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Prof. Leila D. Bram
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Code 432
Office of Naval Research
Department of the Navy
Washington D.C.

Your Ref.: ONR:432:LDB:~~ehb~~ jec

Dear Professor Bram,

I suppose that the first part of my report requested in your letter of 28 ult. is now in your hands. Herewith enclosed please find the second part.

The subject of my research is very difficult to state in language easily understandable to non-specialists and more of course for non-scientists. However I have done my best, even simplifging results to the extent of losing precision somewhat. If you believe necessary any alterations please let me know and I will carry tkem out.

Finally my best thanks for sending me volume II of "Lectures on Mdern Mathematics" edited by Prof. Saaty.

Yours sincerely

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MAIN RESULTS OBTAINED IN THE LAST YEAR OF THE CONTRACT N62558-3079 NR 043-266

Let Δ be the half strip (in the $s = \sigma + it$ plane)

$$\Delta = \{\sigma > \sigma_0, |t| < \pi g\}$$

and

$$\Delta_\alpha = \{s | se^{-i\alpha} \in \Delta\}.$$

We represent by $W(\Delta_\alpha, \{\lambda_n\}, b, A)$ the class of the functions $F(s)$ holomorphic in Δ_α and such that the linear combinations of $\{e^{-\lambda_n s}\}$ represent $F(s)$ in Δ_α with a logarithmic b-precision so that the adherence hypothesis $A(g, p(\sigma), \{\lambda_n\})$ is satisfied. In relation to the hypothesis A (the exact definition of which can be seen in my paper [1] or in my seventh report) we shall only say here that if Φ represents the linear combinations of $\{e^{-\lambda_n s}\}$ and $C(s_0)$ is a circle $\in \Delta_\alpha$ of centre s_0 and of fixed radius R such that $\pi \bar{D}^* < R < \pi g$ then the afirmation of the hypothesis A can be roughly defined saying that

$$E(s_0) = \inf_{\varphi \in \Phi} \sup_{s \in C(s_0)} |F(s) - \varphi(s)| \longrightarrow 0$$

uniformly and very rapidly if $s_0 \longrightarrow \infty$.

With these definitions I have obtained:

THEOREM A.- If

1º $\{\lambda_n\}$ is such that ~~and~~ $\liminf |\arg \lambda_n| \leq \theta < \pi/2$, and

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$D < \infty$, where D is the maximum density of $\{\lambda_n\}$

2º $g > D + \overline{D}^{*}$ where $\overline{D}^{*}$ is the upper mean density of $\{\lambda_n\}$

3º $|\alpha| < \frac{\pi}{2} - \theta$

then $F(s) \in W(\Delta_\alpha, \{\lambda_n\}, b, A)$, where $b > 2\pi\overline{D}^{*}$, if and only if

(I) $F(s)$ is holomorphic in $\Delta_\alpha \cup H$, where H is an angle $|\arg(s - s')| < \frac{\pi}{2} - \theta$ (where s' depends on Δ_α)

(II) There exists a Dirichlet series $\sum a_n e^{-\lambda_n s}$ and a sequence $\{n_k\}$ of natural numbers such that

$$\lim S_{n_k}(s) = F(s)$$

uniformly in every angle $|\arg(s - s'')| \leq \beta$, for any $\beta < \frac{\pi}{2} - \theta$ and where s'' depends on Δ_α and $\{\lambda_n\}$, and where

$$S_m(s) = \sum_1^m a_n e^{-\lambda_n s}$$

and where the sequence $\{n_k\}$ depends only on $\{\lambda_n\}$.

On the contrary if

$$\frac{\pi}{2} - \theta \leq |\alpha| \leq \frac{\pi}{2} + \theta$$

the (I) and (II) are necessary conditions so that $F(s) \in W(\Delta_\alpha, \{\lambda_n\}, b, A)$, but they are not sufficient. In this case I have obtained an interesting result.

THEOREM B.- The same conditions 1º and 2º as in theorem A

3º $\frac{\pi}{2} - \theta \leq |\alpha| \leq \frac{\pi}{2} + \theta$ ~~the~~

then every $F(s) \in W(\Delta_\alpha, \{\lambda_n\}, b, A)$, where $b > 2\pi\overline{D}^{*}$, is holomorphic in the angle P_α

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$$\alpha > \arg(s - s_0) > -\frac{\pi}{2} + \theta \qquad \text{for } \alpha > 0$$

or

$$\alpha < \arg(s - s_0) < \frac{\pi}{2} - \theta \qquad \text{for } \alpha < 0$$

where s_0 depends only on Δ_α and there exists a Dirichlet series $\sum a_n e^{-\lambda_n s}$ and a sequence $\{n_k\}$ of natural numbers such that $\lim S_{n_k}(s) = F(s)$ uniformly in every angle $|\arg(s - s'')| \leq \beta < \frac{\pi}{2} - \theta$ for any $s'' \in P_\alpha$, where $S_m(s) = \sum_1^m a_n e^{-\lambda_n s}$ and where the sequence $\{n_k\}$ depends only on $\{\lambda_n\}$.

When

$$\frac{\pi}{2} + \theta < |\alpha| \leq \pi$$

I have obtained the folloving result:

THEOREM C.- The same conditions 1º and 2º as in theorem A

3º $\frac{\pi}{2} + \theta < |\alpha| \leq \pi$

then $F(s) \in W(\Delta_\alpha, \{\lambda_n\}, b, A)$, where $b > 2\pi \overline{D}^*$, if and only if

(I) $F(s)$ is an entire function

(II) there exists a Dirichlet series $\sum a_n e^{-\lambda_n s}$ and a sequence $\{n_k\}$ of natural numbers such that $\lim S_{n_k}(s) = F(s)$ uniformly in every angle $|\arg(s - s'')| \leq \beta < \frac{\pi}{2} - \theta$ for any s'', where $S_m(s) = \sum_1^m a_n e^{-\lambda_n s}$ and where the sequence $\{n_k\}$ depends only on $\{\lambda_n\}$.

Hence we see that for $\frac{\pi}{2} + \theta < |\alpha| \leq \pi$ the conditions (I) and

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(II) are necessary and sufficient as when $|\alpha| < \frac{\pi}{2} - \theta$.

After I have proved that in the adherence hypothesis in most of the cases (in particular in the hypothesis A of theorems A,B and C) it is not necessary that $E(s_0) \to 0$ when $s_0 \to \infty$ in a half strip, it is sufficient to suppose only that $E(s_0) \to 0$ when for example, $s_0 \to \infty$ on the central line of Δ_α.

The results that I have proved are more precise than these stated here but the precise statements become very complicated and less intuitive.

The comparison of theorems A,B and C with those results previously obtained show the similarities and the differences between the theory when the $\{\lambda_n\}$ is formed by real numbers and when it is formed by complex numbers.

The common idea of theorems A,B and C, and of the most part of the results previously proved is that with the conditions on Δ_α and $\{\lambda_n\}$ in order to know some properties of $F(s)$ it is unnecessary the hypothesis that $F(s)$ can be approximated ~~arbitrarly~~ closely in a fixed domain $d \in \Delta_\alpha$ by linear combinations of $\{e^{-\lambda_n s}\}$, it is sufficient to suppose only that $E(s_0) \to 0$ very rapidly when $s_0 \to \infty$, i.e. that the error of approximation of $F(s)$ by linear combinations of $\{e^{-\lambda_n s}\}$ on $C(s_0)$ tends to zero very rapidly when $s_0 \to \infty$.

The probable applications of these results are to improve the theory of the Dirichlet series of complex exponents and the study of the singular points of the functions represented by these series. In the case of real exponents the results previously obtained by me have enabled me to proove interesting results on the singular points of the functions represented by Dirichlet series of real exponents.

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REFERENCE

1.- Sunyer Balaguer, F. - Aproximaciones de Funciones por sumas de exponenciales (Collectanea Math. vol. V, pag. 241-267, 1952).

F. Sunyer Balaguer