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A Fifteenth-Century Planetary Computer: al-Kāshī's "Ṭabaq al-Manāṭeq"

II. Longitudes, Distances, and Equations of the Planets

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1. Introduction. — This is the second of a series of papers discussing an astronomical instrument, the *Plate of Zones*, invented by the Iranian mathematician and astronomer JAMSHĪD GHĪĀTH ED-DĪN AL-KĀSHĪ (died *c.* 1436). Biographical data concerning al-Kāshī and remarks about a Persian manuscript describing the instrument have been published in a previous note.[1] The first paper [2] of the series described the general construction of the instrument and its application to the problem of finding the true longitude of the sun and moon at a given time. The present paper deals with the solution of the analogous problems for the planets. As in the solar and lunar cases, the basic idea is to reproduce to a suitable scale the Ptolemaic configuration for the given planet at the given time, a graduated ring and alidade then being used to read off the true longitude.

In the course of working over this material it turned out that at least one copy of the *Nuzhat al-Ḥadāʾiq* (*A Fruit-Garden Stroll*) is in existence and available for study. This is the Arabic description of the instrument written by al-Kāshī himself which presumably was used by the anonymous author of the Persian manuscript. It is Number 210 in the *Catalogue of Two Collections of Persian and Arabic Manuscripts Preserved in the India Office Library, E. Denison Ross and E. G. Browne, London, 1902*. In the colophon of this copy of the *Nuzha* al-Kāshī states that he completed the work on 10 Dhū al-Hijja, 818 A.H. (10 February 1416) in Kāshān (his home town), hence before he entered the service of the astronomer-prince Ulugh Beg. Officials of the India Office kindly arranged for the microfilming of this manuscript. It is carelessly written in an easily legible *nastaʿlīq* hand, and the copy is modern, having been completed on 3 Rajab, 1280 A.H. (2 December 1863). Moreover, all the figures and the most extensive tables are left out. In spite of this, the film has been quite useful, for the Persian manuscript is by no means a translation of the *Nuzha*.

In a short communication addressed in 1914 to the Imperial Academy of Sciences of Saint Petersburg [3] the Russian orientalist Wilhelm Barthold noted the existence at Leiden of an uncatalogued tract by al-Kāshī describing an astronomical instrument. Dr. Voorhoeve of the Rijksuniversiteit library has kindly supplied microfilms of this manuscript, bound with Cod. 945 (Warner). It is not the *Nuzha*, but a description of several observational instruments.

Section **2** below lists the parameters used by the inventor in laying out the plate and alidade, and compares them with corresponding Hellenistic and Moslem values. Alternative layouts given in the *Nuzha* but not in the Persian version are discussed in **3**. Since al-Kāshī's arrangement for Mercury differs somewhat from the classical Ptolemaic one, **4** is devoted to a description of the non-circular deferent he uses for this planet. In **5** the configuration for finding the longitudes of all the planets is given. Based on this fundamental configuration, **6** shows how the instrument may be used for finding

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[1] *Al-Kāshī's Plate of Conjunctions*, Isis, Vol. 38, p. 56.

[2] *A Fifteenth Century Planetary Computer. I. Motion of the Sun and Moon in Longitude*, Isis, Vol. 41, pp. 180–183. The reader may consult this paper for an account of the parts of the instrument referred to in the sequel. See also *An Islamic Computer for Planetary Latitudes*, Journ. Am. Or. Soc., Vol. 71, 1951, pp. 13–21; and *A Fifteenth-Century Lunar Eclipse Computer*, Scripta Mathematica, Vol. XVII (1951), pp. 91–97.

[3] *Izvestiya Imperatorskoĭ Akademiĭ Nauk*, 1914, p. 459.

the planetary equation, **7** its use for finding the lunar equations, and **8** for finding the distances of the planets from the earth.

A remark of Sarton's [4] led to the conjecture that ancestral forebears of the *Plate of Zones* were instruments invented by two Spanish Arabs, ibn as-Samḥ (*c.* 1020) and Azarquiel (az-Zarqālī, *c.* 1060). Results of the inquiry thus suggested are given in **9**, while concluding remarks are found in **10**.

Following the usage of the manuscript, numbers are here given in sexagesimals. The semicolon indicating the sexagesimal point and commas separating sexagesimal places are modern notations. All numbers in the manuscripts are given in the Arabic alphabetical (*jummal*) system.

2. Planetary Parameters.—In Chapter II of the Persian manuscript the author gives directions for laying out on the plate of the instrument the *deferent* (*falak-i ḥāmel*) of each planet, the epicycle-carrying eccentric orbit. The author also refers to the deferents as *zones* (*manāṭeq*), hence the name of the instrument.

Column 1 of Table One is taken from the manuscript except that signs of the zodiac have been converted into degrees. Thus we represent the first entry in the table,

PLANET	① Angle in Degrees from Solar Apogee to Planetary Apogee	Angle in Degrees between Apogee of Jupiter and of the other Planets			⑤ Distance from Plate Center to Deferent Center (60ths of plate radius)	⑥ Deferent Radius	⑦ Distance of the Difference Mark (epicycle radius)	Normed Eccentricities			Normed Epicycle Radii			⑭ Norming Coefficients	⑮ = 1,0;0/⑭	⑯ = ⑤ + ⑮
		② al-Kāshī	③ Ptolemy	④ al-Bīrūnī				⑧ al-Kāshī = ⑤ × ⑭	⑨ Ptolemy	⑩ al-Bīrūnī	⑪ al-Kāshī = ⑦ × ⑭	⑫ Ptolemy	⑬ al-Battānī			
Saturn	2,40;28	1,11;12	1,12;0	1,20;5	2;58	52;2	5;38	3;25	3;25	3;22,30	6;30	6;30	6;29,50	1;9,11	52;2	55;0
Jupiter	1,29;16	—	—	—	2;32	55;28	10;38	2;44	2;45	2;45	11;30	11;30	11;30	1;4,55	55;28	58;0
Mars	46;5	5,16;41	5,14;30	5,21;47	4;33	45;27	30;32	6;0	6;0	6;0	40;18	39;30	39;27,22	1;19,12	45;27	50;0
Venus	5,49;15	4,19;51	4,14;0	4,37;56	1;2	58;58	42;25	1;3	1;15	1;2,30	43;10	43;10	43;9	1;1,3	58;58	1,0;0
Mercury	2,2;40	33;24	29;0	37;0	4;52	51;23	18;13	6;0	6;0	6;20	22;27	22;30	22;30,30	1;13,56	48;42	56;0

TABLE ONE

$4^s\,40^\circ\,28'$, by $2,40;28^\circ$. The entries themselves give the zodiacal angles from the apogee of the sun to the apogee of each planet respectively. In the classical Ptolemaic theory the planetary apogees are constant relative to the fixed stars, while the solar apogee is fixed with respect to the vernal point, i.e., it has a constant longitude.[5] The Moslem astronomers, however, recognized that the longitude of the solar apogee is a variable,[6] and al-Kāshī makes specific allowance for this by providing for adjustment of the plate inside the zodiacal ring of the instrument. But since all apogees are permanently marked on the plate, once the apogee of the sun is set, the planetary apsidal lines are also fixed. Hence, in comparing al-Kāshī's parameters with those given by other astronomers, there is no point in dealing with the apsidal longitudes as such. But the relative positions of the apogees are comparable, and Columns 2, 3, and 4 of Table One display for al-Kāshī, Ptolemy, and al-Bīrūnī, respectively, the results of subtracting Jupiter's

[4] *Introd. to the Hist. of Science*, Baltimore, 1927–48, Vol. II, p. 337.

[5] *Almagest*, Book III, Ch. 4 (edition of Halma, Paris, 1813, Vol. I, p. 184); Book IX, Ch. 5 (Halma, v. II, p. 158).

[6] Dreyer, *History of the Planetary Systems from Thales to Kepler*, Cambridge, 1906, p. 251.

apsidal longitude from the longitude of the apogee of each of the other planets.[7] It will be noticed that while no pairs of corresponding values are identical, the differences between them are in general not large, al-Kāshī's agreement with Ptolemy being much closer than with al-Bīrūnī. The India Office copy of the *Nuzha* gives no apogees, but both manuscripts state that the parameters have been taken from al-Kāshī's own planetary tables, the *Zīj-i Khāqānī*. This work is in turn based on the *Zīj-i Īlkhānī*, the tables made by the celebrated Naṣīr ed-Dīn at-Ṭūsī (1230) for the Īlkhān Hulāgū, grandson of Chingīz Khān. It is probable that the positions of the apsidal lines given here correspond to the planetary apogees in these two unpublished works.

Both manuscripts then give tables of the eccentricity of the deferent, and of the length of the radii of the deferent and the *epicycle* (*falak-i tadvīr*) for each planet. These values are reproduced in Columns 5, 6, and 7 of Table One, the units being sixtieths of the plate-radius. It was customary in antiquity to take as a unit for each planet its own deferent radius as sixty (i.e., 1,0) and to express the other parameters for that planet in sixtieths of the unit deferent radius. Al-Kāshī, in order to have all the deferents fall in the interior of the plate, reduces the scales by different ratios for each planet. The ratios are given in a later chapter in the Persian manuscript in which planetary distances are discussed, and make up Column 14 of Table One. The tabular values of the deferent radii can now be checked by computing the reciprocals of the entries in Column 14. The results are shown in Column 15, and in every case except those of Mars and Mercury they equal (to two sexagesimal places) the corresponding deferent radii. The entry for Mars is clearly 55;27, but use of the latter value would throw Columns 8 and 11 badly out of line, so the initial 55 has been altered to a 45 to conform with Column 15. The 55 is probably a copyist's error. The exceptional case of Mercury is discussed in 4 below.

In order to compare the eccentricities and epicycle radii with the values used by other astronomers, Columns 5 and 7 have been "normed," i.e., expressed in sixtieths of the respective deferent radii (correct to two sexagesimal places) and displayed in Columns 8 and 11 respectively. Columns 9 and 10 give the values of the eccentricity used by Ptolemy[8] and al-Bīrūnī.[9] It will be noticed that the correspondence is close, most of al-Kāshī's values being identical with those of Ptolemy. Only in the case of Venus is al-Kāshī closer to al-Bīrūnī's value than to the Ptolemaic one. Columns 12 and 13 list the epicycle radii used by Ptolemy and al-Battānī. (The latter used the same deferent eccentricities as Ptolemy, while al-Bīrūnī used the Ptolemaic epicycle radii.) Al-Kāshī's values again correspond closely to those of the others, more often than not the correspondence with Ptolemy being exact, that with al-Battānī less close.

It should also be remarked that in every case the sum of the eccentricity and deferent radius (Columns 5 and 6) is an integer. This sum (Column 16) measures the nearest approach of the deferent circumference to the periphery of the plate.

The reason for choosing these particular scales is not evident, but it becomes so if the deferents are actually laid out to scale with properly directed apogees. For then it is seen that the deferents of the first four planets are snugly nested, no two intersecting. Evidently the object was the practical one of reducing the chance of confusing one orbit with another. This does not hold for Mercury; its oval deferent intersects the deferents of both Mars and Saturn. Perhaps al-Kāshī thought its distinctive shape was sufficient guarantee against confusion. Clearly, this particular choice of scales is anything but unique, and in the *Nuzha* a second scale for Mars is given in addition to the one in the Persian manuscript. This scale makes the sum of the eccentricity and deferent radius fifty-five, the other parameters being increased in proportion.

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[7] Nallino, *Al-Battānī sive Albatenii opus astronomicum*, Rome, 1899–1907, Vol. I, p. 239; *Almagest*, Halma, Vol. II, pp. 168, 195, 231, 258, 283.

[8] Nallino, *op. cit.*, Vol. II, p. 244.

[9] Wright, R. R., *Elements of Astrology by al-Bīrūnī*, London, 1934, p. 97.

The author of the Persian manuscript directs that the various deferents be drawn permanently on the plate, each circle in a different color, further to distinguish between planets. The *center of the equant* (*markaz-i falak-i mo'addel al-masīr*) of each planet is also to be marked on the plate, on the extension of the line from the plate-center to the deferent-center, the distance between the equant-center and the deferent-center to equal the eccentricity of the deferent.

As in the case of the lunar theory, marks are made on the edge of the alidade known as *difference-marks* (*arqām-i ekhtelāf*), one for each planet, at a distance from the midpoint of the alidade equalling the radius of the planet's epicycle as given in Column 7 of Table One above.

3. ***Alternative Plate Layouts.*** — In the *Nuzha*, al-Kāshī does not content himself with the arrangement of the plate and alidade given above. In a fashion reminiscent of Naṣīr ed-Dīn at-Ṭūsī enumerating all the special cases of a theorem in geometry, he proceeds to describe variants of the basic scheme. For example, instead of letting the plate center represent the center of the universe for all the planets, it is possible to make the deferent centers all coincide with the center of the plate. Then for each planet (except Mercury) the points representing the center of the universe and the equant center are placed symmetrically on the line of apsides on either side of the plate center. A plate laid out thus is said to be of the *parallel-zoned* (*motawāzī al-manāṭeq*) type.

Again, if no objection is made to the deferents intersecting, and if it is wished to retain the plate center as center of the universe for all planets, full advantage may be taken of the size of the plate to choose scales such that all deferents will just touch the edge of the plate. This is equivalent to demanding that the sum of the eccentricity and the deferent radius equal 1,0. On page 27 of the *Nuzha* a table gives the resulting constants shown in Table Two (except for Saturn's epicycle radius, which reads 6;9). All units are sixtieths of the plate radius.

TABLE TWO

Planet	Eccentricity	Deferent Radius	Epicycle Radius
Saturn	3;14	56;46	6;0
Jupiter	2;38	57;22	11;0
Mars	5;27	54;33	36;38
Venus	1;2	58;58	42;25
Mercury	5;13	55;3	19;34

The first and third columns norm to the values given in Columns 8 and 11 of the preceding table.

Thirdly, in the *united-zones* (*mottaḥed al-manāṭeq*) type, the edge of the plate itself is made to serve as the deferent for all the planets whose deferents are circular, just as was the case with the sun. For each planet this again pushes the center of the universe out along its individual apsidal line, the eccentricities and epicycle radii being the normed entries in Columns 8 and 11 of the first table above.

Finally, the plate may be laid out with the same apogee for all the planets, and the position of the apsidal line taken into consideration by a separate setting of the plate within the ring for each planet. Or, since all the apsidal motions are slow, the movable ring may be dispensed with altogether, the apsidal lines being laid out once and for all to a fixed position in the zodiac correct for the time when the instrument is made.

Al-Kāshī enumerates some fifteen combinations of these various possibilities.

4. *The Deferent of Mercury.* — Ptolemy's model for the motion of Mercury [9a] differs somewhat from his theory for the other planets. Instead of assuming a circular deferent around which the epicycle center moves with a constant angular velocity as viewed from the equant, he makes the deferent-center itself (D on Figure 1) move on the arc of a circle with fixed center F and radius equal to three sixtieths of the deferent-radius. The equant is fixed at E in such fashion that E is colinear with FC, where C is the center of the universe. Also, $CE = EF = FD$. Moreover, at all times $\theta_2 = \theta_1$, EP rotating so that it always remains parallel to the radius vector from C to the mean sun. PD, being a radius of the moving deferent, maintains a constant

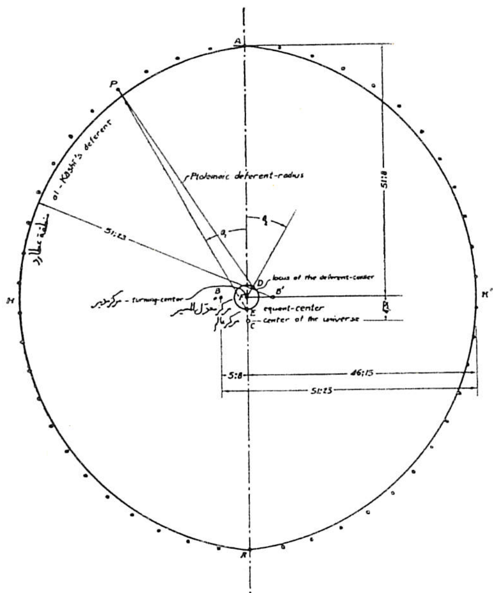

FIG. 1

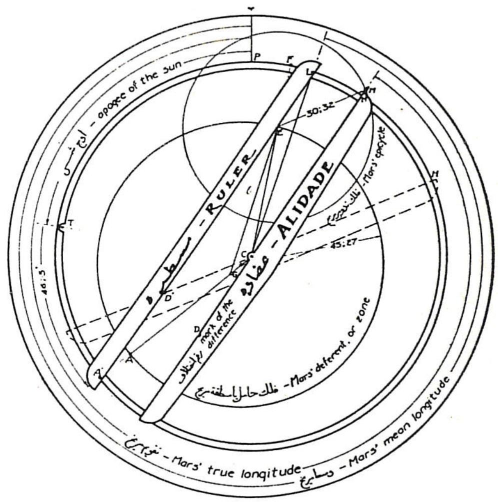

FIG. 2

length, and as FD and EP rotate in opposite senses, P traces out an oval, egg-shaped curve, symmetrical with respect to AR, but slightly pointed in the direction of R. Successive positions of P are indicated by the small circles on the figure.

Apparently al-Kāshī wanted to simplify the construction. He therefore substitutes for the Ptolemaic transcendental locus of P an approximation to it by means of a pair of circular arcs. For the author of the Persian manuscript says:

> . . . then from the *turning-center* (*markaz-i modīr*, F) make to go out from the apogee line (RA) a line perpendicular to it (BB'), and from this center (F) on the two sides of the apogee line, left and right, place two marks (B and B') at a distance of 5;8, and taking each of these two marks as a center and a radius of 51;23, describe two arcs (AHR and $AH'R$) so that an oval-shaped figure results, of which half its larger diameter (FA) is 51;8 and half its shorter diameter (FH) is 46;15. This is the pivot of the center of Mercury's epicycle, and we call it the *zone* (*manṭaqeh*) of Mercury. . . .

An internal check that these figures have been correctly read is given by the computations

$$BH' - BF = 51;23 - 5;8 = 46;15 = FH'$$

$$\sqrt{(BA)^2 - (BF)^2} = \sqrt{(51;23)^2 - (5;8)^2} = \sqrt{43,33;53,45} = 51;8 = FA,$$

rounded off to two sexagesimal places.

[9a] *Almagest*, Bk. X, Halma, Vol. II, pp. 166–193.

In Figure 1 the scale of the Ptolemaic locus was so chosen that apogee (A) and perigee (R) would coincide with the corresponding points on al-Kāshī's deferent. It will be observed that the latter is a good approximation to the Ptolemaic curve, the two loci practically coinciding for wide distances on either side of H and H'. Al-Kāshī puts the center of Mercury's equant (E) midway between the turning center (F) and the center of the universe (C). This corresponds to the Ptolemaic arrangement.

The entry for Mercury under Column 16 of Table One above is 56;0, the sum of the eccentricity, 4;52, and the semi-major axis of the deferent, 51;8. Since the corresponding normed distance in Ptolemy's configuration is 1,9;0 (the distance from the center of the universe to the deferent apogee), al-Kāshī's norming coefficient for Mercury is the ratio of these two numbers, 1,9/56 = 1;13,56.

The value of Mercury's epicycle radius given as 18;13 in Column 7 of the same table could equally plausibly be read 18;18 in the *Nuzha*. This would norm to 22;32, a number about as far above Ptolemy's 22;30 as the given value is below it.

For the alternative plate layouts described above, the shape of Mercury's deferent remains unchanged, but the scale is altered to suit. Thus, for the variant in which all apogees lie on the circumference of the plate the semi-major axis of the deferent is increased to 54;47. This, added to the eccentricity shown in the second table, gives the desired 1,0, the plate radius. The other constants are altered proportionately.

Again, in the "united-zones" type the plate circumference can not be used as the deferent for Mercury as it is for the other planets. Al-Kāshī does the next best thing by making the semi-major axis of the oval equal to 1,0. This forces a corresponding change in the eccentricity to 5;43 and in the epicycle radius to 21;26.

Other, though later, examples of oval deferents actually plotted for Mercury are given by Dreyer.[10] In the instrument of az-Zarqālī mentioned in **9** below, Mercury's deferent is drawn by plotting the Ptolemaic locus pointwise as in Figure 1, and connecting the resulting set with arcs.

5. *True Longitude of the Planets.* — Assuming now the deferents and equant-centers permanently engraved on the instrument, a table of mean longitudes of the sun and the three superior planets, and the *compound anomaly* (*khāṣṣeh-i morakkabeh*, defined below) of the two inferier planets, the instrument can be used to find the true longitude of any one of the planets at any given time. Figure 2 shows the final positions of the alidade and ruler in the solution of a typical such problem involving the planet Mars. Presumably the plate has previously been set and fixed in the ring so that the tongue (T on the figure) is at the longitude of the sun's apogee. Then A, the apogee of Mars, will be at its proper longitude.

Following now the directions of the author of the Persian manuscript, extract from the tables of mean motions (given in the manuscript) the mean longitude of the planet for the time desired, and turn the alidade until its edge shows this mean longitude (L) on the divisions of the ring. Put the ruler so that its edge is parallel to the edge of the alidade and passes through the equant-center, G. Mark E, that intersection of the ruler's edge and Mars' deferent which gives the directed segments CL and GE the same sense. The author calls E the *center-mark* (*'alāmat-i markaz*). It is in fact Mars' epicycle center for the given time.

Now, and in the case of the superior planets generally, put H, the *head* (*ra's*), of the alidade on the *mean longitude of the sun* (*vasaṭ-i shams*), and put a mark D' on the plate at the point where the difference mark (cf. **2** above) then lies. The author's statement is here somewhat ambiguous; by *head* he evidently means that side of the alidade edge opposite the side carrying the difference mark, D. Now put

[10] *Op. cit.*, p. 274.

the ruler so that its edge lies along $D'E$, and turn the alidade until it is parallel to the ruler. Then the intersection M' of the head of the alidade with the divisions of the ring gives the true longitude of Mars. For the vector $D'C$ has been constructed in magnitude and direction equal to the vector EM from the epicycle-center to the planet, and side CM of parallelogram $CD'EM$ gives the direction of the vector sum of CE and $D'C$. And since EM is parallel to CH, EM has the required direction of the mean sun.

For the inferior planets, Venus and Mercury, the construction is of the same sort, bearing in mind that their mean longitude is the mean longitude of the sun. The same figure may be used to illustrate the configuration, although of course it is no longer to scale. Now L will be the sun's mean longitude and arc $PA'H$ the planet's *compound anomaly*. This latter term is not common in Islamic astronomical works.[11] It is defined as the sum of the longitude of the mean sun and the mean epicyclic anomaly of the planet.

6. *The Planetary Equations.* — In the following, the word *equation* is used in the astronomical sense as meaning the amount by which a celestial object varies from its mean position. Thus the (total) equation in longitude of Mars for the situation shown in Figure 2 is the arc LM'. It was customary in antiquity to separate the total equation into two components. The *first equation* (*ta'dīl-i avval*) or *equation of the center* is the part caused by the eccentricity of the deferent orbit. The *second equation* (*ta'dīl-i thānī*) is the irregularity caused by the travel of the planet in the epicycle. Clearly, the algebraic sum of the first and second equations equals the total equation.

The author of the Persian manuscript remarks that a knowledge of these equations is not necessary for finding planetary longitudes with the instrument, but that if anyone does want them he should follow the procedure then described. In the computation of tables of planetary positions the two equations were used, and he may have had in mind the needs of people wanting a quick graphical check for these elements in tables in their possession.

For whatever reason, put a *first mark* (L), he says, on the edge of the ring at the mean of the planet. Rotate the alidade until its edge passes through E, and put the *second mark* (F) at the point where its edge crosses the ring He calls F the *corrected mean* (*vasaṭ-i moa'ddel*). This is also a technical term not previously encountered by the present writer.[12] Then arc LF is the first equation and FM' the second. This is correct, for F is indeed the longitude the planet would have if the only irregularity in its motion were that of the center. And secondly, $LF + FM' = LM'$, the total equation.

In this connection the author remarks that if the longitude of the apogee (A') of a planet is subtracted from its corrected mean (F), the remainder, arc $A'HF$ is the *true center* (*markaz-i moa'ddel, centrum verum*[13]). It is as though he refers the motion in the deferent to the intrinsic coordinates of the planet itself, that is, the system determined by its own apogee rather than by the vernal equinox. The concept of true center is used later in the manuscripts in the process of finding planetary latitudes.

Finally, he states that if the corrected mean of any one of the superior planets is subtracted from the mean of the sun, or if the corrected mean of either of the inferior planets is subtracted from its compound anomaly, the *true anomaly* (corrected or equalized anomaly, *argumentum verum, khāṣṣeh-i moa'ddeleh*) of that planet remains.

[11] But cf. Nallino, *op. cit.*, Vol. II, p. 344, *al-ta'dīl al-morakkab.*

[12] But cf. Wright, *op. cit.*, p. 95.

[13] Cf. Nallino, *op. cit.*, Vol. II, p. 335.

Both these statements are correct. On Figure 2 they are both equivalent to the valid relation between the three arcs

$$PA'H = PA'F + FA'H.$$

7. *The Lunar Equations.* — As for the moon, the nomenclature is in a sense reversed; the *first lunar equation* is the difference in longitude between the mean moon and an object moving on an epicycle whose center is at constant distance from the center of the universe. The *second equation* is the added irregularity caused by the *prósneusis*, the effect of the moving eccentric orbit.

In the passage of both manuscripts in which the planetary equations are discussed a single sentence has to do with the analogous lunar problem. In translation it is "The first equation of the moon shall be to the amount between the second mark and the mark of the beginning of the anomalistic motion. . . ."

This seems to make little sense, for in the lunar theory the two marks coincide if defined as in **6** above, the first being the longitude of the mean moon, the second the longitude of the moon's epicycle center. The *mark of the beginning of the anomalistic motion* is not, as one might suppose, the epicycle apogee; it is *B* in the figure of the first paper of the series. The arc apparently referred to by al-Kāshī is *PB*, which measures the angular displacement in the epicycle apogee caused by the *prósneusis*. This is neither one of the two equations.

The instrument could easily be used for finding the lunar equations, however. Referring again to the figure of the preceding paper, from *P* lay off counterclockwise on the divisions of the ring an arc *PAK'*, say, equal to *a*, the moon's anomaly. Rotate the alidade until the edge is on *K'* and mark *D''*, the new position of the *mark of the difference*. Lay the ruler along *PD''* and rotate the alidade until its edge is parallel to the ruler. Then the intersection of the head of the alidade with the edge of the ring, say *H*, is the longitude the moon would have if it were subject only to the displacement of the apogee, and arc *PH* is the first equation of the moon. Since the longitude of the true moon is *G*, arc *GH* is the second equation.

8. *Planetary Distances.* — Having set up to scale on the instrument the Ptolemaic configuration shown in Figure 2 for finding the true longitude of a planet at a given time, it follows immediately that, if measured to proper scale, the radius vector *CM* gives the distance of the planet from the center of the universe at that time. In practise (as in the present example) *M* will frequently fall outside the edge of the plate. But the author points out that the line *D'E* will always lie wholly on the plate, and its length, which equals *CM*, may be measured in the divisions of the ruler. In order to norm the resulting distance, he directs that it be multiplied by the appropriate coefficient in Column 14 of Table One in **3** above.

9. *"El Libro de las Láminas."* — In the 13th century the astronomically-inclined Don Alfonso el Sabio, King of Castille, had a commission of Jewish and Christian scholars prepare in Spanish a compilation of Arabic astronomical works since known as the *Libros del Saber de Astronomía*. They were edited and published in Madrid in 1864 by Don Manuel Rico y Sinobas. One of the treatises [14] is in two parts, the Arabic original of the first being a work of Abulcacim Abnacahm (Abūlqāsim ibn Moḥammad ibn as-Samḥ [14]) of Grenada. It is probably incomplete, for, having described how to make his "Láminas de las VII Planetas," the author gives no indication of their application beyond saying that they may be used for finding the true positions of the planets. One plate is used for each planet, the deferent being laid

[14] Vol. III, pp. 239–284; cf. also Suter, *Die Math. und Astr. der Araber und ihre Werke*, Leipzig, 1900, p. 85.

out on it much as in the *Plate of Zones.* Values of the eccentricity and epicycle radius are in general identical with those of Ptolemy; the planetary apogees are al-Battānī's with a correction for precession.

The author of the second treatise is Abnizac el Zarquiel (Azarquiel, Ibrāhīm ibn Yaḥyā abū Isḥāq ibn az-Zarqālī [15]). He proposes to dispense with most of the plates used by his predecessor by consolidating all the deferents on one plate and all the epicycles on another. In the archaic Castilian of the publication, "se puede fazer una lámina con que se puedan escusar las VII laminas que auemos dicho en la primera parte."

Hence differing scales are chosen for all the planets so that the deferents nest on the plate in much the same fashion as is described in **2** above. It had been hoped that undoubted influence of the Spanish inventors on al-Kāshī would be shown by the use of the same scales or parameters, but this is not the case. In fact, the scales cannot be comparable because on az-Zarqālī's instrument each deferent has a concentric and slightly smaller circle inside it, the ring space in between being used for marking the divisions into which the deferent is divided. The successive deferents are drawn with such scales that the graduated rings just touch, instead of the deferents themselves touching as in the *Plate of Zones.* To find the true longitude of a planet, mark the position of the epicycle center on the proper point of the deferent; place the small plate with the epicycles on top the deferent plate (both are to be made of parchment or other thin material) so that the epicycle center coincides with the mark just made. Turn the epicycle plate until the epicyclic apogee is in correct position, and mark the proper value of the anomaly on the graduated epicycle circumference. Turn the alidade until it passes through the mark just made; its position then gives the true longitude of the planet.

10. Conclusion. — Al-Kāshī's instrument is now seen to be part of an extensive and more or less continuous development of mechanico-graphical scale models of the Ptolemaic system. This development was already well under way in classical times. Bronze fragments of what was probably a Greek planetarium of about 30 B.C. have been recovered from the Mediterranean sea bottom off the island of Antikythera.[16] The existence of a planetarium invented by Archimedes is well attested.[17] The *Hypotyposis Astronomicarum Positionum* of Proclus Diadochus (*c.* 450 A.D.) contains [18] a careful description of a device for finding the longitude and equation of the sun by a method not differing essentially from that of the *Plate of Zones.* The trend persisted into the sixteenth century; witness the work of Petrus Apianus.[19] In all this the work of al-Kāshī is of considerable merit. His elegant constructions for the latitude and eclipse theory carry the general methods into branches of astronomical theory where they had not previously been applied.

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[15] Suter, *op. cit.*, p. 109.

[16] Svoronos, J. N., *Das Athener Nationalmuseum*, Athens, 1908, Textband 1., pp. 43–51.

[17] See, for example, Schlachter, *Der Globus*, Berlin, 1927, p. 49.

[18] Edition of Manitius, Leipzig, 1909, p. 75.

[19] *Astronomicum Caesareum*, Ingolstadt, 1540.