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DEPARTMENT OF MATHEMATICS
HAMILTON COLLEGE

July 28, 1967

Professor F. Sunyer Balaguer,
Angel Guimera 36, pral. 2º,
Barcelona-17,
SPAIN.

Dear Professor Balaguer:

I had your kind letter of July 4. I am extremely grateful to you to have given the solution of the convexity of log $A_\delta(\sigma)$, where

$$\left\{I_\delta(\sigma,f)\right\}^\delta = A_\delta(\sigma,f) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \left|f(\sigma+i\,t)\right|^\delta dt, \quad 0<\delta<\infty,$$

but certain steps, that you had omitted as reference like (i) almost periodicity of f(s) (which is of course simple), (ii) $\frac{1}{2T}\int_{-T}^{T}\left|f(\sigma+ia+it)\right|^\delta dt$ tending to $A_\delta(\sigma)$ uniformly in 'a' and (iii) a reference to Polya and Szegö: Aufgaben and Lehrsatze, 1º Band, III Absch, Losungen 310, need explanation. Filling up these explanations I have rewritten the proof of this result without changing the proof you had supplied to me.

Secondly, we have also obtained that for any $\eta > 0$

$$A_\delta(\sigma,f^{(s)}) \leq \frac{K_2}{\eta^{2\delta}}\, A_\delta(\sigma+\eta), \quad K_2 \text{ is some constant}$$

I am enclosing these two results for your kind examination and I request you to be kind enough to let me have any of your valuable comments and suggestions and oblige me.

Finally, I have received a letter of my colleague and student Mr. P. K. Jain from India that you have been waiting for my reply to your letter of December 14, 1966. I am still under the impression that you have received the answers of the paper from Mr. Jain and you might have kindly examined the paper with your further comments. If you have not received the answers of your querries about this paper, I will then send the answers.

With best regards,

Sincerely yours

P. K. Kamthan

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PKK:lk
Encl.

P.S. From September 1, my address will be:

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<u>Lemma 1:</u> Let $A_\delta(\sigma, f)$* be defined as above, then $\log A_\delta(\sigma, f)$ (3) is a convex function of σ $(0<\sigma<\infty)$.

(Cor. 2, p. 144, [2])*

<u>Proof</u>: Since the exponential polynomial $\sum_1^n a_n e^{s\lambda_n}$ is almost periodic and converges in the <u>open compact topology</u> to $f(s)$** (lemma 2, [3]) $f(s)$ is therefore almost periodic (Cor. 2, p. 144, [2]).

Now by Minkowski's inequality

$$\left\{\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(\sigma+it)|^\delta dt\right\}^{1/\delta}-\left\{\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(\sigma+ia+it)|^\delta dt\right\}^{1/\delta}$$

$$\leq\left\{\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left\{|f(\sigma+it)|^\delta-|f(\sigma+ia+it)|^\delta\right\}^\delta dt\right\}^{1/\delta}$$

$$\leq\left\{\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(\sigma+it)-f(\sigma+ia+it)|^\delta dt\right\}^{1/\delta}\leq\varepsilon,$$

on account of the almost periodicity of $f(s)$, uniformly relative to a. Therefore

$$\frac{1}{2T}\int_{-T}^{T}|f(\sigma+ia+it)|^\delta dt$$

tends to $A_\delta(\sigma, f)$ uniformly relative to a. Let now $\sigma_1\leq\sigma\leq\sigma_2$, we then claim that

$$A_\delta(\sigma)\leq\max\left(A_\delta(\sigma_1), A_\delta(\sigma_2)\right), \tag{3.1}$$

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* We are very grateful to Professor F. Sunyer Balaguer of Barcelona for the help in the proof of this result (in communication to PKK) in confirmation

* If there is no confusion is likely to occur, with which function $A_\delta(\sigma, f)$ refers we shall $A_\delta(\sigma, f)$ as $A_\delta(\sigma)$ and etc.

* Besicovitch, A. S. Almost Periodic Functions, Dover Publs., 1945.

** converges uniformly in every bounded strip.

for, otherwise exists an σ, such that

$$(3.2)\qquad A_\delta(\sigma) > \max\left(A_\delta(\sigma_1),\ A_\delta(\sigma_2)\right)$$

and it follows from this that there exists a T_0, such that

$$(3.3)\qquad \frac{1}{2T_0}\int_{-T_0}^{T_0}|f(\sigma+it)|^\delta dt > \max\left[\sup_a\left(\frac{1}{2T_0}\int_{-T_0}^{T_0}|f(\sigma_1+ia+it)|^\delta dt\right),\ \sup_a\left(\frac{1}{2T_0}\int_{-T_0}^{T_0}|f(\sigma_2+ia+it)|^\delta dt\right)\right]$$

We show that (3.3) is not true and consequently it will follow that (3.2) is not true. For this, let us define functions $\varepsilon(a)$ and $F(\sigma+it)$ as follows: let $\sigma_1 \le \sigma \le \sigma_2$

$$\varepsilon(a)\{f(\sigma+ia)\}^\delta = |f(\sigma+it)|^\delta, \quad -\infty < a < \infty$$

$$F(\sigma+it) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(\sigma+it+ia)|^\delta\,\varepsilon(a)\,da$$

The function $F(\sigma+it)$ is regular in the strip $[\sigma_1,\sigma_2]$ and it follows from Phragmén–Lindelöf Theorem (p. 137, [2]) that

$$A_\delta(\sigma) = F(\sigma) \le \max\left(\sup_t |F(\sigma+it)|_{\sigma=\sigma_1},\ \sup_t |F(\sigma+it)|_{\sigma=\sigma_2}\right)$$

$$(3.4)\qquad \le \max\left[\sup_t\left(\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(\sigma_1+ia+it)|^\delta da\right),\ \sup_t\left(\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(\sigma_2+ia+it)|^\delta da\right)\right].$$

Hence (3.3) is in contradiction with (3.4).

Finally, as $e^{\frac{\alpha s}{\delta}} f(s)$ is also represented by Dirichlet series, absolutely convergent for every s, we have (α = real)

$$e^{\alpha\sigma} A_\delta(\sigma) \le \max\left(e^{\alpha\sigma_1} A_\delta(\sigma_1),\, e^{\alpha\sigma_2} A_\delta(\sigma_2)\right).$$

And if we suppose α, such that

$$e^{\alpha\sigma_1} A_\delta(\sigma_1) = e^{\alpha\sigma_2} A_\delta(\sigma_2),$$

i.e., $$\alpha = \frac{\log A_\delta(\sigma_2) - \log A_\delta(\sigma_1)}{\sigma_1 - \sigma_2},$$

we shall have

$$\log A_\delta(\sigma) \le \frac{\sigma - \sigma_1}{\sigma_2 - \sigma_1} \log A_\delta(\sigma_2) + \frac{\sigma_2 - \sigma}{\sigma_2 - \sigma_1} \log A_\delta(\sigma_1)$$

and it follows that $\log A_\delta(\sigma)$ is convex in $[\sigma_1, \sigma_2]$ with respect to σ.

Corollary: $\log I_\delta(\sigma)$ is a convex function of σ and for $\sigma_1 < \sigma_2$, $\log I_\delta(\sigma_1) \le \log I_\delta(\sigma_2)$ and that $\log I_\delta(\sigma)/\sigma \to \infty$ as $\sigma \to \infty$.

~~The first part of this corollary follows~~ directly

Since $\log A_\delta(\sigma)$ is convex with respect to σ and so there exists a non-decreasing and almost continuous function $n(x)$ of x ($n(x) \to \infty$ as $x \to \infty$) such that ([5], equation (4), $\varphi(t) = t$)

(3.5) $$\log A_\delta(\sigma) = \int_\alpha^\sigma n(x)\,dx$$

and so

(3.6) $$\log I_\delta(\sigma) = \int_\alpha^\sigma w(x)\,dx, \quad w(x) = \frac{1}{\delta} n(x)$$

and it follows that $\log I_\delta(\sigma)$ is a convex function with respect to σ. From

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Proof for $A_\delta(\sigma, f^{(s)}) \leq \dfrac{K_2}{\eta^{2\delta}} A_\delta(\sigma+\eta)$

Using Cauchy's formula for s-th derivative

$$A_\delta(\sigma, f^{(s)}) = \lim_{T\to\infty} \int_{-T}^{T} \left| \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(\omega)}{(\omega-\sigma-it)^{s+1}}\, d\omega \right|^\delta dt,$$

where ω varies over a circle centred at $\sigma+it$ & radius η. The above relation immediately leads to the following on change of integration

$$A_\delta(\sigma, f^{(s)}) \leq \frac{1}{(2\pi)^\delta} \frac{1}{\eta^{s\delta}} \lim_{T\to\infty} \int_0^{2\pi} \left\{ \frac{1}{2T} \int_{-T}^{T} |f(\sigma+it+\eta e^{i\theta})|^\delta dt \right\} d\theta$$

$$= \frac{1}{(2\pi\eta^s)^\delta} \int_0^{2\pi} \left\{ \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma+it+\eta\cos\theta)|^\delta dt \right\} d\theta$$

$$= \frac{K_1}{\eta^{s\delta}} \int_0^{2\pi} A_\delta(\sigma+\eta\cos\theta)\, d\theta \leq \frac{K_2}{\eta^{s\delta}} A_\delta(\sigma+\eta),$$

since f is almost periodic & $A_\delta(x)$ is non-decreasing