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Washington 25, D.C.
U. S. A.

Dear Professor Saaty,

Subject: First quarterly report of the Contract N62558-3079 NR 043-266.

The objectives of the research are, when the strip Δ and the sequence $\{\lambda_n\}$ have any of the usual properties, to determine the properties of the functions, holomorphic in Δ, which we can represent in Δ by the linear combinations of $\{e^{-\lambda_n s}\}$ with a certain precision which satisfies an adherence hypothesis.

I shall give here the main results I have hitherto obtained. Some definitions are needed first.

Let Δ be the strip $\{\sigma > \sigma_0, |t| < \pi g(\sigma)\}$ in the $s = \sigma + it$ plane, where $g(\sigma)$ is a continuous function of bounded variation for $\sigma > \sigma_0$. We shall assume $g = \liminf g(\sigma) > 0$. Let $\delta(X)$ be the intersection of Δ and the vertical strip $x-b < \sigma < x$. Then if

$$\operatorname*{Inf}_{\varphi \in \Phi} \operatorname*{Sup}_{s \in \delta(x)} |F(s) - \varphi(s)| \leq e^{-p(x)}$$

where Φ is the set of the linear combinations of the $\{e^{-\lambda_n s}\}$ and where $p(x)$ is a non-decreasing function tending to $+\infty$ ($p(x)$ may be equal to $+\infty$ for x sufficiently large), we say that the linear combinations $\varphi(s) \in \Phi$ represent $F(s)$ in Δ with the logarithmic b-precision $p(x)$.

On the other hand, if the sequence $\{\lambda_n\}$ is such that $0 \leq \lambda_n < \lambda_{n+1}$ and $\sum_2^\infty \frac{1}{\lambda_n} < \infty$, and if we set

$$(1) \qquad q(z) = \begin{cases} z\prod_2^\infty \left(1+\frac{z}{\lambda_n}\right) = \sum b_n z^n & \text{if } \lambda_1 = 0 \\ \prod_1^\infty \left(1+\frac{z}{\lambda_n}\right) = \sum b_n z^n & \text{if } \lambda_1 > 0 \end{cases}$$

$$(2) \qquad Q(R) = \int_0^\infty e^{-Rr} q(r)\,dr$$

the theorie of entire functions allow us to state immediatly that the function (1) is entire, and thet the integral (2) converges for any $R>0$

Definition of the adherence hypothesis $\Omega[g(\sigma), p(\sigma), \{\lambda_n\}]$.- We say that the functions $g(\sigma)$ and $p(\sigma)$, and the sequence $\{\lambda_n\}$ satisfy the hypothesis $\Omega[g(\sigma), p(\sigma), \{\lambda_n\}]$ if there exists a continuous non-increasing function $h(\sigma)$, with $\lim h(\sigma) = h$, such that

$$h < g, \qquad \log Q(\pi h(\sigma)) < p(\sigma) + M \qquad (M < \infty)$$

$$\int^\infty \left[p(\sigma) - \log Q(\pi h(\sigma))\right] \cdot \exp\left[-\frac{1}{2}\int^\sigma \frac{du}{g(u)-h(u)}\right] d\sigma = \infty$$

Now using almost the same method that I use in the proof of conclusion β of my theorem V of [2] (see references at the end) we can prove.

THEOREM I.- Supposing the following conditions are satisfied:

1º $\{\lambda_n\}$ is such that $0 \leq \lambda_n < \lambda_{n+1}$ and $\sum_2^\infty \frac{1}{\lambda_n} < \infty$

2º F(s) is holomorphic function in Δ, and the linear combinations $\varphi(s) \in \Phi$ represents F(s) in Δ with the logarithmic b-precision $p(\sigma)$.

3º The hypothesis $\Omega[g(\sigma), p(\sigma), \{\lambda_n\}]$ is satisfied.

Then for any finite rectangle $\chi = \{\sigma_0 < \sigma < a, |t| \leq \alpha\} \subset \Delta$, we have $F(s) \in K(\{\lambda_n\}, \chi)$, where $K(\{\lambda_n\}, \chi)$ denote the closure of the linear combinations $\varphi(s)\in\Phi$ in the rectangle χ, with the topology of uniform convergence.

If, instead of $q(z)$ and $Q(R)$, we had taken $\Lambda(z)$ and $L(R)$ of Mandelbrojt, we should have stated a similar result, but we should have only proved that $F(s)\in K(\{\lambda_n\}+\{-\lambda_n\},\chi)$.

Let $W(\Delta,\{\lambda_n\},b,\Omega)$ be the class of the functions $F(s)$ holomorphic in the strip Δ and such that the linear combinations $\varphi\in\Phi$ represent $F(s)$ in Δ with a logarithmic b-precision $p(x)$ so that the hypothesis $\Omega(g(\sigma),p(\sigma),\{\lambda_n\})$ be satisfied.

Then, combining theorem I and two results of Schwartz [1, pag. 38-39 and 57-58] we obtain.

THEOREM II.- When

1º $\{\lambda_n\}$ is such that $0\leq\lambda_n<\lambda_{n+1}$ and $\sum_2^\infty\frac{1}{\lambda_n}<\infty$

2º $\Delta=\{\sigma>\sigma_0,|t|<\pi g(\sigma)\}$, where $g(\sigma)$ is a continuous function of bounded variation in $\sigma_0<\sigma<\infty$ such that $g(\sigma)>0$ and $\lim g(\sigma)>0$.

Then $F(s)\in W(\Delta,\{\lambda_n\},b,\Omega)$ if, and only if,

(i): $F(s)$ is holomorfic in the half-plane $\sigma>\sigma_0$.

(ii): There exists a Dirichlet series $\sum d_n e^{-\lambda_n s}$ and a sequence $\{n_k\}$ of natural numbers such that

$$\lim_{k=\infty} S_{n_k}(s)=F(s)$$

uniformly in every domain

$$\sigma\geq\sigma_0+\varepsilon \qquad \left|\frac{t}{\sigma-\sigma_0-\varepsilon}\right|\leq C$$

for any $\varepsilon>0$ and any $C>0$, where

$$S_{n_k}(s)=\sum_1^{n_k} d_n e^{-\lambda_n s}$$

and where the sequence $\{n_k\}$ depends only on $\{\lambda_n\}$.

As an immediate consequence, we obtain the following:

COROLLARY.- Let $\{n_k\}$ be a sequence of natural numbers such that $\sum_1^\infty 1/n_k < \infty$. Let Φ_0 be the set of the polynomials $\psi(z)=a_0+\sum_{k=1}^{m} a_k z^{n_k}$.

If the function f(z) is holomorfic in the domain $\{0<|z|<1,\ |\arg z| < \alpha\}$ and if

$$\inf_{\psi\in\Phi_0} \sup_{\substack{\theta x<|z|<x \\ |\arg z|<\alpha}} |f(z)-\psi(z)| < e^{-p(x)} \qquad (0<\theta<1)$$

holds. Then, supposing that for $\varepsilon>0$ sufficiently small

$$\int_0 p(x)\, x^{\frac{\pi}{2\alpha}-1+\varepsilon}\, dx = \infty,$$

the function f(z) is holomorfic in the circle $|z|<1$.

In the next quarter I shall endeavour to treat the similar topics of those above considered but when $\sum_2^\infty \frac{1}{\lambda_n}=\infty$ and the upper density of $\{\lambda_n\}$ is finite.

REFERENCES

Schwartz, L. - Etude des sommes d'exponentielles. (Actualités scientifiques et industrielles 959, Deuxième édition)

Sunyer Balaguer, F. - Aproximación de funciones por sumas de exponenciales (Collectanea Math. vol. V, pag. 241-267, 1952).

Sincerely yours,

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