$$\frac{1}{r}\int_0^r \log M(x)\,dx \leq C(k)T(kr), \quad k>1.$$

Since $T(r)$ is continuous, non-decreasing for $r>0$ and so a proximate order for $T(r)$ can be constructed, satisfying the conditions (i)-(iii) in §2, with $N(r)$ replaced by $T(r)$. Now

$$\begin{aligned} T(kr) &\leq (kr)^{\rho(kr)}, \quad r\geq r_0 \\ &= k^{\rho(kr)} r^{\rho(r)} \exp\{(\rho(kr)-\rho(r))\log r\}. \end{aligned}$$

But

$$\begin{aligned} \rho(kr)-\rho(r) = \int_r^{kr} \rho'(x)\,dx &\leq \varepsilon\int_r^{kr}\frac{dx}{x\log x} \leq \frac{\varepsilon}{\log r}\int_r^{kr}\frac{dx}{x} \\ &\leq \frac{\varepsilon}{\log r}. \end{aligned}$$

Therefore

$$\begin{aligned} T(kr) &\leq k^{\rho(kr)} r^{\rho(r)} \exp(\varepsilon) \\ &= k^{\rho(kr)}\, T(r), \quad r=r_n,\ r_n\to\infty. \end{aligned}$$

Therefore

$$\frac{1}{r}\int_0^r \log M(x)\,dx \leq C(k)k^{\rho}T(r),$$

for arbitrarily large values of $r\to\infty$. This leads to the desired result.

3. Suppose now $f(z)$ is an entire function and $T(r,f)$ be the Nevanlinna characteristic function corresponding to $f(z)$, then

$$T(r,f) \leq \overset{+}{\log} M(r) \leq \frac{R+r}{R-r}T(R,f); \quad 0\leq r<R, \tag{3.1}$$

$$M(r) = \operatorname*{l.u.b.}_{|z|=r}|f(z)|.$$

The example

$$E_\alpha(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(1+\alpha n)}, \quad \alpha=\frac{1}{k}, \quad k>\frac{1}{2}$$

suggests that for finite order entire functions, we need not have necessarily the result that

$$T(r,f)\sim \log M(r), \quad (r\to\infty),$$

since in this case

$$\log M(r,f) \sim r^{k}\ ;\ T(r,f) \sim \frac{r^{k}}{\pi k}\ ,\ (M(r,f) = M(r)).$$

Naturally we enquire if

$$\log\log M(r) \sim \log T(r,f)$$

always holds for entire functions of finite order. This seems possible, though not immediately obvious, since $\log M(r)$ and $T(r,f)$ have the same order. Therefore I state and prove

THEOREM E: Let $f(z)$ be ah entire function of finite order and $T(r,f)$ and $M(r)$ stand as usual for $f(z)$. Then

$$\log\log M(r) \sim \log T(r,f).$$

<u>Proof</u>: As $T(r,f)$ is convex with respect to $\log r$, we fins that

$$T(r,f) = T(r_0,f) + \int_{r_0}^{r} \frac{\omega(x)}{x}\,dx,$$

where $\omega(x)$ is non-decreasing, tending to infinity with x. Now

$$T(r,f) < r^{\rho+\varepsilon} \quad , \text{ all } r \geq r_0 = r_0(\varepsilon)$$

and so for $\lambda > 1$,

$$\int_{r}^{\lambda r} \frac{\omega(x)}{x}\,dx < K r^{\rho+\varepsilon},\ r \geq r_0,\ K = O(1).$$

Now

$$T(R,f) < T(r,f) + \omega(R)\log\left(1+\frac{R-r}{r}\right)$$

$$< T(r,f) + KR^{\rho+\varepsilon}\,\frac{R-r}{r},$$

and let $(R-r)/r = r^{-(\rho+\varepsilon)}$, then

$$T(R,f) < T(r,f) + K\left(1+\frac{R-r}{r}\right)$$
$$= (1+o(1))\,T(r,f).$$

Therefore, using (3.1),

$$\log M(r) \leqslant (1+o(1))(1+2r^{\rho+\varepsilon})\,T(r,f),\ r \geqslant r_0$$

or,

$$\log\log M(r) \leqslant (1+o(1))\,T(r,f),\ r \geqslant r_0$$

and the proof is completed in view of another application of (3.1).

For a non-constant entire function T. Shimizu [6] has proved the following result:

(3.2)

(3.2) $$\lim_{r\to\infty} \frac{\log M(r)}{T(r)\{\log T(r)\}^k} = 0, \quad k>1.$$

The term $(\log T(r))^k$ in (3.2) does not seem to be a sharp term, for example see the remark 1 immediately following Theorem F. If $f(z)$ is an entire function of non-zero finite order, this can be largey improved. Precisely. one has the following

THEOREM F: Let $f(z)$ be an entire function of non-zero finite order, then

$$\lim_{r\to\infty} \frac{\log M(r)}{T(r)\psi(r)} = 0$$

where $\psi(r)$ is non-decreasing, however slow growth be of $\psi(r)$ to increase with r.

Proof: The proof follows on the lines of Theorem D, dince if $R = kr$, $k>1$, then

$$\log M(r) \leq \frac{k+1}{k-1} T(kr).$$

As $T(r)$ is continuous and non-decreasing and so a corresponding proximate order for $T(r)$ exists with the help of which we find that $T(kr) \leq O(1)\,T(r)$, for arbitrarily large values of $r\to\infty$. Hence the result follows.

Remark 1: In the above theorem, one can x choose $\psi(r)$ to be much smaller than $(\log T(r))^k$, for example let $f(z) = e^z$, then $\log M(r) = r$; $T(r) = r/\pi$, and so $\log M(r)/T(r)(\log T(r))^k = \pi/(1+o(1))(\log r)^k \to 0$, $r\to\infty$; while $\log M(r)/T(r)\psi(r)$ $\pi/\psi(r)$. Let $\psi(r) = l_k r$, where k is as large as we please but fixed.

Remark 2: For a class of entire functions of infinite order the result (3.2) can be easily obtained. For, let

(3.3) $$0 < \overline{\lim_{r\to\infty}} \frac{\log T(r)}{r} < \infty ; \quad \overline{\lim_{r\to\infty}} \frac{\log T(r)}{\log r} = \infty.$$

If the relation in (3.3) (the first relation) holds , then we have on following Levin ([3] ,p.52) that

$$T(r) \leq e^{r\lambda(r)}, \text{ all } r \geq r_0$$

$$T(r_n) = e^{r_n \lambda(r_n)}, \quad r_n \to \infty$$

$$r\lambda'(r) \to 0, \quad (r\to\infty)$$

and where

$$\lim_{r\to\infty} \lambda(r) = \overline{\lim_{r\to\infty}} \frac{\log T(r)}{r}.$$

Then

$$T(r+\eta) \leqslant e^{\eta\lambda(r+\eta)+r\lambda(r)} e^{r(\lambda(r+\eta)-\lambda(r))}, \text{ all } r \geqslant r_0$$

$$\leqslant A T(r) e^{r(\lambda(r+\eta)-\lambda(r))}, \quad r = r_n.$$

Also

$$\lambda(r+\eta) - \lambda(r) \leqslant \int_r^{r+\eta} \frac{\varepsilon\, dx}{x} \leqslant \frac{\varepsilon\eta}{r}.$$

Therefore for $r = r_n$,

$$T(r+\eta) \leqslant A_1 T(r).$$

Also $r_n = \frac{1}{\lambda(r_n)} \log T(r_n)$. Therefore

$$\log M(r_n) \leqslant A_1 (1 + o(1)) \frac{r_n}{\eta} T(r_n).$$

Hence for $K > 1$

$$\frac{\log M(r_n)}{T(r_n)\{\log T(r_n)\}^K} \leqslant A_2 (1 + o(1)) r_n^{1-K} \to 0, \ (n \to \infty)$$

and so

$$\underline{\lim_{r\to\infty}} \frac{\log M(r)}{T(r)\{\log T(r)\}^K} = 0.$$

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where for different entire functions only a_n's change and that $0=\lambda_0<\lambda_1<\cdots<\lambda_n\to\infty$ as $n\to\infty$. Let $E=\{x_1, x_2, \cdots\}$, where $x_i \le x_{i+1}$, and $x_i\to\infty$ as $i\to\infty$. We then prove the following:

2. <u>Theorem 1</u>: with the above notation, $\mathfrak{X}$ is a complex FK-Space.

<u>Proof</u>: Let, for $f;g\in\mathfrak{X}$, define the addition of f and g as

$$(f+g)(s)=f(s)+g(s);$$

and the multiplication of $f\in\mathfrak{X}$ with any scalar λ as

$$(\lambda f)(s)=\lambda f(s).$$

The zero element of $\mathfrak{X}$, to be denoted by 0^*, is defined to be that entire function which is zero for all s and that if $f=0^*$ then $a_n=0$ for all $n\ge 1$ and conversely. It is then seen that $\mathfrak{X}$ is a linear space over complex numbers. It is also easily seen that $\mathfrak{X}$ is infinite dimensional and hence we have got a base for $\mathfrak{X}$, namely $x_1,\dots,x_n,\dots$ such that $x_1(s)=e^{s\lambda_1}$, $x_2(s)=e^{s\lambda_2},\dots,x_n(s)=e^{s\lambda_n}$, and so on. Also note that $x_1=(1,0,\dots),\dots,x_n=(0,0,\dots,1,0,0,\cdots)$, where 1 in x_n is at the n-th place. It, therefore, follows that if $x\in\mathfrak{X}$, then

$$x=\left(a_1(x),a_2(x),\dots\right),$$

where $\log|a_n|/\lambda_n \to -\infty$ as $n\to\infty$ and this shows that $\mathfrak{X}$ satisfies (1.1). Now prove that $\mathfrak{X}$ satisfies other conditions.

Let now for each $x_i\in E$, define

$$\|f;x_i\|=\sum_{n=1}^{\infty}|a_n|e^{x_i\lambda_n},$$

where $f\in\mathfrak{X}$ and

$$f(s)=\sum_{n=1}^{\infty}a_n e^{s\lambda_n}$$

is an entire function. Then, as $\log|a_n|^{-1}>K$ for all $n\ge n_0(K)$ where K is arbitrarily taken large number, one gets

$$|a_n|<e^{-K\lambda_n},\ n\ge n_0,$$