Barcelona 12 October 1965

Prof. Leila D. Bram
Mathematics Branch
Code 432
Office of Naval Research
Department of the Navy
Washington D.C.

Dear Professor Bram,

I suppose that the first part of my report requestd in your letter 1 ult. (Your Ref.: ONR:432:LBB:jec) is now in your hands. Herewith enclosed please find the second part

Yours sincerely

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MAIN RESULTS OBTAINED IN THE SECOND, THIRD AND FOURTH REPORT OF THE CONTRACT N62558-4161AND IN THE FIRST REPORT OF THE CONTRACT N62558-4484

In the first place I have improved the King-Lai Hiongś orders, i.e., Ihave proved that for any entire function F(z) of infinite order there exists a real function W(r) such that

1º log W(r) is an increasing convex function of log r,

2º
$$W\left(r\left(1+\left(\frac{\log r}{\log W(r)}\right)^{\varepsilon(r)}\right)\right) \leq (W(r))^{1+o(1)}$$

3º $\varepsilon(r) > 0$ and $\varepsilon(r) \to 0$ as $\to \infty$

4º $\left(\dfrac{\log r}{\log W(r)}\right)^{\varepsilon(r)} \to 0$ monotonously as $r \to \infty$

and

5º $\limsup \dfrac{\log_2 M(r,F)}{\log W(r)} = 1$

If W(r) is a given function which verifies 1º, 2º, 3º and 4º and if t(r) is another function such that $t(r) > 0$ and $t(r) \to 0$ as $r \to \infty$, then if $F(z) = \sum a_n z^n$ verifies

$$\limsup \frac{\log_2 M(r,F)}{\log W(r)} = 1$$

and

$$\log_2\left(\sum |a_n| r^n\right) \leq (1 + t(r)) \log W(r)$$

I shall say that $F \in S_{W,t}$. Moreover I suppose that the topology of $S_{W,t}$ is the compact open topology, and instead of saying "space $S_{W,t}$ with the compact open topology" I shall say briefly "space $S_{W,t}$".

It is known that this space is metrisable and I have proved

THEOREM 1.- $S_{w,t}$ *is of second category*

If for one $F \in S_{w,t}$ there exists an entire function $f(z)$ such that

$$\limsup \frac{\log_2 M(r,f)}{\log W(r)} < 1$$

and

$$\limsup \frac{\log n(r,1/(F-f))}{\log W(r)} < 1$$

where $n(r,1/\varphi)$ is the number of zeros of $\varphi(z)$ in $z < r$, I shall say that $F \in A_{w,t}$. Finally I represent by $B_{w,t}$ the complement of $A_{w,t}$ respect $S_{w,t}$.

With these definitions I have proved:

THEOREM 2.- $B_{w,t}$ *is a dense* G_δ *of second category*

THEOREM 3.- $A_{w,t}$ *is a* F_σ *of firsts category*

Let $n(r,\alpha,\varepsilon,F)$ be the number of zeros of $F(z)$ within the sector

$$|z| < r, \qquad |\arg z - \alpha| < \varepsilon$$

Then if $F \in S_{w,t}$ and for any $\varepsilon > 0$ and every finite value of a except at most for one

$$\limsup \frac{\log n(r,\alpha,\varepsilon,F-a)}{\log W(r)} = 1 \tag{1}$$

holds we shall that the direction $\arg z = \alpha$ is a direction of Borel-Valiron If the equality (1) holds for avery finite value of a without exception we shall say that $\arg z = \alpha$ is a directions of Borel-Valiron without exceptional values.

Let $C_{w,t}$ be the familly containing all the $F \in S_{w,t}$ for which there exist a value of α (which may depend on F) such that $\arg z = \alpha$ is not a direction of Borel-Valiron without exceptional value. Then we shall represent by $D_{W,t}$ the complement of $C_{W,t}$ respect $S_{W,t}$ i.e., a function $F \in D_{W,t}$ if $F \in S_{W,t}$ and any direction arg $z = \alpha$ is a direction of Borel-Valiron without exceptional value for F.

With these notations and following a result of mine [3] we can prove.

THEOREM 4.- $D_{W,t}$ <u>is a dense</u> G_δ <u>of second category</u>

THEOREM 5.- $C_{W,t}$ <u>is a</u> F_σ <u>of first category.</u>

Let $F^{(p)}(z)$ be an integral of $F(z)$, the function $F(z)$ or a derivative of $F(z)$ respectively when $p < 0$, $p = 0$ or $p > 0$.

If $F \in S_{W,t}$ for any p

$$\limsup \frac{\log_2 M(r, F^{(p)})}{\log W(r)} = 1$$

holds and we can define the set $C'_{W,t}$ as follows: $F \in C^{\circ}_{W,t}$ if F $S_{W,t}$ and at least for one p the function $F^{(p)}(z)$ has a direction that it is not a direction of Borel-Valiron without exceptional value. Moreover I define the set $D^{\circ}_{W,t}$ as the complement of $C^{\circ}_{W,t}$ respect $S_{W,t}$ Then I have proved

THEOREM 6.- $D^{\circ}_{W,t}$ <u>is a dense</u> G_δ <u>of second category.</u>

THEOREM 7.- $C^{\circ}_{W,t}$ <u>is a</u> F_σ <u>of first category.</u>

I remark that from theorem 1 and from theorems 3,5 and 7 respectively it follows that the classes of the functions $F \in A_{W,t}$, $F \in C_{W,t}$ and $F \in C^{\circ}_{W,t}$ are exceptional in $S_{W,t}$ in the topological sense.

The probable applications of these results are to extend to the infinite order some results of the theory of the entire functions of finite order. Moreover the application of these methods to the finite order will improve some results of the theory of entire functions of finite and infinite order.

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