{z \in D_3} |r_{n_k}| \leqq H^{1-\theta} \{H_1^\theta (3/4)^{\lambda(1-\theta)}\}^{n_k}.$$

Thus if $\lambda > \lambda_0(\Delta, D_3)$ there is overconvergence in D_3. Similarly there is overconvergence in D_3' if $\lambda > \lambda_0(\Delta, D_3')$. Now since $D_3 + D_3' = D_2 \supset D_1$ the theorem is proved.

REMARK. By the same method similar results are proved for the series of Dirichlet and for the integral of Laplace.

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