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Seminars

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Order Types: Points, from Geometry to Combinatorics

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Prologue

The aim of this talk is to provide a framework that could be used to attack problems like the following ones.

Example 1.0.1 (The Asymptotic Number of Triangulations). It is well known that the number of triangulations of a convex n point set equals the Catalan number C_{n-2}.

What are (upper and lower) bounds for the maximal and minimal numbers of triangulations of any n point set [3]?

Example 1.0.2 ((Empty) Convex k-gons). We say that a point set S contains a convex k-gon if there exist k points of S that are in convex position. A k-gon (contained in S) is called *empty* if the convex hull of the k-gon does not contain any points of S in its interior.

What is the least number of (empty) convex k-gons determined by any set S of cardinality n [8]?

Example 1.0.3 (Reflexivity). Consider a point set S and a simple polygonization P of S. We denote by $\rho(S)$ the minimum number of reflex vertices in P over all polygonizations P of S, and by $\rho(n)$ the maximum of $\rho(S)$ over all sets S with n points.

How large is $\rho(n)$, or what are upper and lower bounds [14]?

Example 1.0.4 (Crossing Families). A crossing family (of size k) of a point set S is a set of k line segments spanned by points in S which pairwise intersect.

What is the minimum number $n(k)$ of points such that any point set of size at least $n(k)$ admits a crossing family of size at least k?

1.1 Introduction

Many problems in computational and combinatorial geometry are based on finite sets of points in the Euclidean plane. Of course, there exist infinitely many different sets of n points, for a given number n.

However, a quite large class of problems is determined already by the combinatorial properties of the underlying n-point set, rather than by its metric properties. To be more specific, look at all the $\binom{n}{2}$ straight-line segments spanned by a given n-point set, as in Figure 1.1. Which of these line segments cross each other, and which do not, turns out to be important: point sets with the same crossing properties give rise to equivalent geometric structures. This is valid for many popular structures, like spanning trees, triangulations, simple polygonizations (crossing-free Hamiltonian cycles), k-sets, and many others.

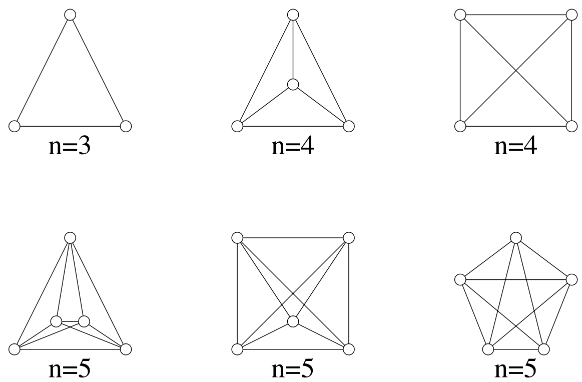


Figure 1.1: Inequivalent sets of small size. There is only one (combinatorial) way to place 3 points, two ways to place 4 points, and three to place 5. The situation changes drastically for larger cardinality.

For several problems, like counting the number of triangulations for a point set, no efficient algorithms are known. For others, like for k-sets, the combinatorial complexity is still unsettled. Sometimes even the existence of a solution has not yet been established, such as the question of whether any two given n-point sets (with the same number of extreme points) can be triangulated in an isomorphic manner.

Remark 1.1.1. The above mentioned question is closely related to the research problem about isomorphic pointed pseudo-triangulations.

To gain insight into the structure of hard problems, the study of instances that are typical and/or extreme is often very helpful. For example, detecting a counterexample prevents from constructing hopeless proofs. Conversely, the non existence of counterexamples of small size gives some evidence for the truth of a conjecture. This motivates the complete enumeration of all problem instances of small size. For the problems mentioned above, this means to investigate all inequivalent sets of points where equivalence is with respect to the crossing properties of the complete geometric graph spanned by the set. It is well known that crossing properties are exactly reflected by the order type of a point set, introduced in Goodmann and Pollack [10]. Some years ago, a complete data base of all realizable order types of size ≤ 11 and of according inequivalent point sets in general position has been established [2].

The goal of this talk is to first explain what order types are and how they can be efficiently represented (Section 1.2). Next, to give a short overview of why it is complicated and how it is possible to create the order type data base (Section 1.3). And finally, to provide some examples of successful application of the data base (Section 1.4).

Remark 1.1.2. Parts of the framework presented here have been developed in the PhD thesis of Hannes Krasser [11], which we refer to for further reading. Most of the figures in this talk are taken from there.

1.2 Order types

The *order type* of a labeled set $S = \{p_1, \ldots, p_n\}$ of points in general position[1] is a mapping that assigns to each (ordered) index triple i, j, k in $\{1, \ldots, n\}$ the orientation —clockwise or counter-clockwise— of the point triple p_i, p_j, p_k.

Two point sets S_1 and S_2 are said to be *(combinatorially) equivalent* if they exhibit the same order type. That is, there is a bijection between S_1 and S_2 such that each triple in S_1 agrees in orientation with the corresponding triple in S_2; see Figure 1.2 for an example.

Now how can we decide whether or not two point sets are equivalent? To answer this question, we need a unique representation of the order type of a point set, meaning one that is independent of the labeling.

Given a labeled point set $S = \{p_1, \ldots, p_n\}$, the λ-*matrix* is an $n \times n$ matrix where $\lambda(i, j)$ contains the number of points in S that lie to the left of

[1] All point sets in this talk are assumed to be in *general position*, meaning that no three points are collinear.

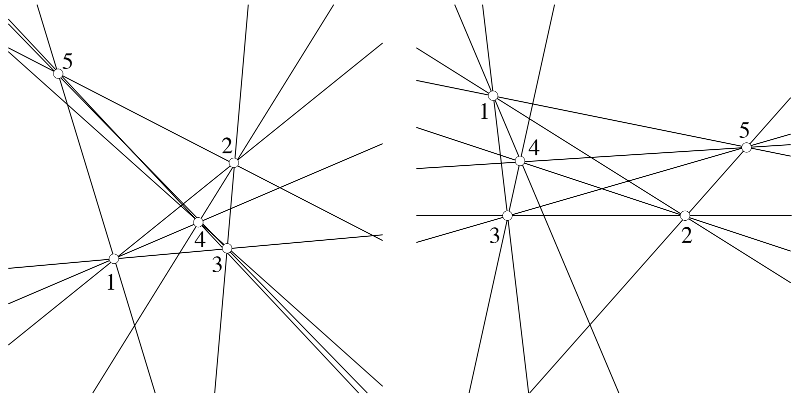
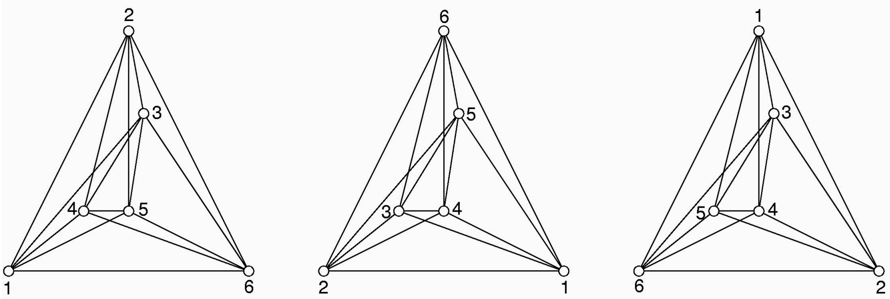


Figure 1.2: Two equivalent sets of 5 points.

the oriented line through p_i and p_j. Goodman and Pollack [10] show that the order type and the λ-matrix encode exactly the same information.

Given a point set S, we (re)label the points of S in a way that p_1 lies on the convex hull and that $p_2, \ldots, p_n$ are sorted clockwise around p_1; see Figure 1.3. From these *natural λ-matrices* (one for each possible choice of p_1) we choose the lexicographically minimal one as the unique representation of the according order type, also called the *fingerprint* of the point set. In the example in Figure 1.3, the leftmost natural λ-matrix is the fingerprint for this set.



–	0	1	2	3	4
4	–	1	3	2	0
3	3	–	2	1	1
2	1	2	–	2	3
1	2	3	2	–	2
0	4	3	2	1	–

–	0	1	2	3	4
4	–	2	3	1	0
3	2	–	2	2	1
2	1	2	–	3	2
1	3	2	1	–	3
0	4	3	2	1	–

–	0	1	2	3	4
4	–	3	2	1	0
3	1	–	1	2	3
2	2	3	–	2	1
1	3	2	2	–	2
0	4	1	3	2	–

Figure 1.3: Labellings and according natural λ-matrices of a point set with 5 points.

Remark 1.2.1. There is an error in one of the matrices in Figure 1.3. Find it!

Observation 1.2.2 (Properties of the natural λ-matrices).

- If $\lambda(i, j) = 0$, then p_i and p_j lie on the convex hull.
- $\lambda(i, j) + \lambda(j, i) = n - 2$.
- p_1 lies on the convex hull of S.
- $p_2, \ldots, p_n$ are sorted clockwise around p_1.
- The fingerprint is the lexicographic minimum of all λ-matrices.

1.3 Generating and realizing order types

If we want to make a data base of point sets for all order types (up to some small constant cardinality), there are two major goals to meet: First we have to guarantee that we do not miss any order type, and second we want to ensure that we do not have any duplicates.

In principle, what we want to do for generating all order types (up to some small constant cardinality) is to use an iterative approach which we call *complete order type extension*: For every order type T_n of n points, we generate all different order types T_{n+1} of $n+1$ points that contain T_n as a sub order type.

1.3.1 Geometric order type extension

The intuitive approach for this would be to do this extension geometrically: For every order type T_n, take a point set S that represents T_n. Consider the *arrangement of lines*[2] spanned by the points of S, and place the additional point in (combinatorially) all possible ways, meaning in each of the cells of the arrangement.

Unfortunately, this approach is *the* canonical erroneous approach: It does not guarantee to result in the complete set of order types T_{n+1}. Two different representations of the same order type can still span line arrangements with different cells; see Figure 1.4.

Stated the other way round, not every extension can be derived from every geometrical representation of the same order type, as you can easily see in Figure 1.5.

[2] A *line arrangement* is the dissection of the plane induced by a set of straight lines. A line arrangement is *simple* if no three lines pass through the same point and no two lines are parallel.

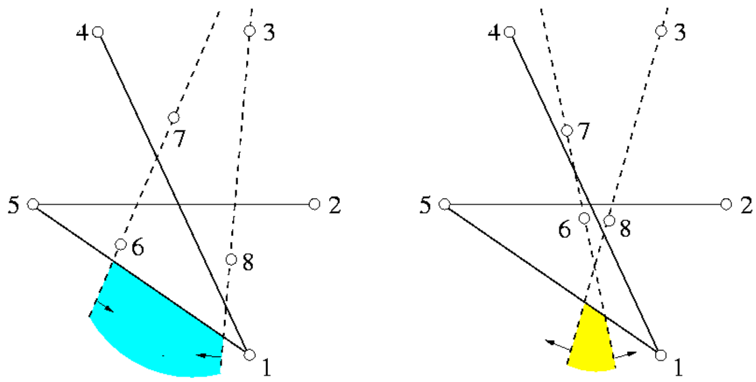


Figure 1.4: Arrangements of two point sets with the same order type. The cell marked blue in the left arrangement does not exist in the right arrangement.

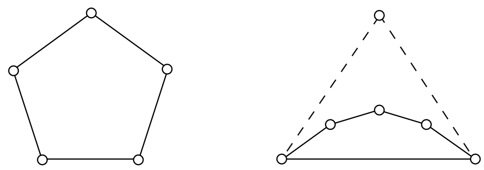


Figure 1.5: Although both shown 5 point sets have the same order type, there is no way to add the sixth point from the right to the set on the left and obtain the same order type.

1.3.2 A duality transformation between points and lines

Our (successful) approach [11] to generate all order types makes use of the duality of point sets (in general position) and (simple) line arrangements in the Euclidean plane. The duality transformation T used is the *unit circle duality*. It matches a point p to the line $T(p) = l$ which fulfills the equality $p^t l = 1$. This transformation has the property that it preserves point-line incidences and order:

$$\begin{aligned} p \in l \quad &\Longleftrightarrow \quad T^{-1}(l) \in T(p) \\ p \in l^+ \quad &\Longleftrightarrow \quad T^{-1}(l) \in T(p)^+ \end{aligned}$$

Remark 1.3.1. This transformation translates order types into local intersection sequences, and order type extension into adding a line to an arrangement (again in all possible ways).

1.3.3 Abstract order type extension

Like in the primal setting, no direct way is known to enumerate all line arrangements with $n + 1$ lines that contain a given line arrangement with n points. Thus, for the extension we first produce all (non-isomorphic) arrangements of pseudolines. A set of *pseudolines* is a set of simple curves which pairwise cross at exactly one point. Figure 1.6 shows an example.

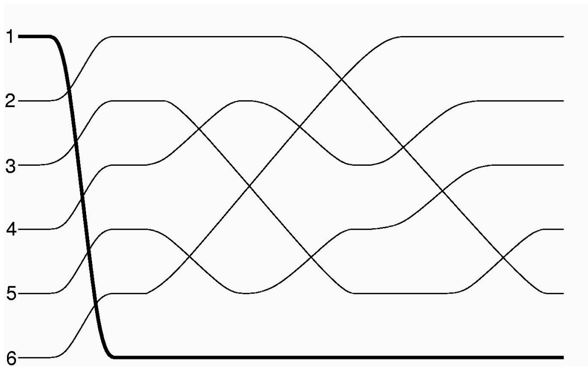


Figure 1.6: A pseudoline arrangement (?) that occurs for order type extension.

Remark 1.3.2. Figure 1.6 happens to contain an error (and thus is not really a pseudoline arrangement). What is wrong?

Handling pseudolines is relatively easy in view of their equivalent description by *wiring diagrams* (see e.g. Goodman [9] and Figure 1.7). We can read off a corresponding *abstract order type* (also called *pseudo order type*) from each wiring diagram: the order in which the wires cross each other determines the orientations for all index triples.

Back in the primal setting, where each wire potentially corresponds to a point, this leads to a list of candidates guaranteed to contain all different order types.

1.3.4 Realization of abstract order types

We are left with the problem of identifying all the *realizable* order types in the obtained list, that is, those for which corresponding point coordinates do exist. Here we enter the world of *oriented matroids*, an axiomatic (combinatorial) abstraction of geometric objects, introduced in the late 1970s.

The question whether or not an abstract order type is realizable is equivalent to the question of whether or not a wiring diagram is *stretchable*. As a known phenomenon, a wiring diagram need not be stretchable to straight lines.

In fact, there exist non-stretchable diagrams already for 8 pseudolines (see e.g. [6]). Moreover, deciding stretchability for a wiring diagram is NP-hard [6].

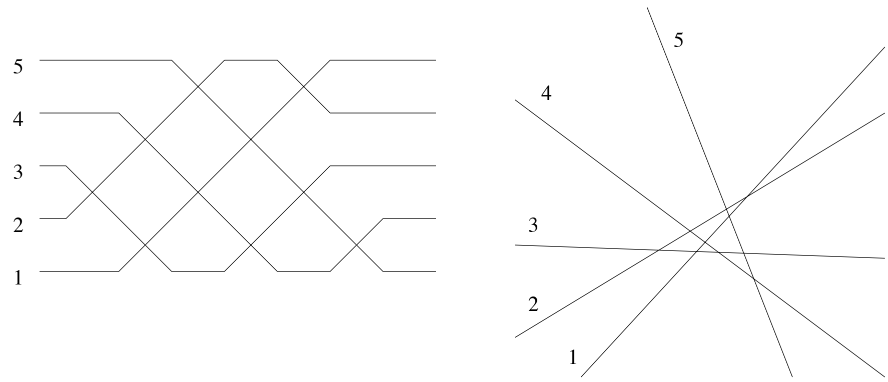


Figure 1.7: A wiring diagram that can be stretched.

As a consequence, our candidate list will contain *non-realizable* abstract order types for size $n \geq 9$. Moreover, even if realizability has been recognized for a particular candidate, we still have to find a corresponding point set.

We have two steps to attack this problem. First, and surprisingly, the problem gets easier by transforming it into projective geometry. Unlike before, order types now directly correspond to arrangements of great circles by duality (and isomorphism classes of pseudo-circle arrangements coincide with reorientation classes of rank 3 oriented matroids). Moreover, the desired order types in the plane can be nicely grouped into *projective classes*, and in every class either each or none of the order types is realizable.

And second, we apply heuristics for both proving realizability (geometrical insertion and local optimization) and non-realizability (linear systems of inequations derived from Grassmann–Plücker relations [7]) to these classes.

1.3.5 The data base

Happy ever after (and after quite some time of calculating), for every instance of the problem (up to 11 points) one of the heuristics provided an answer. As a result, we now have a complete data base of point sets for all (realizable) order types for sets of up to 11 points.

Table 1.1 gives an overview of the resulting numbers of order types. The tremendous growth of order types, especially in the Euclidean case, becomes apparent. By the nature of our approach, we computed a combinatorial description for all the objects counted in Table 1.1, along with a geometric representation if the object is realizable.

number of points	4	5	6	7	8	9	10	11
Euclidean abstract order types	2	3	16	135	3 315	158 830	14 320 182	2 343 203 071
thereof non-realizable						13	10 635	8 690 164
Euclidean order types	2	3	16	135	3 315	158 817	14 309 547	2 334 512 907
projective abstract order types	1	1	4	11	135	4 382	312 356	41 848 591
thereof non-realizable						1	242	155 214
projective order types	1	1	4	11	135	4 381	312 114	41 693 377

Table 1.1: Numbers of different order types of size n.

Up to and including point sets of cardinality 8, the data is stored with 8 bits per coordinate, while for 9 to 11 points 16 bit coordinates are necessary. The whole data base up to and including sets of cardinality 10 needs just 550 MB (which fits on a CD which is available upon request), while for all sets of 11 points the data needs close to 100 GB (sorry, no CDs are provided for this ;-).

1.4 Applications

There are several different motivations one might have for using the data base. It can be useful to find a counterexample to a conjecture (thus preventing from constructing hopeless proofs). The non existence of counterexamples of small size might give some evidence for the truth of a conjecture. Also, knowing the exact answer to a question for small constant problem size can serve as an induction base for proving asymptotic bounds. In any case, examples are helpful to obtain structural insight into a problem, possibly also leading to new observations or conjectures. The list of problems that have been investigated includes the following:

- rectilinear crossing number
- crossing families
- Happy End Problem
- convex k-gons
- empty convex k-gons
- convex covering
- convex partitioning
- convex decomposition
- number of triangulations
- flip distances
- isomorphic triangulations
- sequential triangulations
- minimum pseudo-triangulations
- Hamiltonian cycles
- minimum reflex polygonizations
- spanning trees
- crossing-free matchings
- k-sets
- Hayward Conjecture
- and many more...

1.4.1 Crossing families

Let us come back to Example 1.0.4 from the very beginning: Recall that a *crossing family* (of size k) of a point set S is a set of k line segments spanned by points in S which pairwise intersect. We want to know the minimum number $n(k)$ of points such that any point set of size at least $n(k)$ admits a crossing family of size at least k.

For $k = 2$ the answer is $n(2) = 5$, which can be easily seen from the fact that the complete graph K_5 cannot be drawn without a crossing, while K_4 still admits a plane drawing.

For $k = 3$ it could be shown with the help of the data base that every point set with $n \geq 10$ points spans at least 3 pairwise crossing edges, which is tight (meaning that there exist sets of 9 points without a 3-family). The best previous known bound on this was $n(3) \leq 37$ [13].

This result also plays a crucial role in deriving the first non-trivial (and up to date still the best) asymptotic lower bound on the number of triangulations every n point set must have, which partly solves the problem stated in Example 1.0.1: In [5] the authors prove that for every n point set there always exist $\Omega(2.33^n)$ triangulations.

1.4.2 Ramsey type results

In Example 1.0.2 we look for (empty) convex k-gons in point sets. The question of finding the number $f(k)$ such that every set with $\geq f(k)$ points contains a k-gon was stated by Erdős ans Szekeres in 1937. Erdős conjectured that $f(k) = 2^{k-2} + 1$, and by now the best known bound is $f(k) \leq \binom{2k-5}{k-2} + 2$.

Empty k-gons need not exist for $k \geq 7$: There are arbitrary large sets not containing any empty convex 7-gon. For $k = 6$ it is known that at least

30 points are needed and that 1717 points are for sure sufficient.

If a point set is large enough to contain (empty) k-gons, one could as well ask how many there are. It easy to see that every set with cardinality 5 contains an empty convex 4-gon, and not necessarily more than one (even if it need not be empty).

5-gons are guaranteed to exist in any set with at least 9 points, and there are empty 5-gons in every set of cardinality at least 10. It has been shown with the help of the data base that every set with 10 points in fact contains at least two empty 5-gons, while there are sets with 9 points that contain only one (not necessarily empty) 5-gon.

Let us state two more Ramsey type results of this flavor that have been observed from the data base. For both of them, a human readable proof has been provided afterwards; see [4].

- Every set of 8 points contains either an empty convex pentagon or two disjoint empty convex quadrilaterals.
- Every set of 11 points contains either an empty convex hexagon or an empty convex pentagon and a disjoint empty convex quadrilateral.

Remark 1.4.1. Ramsey type results for small constant size objects might lead to improved results for arbitrary size point sets (e.g. when used as a basis for divide-and-conquer approaches). For example, the results from above have been successfully used to improve asymptotic bounds for convex decompositions.

1.4.3 Reflexivity

Recall Example 1.0.3 from the beginning: Considering simple polygonizations of point sets, we denote by $\rho(S)$ the minimum number of reflex vertices in P over all polygonizations P of S, and by $\rho(n)$ the worst case number for all sets of size n (the maximum of $\rho(S)$ over all sets S with n points).

Question 1.4.2. How large is $\rho(n)$, or what are upper and lower bounds?

First of all, $\rho(S) = 0$ is equivalent to S being a convex set. It has been known that

$$\lfloor \tfrac{n}{4} \rfloor \leq \rho(n) \leq \lceil \tfrac{n}{2} \rceil.$$

Using the data base we could show that $\rho(8) = 2$ and $\rho(11) = 3$ [1]. Again: exact values for small constants are often useful for asymptotic proofs. In this case the upper bound for the reflexivity of point sets could be improved to

$$\rho(n) \leq \frac{5n}{12} + \mathcal{O}(1).$$

Remark 1.4.3. This question is Open Problem number 66 of *The Open Problems Project* [14] (of discrete and computational geometry). If you do not know this project by now, have a look!

Epilogue

We have seen that the order type data base is useful, and that it can be applied

- to get (counter) examples for cardinality ≤ 11, and
- to obtain a basis for asymptotic results.

Now what can we do if we need special examples, say for 12 or 13 points, because 11 is just a little too small?

Here complete order type extension is definitely intractable: For cardinality $n = 12$ there are about 750 billion order types, and for calculating and storing this data one would need at least 30 terabytes of disk space, and at least 200 years of computation time. But there is something that can be done: It is possible to do *partial extension* on *suitable sets*.

Say that, for the problem we want to consider, only sets with special properties. Say further that these properties are such that a set of n points with a required property has at least one subset of $n-1$ points with an according property (we call this the *subset property*). Then we can take all sets for $n = 11$ which fulfill the subset property and extend only these sets to $n = 12, 13, \ldots$

For example, to generate all sets of n points with at most $t(n)$ triangulations, start with $(n-1)$-sets with at most $t(n)/2$ triangulations.

Also, there is no need to realize all the sets we generate. Instead, we only realize special sets like counterexamples or extreme configurations.

Remark 1.4.4. The above approach can easily be modified to use *distributed computing* and take advantage of the tremendous computing power of thousands of computers all over the world. For example, for the rectilinear crossing number it was possible to process all relevant sets up to 20 points.

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Seminar 2

Roots of the Steiner Polynomial

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1.1 Some basic concepts

We introduce the following notation and terminology:

1. $\mathcal{K}^n$ denotes the set of n-dimensional convex bodies.
2. K and E denote elements of $\mathcal{K}^n$.
3. B_n denotes the unit ball.
4. $V(K)$ denotes the usual volume of K.
5. $+$ is the Minkowski addition.
6. For a convex body $K \in \mathcal{K}^n$ and $\lambda \geq 0$, we define $K_\lambda = K + \lambda B_n$, the *outer parallel body* of K at distance λ.
7. We have that $V(K_\lambda) = \sum_{i=0}^n \binom{n}{i} W_i(K)\lambda^i$, and call it the *Steiner polynomial*, with $W_i(K)$ the *quermassintegrals* of the body K. Some of these quermassintegrals have a special meaning:
   (a) $W_0(K) = V(K)$.
   (b) $nW_1(K) = S(K) =$ Surface area of K.
   (c) $nW_2(K) = M(K) =$ Integral mean curvature of K.
   (d) $W_n(K) = V(B_n)$.

Example 1.1.1. For $n = 3$, $V(K_\lambda) = V(K) + S(K)\lambda + M(K)\lambda^2 + \frac{4}{3}\pi\lambda^3$. Because of this, we can see the coefficients as parts of the Minkowski sum.

It is useful to know that some constructions can be evaluated not only in B_n, but in a general convex body E, hence obtaining

$$V(K + \lambda E) = \sum_{i=0}^{n} \binom{n}{i} W_i(K; E)\lambda^i,$$

where $W_i(K; E) = V(K, \dots, K, E, \dots, E)$ are the *relative quermassintegrals* of K.

The *relative inradius* is

$$r(K; E) = \max\{r > 0 \mid rE \subseteq K\} \quad \text{for some translation of } K,$$

and the *relative circumradius* is

$$R(K; E) = \min\{R > 0 \mid K \subseteq RE\} \quad \text{for some translation of } K.$$

Then the following relation holds:

$$R(K; E) = \frac{1}{r(E; K)}.$$

With $E = B_n$, we have the classical inradius and circumradius.

1.2 Tessier's problem

Problem 1.2.1. Look for relations between zeros of the Steiner polynomial $V(K + \lambda E) = \sum_{i=0}^{n} \binom{n}{i} W_i(K; E)\lambda^i$ and the inradius $r(K; E)$ and the circumradius $R(K; E)$.

First of all, remembering the isoperimetric inequality $p^2 - 4\pi A \geq 0$, and because $W_1(K, B_2) = \frac{1}{2}p(K)$ and $A(B_2) = \pi$, we obtain

$$W_1(K; E)^2 - A(E)A(K) \geq 0,$$

the *generalized isoperimetric inequality.*

The *Bonnesen* (Blaschke) *inequality* says that

$$W_1(K; E)^2 - A(E)A(K) \geq \frac{A(E)^2}{4}(R(K; E) - r(K; E))^2.$$

Then, if we study the Steiner polynomial for dimension 2, we ask if it is true that

$$A(K) + 2W_1(K; E)\lambda + A(E)\lambda^2 \leq 0 \quad \text{when} \quad -R(K; E) \leq \lambda \leq -r(K; E).$$

If we study its roots, we see that

$$\lambda = \frac{-W_1(K;E) \pm \sqrt{W_1(K;E)^2 - A(K)A(E)}}{A(E)}$$

and, observing that the roots are real (by the generalized isoperimetric inequality), it is obvious that $\lambda_1, \lambda_2 \leq 0$ and in dimension 2 it is true that $\lambda_1 \leq -R \leq -r \leq \lambda_2 \leq 0$, because of the Bonnesen inequality.

Teissier raised the problem of determining when an extension of this fact can be stated in arbitrary dimension:

Question 1.2.2. For which convex bodies do the real parts of the roots of the Steiner polynomial $\sum_{i=0}^{n} \binom{n}{i} W_i(K;E)\lambda^i$, say $\mathrm{Re}(\gamma_1) \leq \cdots \leq \mathrm{Re}(\gamma_n)$, satisfy

- $\mathrm{Re}(\gamma_1) \leq \cdots \leq \mathrm{Re}(\gamma_n) \leq 0$;
- $\mathrm{Re}(\gamma_1) \leq -R \leq -r \leq \mathrm{Re}(\gamma_n) \leq 0$?

The Bonnesen inequality says that this conjecture is true for $n = 2$.

1.3 The Hurwitz criterion

Definition 1.3.1. A polynomial $f(\lambda)$ is called *Hurwitz* or *stable* if all its roots lie in the left half-plane $\mathrm{Re}(\lambda) < 0$.

So the negativity part of the conjecture is equivalent to asking if the Steiner polynomial is a Hurwitz polynomial.

In dimensions 3, 4 and 5, we can apply the *Hurwitz criterion* by hand. If we have $f(\lambda) = \lambda^n + A_1\lambda^{n-1} + \cdots + A_n$, let us form the determinants $\delta_1 = A_1$ and, for $k = 2, 3, \ldots, n$,

$$\delta_k = \begin{vmatrix} A_1 & A_3 & A_5 & \ldots & A_{2k-1} \\ 1 & A_2 & A_4 & \ldots & A_{2k-2} \\ 0 & A_1 & A_3 & \ldots & A_{2k-3} \\ 0 & 1 & A_2 & \ldots & A_{2k-4} \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ 0 & 0 & 0 & \ldots & A_k \end{vmatrix}$$

with $A_j = 0$ for $j > n$. If $\delta_k > 0$ for all k, then $f(\lambda)$ is Hurwitz.

In arbitrary dimension, we can find counterexamples as follows.

1.4 Counterexamples

Definition 1.4.1. We say that x is an *r-singular point* of K if

$$\dim N(K, x) \geq n - r.$$

Definition 1.4.2. K is a *p-tangential body* of E, where $0 \leq p \leq n-1$, if each support plane of K that is not a support plane of E contains only $(p-1)$-singular points of K.

For example, the 1-tangential bodies of B_3 are the cap-bodies.

Theorem 1.4.3 (Favard)**.** *Let K, E have non-empty interior. Then K is a p-tangential body of E for all $p = 1, \dots, n-1$ if and only if*

$$V(K) = W_0(K; E) = W_1(K; E) = \cdots = W_{n-p}(K; E).$$

From this fact we can obtain a counterexample for the negativity part, as follows:

Theorem 1.4.4. *There exist 2-tangential bodies of B_{15} whose Steiner polynomial has complex roots with strictly positive real part.*

For this, we have the following:

- $V(K) = W_i(K)$ for $1 \leq i \leq n-2$.
- After doing some changes in the Steiner polynomial, we obtain that

$$\sum_{i=0}^{n} \binom{n}{i} W_i(K; E)\lambda^i = V(K)\left[\sum_{i=2}^{n} \binom{n}{i}\mu^i + n\beta(K)\mu + \alpha(K)\right]\frac{1}{\mu^n}$$

with α and β constants and $\mu = 1/\alpha$.

Then $\sum_{i=2}^{n} \binom{n}{i}\mu^i$ has roots with positive real part when $n = 15$.

Because of this, we construct a counterexample in dimension 15 by a 2-tangential body, trying to verify that $\alpha, \beta \to 0$ and remembering that zeros of polynomials are continuous functions of the coefficients. Hence, the claim that $\mathrm{Re}(\gamma_n) \leq 0$ is false.

Problem 1.4.5. Related with the circumradius bound, now $E = B_3$ in dimension 3, if K is a planar convex body with area A and perimeter p, then all the roots of the Steiner polynomial have real part greater than $-R(K)$ if and only if $p(K)^2 < 128A(K)/(3\pi)$ and $p(K) < 16R(K)/3$.

However, many symmetric lenses verify the problem, so the circumradius conjecture is false.

Problem 1.4.6. Finally, the inradius bound is also false. Another counter-example in dimension 3 can be found by taking

$$\mathrm{conv}(\{\text{planar square}\}, B_3, \text{external point to } \mathrm{conv}(\{\text{planar square}\}, B_3)).$$

But not everything is false, as we can find some families of sets that verify the conjecture:

- In dimension 3: cylinders, orthogonal boxes...;
- regular n-cubes, n-simplices...;
- also the cap bodies, that satisfy Teissier's conjecture.

Finally, another bound is known for the roots, as follows:

1. They lie in the ring $\frac{1}{n}r(K;E) \leq |\gamma_i| \leq nR(K;E)$.
2. $|\mathrm{Re}(\gamma_1)| + \cdots + |\mathrm{Re}(\gamma_n)| \geq nr(K;E)$.
3. $|\mathrm{Re}(\gamma_1)| + \cdots + |\mathrm{Re}(\gamma_n)| \leq nR(K;E)$ if $\mathrm{Re}(\gamma_i) \leq 0$ for all i.

Seminar 3

Paths with no Small Angles

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1.1 Introduction

For a finite set $X \subset \mathbb{R}^2$, each linear ordering of X corresponds to a polygonal path with vertex set X.

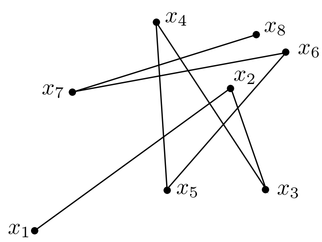


Figure 1.1: The ordering $x_1, x_2, \dots, x_8$ yields this polygonal path.

In 1992, S. Fekete raised the following question in his thesis [4]: Given a finite set of points in the plane, can you always find an ordering of the points such that all angles of the corresponding path are at least 30°? An angle of 30° degrees is a natural bound because of the 4-point example shown in Figure 1.2. Figure 1.3 shows that this problem can occur with any number of points.

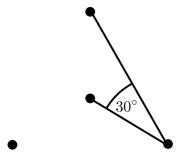


Figure 1.2: One edge has to connect the inner point with an outer point. The angle of this edge with any incident edge will be 30°.

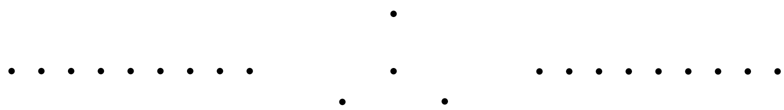


Figure 1.3: For every number of points, an angle of 30° can occur.

Figure 1.4 shows that you cannot restrict the question to non-self-intersecting paths. Any path connecting the points has to start and end on the outermost points to avoid small angles. In order to reach the upper point, the path is forced to cross itself.

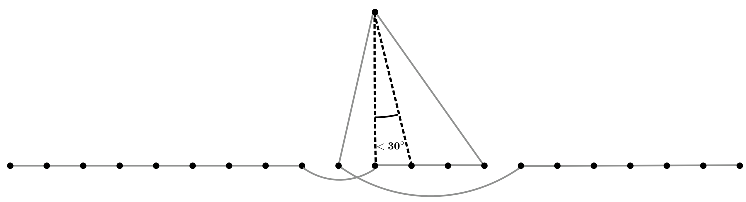


Figure 1.4: Any good path connecting these points has to intersect itself.

The question reappeared in a paper by S. Fekete and G. J. Woeginger in 1997 [3]. In 2005, A. Dumitrescu suggested to consider the question for an arbitrary $\alpha > 0$. The first answer is the following theorem by I. Bárány, A. Pór and P. Valtr from 2008 [2, 3].

Theorem 1.1.1 (Bárány–Pór–Valtr [2, 3])**.** *For every finite set $X \subset \mathbb{R}^2$, there is a polygonal path with vertex set X such that all angles between consecutive edges are at least* $20°$*.*

Very recently, J. Kyncl proved the statement for $\alpha = 30°$ with similar methods.

1.2 Proof of the theorem

For $\alpha > 0$, we call a path α-*good* if all angles between consecutive edges are at least α. Thus, Theorem 1.1.1 states the existence of a $\pi/9$-good path for each finite $X \subset \mathbb{R}^2$.

Here is an inductive idea for a proof that does not work: Choose a large triangle, with all angles large, and take its vertices out of X. Then find a path on the remaining vertices. The hope would be to find a vertex among the remaining ones that you can connect to the triangle. This method fails if all triangles are flat!

Definition 1.2.1. A set $X \subset \mathbb{R}^2$ is called *α-flat* if every triangle with vertices in X has an angle $< \alpha$.

Figure 1.5 shows an example of an α-flat set (for $\alpha = 20°$). One can see that any path connecting the points without small angles has to start and end at the outermost points.

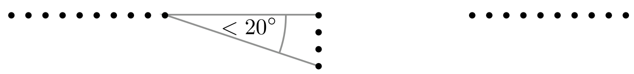


Figure 1.5: A 20°-flat set.

A question one might ask is whether every α-flat set looks like this. A more precise statement of this question involves the following definition:

Definition 1.2.2. A set $X \subset \mathbb{R}^2$ is *β-separable* if and only if $X = U \cup V$ and $U \cap V = \emptyset$ with $U, V \neq \emptyset$ such that for all $u_1, u_2 \in U$ and $v_1, v_2 \in V$ the angle between $u_1 v_1$ and $u_2 v_2$ is $< \beta$.

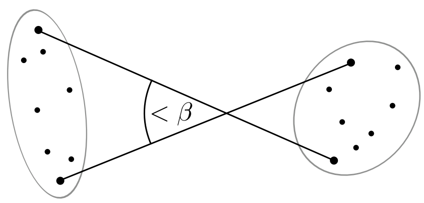


Figure 1.6: A β-separable set.

Now, one could ask whether for every α there exists a β such that every α-flat set is β-separable. Both questions are open!

Instead of proving Theorem 1.1.1 directly, a stronger statement will be proven. To formulate this, we need the following definitions:

Definition 1.2.3. Let $\alpha > 0$.

1. Let $P = x_1 \dots x_n$ be a path connecting n given vertices in $\mathbb{R}^2$. Then $\overline{x_2 x_1}, \overline{x_{n-1} x_n} \in S^1$ are called the *end directions* of P.

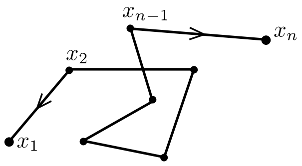


Figure 1.7: The end directions of a path.

2. A set $R \subset S^1$ is called an *α-restriction* if it is the union of two arcs $R_1, R_2 \subset S^1$ such that $|R_1|, |R_2| \geq 4\alpha$ and $\text{dist}(R_1, R_2) \geq 2\alpha$. (See Figure 1.8.)

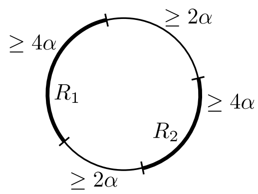


Figure 1.8: A restriction.

3. A path P is *R-avoiding* for an α-restriction R if it is α-good and not both end directions are in the same R_i.

Theorem 1.2.4 (Bárány–Pór–Valtr [2, 3])**.** *For every $\pi/9$-restriction $R \subset S^1$ and every finite $X \subset \mathbb{R}^2$ there is an R-avoiding path on X.*

The advantage of this approach is that induction works better.

Sketch of proof. Let $\alpha = \pi/9$. The proof proceeds by induction on $|X|$.

If $|X| = 2$, the end directions of any path are antipodal on S^1 and can therefore not both be contained in any arc of length $4\alpha = \frac{4}{9}\pi < \pi$. Thus, there is an R-avoiding path for any α-restriction R.

Now, let $|X| > 2$ and set $K = \text{conv}(X)$. There are several cases to consider:

Case 1: K has an angle $< 2\alpha$. Let z be the vertex at which this small angle occurs. The idea is to construct a path on $X \setminus z$ and extend it at one end.

We can assume that $X \setminus z$ lies in a cone of angle 2α with apex z that is symmetric to the x-axis. Set $I = (\pi - \alpha, \pi + \alpha)$. We then choose a new restriction Q as follows: Let Q_1 be the interval $[-2\alpha, 2\alpha]$. If $R \cap I = \emptyset$, set $Q_2 = [\pi - 2\alpha, \pi + 2\alpha]$, otherwise assume without loss of generality that $R_2 \cap I \neq \emptyset$ and set $Q_2 = R_2$.

One can show that Q is an α-restriction. Thus, there is a Q-avoiding path $x_1 \dots x_{n-1}$ on $X \setminus z$. We can assume that $\overline{x_2 x_1} \notin Q_1$ and $\overline{x_{n-1} x_n} \notin Q_2$. Then the extended path $zx_1 \dots x_{n-1}$ is α-good. Why is it furthermore R-avoiding? The end direction $\overline{x_1 z}$ has to be in I. If $R \cap I = \emptyset$, this end direction is therefore neither in R_1 nor in R_2 and the path is R-avoiding. If not, we assumed that $R_2 \cap I \neq \emptyset$ and have $Q_2 = R_2$. Thus, $\overline{x_1 z} \neq R_1$ and, by assumption, $\overline{x_{n-1} x_n} \notin Q_2 = R_2$.

Case 2: All angles of K are $\geq 2\alpha$. Let V be the set of vertices of K. One considers three cases depending on the size of $Y = x \setminus V$.

Case 2a: $Y = \emptyset$. After one edge is deleted from it, the boundary of K gives an α-good path. If there is an edge on the boundary that can be oriented so that the corresponding direction does not lie in R, one can get an R-avoiding path like this. If there is no such edge, it is possible to construct an R-avoiding path that does not only use edges of K.

(**Case 2b:** $Y = \{z\}$) and (**Case 2c:** $|Y| \geq 2$) are treated independently. Case 2c is long and more complicated. □

J. Kyncl's proof for $\alpha = 30°$ uses the same idea but adds "individually tailored R-avoidance": Every single point has its own restriction. This solves the question whether for any finite set of points in the plane one can always find an ordering for which all angles in the corresponding path are at least 30°. There are several ways to generalize this question:

1. If one defines restrictions in an appropriate way as caps on the sphere, the same method works in any dimension for an angle of $\alpha = \pi/42 \approx 4.4°$. See [3].

2. Can one solve the problem with infinitely many points in the plane?

3. What happens with finitely many points on S^2? (Paths of great arc pieces.)

4. What on other surfaces with positive curvature? (The hyperbolic plane would be a bad choice because it has a triangle with all angles of 0°.)

Questions 3 and 4 are open. Question 2 will be tackled in the next part.

1.3 The infinite case

What about infinitely many points in the plane? Let $X \subset \mathbb{R}^2$ be a countably infinite set. We would like to find an ordering of X such that all angles are at least 30°. But how should we define angles? For example, for $\mathbb{Q}$, which is a countable set, there does not seem to be a reasonable definition of angles. Therefore, we restrict our attention to infinite, but discrete sets. A set X is *discrete* if for all $x \in \mathbb{R}^2$ there is $\varepsilon > 0$ such that the ε-ball with center x contains at most one point from X.

An example like in Figure 1.4 with infinitely many points shows that we should not restrict the question to non-self-intersecting paths and that we should furthermore allow the ordering to be indexed by $\mathbb{Z}$ (and not by $\mathbb{N}$).

I. Bárány and A. Pór [1] recently answered the question for small angles with the following theorem. The proof of this theorem relies on several lemmata which will be presented before a sketch of the proof.

Theorem 1.3.1 (Bárány–Pór [1]). *For all* $0 < \alpha < \pi/18$ *and all discrete sets* $X \subset \mathbb{R}^2$*, there exists an* α*-good path on* X*.*

In what follows we will assume that X does not contain the origin, $0 \notin X$, and that no two points in X have the same norm: $\|x\| \neq \|y\|$ for $x, y \in X$, $x \neq y$.

We call $a \in X$ *α-sharp* if, for all $x \in X$ with $\|x\| < \|a\|$, we have $\sphericalangle a0x < \alpha$.

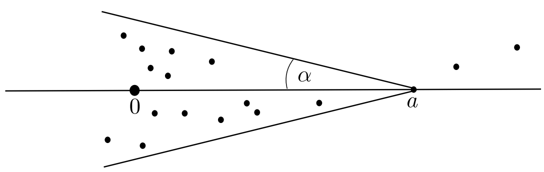


Figure 1.9: An α-sharp point.

Lemma 1.3.2 (Sharp Lemma). *If all but finitely many points of* X *are sharp, then there exists a* $\pi/9$*-good path on* X*.*

Sketch of proof. Let D_r be a disk of radius r that contains all non-sharp points of X and set $X_0 = X \cap D_r$. Because X is discrete, D_r can only contain finitely many points of X, otherwise X would contain a limit point. Order the points of $X \setminus X_0$ by their norm: $X \setminus X_0 = \{x_0, x_1, \ldots\}$ such that $\|x_j\| < \|x_{j+1}\|$. Furthermore, set $X_n = X_0 \cup \{x_0, \ldots, x_n\}$.

For each $n \in \mathbb{N}$ we get a $\pi/9$-good path P_n on X_n, by Theorem 1.1.1. One can observe that x_n always has to be one of the endpoints of P_n (and of every path on X_n).

Now we construct a $\pi/9$-good path on all of X using these finite paths.

Let $k \in \mathbb{N}$. For each $n \in [k]$ we define a path $P_k(n)$ on X_n in the following way: The path $P_k(k)$ is just P_k. If $P_k(n)$ is defined, we let $P_k(n-1)$ be the path $P_k(n)$ with its endpoint x_n deleted.

Now let $L \subset \mathbb{N}$ be infinite and $n \in \mathbb{N}$. For $k \in L$ with $k \geq n$, the path $P_k(n)$ is a path on X_n. Because there are only finitely many distinct paths on X_n, there exists an infinite $L' \subset L$ such that $P_k(n) = P_j(n)$ for all $k, j \in L'$.

Thus, we can construct a series of infinite sets $\mathbb{N} = L_0 \supset L_1 \supset \ldots$ such that $P_k(n) = P_j(n)$ for all $k, j \in L_n$.

For each $n \in \mathbb{N}$, choose some $k_n \in L_n$ and let $Q_n = P_{k_n}(n)$. For $n < m$, we have $L_m \subset L_n$ and $k_n, k_m \in L_n$. Thus, $Q_n = P_{k_n}(n) = P_{k_m}(n)$ is a subpath of $Q_m = P_{k_m}(m)$.

Hence, by setting $Q = \bigcup_{i \geq 1} Q_i$ we get a $\pi/9$-good path on X. □

One might observe that the above proof does not involve a lot of geometry; only the finite case, which is heavily used here, and the observation on the endpoints of the finite paths involve geometrical reasoning.

In what follows, let $0 < \alpha < \pi/18$.

Lemma 1.3.3 (Cone Lemma)**.** *Let $\beta \in (0, \pi/18)$ and K be the set of points (x, y) in the plane for which the angle between (x, y) and $(1, 0)$ is in $(-\beta, \beta)$. Set $K^* = K \cup -K$. If $X \setminus K^*$ is finite, there is an α-good path on X.*

For a point z in the plane, denote by $\bar{z}$ the point on the unit sphere $z/\|z\|$. A point z is a *limit direction* of X if there is a sequence of distinct points $x_i \in X$ with $\lim \bar{x}_i = z$. We denote the set of limit directions by $\Delta(X)$. Then $\Delta(X)$ is closed.

Lemma 1.3.4 (Cone Lemma, 2nd version)**.** *Let $I \subset S^1$ be an open arc of length $\pi/9$ and $I^* = I \cup -I$. If there exists such an I with $\Delta(X) \subset I^*$, then there exists an α-good path on X.*

It can be easily seen that Lemma 1.3.3 implies Lemma 1.3.4: $\Delta(X)$ is closed while I is open. Thus, if $\Delta(X) \subset I^*$, there has to be a closed arc $J \subset I$ such that $\Delta(X) \subset J^*$. This means that almost all points of X lie in K^* where K is the cone hull of J. Thus, Lemma 1.3.3 applies.

The proof of Lemma 1.3.3 is the longest of all proofs involved here and makes use of the finite case of the theorem (Theorem 1.1.1).

Two points $a, b \in \mathbb{R}^2$ are a *fat pair* if all three angles in the triangle $0ab$ are at least $\pi/18$.

Proposition 1.3.5. *If there are infinitely many fat pairs among X, then there is an α-good path on X.*

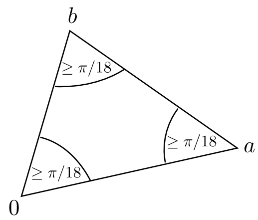


Figure 1.10: A fat pair a, b.

Sketch of proof. Among the infinitely many fat pairs of X we can find a sequence of fat pairs (a_k, b_k) such that the sequences $(\bar{a_k}), (\bar{b_k})$ converge:

$$\lim a_k = a \in S^1, \quad \lim b_k = b \in S^1.$$

Then it is possible to construct an α-good path

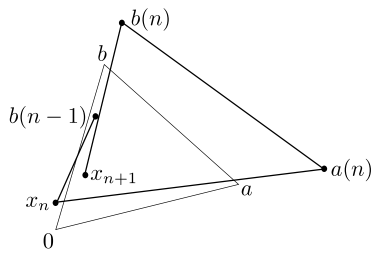
$$P = x_1\, a(1)\, b(1)\, x_2\, a(2)\, b(2)\, x_3 \ldots$$

on X, where $(a(n), b(n))$ are fat pairs from the sequence (a_k, b_k), such that the following condition is fulfilled:

All $x \in X$ with $\|x\| < \|x_n\|$ come before x_n in P for all $n \in \mathbb{N}$.

First, let x_1 be the point with smallest norm in X. Now, assume that $P_n = x_1\, a(1)\, b(1)\, x_2 \ldots a(n-1)\, b(n-1)\, x_n$ is already constructed, fulfilling the condition above, and such that $\overline{(b(n-1) - x_n)}$ is close to b on S^1.

Then x_{n+1} has to be the shortest unused element of X. Choosing $a(n)$ and $b(n)$ such that $\overline{(a(n) - x_n)}$ is close to a and $\overline{(b(n) - x_{n+1})}$ is close to b on S^1 yields a good path $P_{n+1} = P_n\, a(n)\, b(n)\, x_{n+1}$. $\square$



Two points $a, b \in \mathbb{R}^2$ are a *balanced pair* if the angle $\sphericalangle a0b$ is $< \pi/18$ and the angles $\sphericalangle 0ba$, $\sphericalangle ba0$ are $\geq \pi/18$.

Proposition 1.3.6. *If there are infinitely many balanced pairs among X, then there is an α-good path on X.*

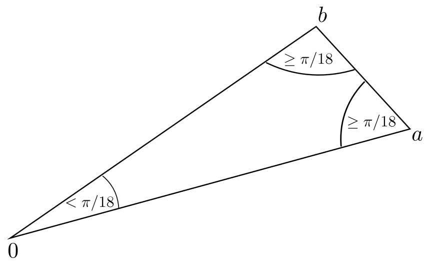


Figure 1.11: A balanced pair a, b.

The proof of Proposition 1.3.6 splits in two cases. One involves similar ideas as the proof of Proposition 1.3.5, and the other uses the second version of the Cone Lemma.

Sketch of proof for Theorem 1.3.1. Propositions 1.3.5 and 1.3.6 reduce the problem to the case in which there are only finitely many fat and finitely many balanced pairs. One can show that in this case all points of X that lie outside a disk containing all these pairs have to be α-sharp. Since this disk can contain only finitely many points of the discrete set X, the Sharp Lemma 1.3.2 yields the existence of an α-good path on X. □

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Seminar 4

Orientations of Planar Graphs

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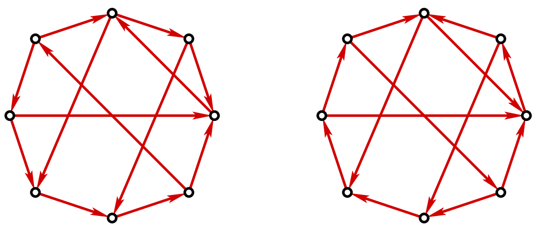
Definition 1.0.1. Given $G = (V, E)$ and $\alpha\colon V \to \mathbb{N}$, an *$\alpha$-orientation* of G is an orientation with $\operatorname{outdeg}(v) = \alpha(v)$ for all v.

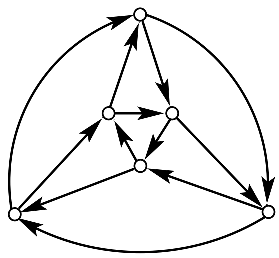
For example, the following are two orientations for the same α:



In these notes, we will mostly consider α-orientations for planar graphs.

1.0.1 Example: Eulerian orientations

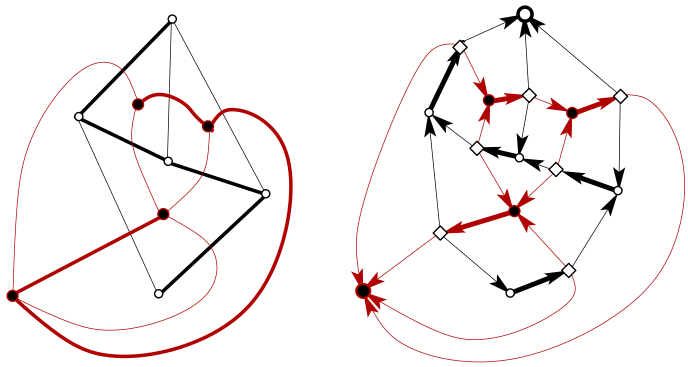
These are orientations with $\operatorname{outdeg}(v) = \operatorname{indeg}(v)$ for all v, i.e., $\alpha(v) = \frac{1}{2}d(v)$.



1.0.2 Example: Spanning trees of planar graphs

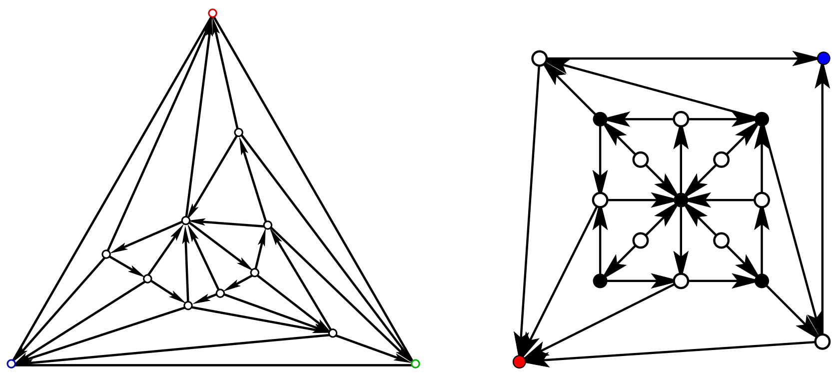
Let G be a planar graph. Then the spanning trees of G are in bijection with the α_T-orientations of a rooted primal-dual completion $\widetilde{G}$.

A primal-dual completion $\widetilde{G}$ is a bipartite graph with the first (resp. the second) partition represented by edges of G (resp. by vertices and faces in a given drawing of G). The edges of $\widetilde{G}$ are represented by adjacencies of edges and vertices (resp. faces). We define $\alpha_T(v) = 1$ for a non-root vertex v; and $\alpha_T(v_e) = 3$ for an edge-vertex v_e; and $\alpha_T(v_r) = 0$, $\alpha_T(v_r^*) = 0$.



1.0.3 Example: 3-orientations

For a planar triangulation G, let $\alpha(v) = 3$ for each inner vertex and $\alpha(v) = 0$ for each outer vertex (left picture below).



1.0.4 Example: 2-orientations

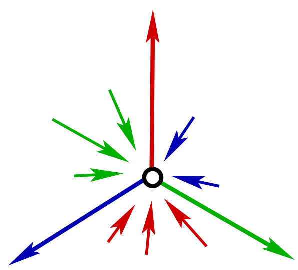
For a planar quadrangulation G, let $\alpha(v) = 0$ for an opposite pair of outer vertices and $\alpha(v) = 2$ for each other vertex (right picture above).

1.1 Sample applications

1.1.1 Schnyder woods

Let $G = (V, E)$ be a plane triangulation and $F = \{a_1, a_2, a_3\}$ the outer triangle. A coloring and orientation of the interior edges of G with colors 1, 2, 3 is a *Schnyder wood* of G if the following conditions hold:

- Inner vertex condition:

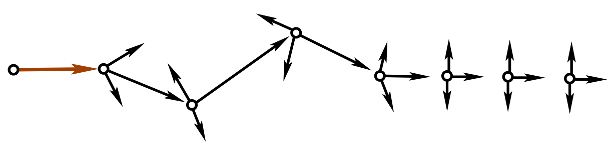


- The edges $\{v, a_i\}$ are oriented $v \to a_i$ in color i.

1.1.2 Schnyder woods and 3-orientations

Theorem 1.1.1. *Schnyder woods and* 3*-orientations are equivalent.*

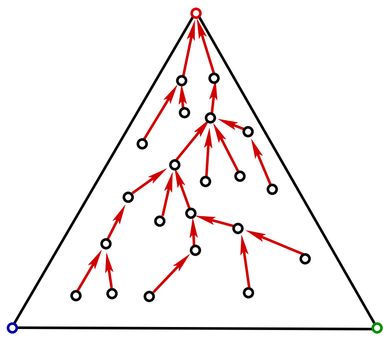
Proof. From a 3-orientation to a Schnyder wood, define the path of an edge:



If the path were not simple, it would form a boundary of a region where, by applying the Euler relation, we would get a contradiction. Hence, it ends at some a_i. We define this edge to have color i. □

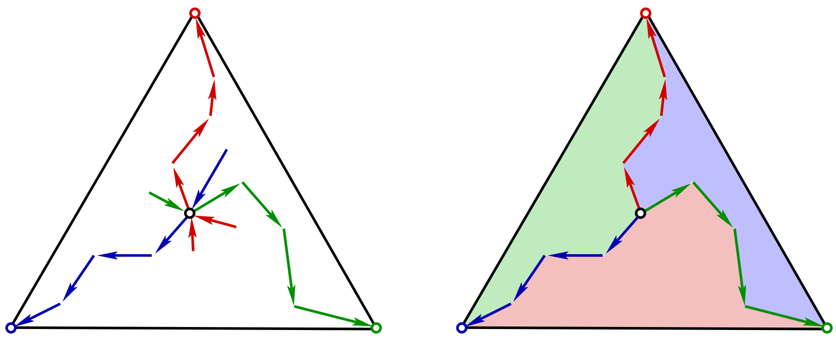
1.1.3 Schnyder woods: trees

The set T_i of edges colored i is a tree rooted at a_i. To prove this claim, similarly as before, if there were two different paths $e \to a_i$, they would form a boundary of a region where applying the Euler relations yields a contradiction.



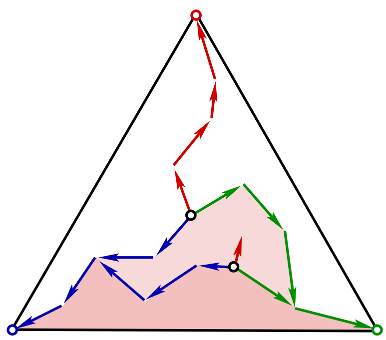
1.1.4 Schnyder woods: paths and regions

- Paths of different color have at most one vertex in common.
- Every vertex has three distinguished regions.



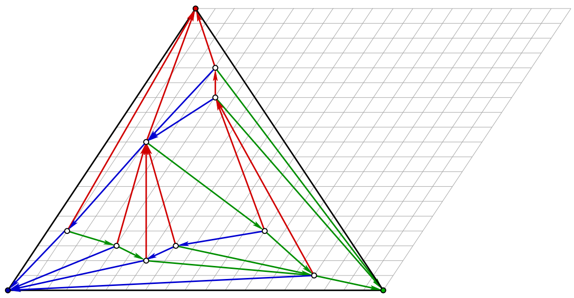
1.1.5 Schnyder woods: regions

If $u \in R_i(v)$, then $R_i(u) \subset R_i(v)$.



1.1.6 Grid drawings

The count of faces in the green and red region yields two coordinates (v_g, v_r) for a vertex v.

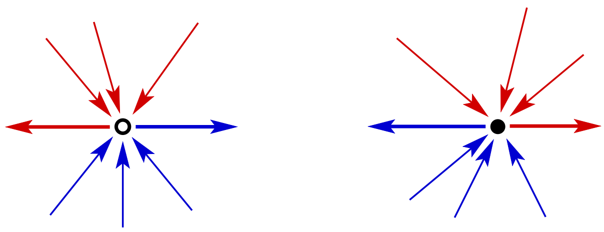


The picture shows a straight line drawing on a $(2n-5)\times(2n-5)$ grid. (In fact, the drawing can be done on an $(n-1)\times(n-1)$ grid.)

1.1.7 Separating decompositions

Let $G=(V,E)$ be a plane quadrangulation and $F=\{a_0, x, a_1, y\}$ the outer face. A coloring and orientation of the interior edges of G with colors 0, 1 is a *separating decomposition* of G if the following conditions hold:

- Inner vertex condition:

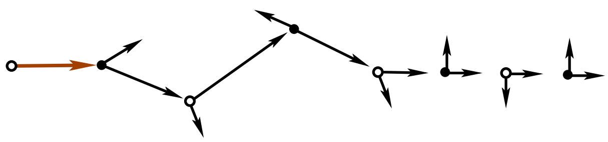


- The edges incident to a_0 and a_1 are oriented $v \to a_i$ in color i.

1.1.8 Separating decompositions and 2-orientations

Theorem 1.1.2. *Separating decompositions and* 2*-orientations are equivalent.*

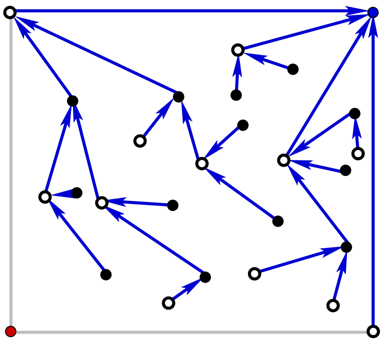
Proof. Define the path of an edge:



The path is simple (Euler). Hence, it ends at some a_i. □

1.1.9 Separating decompositions: trees

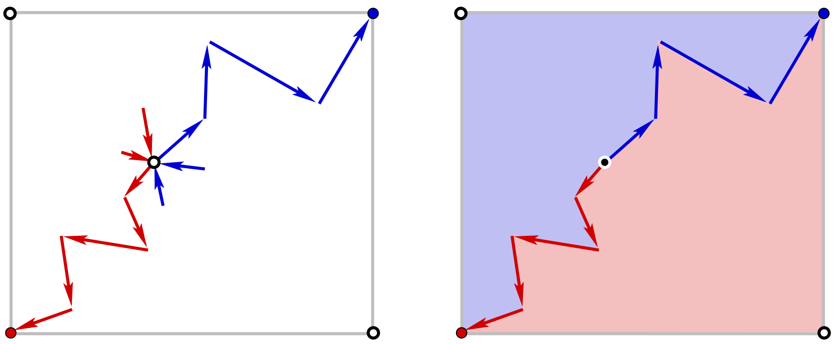
The set T_i of edges colored i is a tree rooted at a_i.



Proof. The path $e \to a_i$ is unique. □

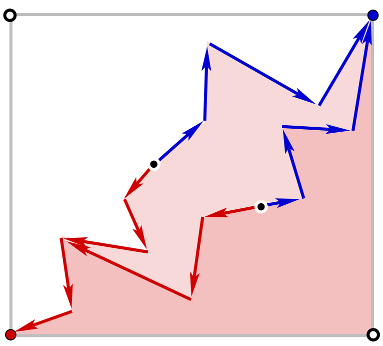
1.1.10 Separating decompositions: paths and regions

- Paths of different color have at most one vertex in common.
- Every vertex has two distinguished regions.



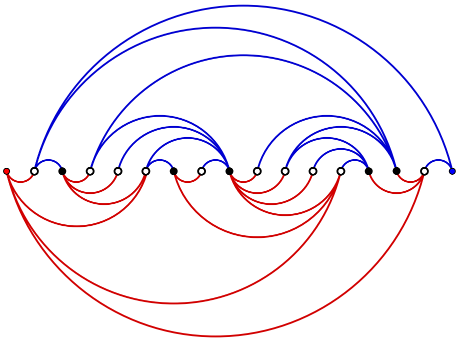
1.1.11 Separating decompositions: regions

If $u \in R_0(v)$ then $R_0(u) \subset R_0(v)$.



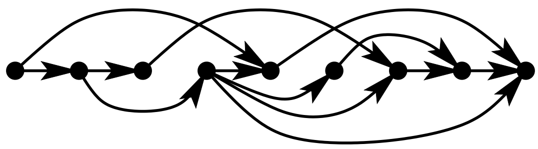
1.1.12 2-book embedding

The count of faces in the red region yields a number v_r for each vertex $v \neq s, t$.

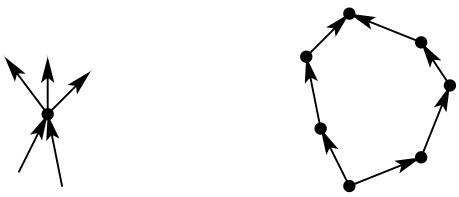


1.1.13 Bipolar orientations

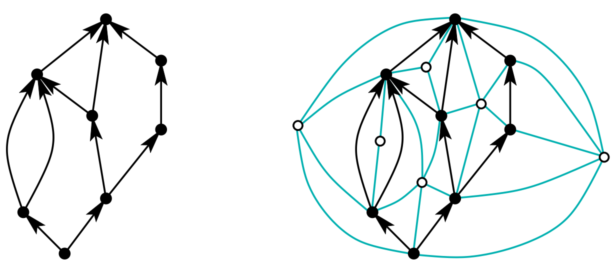
Definition 1.1.3. A *bipolar orientation* is an acyclic orientation with a unique source s and a unique sink t.



Plane bipolar orientations with s and t on the outer face are characterized by:

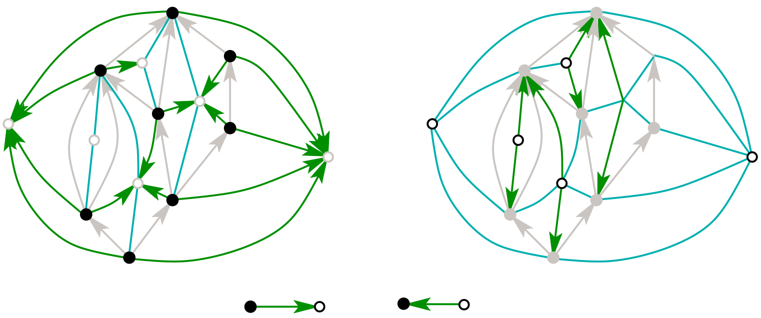


1.1.14 Plane bipolar orientations and 2-orientations



A plane bipolar orientation and its angular map. (We add one vertex for each face and two for the outer face.)

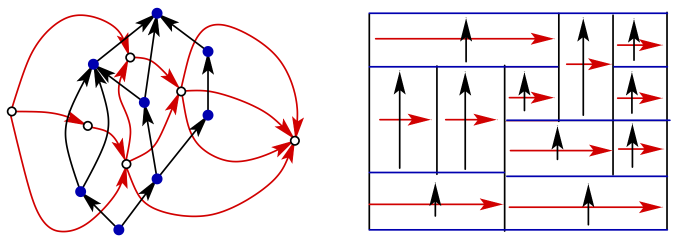
1.1.15 Orienting the angular map



Angular edges oriented by vertices and faces.

1.1.16 Plane bipolar orientations and rectangular layouts

A plane bipolar orientation and its dual orientation yield a rectangular layout (visibility representation).



Primal vertices correspond to the horizontal direction and dual vertices correspond to the vertical direction.

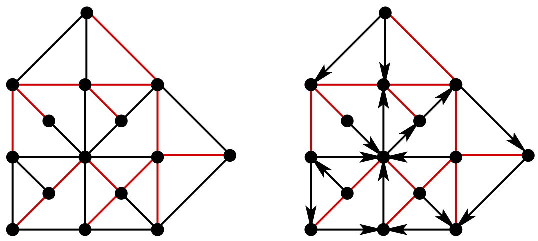
1.2 Counting I: Bounds

1.2.1 How many?

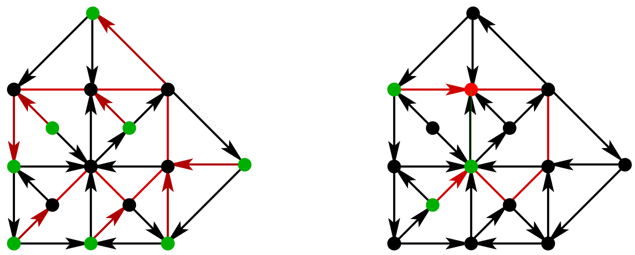
Suppose given a plane graph G and $\alpha\colon V \to \mathbb{N}$. How many α-orientations can G have?

1.2.2 Towards an upper bound

We aim at giving a better bound than 2^m. Choose a spanning tree T of G and orient the edges not in T randomly.



If at all, the orientation on $G \setminus T$ is uniquely extendible.



Therefore, there are at most $2^{m-(n-1)}$ α-orientations.

1.2.3 Improve on one color

- An orientation can be extended only if $\text{outdeg}(v) \in \{\alpha(v), \alpha(v) - 1\}$ for all v.
- Let I be an independent set of size greater than or equal to $\frac{1}{4}n$ (4CT).
- Choose a tree T such that $I \subset \text{leaves}(T)$.
- Each $v \in I$ can independently obstruct extendability.
- There are $\binom{d(v)-1}{\alpha(v)} + \binom{d(v)-1}{\alpha(v)-1} = \binom{d(v)}{\alpha(v)} \leq \binom{d(v)}{\lfloor d(v)/2 \rfloor}$ good choices for the orientations of edges at v.

1.2.4 The result

Since

$$\text{Prob}(d(v) = \alpha(v)) \leq \frac{1}{2^{d(v)-1}} \binom{d(v)}{\lfloor d(v)/2 \rfloor} \leq \frac{3}{4},$$

we conclude:

Theorem 1.2.1. *The number of α-orientations of a plane graph on n vertices is at most*

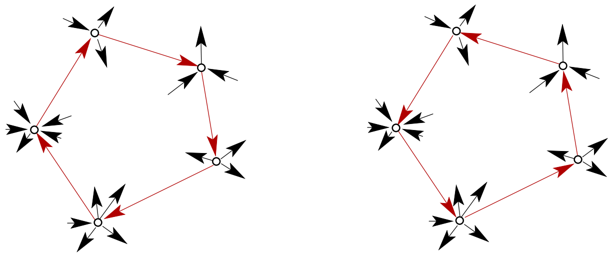
$$2^{m-n} \left(\frac{3}{4}\right)^{n/4} \leq 2^{2n} \left(\frac{3}{4}\right)^{n/4} \approx (3.73)^n.$$

Since $2^m \sim 8^n$, $2^{m-n} \sim 4^n$, this result gives a nontrivial improvement.

1.2.5 Towards a lower bound

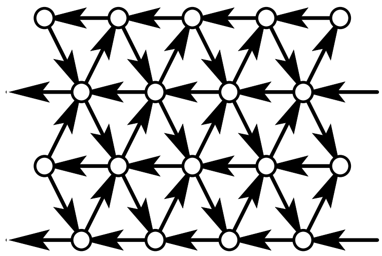
Here we want to find a graph G and an α such that there are many α-orientations.

Observation 1.2.2. Flipping cycles preserves α-orientations:



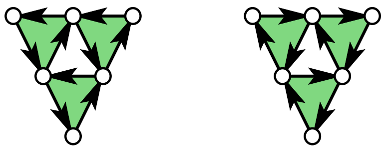
We show that there are *many* 3-orientations of the triangular lattice.

1.2.6 The initial orientation



Any subset of the green triangles can be flipped.

1.2.7 Green and white flips



If 0 or 3 of the green neighbors are flipped, a white triangle can be flipped. In what follows, the second inequality follows from Jensen's inequality:

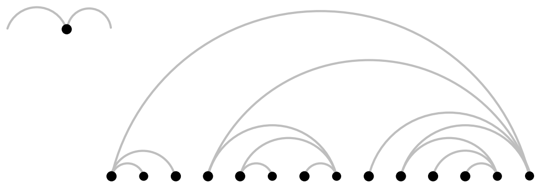
$$\#\,\text{3-orientations} \geq 2^{\#\,\text{f-green}}\, \mathbf{E}\left(2^{\#\,\text{f-white-flippable}}\right) \geq$$
$$2^n\, 2^{\mathbf{E}(\#\,\text{f-white-flippable})} = 2^n\, 2^{\frac{2}{8}\#\,\text{f-white}} = 2^{\frac{5}{4}n} = (2.37)^n,$$

where $\#\,$f-green $\sim n$ is the number of green faces, $\#\,$f-white $\sim n$ is the nuber of white faces and $\#\,$f-white-flippable $\sim n$ is the number of white faces surrounded by all-flipped or all-unflipped green faces.

1.3 Counting II: Exact

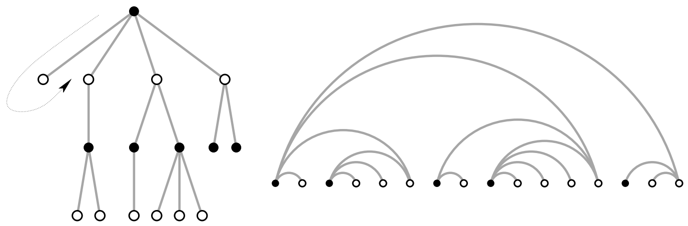
1.3.1 Alternating layouts of trees

Definition 1.3.1. A numbering of the vertices of a tree is *alternating* if it is a 1-book embedding with no *double-arc*.



1.3.2 Alternating layouts of trees

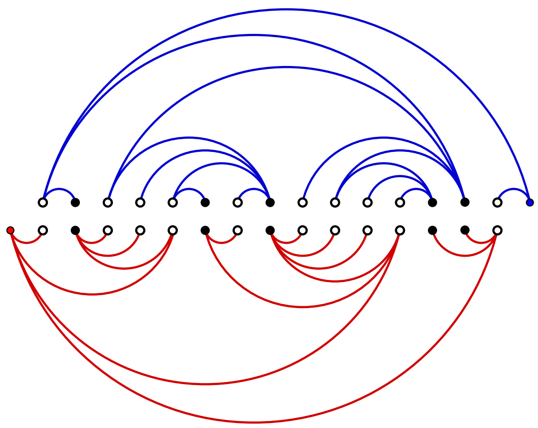
Proposition 1.3.2. *A rooted plane tree has a unique alternating layout with the root as leftmost vertex.*



Label black vertices at first visit and white vertices at last visit.

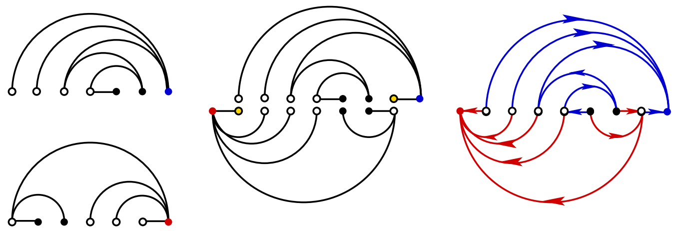
1.3.3 Separating decompositions and alternating trees

Proposition 1.3.3. *The 2-book embedding induced by a separation decomposition splits into two alternating trees.*



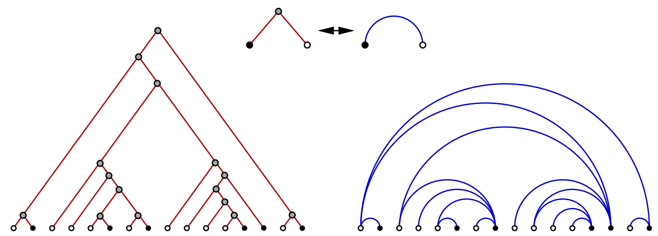
1.3.4 A bijection

Theorem 1.3.4. *There is a bijection between pairs (S, T) of alternating trees on n vertices with reverse fingerprints and separating decompositions of quadrangulations with $n+2$ vertices.*



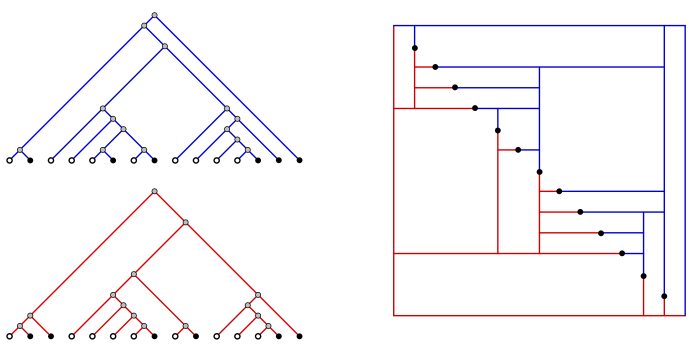
1.3.5 Alternating and full binary trees

Proposition 1.3.5. *There is a bijection between alternating and binary trees that preserves fingerprints.*



1.3.6 Rectangular dissections

Theorem 1.3.6. *There is a bijection between pairs (S, T) of binary trees with n leaves and reverse fingerprints and rectangular dissections*[1] *of the square based on $n-2$ diagonal points.*



[1]This *is* again the rectangular layout associated to the bipolar orientation.

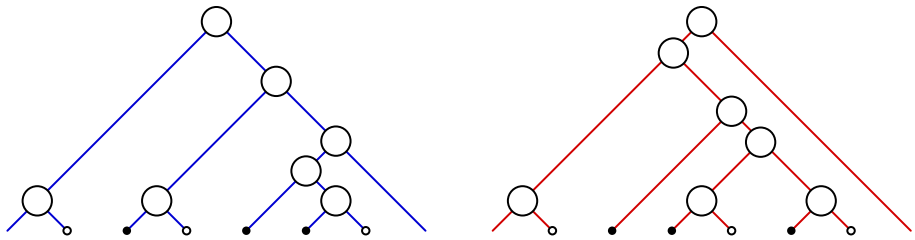
1.3.7 Summary

We have the following bijection relations:

$$\begin{pmatrix}\text{bipolar}\\ \text{orientations}\end{pmatrix} \leftrightarrow \begin{pmatrix}\text{2-orientations}\\ \text{of angular maps}\end{pmatrix} \leftrightarrow \begin{pmatrix}\text{sep. decompositions}\\ \text{of ang. maps}\end{pmatrix} \leftrightarrow$$

$$\leftrightarrow \begin{pmatrix}\text{pairs of binary}\\ \text{trees}\end{pmatrix} \leftrightarrow \begin{pmatrix}\text{rectangular}\\ \text{layout}\end{pmatrix} \leftrightarrow \begin{pmatrix}\text{bipolar}\\ \text{orientations}\end{pmatrix}$$

1.3.8 Permutations and trees



Proposition 1.3.7. *For a permutation π of $[n-1]$, the pair $(\textsc{Max}(\pi), \textsc{Min}(\overline{\pi}))$ is a pair of binary trees with n leaves and reverse fingerprints. The relation is not bijective:*

$$(\textit{permutations}) \xleftrightarrow{k:1} (\textit{tree pairs})\,.$$

1.3.9 Baxter permutations

Definition 1.3.8. A permutation is *Baxter* if it avoids the patterns 3142 and 2413.

Example 1.3.9. A non-Baxter permutation with a 2413 pattern:

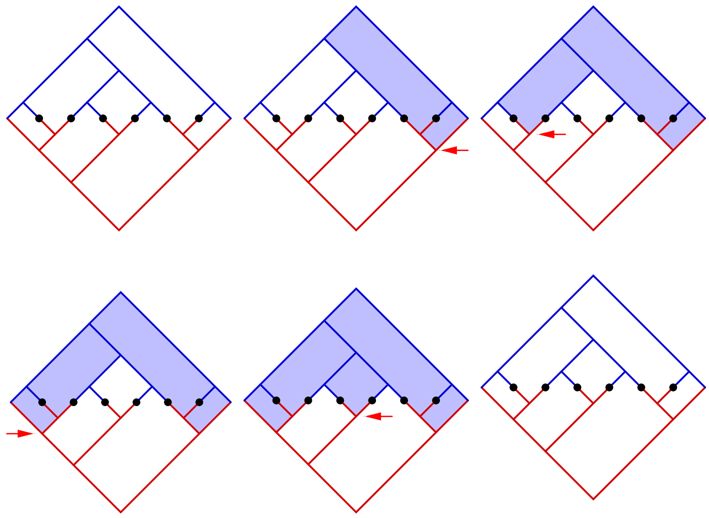
$$\pi = 6, 3, 8, 7, 2, 9, 1, 5, 4$$

Theorem 1.3.10. *The mapping $\pi \mapsto (\textsc{Max}(\pi), \textsc{Min}(\overline{\pi}))$ is a bijection between Baxter permutations of $[n-1]$ and binary trees with n leaves and reverse fingerprints, i.e., rectangular dissections of the square based on $n-2$ diagonal points.*

1.3.10 Constructing the permutation

Rule: If the south-corner of $R(k)$ is

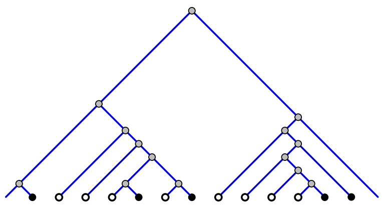
i.e., a left child in tree T, then $R(k-1)$ is the next-left; otherwise, next-right.



1.3.11 Encoding a binary tree

α: Fingerprint extended by a leading 1 for left leaf.

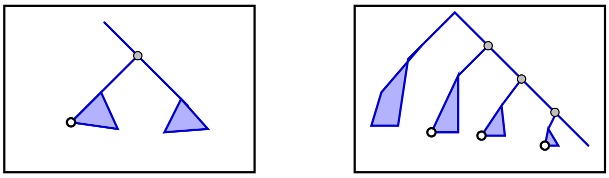
β: Inner nodes in in-order represented by 0 (left) and 1 (right) with the root being a 1.



α: Fingerprint including the left extreme leaf.

β: Inner nodes in in-order represented by 0 (left) and 1 (right) with the root being a 1.

Lemma 1.3.11. $\sum_{i=1}^{n-1} \alpha_i = \sum_{i=1}^{n-1} \beta_i$ *and* $\sum_{i=1}^{k} \alpha_i \geq \sum_{i=1}^{k} \beta_i$.



Lemma 1.3.12. *The tree can be reconstructed.*

Proof. The minimal k with $\sum_{i=1}^{k} \alpha_i = \sum_{i=1}^{k} \beta_i$ and $\sum_{i=1}^{k+1} \alpha_i = \sum_{i=1}^{k+1} \beta_i$ determines the position of the root. □

1.3.12 Counting binary trees

Proposition 1.3.13. *The number of binary trees with* $i+1$ *left leaves and* $j+1$ *right leaves equals the number of nonintersecting lattice paths* α' *and* β' *where* $\alpha' \colon (0,1) \to (j, i+1)$ *and* $\beta' \colon (1,0) \to (j+1, i)$.

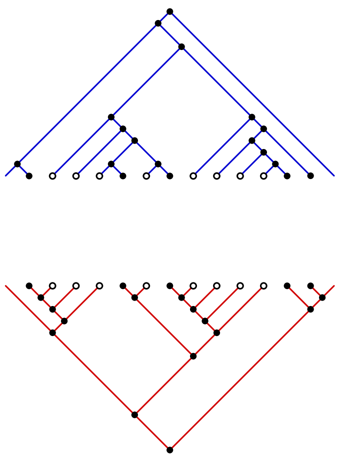
From the Lemma of Gessel Viennot we deduce that their number is

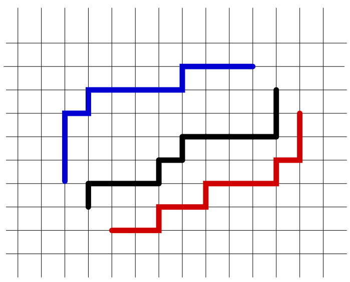
$$\det \begin{pmatrix} \binom{j+i}{j} & \binom{j+i}{j-1} \\ \binom{j+i}{j+1} & \binom{j+i}{j} \end{pmatrix} = \frac{1}{i+j+1} \binom{i+j+1}{j} \binom{i+j+1}{j+1}.$$

This is the *Narayana number* $N(i+j+1, j)$.

1.3.13 Three paths

Proposition 1.3.14. *Baxter permutations of* $[n-1]$ *with a fixed number* i *of increases can be encoded by triples of disjoint lattice paths.*





1.3.14 Counting Baxter

Theorem 1.3.15. *The number of Baxter permutations of* $[n-1]$*, separating decompositions and* 2*-orientations on* $n+2$ *vertices, rectangular dissections on* $n-2$ *diagonal points, etc., is given by*

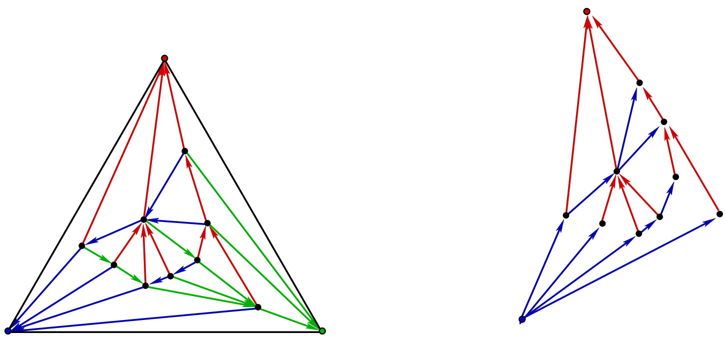
$$\sum_{i=0}^{n-2} \frac{2n!(n-1)!(n-2)!}{i!(i+1)!(i+2)!(n-i)!(n-i-1)!(n-i-2)!} =$$

$$\frac{2}{n(n-1)^2} \sum_{i=0}^{n-2} \binom{n}{i}\binom{n}{i+1}\binom{n}{i+2}.$$

1.3.15 Schnyder woods and bipolar orientations

Proposition 1.3.16. *There is a bijection between Schnyder woods with* $n+3$ *vertices and bipolar orientations with* $n+2$ *vertices and the following special property:*

($\star$) *The right side of every bounded face is of length two.*



Sketch of proof. Delete green edges and reverse blue edges. □

1.3.16 Special property

Let T^b and T^r be the blue and red trees corresponding to a Schnyder wood. From ($\star$) we get some crucial properties of the fingerprint and the bodyprints of the trees:

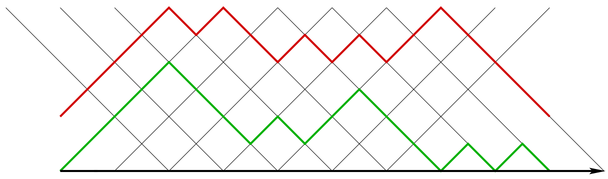
Fact 1. The extended fingerprint $1+\alpha$ is a Dyck word; in symbols,

$$(01)^n \leq_{\text{dom}} 1+\alpha.$$

Fact 2. The fingerprint uniquely determines the bodyprint of the blue tree, precisely $\overline{\beta^b} = 1+\alpha$.

1.3.17 Schnyder woods and Dyck paths

By tracing the special property ($\star$) during the bijective correspondence, we get a Dyck path:

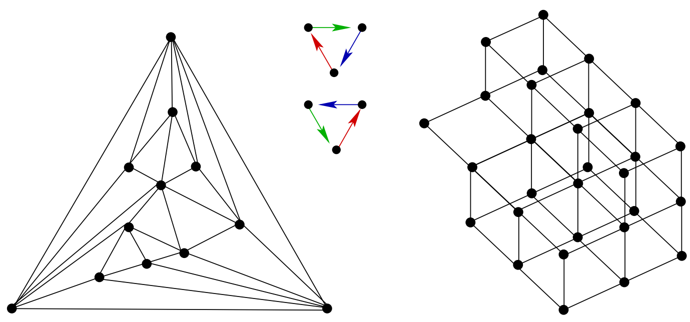


Theorem 1.3.17 (Bonichon)**.** *The number of Schnyder woods on plane triangulations on $n+3$ vertices equals the pairs of non-crossing Dyck paths of length $2n$, which is $C_{n+2}C_n - C_{n+1}^2$.*

1.4 Lattices

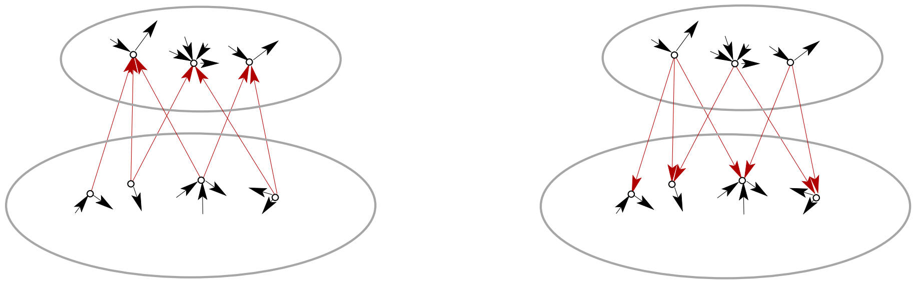
1.4.1 Distributive lattices

Theorem 1.4.1. *The set of α-orientations of a planar graph G has the structure of a distributive lattice.*



1.4.2 A dual construction

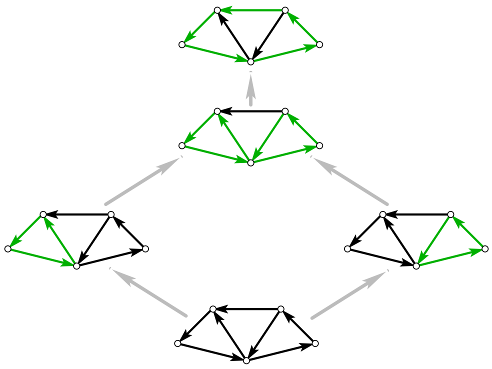
Reorientations of directed cuts preserve flow-differences along cycles, i.e., for each oriented cycle the sum of flows along its edges with negative sign, when the edge is oriented backwards, is preserved.



Theorem 1.4.2 (Propp, 1993)**.** *The set of all orientations of a graph G with prescribed flow-differences for all cycles has the structure of a distributive lattice.*

1.4.3 Circulations in planar graphs

Theorem 1.4.3 (Khuller–Naor–Klein, 1993)**.** *The set of all integral flows respecting capacity constraints ($\ell(e) \leq f(e) \leq u(e)$) of a planar graph has the structure of a distributive lattice.*



1.4.4 $\mathbf{\Delta}$-bonds

Let $G = (V, E)$ be a connected graph with a prescribed orientation. With $x \in \mathbb{Z}^E$ and a cycle C, we define the circular flow difference

$$\Delta_x(C) = \sum_{e \in C^+} x(e) - \sum_{e \in C^-} x(e).$$

With $\Delta \in \mathbb{Z}^{\mathcal{C}}$ and $\ell, u \in \mathbb{Z}^E$, let $\mathcal{B}_G(\Delta, \ell, u)$ be the set of $x \in \mathbb{Z}^E$ such that $\Delta_x = \Delta$ and $\ell \leq x \leq u$.

Theorem 1.4.4 (Felsner–Knauer, 2007)**.** *$\mathcal{B}_G(\Delta, \ell, u)$ is a distributive lattice. The cover relation is vertex pushing.*

1.4.5 Δ-bonds as a generalization

$\mathcal{B}_G(\Delta, \ell, u)$ is the set of $x \in \mathbb{R}^E$ such that:

- $\Delta_x = \Delta$ (circular flow difference), and
- $\ell \leq x \leq u$ (capacity constraints).

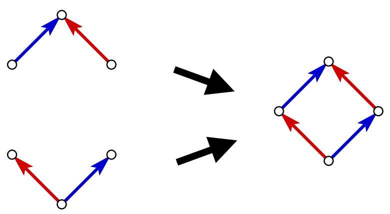
The following are special cases:

- c-orientations are $\mathcal{B}_G(\Delta, 0, 1)$, where $\Delta(C) = |C^+| - c(C)$.
- Circular flows on planar G are $\mathcal{B}_{G^*}(0, \ell, u)$, where G^* is the dual of G.
- α-orientations.

1.4.6 Diagrams of distributive lattices: A characterization

A coloring of the edges of a digraph is a *D-coloring* if

- arcs leaving a vertex have different colors, and
- it has the *completion property*:



Theorem 1.4.5. *A digraph D is connected, acyclic and admits a D-coloring if and only if D is the diagram of a distributive lattice.*

Seminar 5

Geometry of Numbers and Modern Developments

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1.1 Introduction

The Geometry of Numbers can be traced back to Kepler, who asked about the densest way of packing balls. Two hundred years later, Gauss found the answer to Kepler's question.

Authors like Kepler, Lagrange, Gauss, Korkin and Zolotarev or Fedorov contributed with sporadic results to the field. Let us point out that Fedorov was a crystallographer; this exemplifies a common fact in the Geometry of Numbers, namely, that many contributions come from foreigners to the field.

It was not until the end of the ninetieth century when Minkowski started a systematic research. In the first decades of the twentieth century, Voronoi continued Minkowski's work, focused on the geometric theory of positive quadratic forms.

During the twentieth century the field continued its expansion in various ways. In the first decades, the research on particular problems, like diophantine approximation and algebraic number theory, continued. Furthermore, Siegel and Mahler, among others, continued the systematic approach.

In the 1940's and 1950's the systematic approach turned hard and the main contributions were on particular problems.

Among the eminent contributors with sporadic results to the area we can name: Weyl, Siegel (who only wrote two papers on the area, but fundamental), Weil, Van der Waerden, Erdős, Kneser, Zassenhaus, Bombieri, McMullen.

We can also signal several groups of people who worked on the field:

- English school: Mahler, Davenport, Rankin, Cassels, Watson, Rogers, Chalk, Macbeath. They gave careful estimates for difficult questions, focusing on the arithmetic part.
- Vienna school: Furtwängler, Hofreiter, Hlawka, Schmidt, Gruber. They made special emphasis in geometry.
- Russian school: Delone, Skubenko, Ryshkov, Dolbilin. They mainly continued Voronoi's work.
- Indian school: Bambah, Hans-Gill, Dumir.
- American number geometers: Groemer, Leech, Sloane (using error-correcting codes and convexity), Schmidt, Conway, Blichfeldt.
- French school: Martinet, Coulangeon, Nebe, Bachoc, Bavard, Bergé. They focused on the theory of quadratic forms.
- German number geometers: Kneser, Siegel, Wills, Henk.

Some of the modern areas in Mathematics are rooted in the Geometry of Numbers. One remarkable example is Discrete Geometry, which, although can be viewed as its sister, has become predominant over the Geometry of Numbers.

We can signal out three/four big systematic areas, or large research projects in the Geometry of Numbers:

- Lattice packing of balls and convex bodies: density, kissing number, non-lattice packing.
- Lattice covering of balls and convex bodies: density, star number, non-lattice covering.
- Lattice tiling with convex polytopes, non-lattice tiling.
- Quadratic forms.

There are also several particular, or small, and difficult problems which have grabbed the attention of many mathematicians, for example the product of non-homogeneous linear forms.

1.2 The lattice packing problem

Consider C a 0-symmetric convex body. Let L be a *lattice*: the set of integer linear combinations of a set of d independent vectors in the d-dimensional Euclidean space, $\{b_1, \ldots, b_d\} \subset \mathbb{E}^d$. Thus, we can define $d(L)$, the *determinant* of the lattice L, as the determinant of the matrix formed by $\{b_1, \ldots, b_d\}$.

Let $C + l$, $l \in L$, be the translates of C by L. We say that $C + l : l \in L$ form a *packing* of C with *packing lattice* L if any two distinct translates of C have disjoint interiors.

We can define $\delta(C, L) = V(C)/d(L)$ as the *density* of the packing. One may think of the density as the proportion of space covered by the bodies $C+l$, $l \in L$. Let $\delta(C)$ be the maximum density among all the possible lattices, for a given C.

Given any C and L forming a packing, C can be dilated to a new body C', forming a packing with L but with non-empty border intersection between some translates. Let $k(C, L)$ denote the *kissing number* or the number of translates having a non-empty border intersection with a given one. As above, $k(C)$ stands for the maximal of the kissing numbers among all the lattices.

The main problems that arise in this context are the determination of $\delta(C)$ and $k(C)$ for a given C, and the description of the properties of $\delta(C, \cdot)$, $k(C, \cdot)$.

Let us mention some of the pioneers who contributed to this problem: Kepler, Gauss, Korkin and Zolotarev, Minkowski, Voronoi, Blichfeldt, Hlawka, Leech, Sloane.

Here we expose the major results for a general convex body C related to δ. Mahler proved that $\delta(C)$ is attained. The known bounds for $\delta(C)$ are $2^{-d+o(d)} \leq \delta(C) \leq 1$; they are due to Minkowski–Hlawka and Minkowski, respectively. Although the lower bound seems small, it is conjectured to be close to the true value. An algorithm for $\delta(P)$, where P is a convex polytope in $\mathbb{R}^3$, has been given by Betke–Henk.

Minkowski gave the general upper bound for the kissing number, $k(C) \leq 3^d - 1$, which can be shrunk to $k(C) \leq 2^{d+1} - 2$ if C is strictly convex. The lower bound $k(C, L) \geq d(d + 1)$, when the lattice packing $C + l : l \in L$ has maximum density, is due to Swinnerton-Dyer. Gruber showed that $k(C, L) \leq 2d^2$ for most C if the lattice packing $C + l : l \in L$ has maximum density. He also conjectured the equality for most C.

There are improved results for the particular case of the d-dimensional ball B^d. Concerning $\delta(B^d)$, Blichfeldt proved the upper bound $\delta(B^d) \leq 2^{-0.5d+o(d)}$, which was tightened to $\delta(B^d) \leq 2^{-0.599d+o(d)}$ when $d \to \infty$ by Kabatjanski–Levenstein. Rush built lattices such that $\delta(B^d, L) = 2^{-d+o(d)}$ by using error-correcting codes.

Let us observe that the gaps get much closer. The last result by Rush shows that one can, by an appropriate error-correcting code, construct a dense ball packing. However, the proof of the existence of this code is non-constructive: the construction of such codes is an issue in coding theory.

Regarding $k(B^d)$, Coxeter proved that $k(B^d) \leq 2^{0.5d+o(d)}$, which was improved to $k(B^d) \leq 2^{0.4d+o(d)}$ by Kabatjanski–Levenstein. On the other hand, Wyner showed that $k(B^d) \geq 2^{0.207d+o(d)}$.

Some of the most prominent open problems and questions in this area are the following:

- What is $\delta(C)$ for an "average" convex body? First, one has to specify a measure on the locally compact space of convex bodies to talk about average.
- Is the Minkowski–Hlawka bound for $\delta(C)$ optimal? Is it the right order of magnitude?
- Specify an effective algorithm to determine $\delta(C)$.
- Is for "most" convex bodies the densest lattice packing unique? Is it connected?
- Which is the right asymptotic function for $\delta(B^d)$, $k(B^d)$ as $d \to \infty$?
- As the dimension grows, the size of the gaps in packing using balls becomes larger. Are non-lattice packings better to fill $\mathbb{E}^d$?

1.3 The lattice covering problem

As in the previous section, let C be a 0-symmetric convex body and L a lattice. We say that $C + l : l \in L$ is a *lattice covering* of C with covering lattice L if each point in $\mathbb{E}^d$ belongs to, at least, one translate of C.

We define $\mathcal{V}(C, L) = V(C)/d(L)$ as the *density* of the lattice covering and $\mathcal{V}(C)$ as the minimum density.

The *star number*, $st(C, L)$, and the minimum star number, $st(C)$, are the lattice covering analogues to the kissing number: without losing the covering property, shrink C to C' and determine the number of translates of C' that a given copy intersects.

The main problems here are the determination of $\mathcal{V}(C)$, $\mathcal{V}(C)$, $st(C, L)$ and $st(C)$. Some of the pioneers in studying these problems are Kershner, Fejes Tóth, Ryshkov and Baranovskii, and Rogers.

The first major result for $\mathcal{V}(C)$ is due to Mahler, who found that $\mathcal{V}(C)$ is attained. Rogers gave the bounds $1 \leq \mathcal{V}(C) \leq d^{\log_2(d+o(d))}$. In a joint work with Erdős, Rogers proved that $st(C) \geq 2^{d+1} - 1$. Let us mention that Hadwiger–Wills provided some criteria to be fulfilled by a convex body and a lattice to be a covering or a packing.

If we concretize $C = B^d$, Coxeter–Few–Rogers found that

$$\mathcal{V}(B^d) \gtrsim \frac{d}{e\sqrt{e}}.$$

Before proceeding further, let us list some problems and questions that remain open:

- What are the bodies C with maximum $\mathcal{V}(C)$? Are they ellipsoids?
- Is the thinnest covering with C unique for "most" convex bodies?
- What is $\mathcal{V}(C)$ for an "average" C?
- Can Rogers' bound be improved?
- What is the right asymptotic function for $\mathcal{V}(B^d)$ as $d \to \infty$?

1.4 The lattice and non-lattice tiling problem

A lattice tiling is a family of proper convex or unbounded proper convex bodies in $\mathbb{E}^d$ that is both a lattice packing and a lattice covering.

Two examples are the Dirichlet–Voronoi tiling and its dual, the Delone tiling. In the Dirichlet–Voronoi lattice tiling, each tile is formed by the elements in $\mathbb{E}^d$ nearer to a lattice point.

Some of the authors who first studied this kind of problems are Dirichlet, Fedorov, Voronoi, Hajós, Delone, Venkov, Alexandrov, McMullen.

In this area there are mostly particular results with a few systematic and basic results. Venkov–Alexandrof–McMullen proved that a convex polytope that tiles by translations also tiles by a lattice, i.e., it is a parallelohedron. Gruber–Ryshkov showed that a locally finite facet-to-facet tiling is face-to-face and, thus, gives rise to a polyhedral cell complex.

The list of open problems related with lattice tilings includes:

- Describe the parallelohedra space fillers. This has been done for $d = 2, 3, 4$.
- Solve the conjecture of Voronoi: all the space fillers are, in essence, Voronoi cells. It has also been shown for $d = 2, 3, 4$.

1.5 An idea of Voronoi

Let q be a quadratic form. Then

$$q = \sum a_{ik}\, x_i\, x_k,$$

and we can identify q with the matrix $A = (a_{ik})$ with $a_{ik} = a_{ki}$. Thus, A can be viewed as the point $(a_{11}, \dots, a_{1d}, a_{22}, \dots, a_{dd}) \in \mathbb{E}^{\frac{1}{2}d(d+1)}$.

In this way the problems on quadratic forms can be seen as geometric problems in $\mathbb{E}^{\frac{1}{2}d(d+1)}$ and viceversa. An example of this translation is the Voronoi criterion for extreme forms.

Let q be a positive definite quadratic form. We say that q is *extreme* if

$$\frac{\min\left\{q(u) \,:\, u \in \mathbb{Z}^d \setminus \{0\}\right\}}{\det^{2/d} q} \geq \frac{\min\left\{p(u) \,:\, u \in \mathbb{Z}^d \setminus \{0\}\right\}}{\det^{2/d} p}$$

for all positive definite quadratic forms p close to q in $\mathbb{E}^{\frac{1}{2}d(d+1)}$. The expression $\det p$ denotes the determinant of the matrix B that is associated to the quadratic form p.

Theorem 1.5.1 (Voronoi)**.** *Let q be a positive definite quadratic form. Then q is extreme if and only if q is perfect and eutactic.*

The proof presented here will use modern geometric tools. The first one is the *Ryshkov polyhedron:*

$$\mathcal{R}^d(m) = \bigcap_{u \in \mathbb{Z}^d \setminus \{0\}} \left\{p = (b_{11}, \dots, b_{dd}) \in \mathbb{E}^{\frac{1}{2}d(d+1)} \;:\; \sum b_{ik} u_i u_k \geq m, p \text{ pos. def.}\right\}.$$

It is the family of all the positive definite quadratic forms with the arithmetic minimum larger than m.

We can also define the *discriminant body*,

$$\mathcal{D}^d(\delta) = \{p = (b_{11}, \dots, b_{dd}) \in \mathbb{E}^{\frac{1}{2}d(d+1)} \text{ pos. def. } : \det p \geq \delta\},$$

which is a smooth, strictly convex, unbounded body that consists of all the elements with a determinant larger than δ.

A positive definite quadratic form q is *perfect* if it is uniquely determined by its minimum vectors and the value of the minimum, which is equivalent to being a vertex of the Ryshkov polyhedron. The form q is said to be *eutactic* if the following holds: Let $(b_{ik}) = (a_{ik})^{-1}$; then

$$(b_{11},\, 2b_{12}, \dots, 2b_{1d},\, b_{22},\, 2b_{23}, \dots, b_{dd}),$$

the normal vector of the discriminant surface $\mathcal{D}^d(\delta)$ at its point q, is a linear combination with positive coefficients of the vectors

$$(u_1^2,\, 2u_1u_2, \ldots, 2u_1u_d,\, u_2^2,\, 2u_2u_3, \ldots, u_d^2),$$

where $\pm(u_1, \ldots, u_d) \in \mathbb{Z}^d$ ranges over the minimum vectors of q. These vectors are the normal vectors of the facets of the Ryshkov polyhedron $\mathcal{R}(m)$ which contains the boundary point q of $\mathcal{R}(m)$ and thus generate the normal cone of $\mathcal{R}(m)$ at q.

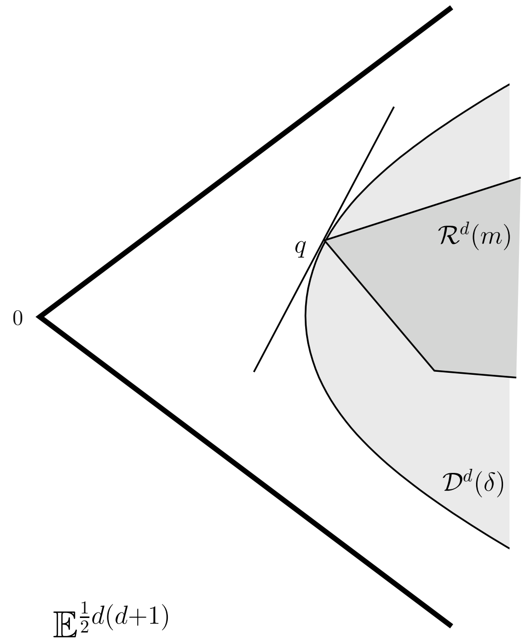

Figure 1.1: Relation between $\mathcal{R}$ and $\mathcal{D}^d$ at q.

Proof of Theorem 1.5.1. Let q be a d-dimensional extreme positive definite quadratic form with minimum m and determinant δ. Then q belongs to the intersection of the boundaries of $\mathcal{R}^d(m)$ and $\mathcal{D}^d(\delta)$, $q \in \operatorname{bd} \mathcal{R}^d(m) \cap \operatorname{bd} \mathcal{D}^d(\delta)$ (see Figure 1.1).

This happens if and only if all points $p \in \operatorname{bd} \mathcal{R}^d(m)$ close to q have a larger determinant than q. This means that, close to q, $\mathcal{R}^d(m)$ is contained in $\mathcal{D}^d(\delta)$.

As $\mathcal{D}^d(\delta)$ is smooth and strictly convex, q is a vertex of $\mathcal{R}^d(m)$ and the tangent hyperplane of $\mathcal{D}^d(\delta)$ at q is a support hyperplane of $\mathcal{R}^d(m)$ meeting $\mathcal{R}^d(m)$ precisely at q. These are the notions perfect and eutactic for q. $\square$

A list of some of the modern applications of this idea include: quadratic forms (mostly studied by the French school), the Epstein zeta function (English and Russian schools and others), Riemannian manifolds (mainly Swiss and French mathematicians), minimum position problems, lattice packings of convex bodies, and lattice zeta functions.

1.6 Lattice packing of balls

Let L be a lattice and $d(L)$ its determinant. Let B^d denote the unit ball. The *packing radius* can be defined as

$$\rho(B^d, L) = \max\left\{\rho > 0 \;:\; \left\{\rho B^d + l : l \in L\right\} \text{ is a lattice packing}\right\}.$$

Hence, $\left\{\rho(B^d, L)\; B^d + l : l \in L\right\}$ is a lattice packing of balls corresponding to L and $\delta(B^d, L) = \rho(B^d, L)^d\, V(B^d)/d(L)$ denotes its density.

We say that L is an *extreme lattice* if $\delta(B^d, (I + A)L) \leq \delta(B^d, L)$ for all $A \in \mathbb{E}^{\frac{1}{2}d(d+1)}$ in a neighborhood of 0. This is equivalent to just looking for the matrices $A \in \mathbb{E}^{\frac{1}{2}d(d+1)}$ such that $\operatorname{tr} A = 0$ in a neighborhood of 0.

The *minimum points* of a lattice are the elements in the set

$$M = \{\pm l_1, \ldots, \pm l_k\} \subset L \setminus \{0\}$$

such that $\|l_i\| = \min\{\|l\| \,:\, l \in L \setminus \{0\}\}$.

Let $l \otimes l = l\, l^t \in \mathcal{M}_{d\times d}$ denote the tensor product. We say that L is *perfect* if $\mathbb{E}^{\frac{1}{2}d(d+1)} = \operatorname{lin}\{l_1 \otimes l_1, \ldots, l_k \otimes l_k\}$, where "lin" stands for the linear expansion of the matrices. A lattice L is *eutactic* if $I = \lambda_1 l_1 \otimes l_1 + \cdots + \lambda_k l_k \otimes l_k$, with $\lambda_i > 0$.

With these notions we can state Voronoi's theorem in the ball packing version:

Theorem 1.6.1. *Let L be a lattice. Then L is extreme if and only if L is perfect and eutactic.*

We can define refined extremum properties by slightly modifying the extreme notion. Let $\mathcal{T} = \{A \subseteq \mathbb{E}^{\frac{1}{2}d(d+1)} \,:\, \operatorname{tr} A = 0\}$. A lattice L is:

- *semi-stationary* if

$$\delta\left(B^d, (I + A)\, L\right) \leq \delta(B^d, L)\,(1 + o\,(\|A\|)) \text{ as } A \to 0,\; A \in \mathcal{T};$$

- *stationary* if

$$\delta\left(B^d, (I + A)\, L\right) = \delta(B^d, L)\,(1 + o\,(\|A\|)) \text{ as } A \to 0,\; A \in \mathcal{T};$$

- *extreme* if

$$\delta\left(B^d, (I+A)\,L\right) \leq \delta(B^d, L) \text{ as } A \to 0,\ A \in \mathcal{T};$$

- *ultra extreme* if, for some constant c,

$$\delta\left(B^d, (I+A)\,L\right) \leq \delta(B^d, L)\,(1 - c\|A\|) \text{ as } A \to 0,\ A \in \mathcal{T},$$

where $\|A\| = (\sum_{i,k}^d a_{ik}^2)^{\frac{1}{2}}$, and an inequality or an equality holds as $A \to 0$ if it holds for all A close to 0.

The semi-stationary lattices in dimension 2 are the square lattice (tp) and the hexagonal lattice (hp). In dimension 3, the semi-stationary lattices are the following: the cubic primitive (cP), the hexagonal primitive (hP), the cubic face centered (cF), the cubic body centered (cI), and the tetragonal body centered (tI). The short form in brackets corresponds to the Bravais type.

For general d, there are finitely many similarity classes of semi-stationary lattices. It can be shown that:

Theorem 1.6.2. *Let L be a lattice. Then L is ultra extreme if and only if L is perfect and eutactic.*

Consequently, each extreme lattice is also ultra extreme.

In dimension 2, the only ultra extreme lattice is the hexagonal one (hp). The only ultra extreme lattice in dimension 3 is the cubic face centered one (cF). The graph of $\delta(B^3, \cdot)$ is shown in Figure 1.2. Observe that it is not smooth at the semi-stationary lattices.

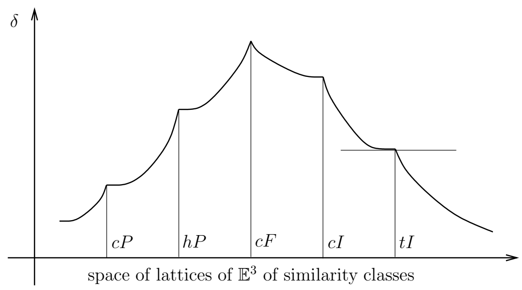

Figure 1.2: Graph of $\delta(B^3, \cdot)$.

Let us point out that some sufficient conditions for ultra extremality have been given using spherical designs and symmetric groups. Other interesting results in this setting include extensions of this theory to lattice packings of smooth convex bodies.

1.7 The Epstein zeta function

Let L be a lattice with $d(L) = 1$. We define

$$\zeta(L,s) = \sum_{l\in L\setminus\{0\}} \frac{1}{\|l\|^s}, \quad \text{for } s > d,$$

as the *Epstein zeta function*.

The Epstein zeta function has important applications in crystal physics, hydrodynamics, numerical integration and many other areas. The main problem is to determine, for fixed $s > d$, for all sufficiently large s or for all $s > d$, those lattices of determinant 1 for which $\zeta(L,s)$ is minimum.

A layer $\mathcal{L} = \{\pm l_1, \dots, \pm l_k\}$ of L is said to be *equi eutactic* if $\lambda I = \sum_i l_i \otimes l_i$, which is equivalent to

$$\sum_i (l_i \cdot x)^2 = \lambda \|x\|^2 \ \text{ for } x \in \mathbb{E}^d.$$

Furthermore, we said it is *Equi Eutactic* if

$$\sum_i (l_i \cdot x)^4 = \mu \|x\|^4 \ \text{ for } x \in \mathbb{E}^d.$$

The lattice L is said to be *total equi eutactic* for s if

$$\sum_{l\in L\setminus\{0\}} \frac{(l\cdot x)^2}{\|l\|^{s+2}} = \frac{\zeta(L,s)}{d}\|x\|^2 \ \text{ for } x \in \mathbb{E}^d,$$

or *Total Equi Eutactic* for s if

$$\sum_{l\in L\setminus\{0\}} \frac{(l\cdot x)^4}{\|l\|^{s+4}} = \frac{3\,\zeta(L,s)}{d(d+2)}\|x\|^4 \ \text{ for } x \in \mathbb{E}^d.$$

Once again, we define refined extremum (minimum) properties. Letting $\mathcal{T} = \{A \subseteq \mathbb{E}^{\frac{1}{2}d(d+1)} : \ \operatorname{tr} A = 0\}$, we say that a lattice L is:

- *stationary* if

$$\zeta\left(\frac{I+A}{\det(I+A)^{\frac{1}{d}}}\,L, s\right) = \zeta(L,s)\left(1 + o\left(\|A\|\right)\right) \ \text{ as } A \to 0, \ A \in \mathcal{T};$$

- *semi-stationary* if

$$\zeta\left(\frac{I+A}{\det(I+A)^{\frac{1}{d}}}\,L, s\right) \geq \zeta(L,s)\left(1 + o\left(\|A\|\right)\right) \ \text{ as } A \to 0, \ A \in \mathcal{T};$$

- *minimum* if

$$\zeta\left(\frac{I+A}{\det(I+A)^{\frac{1}{d}}}\,L,s\right)\geq\zeta(L,s)\text{ as }A\to 0,\ A\in\mathcal{T};$$

- *quadratic minimum* if, for some constant c,

$$\zeta\left(\frac{I+A}{\det(I+A)^{\frac{1}{d}}}\,L,s\right)\geq\zeta(L,s)\left(1+cs^2\|A\|^2\right)\text{ as }A\to 0,\ A\in\mathcal{T},$$

where an inequality or an equality holds as $A \to 0$ if it holds for all A close to 0.

At this point we can state the following result.

Theorem 1.7.1. *Let L be a lattice with $d(L) = 1$. Then L is stationary for s if and only if L is total equi eutactic for s.*

Consequently, each semi-stationary lattice is also stationary. The stationary lattices for dimension 2 are the square lattice (tp) and the hexagonal lattice (hp). In dimension 3 we can list the cubic primitive (cP), the cubic face centered (cF), and the cubic body centered lattice (cI).

It can also be shown that:

Theorem 1.7.2. *Let L be a lattice. Then L is a quadratic minimum for s if and only if L is total eutactic for s and*

$$\sum\frac{(A\cdot l\otimes l)^2}{\|l\|^{s+4}}>\frac{2\zeta}{d(s+2)}\|A\|^2\quad\textit{for } A\in\mathcal{T}.$$

For dimension 2 and 3, the quadratic minimum lattices are the hexagonal (hp) and the cubic face centered lattice (cF).

Figure 1.3 illustrates the graph of $\zeta(\cdot, s)$ for dimension 3. We can see the quadratic minimum point at the cubic face centered lattice.

These results help us to show that many of the lattices in the literature are stationary or quadratic minimum for all $s > d$. Some notorious examples are the lattices which, for $d = 2, 3, \ldots, 8$ and 24, provide the densest lattice packings of balls. For $d = 24$, this lattice is the Leech lattice, which is an ultra extreme and quadratic minimum lattice and was constructed using error-correcting codes.

Various sufficient conditions for stationary and quadratic minimality use spherical designs and symmetry groups.

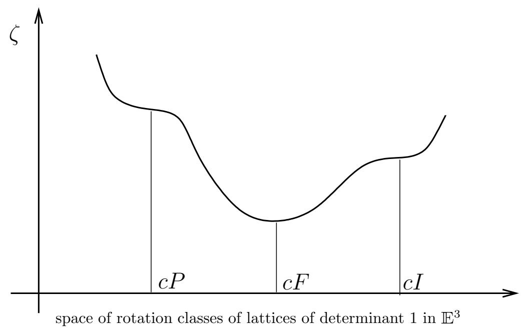

Figure 1.3: Graph of $\zeta(\cdot, s)$ for $d = 3$.

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Seminar 6

Ehrhart Polynomials and Frobenius Numbers

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Introduction

This talk is split into two parts. The first one covers recent developments in the realm of Ehrhart Theory and the second one investigates the average behavior of large Frobenius numbers. These topics are related via the concept of Minkowski's successive minima.

1.1 Ehrhart polynomials

1.1.1 Basic definitions and notations

To introduce the audience to the theory, we first have a look at the basic definitions and notations.

For linearly independent $b_1, \dots, b_k \in \mathbb{R}^n$, the set

$$\Lambda = \left\{ \sum_{i=1}^{k} b_i z_i \ : \ z_i \in \mathbb{Z} \right\}$$

is called a *(k-dimensional) lattice*, and $B = (b_1, \dots, b_k)$ is said to be a *basis* of the lattice $\Lambda = B\mathbb{Z}^k$. The k-dimensional volume of the parallelepiped

$$\left\{\sum_{i=1}^{k} \mu_i b_i \ : \ 0 \leq \mu_i \leq 1\right\}$$

is called the *determinant* of Λ and denoted by $\det \Lambda$. This quantity is independent of the chosen basis B of Λ, and $\det \Lambda = \sqrt{\det B^{\mathrm{t}} B}$.

We will mostly stick to the *integral lattice* $\mathbb{Z}^n$, which has determinant 1. Another example is the hexagonal lattice Λ_{hex} in the plane, that can be described by the basis vectors $b_1 = \binom{2}{0}$ and $b_2 = \binom{1}{\sqrt{3}}$. Its determinant is equal to $2\sqrt{3}$.

We write $\mathcal{P}^n$ for the set of *lattice polytopes* $P \subset \mathbb{R}^n$ with non-empty interior, i.e.,

$$P = \operatorname{conv}\{v_1, \ldots, v_m\} \text{ for some } v_1, \ldots, v_m \in \mathbb{Z}^n, \text{ and } \operatorname{int}(P) \neq \emptyset.$$

The standard simplex $T_n = \operatorname{conv}\{0, e_1, \ldots, e_n\}$, the unit cube $C_n = [-1, 1]^n$ and its polar body, and the crosspolytope $C_n^\star = \operatorname{conv}\{\pm e_1, \ldots, \pm e_n\}$ are well-known lattice polytopes.

The main object under consideration in Ehrhart Theory is the so-called *lattice point enumerator* $G(S) = \#(S \cap \mathbb{Z}^n)$, that counts the integral points in a given subset S of $\mathbb{R}^n$. The notation has historical reasons, since it goes back to the famous “Gaußsches Kreisproblem” (Gauss’ circle problem), which asks for the number of lattice points in a circle of given radius.

1.1.2 Ehrhart’s Theorem

We are interested in the lattice point enumerator for integer dilates of a given lattice polytope $P \in \mathcal{P}^n$, i.e., in the function

$$k \longmapsto G(kP) = \#(kP \cap \mathbb{Z}^n), \ k \in \mathbb{N}.$$

The French high school teacher Eugène Ehrhart proved that this function is a polynomial. To his honour it is called the *Ehrhart polynomial* of P, and its coefficients only depend on the polytope.

Theorem 1.1.1 (Ehrhart, 1967)**.** *Let $P \in \mathcal{P}^n$ and $k \in \mathbb{N}$. Then*

$$G(kP) = \sum_{i=0}^{n} G_i(P) k^i.$$

Proof. The proof splits into the following three principal steps and is not given in detail here.

1. Prove the claim for lattice simplices $\Gamma = \operatorname{conv}\{0, v_1, \dots, v_n\}, v_i \in \mathbb{Z}^n$.

2. Triangulate P into lattice simplices $T_i, 1 \le i \le m$. In fact, there is always a triangulation on the vertex set of P only.

3. Apply the inclusion-exclusion principle to handle multiple counting of lattice points that lie in faces of the simplices T_i to obtain

$$G(kP) = G\left(\bigcup_{i=1}^{m} kT_i\right) = \sum_{I \subseteq \{1,\dots,m\}} (-1)^{\# I - 1} G\left(\bigcap_{j \in I} kT_j\right).$$

Since in the last term only simplices occur as sets to count in, we can derive the statement of the theorem by plugging in the corresponding polynomials from the first step. □

A polynomial is determined by its coefficients; therefore, the first natural question is whether we can describe the $G_i(P)$.

- $G_0(P) = 1$, since plugging in $k = 0$ leads to the set containing only the origin.

- $G_n(P) = \operatorname{vol}(P)$; thus, for large dilates of P the number of lattice points approximates the volume of P. This identity can be seen as follows:

$$\operatorname{vol}(P) = \lim_{k\to\infty} \left(\tfrac{1}{k}\right)^n \#(P \cap \tfrac{1}{k}\mathbb{Z}^n) = \lim_{k\to\infty} \left(\tfrac{1}{k}\right)^n \#(kP \cap \mathbb{Z}^n) = G_n(P).$$

- Also due to Ehrhart, the second last coefficient is given by the *normalized (lattice) surface area*, i.e.,

$$G_{n-1}(P) = \frac{1}{2} \sum_{F \text{ facet of } P} \frac{\operatorname{vol}_{n-1}(F)}{\det(\operatorname{aff}(F) \cap \mathbb{Z}^n)},$$

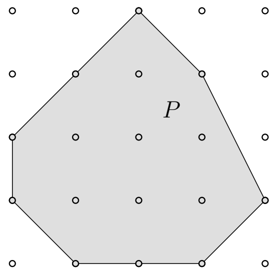
where $\operatorname{aff}(F)$ denotes the affine hull of F.

For $n = 2$ we therefore know all the coefficients and we can state a formula, which was already known to Pick in 1899 (*Pick's Theorem*):

$$G(kP) = \operatorname{vol}(P)k^2 + \frac{1}{2}\#(\operatorname{bd}(P) \cap \mathbb{Z}^2)k + 1.$$

To illustrate this, consider the following example: By Pick's theorem, the lattice point enumerator of this polygon is given by

$$G(kP) = \frac{21}{2}k^2 + \frac{9}{2}k + 1, \text{ and therefore } G(P) = 16.$$



There are formulas in the literature describing the remaining coefficients via generating functions or via the Todd differential operator. But the question if they also admit a geometric interpretation is open.

From the computational point of view, there is an interesting result by Barvinok from 1994. For fixed dimension, he gives a polynomial time algorithm to compute $G(P)$.

Next, we see what is known about the coefficients for particular classes of lattice polytopes. The first result is due to Stanley (1980) and Betke and Gritzmann (1986). They obtained a geometric description for all the coefficients $G_i(Z)$ for lattice zonotopes Z.

Theorem 1.1.2. *Let $Z \in \mathcal{P}^n$ be a lattice zonotope, i.e., there exist integer vectors $a_i \in \mathbb{Z}^n, 1 \le i \le m$, such that*

$$Z = \mathrm{conv}\{0, a_1\} + \mathrm{conv}\{0, a_2\} + \cdots + \mathrm{conv}\{0, a_m\}.$$

Then

$$G_i(Z) = \sum_{1 \le j_1 < \cdots < j_i \le m} \frac{\sqrt{\det((a_{j_1} \ldots a_{j_i})^{\mathrm{t}}(a_{j_1} \ldots a_{j_i}))}}{\det(\mathrm{lin}\{a_{j_1}, \ldots, a_{j_i}\} \cap \mathbb{Z}^n)}.$$

In 2005, Liu showed that the coefficients of cyclic polytopes are given by volumes of projections on the first i coordinates. This proved a conjecture by Beck, De Loera, Develin, Pfeifle and Stanley from 2004. She also generalized this statement in 2008 to a bigger class of polytopes, the so-called *lattice face polytopes*.

Theorem 1.1.3. *Let $C(n, m) \in \mathcal{P}^n$ be a cyclic polytope with m integral vertices on the moment curve $t \mapsto (t, t^2, \ldots, t^n)$. Then*

$$G_i(C(n, m)) = \mathrm{vol}_i(C(n, m)|\mathbb{R}^i).$$

The results for lattice zonotopes and cyclic polytopes suggest that the coefficients might be non-negative. However, the Reeve simplices show that this is not the case in general. For $n = 3$ and $l \in \mathbb{N}$, consider the simplex

$R(l) = \mathrm{conv}\{0, e_1, e_2, (1, 1, l)^{\mathrm{t}}\}$. For all values of l it contains exactly 4 lattice points and its volume is given by $\mathrm{vol}(R(l)) = l/6$. Since we know that G_2, as the lattice surface area, is non-negative, and $G_0 = 1$, the second coefficient $G_1(R(l))$ has to be negative for large l, and indeed this happens for $l \geq 13$, since $G_1(R(l)) = (12 - l)/6$.

It would be interesting to find at least partial answers to the following problems.

Open Problems 1.1.4.

1. Characterize all lattice polytopes with non-negative Ehrhart coefficients (seems to be intractable!).

2. Find more classes of "non-negative" lattice polytopes.

Since $G(kP)$ is a polynomial in k, it is also defined on negative values. And, in fact, when we plug in negative integers, we get another geometric quantity, as again already Ehrhart discovered.

Theorem 1.1.5 (Ehrhart's Reciprocity Law, 1967)**.** *Let $P \in \mathcal{P}^n$ and $k \in \mathbb{N}$. Then*

$$G(\mathrm{int}(kP)) = (-1)^n \sum_{i=0}^{n} G_i(P)(-k)^i.$$

For example, for $n = 2$ we get

$$G(\mathrm{int}(kP)) = \mathrm{vol}(P)k^2 - \frac{1}{2}\#(\mathrm{bd}(P) \cap \mathbb{Z}^2)k + 1.$$

As Betke and Kneser showed, the Ehrhart coefficients serve as a basis for a special class of functions on the set $\mathcal{P}^n$.

Definition 1.1.6. A function $\phi : \mathcal{P}^n \to \mathbb{R}$ is said to be

- a *valuation* if for all $P, Q \in \mathcal{P}^n$ with $P \cup Q \in \mathcal{P}^n$ it is $\phi(P \cup Q) = \phi(P) + \phi(Q) - \phi(P \cap Q)$;

- *unimodular invariant* if $\phi(AP) = \phi(P)$ for all $A \in \mathbb{Z}^{n \times n}$, $\det(A) \neq 0$.

Theorem 1.1.7 (Betke–Kneser, 1985)**.** *Let ϕ be a unimodular invariant valuation on $\mathcal{P}^n$. Then there are constants $\alpha_i \in \mathbb{R}$ such that*

$$\phi(P) = \sum_{i=0}^{n} \alpha_i G_i(P) \quad \textit{for all } P \in \mathcal{P}^n.$$

As we have seen above, we lack of exact descriptions for the coefficients. This is why people are interested in the task to at least bound them by some other geometric quantities. The best known bounds in terms of the volume are stated in the next theorem, where the symbols $s(n,i)$ denote the Stirling numbers of the second kind.

Theorem 1.1.8. *For $P \in \mathcal{P}^n$ and $1 \le i \le n$, we have:*

1. [Betke–McMullen, 1985]

$$G_i(P) \le (-1)^{n-i} s(n,i)\,\mathrm{vol}(P) + (-1)^{n-i-1}\frac{s(n,i+1)}{(n-1)!};$$

2. [Henk–Tagami, 2008]

$$G_i(P) \ge \frac{1}{n!}\left((-1)^{n-i} s(n+1,i+1) + (n!\,\mathrm{vol}(P)-1)M_{i,n}\right),$$

where the $M_{i,n}$ are some constants only depending on i and n.

The *a*-vector of a lattice polytope

Examples like the standard crosspolytope (see below) suggest that sometimes it is more convenient to consider coefficients of the Ehrhart polynomial subject to another basis. We exchange the monomial basis $\{x^i \mid i = 0,\dots,n\}$ for a binomial basis $\left\{\binom{x+n-i}{n} \mid i = 0,\dots,n\right\}$ and write

$$G(kP) = \sum_{i=0}^{n} G_i(P)k^i = \sum_{i=0}^{n} a_i(P)\binom{k+n-i}{n}.$$

The vector $a = (a_0, a_1, \dots, a_n)$ is called the *a-vector* of P. The $G_i(P)$ are homogeneous of degree i, i.e., $G_i(mP) = m^i G_i(P)$ for $m \in \mathbb{N}$, which is no longer the case for the $a_i(P)$. But in exchange we get some other nice properties.

Theorem 1.1.9. *Let $P \in \mathcal{P}^n$. Then*

1. $a_0 = 1$, $a_1 = G(P) - (n+1)$, $a_n = G(\mathrm{int}P)$ *and* $a_0 + a_1 + \cdots + a_n = n!\,\mathrm{vol}(P)$.

2. $a_i(P)$ *is integral for all* $0 \le i \le n$.

3. [Stanley's non-negativity theorem, 1987] $a_i(P) \ge 0$, $0 \le i \le n$.

4. [Stanley's monotonicity theorem, 1993] *For all* $Q \in \mathcal{P}^n$ *such that* $P \subseteq Q$, *we have* $a_i(P) \leq a_i(Q), 0 \leq i \leq n$.

Let us have a look at some examples to contrast the two types of coefficients. What we see is that it depends on the polytope which coefficients are the more convenient to deal with.

- For the *standard simplex* $T_n = \mathrm{conv}\{0, e_1, \ldots, e_n\}$, $G(kT_n) = \binom{k+n}{n}$ and therefore
$$a_0(T_n) = 1,\ a_i(T_n) = 0,\ 1 \leq i \leq n,$$
and
$$G_i(T_n) = (-1)^{n-i}\,\frac{s(n+1, i+1)}{n!},\ 1 \leq i \leq n.$$
- For the *cube* $C_n = [-1, 1]^n$, we have $G(kC_n) = (2k+1)^n$ and thus
$$G_i(C_n) = 2^i \binom{n}{i},\ 0 \leq i \leq n.$$
- The *crosspolytope* $C_n^\star = \mathrm{conv}\{\pm e_1, \ldots, \pm e_n\}$, as the polar body to the cube, has Ehrhart polynomial $G(kC_n^\star) = \sum_{i=0}^{n} \binom{n}{i} \binom{k+n-i}{n}$ and so
$$a_i(C_n^\star) = \binom{n}{i},\ 0 \leq i \leq n.$$

1.1.3 Roots of Ehrhart polynomials

To get more information about the Ehrhart coefficients of a lattice polytope $P \in \mathcal{P}^n$, people are interested in the roots of the Ehrhart polynomial considered as a function on the whole complex plane

$$G(s, P) = \sum_{i=0}^{n} G_i(P) s^i = \prod_{i=1}^{n} \left(1 + \frac{s}{\gamma_i(P)}\right),\ s \in \mathbb{C}.$$

With this representation, the roots of $G(s, P)$ are $-\gamma_i(P), 1 \leq i \leq n$, and the Ehrhart coefficients are given by the elementary symmetric functions of $1/\gamma_i(P)$, $1 \leq i \leq n$, i.e.,

$$G_n(P) = \prod_{i=1}^{n} \frac{1}{\gamma_i(P)},\ G_{n-1}(P) = \sum_{i=1}^{n} \prod_{j \neq i} \frac{1}{\gamma_j(P)}, \ldots, G_1(P) = \sum_{i=1}^{n} \frac{1}{\gamma_i(P)},\ G_0(P) = 1.$$

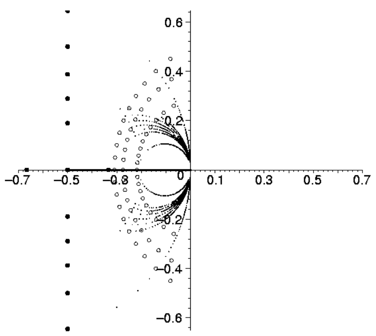
We have seen that the standard simplex T_n has Ehrhart polynomial $G(kT_n) = \binom{k+n}{n}$, so the roots are $-\gamma_i(T_n) = -i,\ 1 \leq i \leq n$. This can also be seen more geometrically by using the reciprocity law. Since $\mathrm{int}(kT_n) \cap \mathbb{Z}^n = \emptyset$ for $1 \le k \le n$, we have

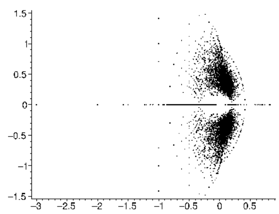
$$0 = G(\mathrm{int}(kT_n)) = (-1)^n \sum_{i=0}^{n} G_i(T_n)(-k)^i = (-1)^n G(-k, T_n)$$

and again we can extract the roots as $-\gamma_i(T_n) = -i,\ 1 \le i \le n$.

For the unit cube, we easily get from $G(s, C_n) = (2s+1)^n$ that the roots are given by $-\gamma_i(C_n) = -\frac{1}{2}, 1 \le i \le n$. For the crosspolytope, things get already complicated, since it is non-obvious how the roots of $G(s, C_n^\star) = \sum_{i=0}^{n} \binom{n}{i}\binom{s+n-i}{n}$ look like.

The first systematic study of the roots of Ehrhart polynomials was done in 2001 by Beck, de Loera, Develin, Pfeifle and Stanley. The following two pictures are taken from their paper and depict that there is a lot of structure in the set of roots of $G(s, P)$. The first one shows roots of 2-dimensional lattice polygons with real part $\ge -\frac{2}{3}$, and the second one roots of 100,000 random lattice simplices in $\mathbb{R}^3$.





In joint work with C. Bey and J. Wills, we found sharp bounds on the roots in the three-dimensional case. Here one can use Ehrhart's reciprocity law to overcome the lack of knowledge about the coefficient $G_1(P)$.

Theorem 1.1.10 (Bey–Henk–Wills, 2006)**.** *The roots of Ehrhart polynomials of* 3*-dimensional lattice polytopes are contained in*

$$[-3, -1] \cup \{a + ib \,:\, -1 \le a < 1,\, a^2 + b^2 \le 3\},$$

and the bounds on a *and* $a^2 + b^2$ *are tight.*

In a recent work from 2008, Pfeifle used Gale duality bounds for roots of polynomials with nonnegative coefficients to explain the behavior and the locations of the roots in the above pictures.

Beck et al. obtained the following general bounds on the roots of a given polytope $P \in \mathcal{P}^n$.

Theorem 1.1.11 (Beck et al., 2001)**.**

1. *The real roots of Ehrhart polynomials of n-dimensional lattice polytopes are contained in the interval $[-n, -n/2]$ and also the upper bound is (up to a constant) best possible.*

2. *Let s be a root of an Ehrhart polynomial of some $P \in \mathcal{P}^n$. Then*

$$|s| \leq (n+1)! + 1.$$

Braun improved on the second statement and derived a quadratic upper bound on the norm of the roots.

Theorem 1.1.12 (Braun, 2006)**.** *Let s be a root of an Ehrhart polynomial of some $P \in \mathcal{P}^n$. Then*

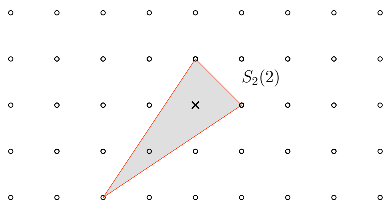
$$\left| s + \frac{1}{2} \right| \leq n \left(n - \frac{1}{2} \right).$$

Special lattice simplices

In the following, consider the lattice simplices

$$S_n(l) = \operatorname{conv} \left\{ e_1, \ldots, e_n, \; -l \sum_{i=0}^{n} e_i \right\}, \; l \in \mathbb{N}_0.$$

An instance in the planar case is shown in the picture below:



The simplex $S_n(l)$ contains exactly l interior lattice points and, as the subsequent theorem states, it minimizes the volume among all lattice polytopes with this number of interior lattice points.

Theorem 1.1.13 (Bey–Henk–Wills, 2006)**.** *Let $P \in \mathcal{P}^n$. Then*

$$\operatorname{vol}(P) \geq \frac{n\, G(\operatorname{int} P) + 1}{n!},$$

and the bound is best possible for any number of interior lattice points. For $G(\operatorname{int} P) = 1$, equality holds if and only if P is unimodularly equivalent to the simplex $S_n(1)$.

In 2008, Duong characterized the equality case completely for $n \geq 3$ and showed that volume minimal lattice polytopes with exactly l interior lattice points are unimodularly equivalent to $S_n(l)$.

Moreover, for $l = 1$ these simplices show that Braun's quadratic bound on the norm of roots of Ehrhart polynomials of n-polytopes is asymptotically best possible.

Theorem 1.1.14. *All roots of the polynomial $G(s, S_n(1))$ have real part $-\frac{1}{2}$. If $-\gamma_n$ is a root of $G(s, S_n(1))$ with maximal norm, then*

$$\left| -\gamma_n + \frac{1}{2} \right| = \frac{n(n+2)}{2\pi} + O(1) \text{ as } n \to \infty.$$

Remarks 1.1.15.

- Rodríguez-Villegas showed in 2002 that, if for a polytope $P \in \mathcal{P}^n$ all roots of the polynomial $\sum_{i=0}^{n} a_i(P)x^i$ have norm 1, then all roots $-\gamma_i$ of its Ehrhart polynomial have real part $-\frac{1}{2}$.
- For $n = 2, 3$, the polynomial $G(s, S_n(1))$ has roots of maximal norm among all Ehrhart polynomials of polytopes with interior lattice points.
- The a-vector of $S_n(l)$ is given by $a_i(S_n(l)) = l$, $1 \leq i \leq n$, and for any lattice polytope $P \in \mathcal{P}^n$ with $G(\text{int}P) = l$ it holds $a_i(P) \geq l$, $1 \leq i \leq n$.

Braun and Develin studied in 2007 roots of Ehrhart polynomials in the more general context of *S(tanley)N(on)N(egative)-polynomials.*

Zero symmetric lattice polytopes

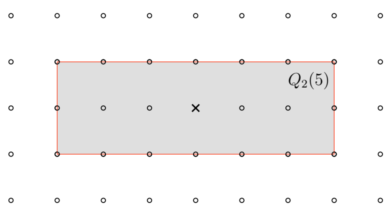
A natural question that arises is to ask for the zero symmetric relatives of the simplices $S_n(l)$, i.e., lattice polytopes $P \in \mathcal{P}^n$ with $P = -P$ that minimize the volume among all such polytopes with given number of interior lattice points. Maximizers are known in the zero symmetric case since Blichfeldt and van der Corput.

Theorem 1.1.16 (Blichfeldt, 1921; van der Corput, 1935)**.** *Let $P \in \mathcal{P}^n$ be zero symmetric. Then*

$$\text{vol}(P) \leq 2^{n-1}(G(\text{int}P) + 1).$$

Equality is attained for lattice boxes of the form

$$Q_n(2l-1) = \{x \in \mathbb{R}^n \ : \ |x_1| \leq l,\, |x_i| \leq 1,\, 2 \leq i \leq n\},\ l \in \mathbb{N}.$$

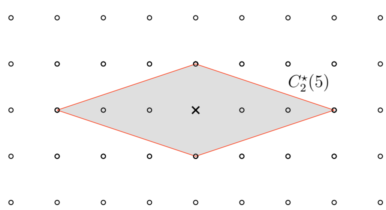


See the figure for a planar example.

In combinatorial type questions it is hard to exploit the additional property of zero symmetry and though it is still an open problem to give a sharp lower bound on the volume in terms of the number of interior lattice points. We conjecture that the lattice crosspolytopes

$$C_n^\star(2l-1) = \mathrm{conv}\{\pm l e_1, \pm e_2, \ldots, \pm e_n\},\ l \in \mathbb{N}$$

are minimizers.



Problem 1.1.17. Let $P \in \mathcal{P}^n$ be zero symmetric. Is it true that

$$\mathrm{vol}(P) \geq \frac{2^{n-1}}{n!}(G(\mathrm{int}P)+1)\ ?$$

Formulated in terms of a-vectors, this conjecture can be stated in a stronger version as

Conjecture 1.1.18. *Let $P \in \mathcal{P}^n$ be zero symmetric. Then*

$$a_i(P) \geq \binom{n}{i} + \binom{n-1}{i-1}(a_n - 1),\ 0 \leq i \leq n.$$

We have been able to prove this for the class of lattice crosspolytopes.

Proposition 1.1.19. *Let $P = \mathrm{conv}\{\pm v_1, \ldots, \pm v_n\} \in \mathcal{P}^n$ for some vectors $v_i \in \mathbb{Z}^n$. Then the preceding conjecture holds for P.*

As a corollary, the only known a-vector inequality for the zero symmetric case can be derived using Stanley's monotonicity theorem. Note that this inequality also follows from early works by Stanley (1987) and Betke and McMullen (1985).

Corollary 1.1.20. *Let $P \in \mathcal{P}^n$ be zero symmetric. Then $a_i(P) \geq \binom{n}{i}$ for all $0 \leq i \leq n$.*

We have already seen that the volume minimizers in the general case are known. Hibi deduced a theorem that covers the a-vector variant of this statement.

Theorem 1.1.21 (Hibi, 1992)**.** *Let $P \in \mathcal{P}^n$ with $a_n(P) = G(\mathrm{int}P) \geq 1$. Then $a_i(P) \geq a_1(P)$ for all $1 \leq i \leq n-1$.*

Note that, in general, it is not true that $a_i(P) \geq a_1(P)$ for all $1 \leq i \leq j$, if $a_j(P) > 0$.

Reflexive polytopes

The starting point for the subsequent discussion of reflexive polytopes will be a partial answer to the former question how the roots of the Ehrhart polynomial of the standard crosspolytope $C_n^\star$ look like. In different recent works (Kirschenhofer, Pethoe, Tichy, 1999; Bump, Choi, Kurlberg, Vaaler, 2000; Rodríguez-Villegas, 2002) it was shown that

$$\mathrm{Re}(-\gamma_i(C_n^\star)) = -\frac{1}{2},\ 1 \leq i \leq n.$$

In order to see how this result relates here, we call a lattice polytope $P \in \mathcal{P}^n$ *reflexive* if its polar $P^\star = \{y \in \mathbb{R}^n : x^\intercal y \leq 1,\ \forall x \in P\}$ is also a lattice polytope. Now, we have the following:

Proposition 1.1.22. *Let $P \in \mathcal{P}^n$. If all roots of $G(s, P)$ have real part $-\frac{1}{2}$, then, up to unimodular translation, P is a reflexive polytope of volume $\leq 2^n$.*

Does the converse also hold? In three dimensions the answer is affirmative, but for $n \geq 4$ we need more conditions.

Proposition 1.1.23. *Let $P \in \mathcal{P}^n$ be a reflexive polytope. Then all roots of $G(s, P)$ have real part $-\frac{1}{2}$*

- *if and only if* $\mathrm{vol}(P) \leq 2^n$ *and* $n \leq 3$,
- *if and only if* $(G(P) - 1 - 4\,\mathrm{vol}(P))^2 \geq 16\,\mathrm{vol}(P)$, $2G(P) \leq 9\,\mathrm{vol}(P) + 18$ *and* $n = 4$.

So, it is an interesting problem to characterize all those lattice polytopes having all the roots with real part $-\frac{1}{2}$. Kreuzer and Skarke (1998 and 2000) classified all reflexive polytopes in dimensions $n = 3, 4$, and a computer-aided computation of their roots suggests that there are only a few among them having a root with real part different from $-\frac{1}{2}$.

- For $n = 2$ there are 16 reflexive polytopes, and 15 of them have all their roots on the line with real part $-\frac{1}{2}$.
- For $n = 3$ there are 4,319 reflexive polytopes, and 4,255 of them have all their roots on the line with real part $-\frac{1}{2}$.
- For $n = 4$ there are 473,800,776 reflexive polytopes, and...

For further information and work on reflexive polytopes, we refer to Benjamin Nill, who devoted many studies to this special class.

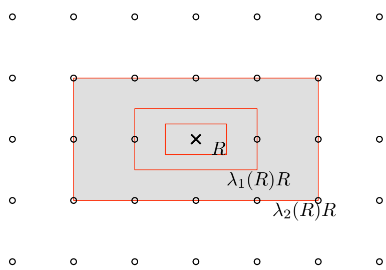
1.1.4 Successive minima and Ehrhart polynomials

This last section is devoted to quantities that Hermann Minkowski already introduced and their relations to Ehrhart theory. For a zero symmetric convex body K in $\mathbb{R}^n$, we define

$$\lambda_i(K) = \min\{\lambda > 0 \,:\, \dim(\lambda K \cap \mathbb{Z}^n) \geq i\}$$

as the *i-th successive minimum* of K, $1 \leq i \leq n$.

A first easy example is given by the Euclidean unit ball B^n, whose successive minima all equal $\lambda_i = 1$, since all unit vectors are contained in the boundary of B^n. For another one, consider the planar box $R = \{x \in \mathbb{R}^2 : |x_1| \leq \frac{1}{2}, |x_2| \leq \frac{1}{4}\}$. Here we have to multiply R by a factor of two to get the first non-trivial lattice point contained, i.e., $\lambda_1(R) = 2$, and similarly the fourth dilate gives two linear independent lattice points for the first time and therefore $\lambda_2(R) = 4$ (compare the picture below).



Minkowski proved two fundamental theorems for these successive minima that have many applications in various fields of mathematics. For example, the first one implies that any integer is representable as the sum of four squares.

Theorem 1.1.24 (Minkowski's 1st Theorem, 1896)**.** *Let $K \subset \mathbb{R}^n$ be a zero symmetric convex body. Then*

$$\mathrm{vol}(K) \leq \left(\frac{2}{\lambda_1(K)}\right)^n,$$

or, equivalently,

$$\textit{if } \operatorname{vol}(K) \geq 2^n \textit{ then } K \cap \mathbb{Z}^n \setminus \{0\} \neq \emptyset.$$

A reformulation for zero symmetric lattice polytopes $P \in \mathcal{P}^n$ can be given in terms of the roots of their Ehrhart polynomial $G(s, P)$ as

$$\left(\prod_{i=1}^{n} \gamma_i(P)\right)^{\frac{1}{n}} \geq \frac{\lambda_1(P)}{2}.$$

His second theorem generalizes the first one and also gives a lower bound on the volume in terms of the successive minima. Note that equality cases are given by the unit cube C_n and the standard crosspolytope $C_n^\star$ for the upper and the lower bound, respectively.

Theorem 1.1.25 (Minkowski's 2nd Theorem, 1896)**.** *Let $K \subset \mathbb{R}^n$ be a zero symmetric convex body. Then*

$$\frac{1}{n!}\prod_{i=1}^{n} \frac{2}{\lambda_i(K)} \leq \operatorname{vol}(K) \leq \prod_{i=1}^{n} \frac{2}{\lambda_i(K)}.$$

Again, this can be stated via roots of Ehrhart polynomials:

$$\left(n!\prod_{i=1}^{n} \frac{\lambda_i(P)}{2}\right)^{\frac{1}{n}} \geq \left(\prod_{i=1}^{n} \gamma_i(P)\right)^{\frac{1}{n}} \geq \left(\prod_{i=1}^{n} \frac{\lambda_i(P)}{2}\right)^{\frac{1}{n}}.$$

Therefore, the geometric mean of the negatives of the roots of $G(s, P)$ is bounded in terms of the geometric mean of the successive minima of P.

This leads us to the question if there is a corresponding bound on the arithmetic mean of the roots in terms of the arithmetic mean of the successive minima.

Theorem 1.1.26 (Henk–Schürmann–Wills, 2005)**.** *Let $P \in \mathcal{P}^n$ be a zero symmetric lattice polytope. Then*

$$\frac{1}{n}\left(\sum_{i=1}^{n} \gamma_i(P)\right) \leq \frac{1}{n}\left(\sum_{i=1}^{n} \frac{\lambda_i(P)}{2}\right),$$

and the bound is best possible, e.g., for the cube C_n and for the crosspolytope $C_n^\star$.

An equivalent version of this inequality is

$$\frac{G_{n-1}(P)}{\mathrm{vol}(P)} \leq \sum_{i=1}^{n} \frac{\lambda_i(P)}{2},$$

and, in contrast to Minkowski's 2nd theorem, there exists no lower bound on the arithmetic mean of the roots.

A main ingredient for the proof of this result is the following lemma, which is interesting on its own.

Lemma 1.1.27. *Let P be a zero symmetric polytope with facets*

$$F_i = P \cap \{x \in \mathbb{R}^n \,:\, a_i^{\mathrm{t}} x = b_i\}$$

for $|a_i| = 1$. Furthermore, let L be a k-dimensional linear subspace and let $I_L = \{i \,:\, a_i \in L\}$. Then

$$\mathrm{vol}(P) \geq \frac{1}{k} \sum_{i \in I_L} \mathrm{vol}_{n-1}(F_i) b_i.$$

Towards a generalization of Minkowski's 2nd theorem

The fundamental theorems of Minkowski relate the successive minima to the volume of the zero symmetric convex body K. We ask if there is a similar relation to the "discrete volume", i.e., the lattice point enumerator $G(K)$.

Problem 1.1.28 (Betke–Henk–Wills, 1993). Let $K \subset \mathbb{R}^n$ be a zero symmetric convex body. Is it true that

$$G(K) \leq \prod_{i=1}^{n} \left(\frac{2}{\lambda_i(K)} + 1 \right) ?$$

The additive constant 1 that appears in the factors on the right-hand side could be exchanged for any other constant. An important observation is that this inequality would imply Minkowski's 2nd theorem. This can be seen again by using $\mathrm{vol}(K) = \lim_{k \to \infty} \left(\frac{1}{k}\right)^n G(kK)$ and the additional fact that the successive minima are homogeneous of degree -1, i.e., $\lambda_i(kK) = \frac{1}{k}\lambda_i(K),\ 1 \leq i \leq n$.

The state of the art concerning this problem is the following.

- It holds in the planar case $n = 2$.
- It is true up to a factor of $\sim \sqrt{3}^{\,n-1}$ [Malikiosis, 2008/09].

- The weaker and analogue inequality to Minkowski's 1st theorem holds:

$$G(K) \leq \left(\frac{2}{\lambda_1(K)} + 1\right)^n.$$

- It suffices to prove the inequality for lattice polytopes.

Next, let us introduce another polynomial in $s \in \mathbb{C}$ given by

$$L(s, K) = \prod_{i=1}^{n} \left(\frac{2}{\lambda_i(K)} s + 1\right) = \sum_{i=0}^{n} L_i(K) s^i.$$

Thus, our problem is to show that, for zero symmetric lattice polytopes $P \in \mathcal{P}^n$, we have

$$G(1, P) = \sum_{i=0}^{n} G_i(P) \leq \sum_{i=0}^{n} L_i(P) = L(1, P).$$

Since Minkowski's 2nd theorem can be stated as

$$G_n(P) \leq L_n(P)$$

and our inequality for the arithmetic means as

$$G_{n-1}(P) \leq L_{n-1}(P),$$

we suggest that maybe a componentwise approach to the problem could work. Joint work is in progress with Matthias Henze and Eva Linke on the class of lattice zonotopes, and partial results so far are the following:

- The componentwise inequalities hold for arbitrary parallelepipeds.

- Let Z be a lattice zonotope generated by $a_1, \ldots, a_m$. Then

$$G_i(Z) \leq \frac{n!}{i!} L_i(Z).$$

- If the generators a_i are furthermore primitive and in general position, then $G_1(Z) \leq L_1(Z)$.

1.2 Average behavior of Frobenius numbers

1.2.1 Introduction and definition

This second part of the talk is about joint work with Iskander Aliev from Cardiff University and deals with an old Diophantine problem posed by Frobenius. It is about the following setting.

Let $a = (a_1, \ldots, a_n) \in \mathbb{N}^n_{>0}$ with $\gcd(a) = 1$. Then the problem is to determine the largest integer $F(a)$, from now on referred to as the *Frobenius number* of a, which cannot be written as a non-negative integer combination of the given numbers a_i, i.e.,

$$F(a) = \max\left\{b \in \mathbb{Z} \,:\, b \neq a^{\mathrm{t}} z \text{ for all } z \in \mathbb{N}^n\right\}.$$

As an introductory example, consider $a = (3, 10)$. Here we have

$$\{a^{\mathrm{t}} z : z \in \mathbb{N}^n\} = \{0, 3, 6, 9, 10, 12, 13, 15, 16, 18, 19, 20, \ldots\},$$

and therefore $F(a) = 17$.

One should note that such a maximal non-expressible number always exists when the a_i have greatest common divisor 1. Indeed, then there are integers $x_1, \ldots, x_n \in \mathbb{Z}$ such that $1 = x_1 a_1 + \cdots + x_n a_n$ and therefore all natural numbers $\geq |x_1| a_1 + \cdots + |x_n| a_n$ are expressible as non-negative integer combinations of the a_i.

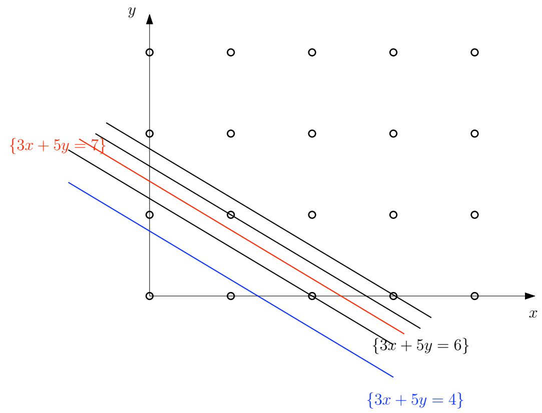
The Frobenius problem is also sometimes known as the *Coin/Money changing problem*, because the values a_i can be seen as denominations of n different coins and the Frobenius number is then the largest non-representable amount of money using these coins only.

A related Google search furthermore leads to *McNugget numbers*, which are positive integers that can be obtained by adding orders of McDonald's © Chicken McNuggets™ that used to come in boxes of $4, 6, 9$ and 20 each. And it turns out that all integers except $1, 2, 3, 5, 7$ and 11 are such McNugget numbers, i.e., $F(4, 6, 9, 20) = 11$.

A geometric approach to the Frobenius problem can be done by considering the so-called *Knapsack polytope*

$$P(a, b) = \{x \in \mathbb{R}^n_{\geq 0} \,:\, a^{\mathrm{t}} x = b\}$$

for given $b \in \mathbb{R}_{>0}$. Now, the equivalent question is whether $P(a, b)$ contains an integer point. In this setting, the existence of the Frobenius number is shown by proving the containment of a sufficiently huge cube or ball in the simplex $P(a, b)$ for big values of b. A planar example is shown in the figure below:



Diophantine equations are usually hard to solve in whole generality, and in fact this also holds for the Frobenius problem. But still, some exact results are known.

- Solving a question by Sylvester from 1884, Curran Sharp proved in the same year an exact formula for the planar case $n = 2$. For $a = (a_1, a_2)$ with $\gcd(a) = 1$, the following holds:

  $$F(a) = a_1 a_2 - (a_1 + a_2).$$

  The proof is nice and simple and uses the fact that, for two relatively prime numbers, every integer b has a unique integral representation

  $$b = a_1 x + a_2 y \text{ with } 0 \le x < a_2 \text{ and } y \in \mathbb{Z}.$$

  Therefore, assuming $a_1 \le a_2$, the largest non-representable integer is obtained by taking $x = a_2 - 1$ and $y = -1$, which directly leads to the above formula.

- Already in dimension $n = 3$ no exact formula is known and in fact there are "only" algorithms to compute the Frobenius numbers.

- In 1992, Kannan found a polynomial time algorithm to compute $F(a)$ in fixed dimension n.

- Ramírez Alfonsín showed in 1996 that the Frobenius problem belongs to the class of $\mathcal{NP}$-hard problems. He also wrote a book, published in 2005, surveying the current knowledge on the problem.

- By restricting the integer vector a to special cases, exact formulas are possible and an example is given by geometric sequences (Ong and

Ponomarenko, 2008). If $\gcd(p,q) = 1$, then

$$F(q^{n-1}, pq^{n-2}, \ldots, p^{n-1}) = p^{n-2}(qp - q - p) + \frac{(p-1)q^2(q^{n-2} - p^{n-2})}{q - p}.$$

1.2.2 Lower and upper bounds

We have seen that for $n \geq 3$ we cannot give exact formulae to compute the Frobenius number of a given integer vector. Thus, a natural question is to ask if we can at least bound it. In fact, many people were interested in this task and there are many results in the literature.

For the upper bounds, we assume $n \geq 3$ and $a_1 \leq a_2 \leq \cdots \leq a_n$.

- [Erdős–Graham, 1972]

$$F(a) \leq 2a_n \left[\frac{a_1}{n}\right] - a_1.$$

- [Vitek, 1975]

$$F(a) \leq \left[\frac{(a_2 - 1)(a_n - 2)}{2}\right] - 1.$$

- [Selmer, 1977]

$$F(a) \leq 2a_{n-1} \left[\frac{a_n}{n}\right] - a_n.$$

- [Beck–Díaz–Robins, 2002]

$$F(a) \leq \frac{1}{2}\left(\sqrt{a_1 a_2 a_3 (a_1 + a_2 + a_3)} - (a_1 + a_2 + a_3)\right).$$

  Note that this bound only depends on the three smallest integers a_1, a_2 and a_3, but still holds in general.

- [Fukshansky–Robins, 2007]

$$F(a) \leq \frac{(n-1)^2}{\Gamma\left(\frac{n}{2} + 1\right) \pi^{\frac{n}{2}}} \sum_{i=1}^{n} a_i \sqrt{\|a\|^2 - a_i^2}.$$

For our considerations, the important observation is the following. If the a_i's are all of the "same size", then all these upper bounds have the same order of magnitude $|a|_\infty^2$. Evidence that this is also best possible can be found in Erdős–Graham (1972), Schlage–Puchta (2005), and Arnold (2006).

Lower bounds are not that frequent and the first one was given in a work by Rödseth (1990), who proved that

$$F(a) \geq (n-1)!^{\frac{1}{n-1}}(a_1 a_2 \cdot \dots \cdot a_n)^{\frac{1}{n-1}} - (a_1 + a_2 + \dots + a_n).$$

For $n = 3$, the factor $(n-1)!^{\frac{1}{n-1}}$ was improved by Davison (1994) to $\sqrt{3}$. With geometric reasoning, Aliev and Gruber found a best possible factor in 2007. Let $\bar{\mu}_k$ be the absolute inhomogeneous minimum of the k-dimensional standard simplex. We have $k > \bar{\mu}_k > (k!)^{\frac{1}{k}} > \frac{k}{e}$, which shows that their bound

$$F(a) \geq \bar{\mu}_{n-1}(a_1 a_2 \cdot \dots \cdot a_n)^{\frac{1}{n-1}} - (a_1 + a_2 + \dots + a_n)$$

extends the one by Rödseth.

The order of magnitude for this lower bound is $|a|_\infty^{1+\frac{1}{n-1}}$ if the a_i's are of the "same size". Our main concern now is the typical behavior of $F(a)$, i.e., given a random integer vector a, is the expected order of magnitude of its Frobenius number $|a|_\infty^{1+\frac{1}{n-1}}$ or $|a|_\infty^2$? The first systematic study of this problem was done by Arnold (1999/2006) and he conjectured that $F(a)$ grows like $T^{1+\frac{1}{n-1}}$ for a generic vector a with $|a|_1 = T$. A first result concerning this assumption is the following:

Theorem 1.2.1 (Bourgain–Sinai, 2007)**.** *Let* $0 < \alpha < 1$ *and*

$$G_\alpha(n, T) = \{a \in \mathbb{N}_{>0}^n : \gcd(a) = 1, |a|_\infty \leq T, a_i \geq \alpha T, 1 \leq i \leq n\}.$$

Then, with respect to the uniform distribution of all points in the set $G_\alpha(n, T)$*,*

$$\mathrm{Prob}_\alpha\left(\frac{F(a)}{T^{1+\frac{1}{n-1}}} \geq D\right) \leq \epsilon(D).$$

The constant $\epsilon(D)$ *does not depend on* T *and tends to* 0 *as* D *goes to infinity.*

Note that we actually would like to have a result considering all integer points with bounded maximum norm, i.e., all points in

$$G(n, T) = \{a \in \mathbb{N}_{>0}^n : \gcd(a) = 1, |a|_\infty \leq T\} = G_0(n, T).$$

Theorem 1.2.2 (Aliev–Henk, 2008)**.** *Let* $T > 0$*. Then, with respect to the uniform distribution among all points in* $G(n, T)$*, we have*

$$\mathrm{Prob}\left(\frac{F(a)}{|a|_\infty^{1+\frac{1}{n-1}}} \geq D\right) \ll_n D^{-2}.$$

Here, $\ll_n$ *denotes the Vinogradov symbol with the constant only depending on* n*.*

This result is not straightforward, since the next statement shows that the ratio $F(a)/|a|_\infty^{1+\frac{1}{n-1}}$ is unbounded along a given vector $\alpha \in \mathbb{R}^n$.

Theorem 1.2.3 (Aliev–Henk, 2008). *For any $\epsilon > 0$, $M > 0$, and any given vector $\alpha = (\alpha_1, \dots, \alpha_{n-1}, 1) \in \mathbb{R}^n$ with $0 \le \alpha_1 \le \dots \le \alpha_{n-1} \le 1$, there exists an integer vector $a = (a_1, \dots, a_n) \in \mathbb{N}^n_{>0}$ with $\gcd(a) = 1$ and $|\alpha - \frac{1}{a_n} a|_\infty < \epsilon$ such that*

$$F(a) > M \cdot |a|_\infty^{1+\frac{1}{n-1}}.$$

As a corollary to Theorem 1.2.2, we get the following:

Theorem 1.2.4 (Aliev–Henk, 2008).

$$\frac{1}{\#G(n,T)} \sum_{a \in G(n,T)} \frac{F(a)}{|a|_\infty^{1+\frac{1}{n-1}}} \ll_n 1.$$

In words, this means that the "average" Frobenius number does not essentially exceed $|a|_\infty$ as n tends to infinity. Looking back to the terms appearing in the best lower bounds on the Frobenius number of a given integer vector, we pose the following question.

Problem 1.2.5. Can we replace the term $|a|_\infty^{1+\frac{1}{n-1}}$ in the subsequent theorem by the geometric mean $(a_1 a_2 \cdots a_n)^{\frac{1}{n}}$?

Beierhofer et al. made extensive computer experiments in 2005 that allow to suspect an affirmative answer to this question.

Reversed arithmetic-geometric-mean inequality

A possible way to a positive answer for Problem 1.2.5 is via a probabilistic reversed arithmetic-geometric-mean inequality. Note that a reversed AGM inequality is not possible in general. Precisely, we ask to determine the order of decay $\sigma(n)$ such that, for large T, with respect to uniform distribution of points in $G(n,T)$, it is

$$\operatorname{Prob}\left(\frac{\frac{1}{n}|a|_1}{\left(\prod_{i=1}^n a_i\right)^{\frac{1}{n}}} > \alpha \right) \ll_n \alpha^{-\sigma(n)}.$$

Hinrichs showed in 2008/09 that it is possible to choose $\sigma(n) = 1$, which implies that

$$\operatorname{Prob}\left(\frac{F(a)}{\left(\prod_{i=1}^n a_i\right)^{\frac{1}{n}}} \ge D \right) \ll_n D^{-R\frac{n-1}{2n}},$$

for some constant $R > 0$.

In the continuous case, meaning that we do not restrict the set $G(n, T)$ only to primitive integer vectors, a linear order of decay is known and the aim is to find a similar result in the integer case.

Theorem 1.2.6 (Gluskin–Milman, 2003)**.**

$$\mathrm{Prob}\left(\frac{\frac{1}{n}|x|_1}{\left(\prod_{i=1}^n x_i\right)^{\frac{1}{n}}} > \alpha \ : \ x \in \mathbb{R}^n_{>0}\right) \ll_n \alpha^{-\frac{n}{2}}.$$

Ideas and ingredients of the proof

In this final part, we would like to give the main ideas and ingredients of our proof of Theorem 1.2.2. To a large extent, it is a combination of various results in the field of geometry of numbers.

Let $B^{n-1} \subset \mathbb{R}^{n-1}$ be the $(n-1)$-dimensional unit ball. For given $a \in \mathbb{Z}^n$ we consider the lattice

$$\Lambda_a = \frac{1}{\|a\|^{\frac{1}{n-1}}} \left\{x \in \mathbb{Z}^n \ : \ a^{\mathrm{t}} x = 0\right\},$$

which has determinant $\det(\Lambda_a) = 1$. The so-called *inhomogeneous minimum* of Λ_a is defined as

$$\mu(\Lambda_a) = \min\{\mu > 0 \ : \ \Lambda_a + \mu B^{n-1} = \mathbb{R}^{n-1}\}.$$

Kannan connected in 1982 this geometric quantity with the Frobenius numbers and using further results of Fukshansky and Robins from 2007 (see also Arnold, 2006), one can show that

$$F(a) \le n^3 |a|_\infty^{1+\frac{1}{n-1}} \mu(\Lambda_a).$$

Jarnik found in 1941 an inequality that relates the inhomogeneous minimum to Minkowski's successive minima. For our purposes, we need them to be defined with respect to the lattice Λ_a and the unit ball, i.e.,

$$\lambda_i(\Lambda_a) = \min\{\lambda > 0 \ : \ \dim(\lambda B^{n-1} \cap \Lambda_a) \ge i\}, \ 1 \le i \le n-1.$$

From the previous inequality we obtain

$$\frac{F(a)}{|a|_\infty^{1+\frac{1}{n-1}}} \le n^4 \lambda_{n-1}(\Lambda_a).$$

In the next step we apply Minkowski's fundamental theorems on the successive minima to prove the existence of an $i \in \{1, \ldots, n-2\}$ such that

$$\frac{\lambda_{i+1}(\Lambda_a)}{\lambda_i(\Lambda_a)} > c_n \left(\frac{F(a)}{|a|_\infty^{1+\frac{1}{n-1}}} \right)^{\frac{2}{n-2}},$$

for some constant $c_n > 0$ only depending on n.

Finally, we use a series of fundamental results by Schmidt (1998). His work deals with the distribution of primitive sublattices and enables us to show that the left-hand side of the former inequality is small with high probability, which leads to the result stated in Theorem 1.2.2.

As a conclusion of this second part, we would like to advertise the study of a (probabilistic) reversed arithmetic-geometric-mean inequality in order to answer Problem 1.2.5.

Seminar 7

Tropical Convexity

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1.1 The tropical semiring

Definition 1.1.1. The triple $(\mathbb{R}, \oplus, \odot)$ is called the *tropical semiring*, where

$$x \oplus y = \min(x, y) \quad \text{and} \quad x \odot y = x + y.$$

Remark 1.1.2. Taking the maximum instead of the minimum gives an isomorphic object. Also, sometimes one uses $(\mathbb{R}^+, \min, \cdot)$ or the above operations over the set $\mathbb{R} \cup \{\infty\}$.

Example 1.1.3.

$$(3 \oplus 5) \odot 2 = 3 \odot 2 = 5 = \min(5, 7) = (3 \odot 2) \oplus (5 \odot 2).$$

This hints that we have distribution. And, in fact, this is the case. On the downside, every $x \in \mathbb{R}$ is idempotent (i.e., $x \oplus x = x$), so there is no inverse for the addition.

Consider the polynomial $6x^3 + 2x + 3 \in \mathbb{R}[x]$. *Tropical evaluation* means to map $x \in \mathbb{R}$ to

$$(6 \odot x^{\odot 3}) \oplus (2 \odot x) \oplus 3 = \min(3x + 6, 2x, 3).$$

In Figure 1.1 we see the three linear functions given by the monomials of the polynomial. The tropical polynomial is therefore a piecewise linear function. We are especially interested in the points where the function is not linear.

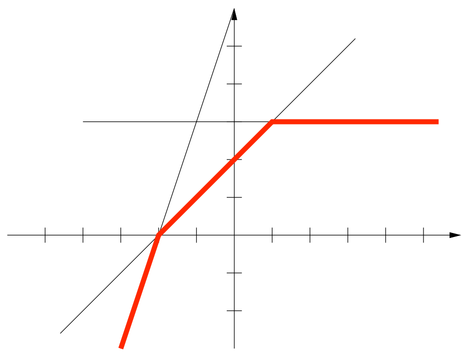

Figure 1.1: The tropical polynomial $(6 \odot x^{\odot 3}) \oplus (2 \odot x) \oplus 3$ and the three linear functions.

Remark 1.1.4. The arithmetic extends to higher dimensions, i.e., to $(\mathbb{R}^d, \oplus, \odot)$, where $\oplus$ is defined componentwise and we have the tropical scalar multiplication

$$\lambda \odot \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} = \begin{pmatrix} \lambda \odot x_1 \\ \vdots \\ \lambda \odot x_d \end{pmatrix} = \begin{pmatrix} x_1 + \lambda \\ \vdots \\ x_d + \lambda \end{pmatrix}.$$

This means that we can look at multivariate polynomials.

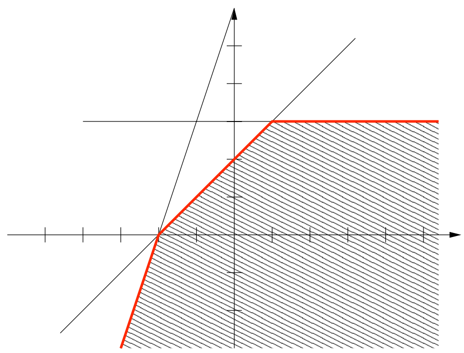

Figure 1.2: The shaded part indicates $\mathcal{P}(6x^3 + 2x + 3)$.

Proposition 1.1.5. *For $f \in \mathbb{R}[t_1, \ldots, t_d]$ a polynomial with tropical evaluation function $f^* \colon \mathbb{R}^d \to \mathbb{R}$, the set*

$$\mathcal{P}(f) = \left\{ (t, f^*(t)) \;\middle|\; t \in \mathbb{R}^d \right\} \subseteq \mathbb{R}^{d+1}$$

is the boundary of an unbounded convex polyhedron.

This shows that it is enough to look at convex polyhedra to know tropical polynomials.

As said before, we will be especially interested in the "non-linearity" points of tropical polynomials.

Definition 1.1.6. A polynomial $p = \sum_i c_i t^i \in \mathbb{R}[t_1, \ldots, t_d]$, where we define $t^i = (t_1^{i_1}, \ldots, t_d^{i_d})$, *vanishes tropically* at t if the minimum

$$\bigoplus_i c_i \odot t^{\odot i}$$

is attained at least for two different values of i.

1.2 Tropical hyperplanes

Consider the homogeneous linear polynomial

$$f = 2t_1 - 2t_2 + t_3 \in \mathbb{R}[t_1, t_2, t_3].$$

The tropical vanishing locus is a 2-dimensional piecewise linear subset of $\mathbb{R}^3$. But if f tropically vanishes at x, then it also vanishes (in the tropical sense) at $\lambda\mathbf{1} + x = \lambda \odot x$ for all $\lambda \in \mathbb{R}$:

$$\begin{aligned} f^*(\lambda\mathbf{1} + x) &= (2 \odot \lambda \odot x_1) \oplus (-2 \odot \lambda \odot x_2) \oplus (1 \odot \lambda \odot x_3) \\ &= \lambda \odot f^*(x). \end{aligned}$$

Hence it is useful to factor out tropical scalar multiplication and define the *tropical torus*

$$\mathbb{T}^{d-1} = \mathbb{R}^d / \mathbb{R}\mathbf{1}.$$

Remark 1.2.1. The tropical torus $\mathbb{T}^{d-1}$ is topologically equivalent to $\mathbb{R}^{d-1}$:

$$(x_1, \ldots, x_d) + \mathbb{R}\mathbf{1} = (0, x_2 - x_1, \ldots, x_d - x_1) + \mathbb{R}\mathbf{1} \mapsto (x_2 - x_1, \ldots, x_d - x_1),$$

and this is a bijection.

Now we can draw the image of the vanishing locus in $\mathbb{R}^d/\mathbb{R}\mathbf{1}$. For the above example f, the result is depicted in Figure 1.3. Note that the zero-value $(-2, 2, -1)$ of f equals $(0, 4, 1)$ in $\mathbb{T}^{d-1}$.

Definition 1.2.2. A *tropical hyperplane* is the vanishing locus of a homogeneous linear polynomial in $\mathbb{T}^{d-1}$.

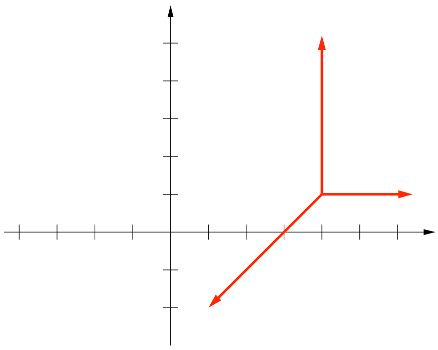

Figure 1.3: The downward arrow looks skew due to the projection.

1.3 Tropical determinants

Consider the permanent

$$p = \sum_{\sigma \in \mathrm{Sym}(d)} t_{1,\sigma(1)} \cdot \ldots \cdot t_{d,\sigma(d)} \in \mathbb{R}[t_{11}, \ldots, t_{dd}].$$

It is convenient to think of the t_{ij} as entries of a $d \times d$-matrix of indeterminates.

Definition 1.3.1. The *tropical determinant* is defined as

$$\mathrm{tdet} = p^* = \min_{\sigma \in \mathrm{Sym}(d)} t_{1,\sigma(1)} + \cdots + t_{d,\sigma(d)}.$$

Remark 1.3.2. The computation of tdet equals the assignment problem in combinatorial optimization. This can be solved via the Hungarian method, which has complexity $O(d^3)$.

It is a curious fact that this is the complexity of the Gaussian elimination for the normal determinant. Whether or not this is a coincidence is not known so far.

Theorem 1.3.3. *The d points $x_1, \ldots, x_d \in \mathbb{T}^{d-1}$ lie in a common tropical hyperplane if and only if*

$$\mathrm{tdet} \begin{pmatrix} x_{11} & \cdots & x_{1d} \\ \vdots & & \vdots \\ x_{d1} & \cdots & x_{dd} \end{pmatrix} \textit{ vanishes,}$$

where $x_i = \begin{pmatrix} x_{i1} \\ \vdots \\ x_{id} \end{pmatrix}$.

1.4 Tropical polytopes

Definition 1.4.1.

- For $X \subseteq \mathbb{T}^{d-1}$, let

  $$\operatorname{tconv} X = \{(\lambda_1 \odot x_1) \oplus \cdots \oplus (\lambda_k \odot x_k) \mid x_1, \ldots, x_k \in X,\ \lambda_i \in \mathbb{R}\}$$

  be the *tropical convex hull* of X. This is well defined because it is the tropical evaluation of a *homogeneous* polynomial.

- A *tropical polytope* is the tropical convex hull of finitely many points.

- X is *tropically convex* if and only if $X = \operatorname{tconv} X$.

Examples of tropically convex sets:

- Tropical hyperplanes.

- Open sectors, i.e., the connected components of the complement of a tropical hyperplane (see Figure 1.4).

- See Figure 1.5 for more examples.

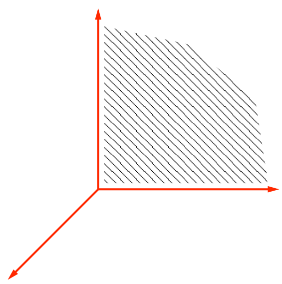

Figure 1.4: The shaded area indicates an open sector in $\mathbb{T}^2$.

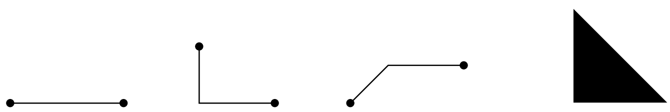

Figure 1.5: The first three types of sets are called *segments*. The fourth is an example of a tropically convex set that is not a tropical polytope.

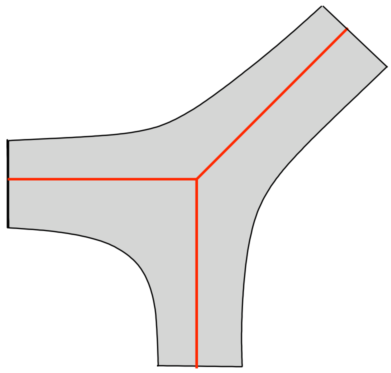

Figure 1.6: An amoeba and its limit for $k \to \infty$.

So what is the connection between the locus of a polynomial and the locus of its tropical evaluation?

Let $f \in \mathbb{C}[t_1, \dots, t_d]$ and define

$$V(f) = \left\{ x \in \mathbb{C}^d \;\middle|\; f(x) = 0 \right\}$$

as the vanishing locus of f. Then the map from $V(f)$ to $\mathbb{R}^d$ with

$$x = (x_1, \dots, x_d) \longmapsto (\log_k |x_1|, \dots, \log_k |x_d|) \in \mathbb{R}^d$$

defines an *amoeba* of f, where $k > 1$ (see Figure 1.6).

Proposition 1.4.2. *Tropical line segments in $\mathbb{T}^{d-1} \approx \mathbb{R}^{d-1}$ are the concatenation of at most $d-1$ ordinary line segments.*

The proof is easy and left to the reader.

Consider a finite sequence $V = (v_1, \dots, v_n)$ of points in $\mathbb{T}^{d-1}$. Then for $x \in \mathbb{T}^{d-1}$ we define

$$\text{type}_V(x) = (T_1, \dots, T_d), \quad \text{with}$$

$T_k = \{i \in [n] \mid v_{ik} - x_k \leq v_{ij} - x_j \text{ for all } j\}$ (and, as before, $v_i = \begin{pmatrix} v_{i1} \\ \vdots \\ v_{id} \end{pmatrix}$).

Example 1.4.3. Let $V = (v_1, \dots, v_4)$ with

$$v_1 = \begin{pmatrix} 0 \\ 3 \\ 6 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix}.$$

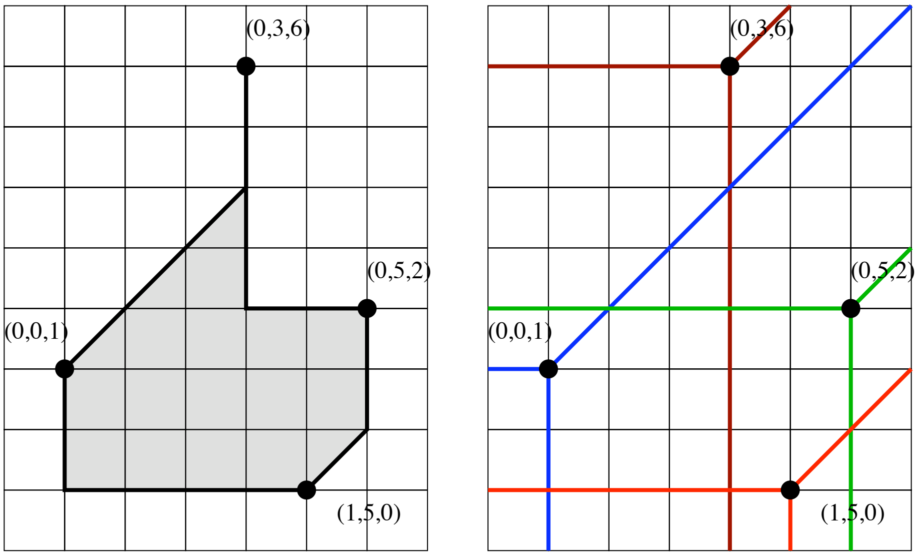

Figure 1.7: Example of a tropical polytope with a minimal generating set indicated, and the decomposition of $\mathbb{T}^2$ for this pointset, as described in Lemma 1.4.4.

Then

$$\text{type}_V \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} = (\{1\}, \{1,3\}, \{2,3,4\}).$$

To see this, we compute $v_1 - x = \left(\begin{smallmatrix} 0 \\ 0 \\ 2 \end{smallmatrix}\right)$, where the two 0s tell us that $1 \in T_1$ and $1 \in T_2$. Furthermore, we have $v_2 - x = \left(\begin{smallmatrix} 0 \\ 2 \\ -2 \end{smallmatrix}\right)$, so $2 \in T_3$; proceed in this manner for the remaining v_is.

Lemma 1.4.4.

$$\begin{aligned} X_S &= \left\{x \in \mathbb{T}^{d-1} \mid S = (S_1, \ldots, S_d) \subseteq \text{type}(x) \ \text{(componentwise)}\right\} \\ &= \left\{x \in \mathbb{T}^{d-1} \mid x_j - x_k \leq v_{ij} - v_{ik} \text{ for all } j,k \text{ and all } i \in S_k\right\} \end{aligned}$$

is an ordinary convex polyhedron (or empty).

Note that the constraints in the second description of X_S are linear, as V is fixed. This gives us a decomposition of $\mathbb{T}^{d-1}$; see Figure 1.7.

Theorem 1.4.5. *The tropical convex hull* $\text{tconv}\{v_1, \ldots, v_n\}$ *equals*

$$\bigcup_{S \in (2^{[n]})^d} \{X_S \mid X_S \text{ is bounded}\},$$

so it is an ordinary polytopal complex.

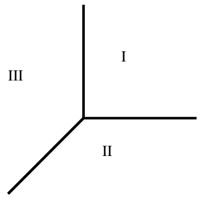

Figure 1.8: Basic idea of the proof: If v_i is in the closure of the kth sector, then $i \in T_k$.

Remark 1.4.6. $\operatorname{tconv}(V)$ is dual to the regular subdivision of $\Delta_{n-1} \times \Delta_{d-1}$ (shown by Develin and Sturmfels).

Note that we know from ordinary polytope theory that $\Delta_{n-1} \times \Delta_{d-1} \cong \Delta_{d-1} \times \Delta_{n-1}$. So we can also consider d points in $(n-1)$-space instead of n points in $(d-1)$-space and therefore have an isomorphism between tropical polytopes in different dimensions.

1.5 Signs of tropical determinants

Consider

$$\operatorname{tdet}\begin{pmatrix} 0 & -1 & 2 \\ 0 & -2 & -2 \\ 0 & 2 & 0 \end{pmatrix} = (-1) + (-2) + 0 = -3.$$

This matrix is tropically non-singular, and the unique optimal permutation is $\sigma = (123)$ with sign $= +1$ (as a permutation).

IF $M \in \mathbb{R}^{d \times d}$, then

$$\operatorname{tsign} M = \begin{cases} \varepsilon & \text{if all optimal } \sigma \text{ share the sign } \varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 1.5.1. The computation of tsign corresponds to the even-dicycle-problem for some (directed) graphs. Although this was open for about 30 years, we now know that there is a polynomial-time algorithm for this.

In normal geometry, we can use the determinant to determine whether a point lies on a given hyperplane, and, if not, then on which side of the hyperplane the point is. Now we would like to have a tropical version of this.

Let H be a fixed tropical hyperplane in $\mathbb{T}^{d-1}$. A *closed sector* is the closure of an open sector, and a closed *tropical half-space* is the non-trivial union of closed sectors of a fixed hyperplane (where a trivial union is the empty union or the union over all the sectors).

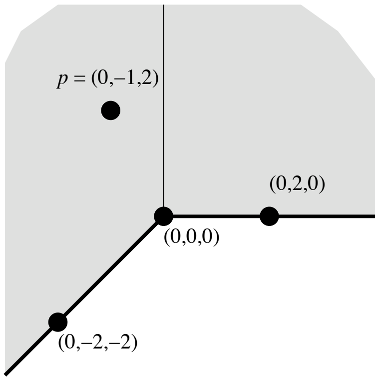

Figure 1.9: The points (without the zero) correspond via Theorem 1.5.2 to the matrix at the beginning of the section and define the shaded half-space.

Theorem 1.5.2. *Pick points $x_2, x_3, \ldots, x_d \in \mathbb{T}^{d-1}$ in sufficiently generic position. Then*

$$\left\{ p \in \mathbb{T}^{d-1} \ : \ \operatorname{tsign} \begin{pmatrix} p_1 & p_2 & \cdots & p_d \\ x_{21} & x_{22} & \cdots & x_{2d} \\ \vdots & & & \vdots \\ x_{d1} & x_{d2} & \cdots & x_{dd} \end{pmatrix} = +1 \right\}$$

is a tropical half-space. Conversely, each tropical half-space arises in this way.

1.6 Further results and concluding remarks

We give tropical versions of classical results:

- Let P be a tropical polytope and $x \notin P$. Then there exists a tropical half-space H^+ with $P \in H^+$ and $x \notin H^+$. In other words, we have a tropical separation theorem.
- Each tropical polytope is the bounded intersection of finitely many tropical half-spaces. Conversely, each such intersection is a tropical polytope.

Furthermore, we have:

- Tropical polytopes that are also classical polytopes are called *polytropes*.
- The type decomposition of a tropical polytope is a decomposition into polytropes.

- Each polytrope is generated by d vertices and can therefore be regarded as a tropical simplex.
- Based on these facts, one can derive a tropical version of Carathéodory's theorem, which says that each point in a d-polytope is in the convex hull of at most $d + 1$ vertices.
- Tropical polytopes can be seen as tropical linear spaces. In ordinary geometry, the Grassmannian parametrizes linear spaces, and, as one would hope, the tropicalization of the Grassmannian does indeed parametrize tropical linear spaces.

Seminar 8

Helly-type Theorems

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1.1 Introduction

Helly's theorem is one of the most important and classic theorems in discrete geometry. This theorem tells that, given a collection of convex sets in $\mathbb{R}^d$, if every choice of $d+1$ convex sets have a common intersection, then the whole collection also have a common intersection. In this talk, we discuss the proof of Helly's theorem and its relative theorems.

1.2 Helly's theorem and its relatives

1.2.1 Helly's theorem and its proof

Let us start with the main theorem:

Theorem 1.2.1 (Helly's Theorem)**.** *Let $K_1, \ldots, K_n$ be a family of convex sets in $\mathbb{R}^d$, where $n \geq d+1$. If every $d+1$ of these sets have a point in common, then all sets have a point in common.*

We will introduce two proofs. The first one is Radon's proof. In his proof, he used a lemma which is now known as Radon's theorem.

Theorem 1.2.2 (Radon's Theorem)**.** *If $x_1, \ldots, x_n \in \mathbb{R}^d$ with $n \geq d+2$, then it is possible to find a partition $S, T \subset \{1, \ldots, n\}$ such that*

$$\operatorname{conv}(x_i,\, i \in S) \cap \operatorname{conv}(x_j,\, j \in T) \neq \emptyset.$$

Proof. Since $n > d+1$, the points $x_1, \ldots, x_n$ are affinely dependent. That is,

$$\sum_{i=1}^{n} \alpha_i x_i = 0, \quad \sum_{i=1}^{n} \alpha_i = 0, \quad \exists \alpha_i \neq 0.$$

Divide the sum into the positive coefficient part and the negative coefficient part:

$$\sum_{i\,:\,\alpha_i>0} \alpha_i x_i = \sum_{j\,:\,\alpha_j<0} (-\alpha_j) x_j, \qquad \sum_{i\,:\,\alpha_i>0} \alpha_i = \sum_{j\,:\,\alpha_j<0} (-\alpha_j) = t.$$

Divide both sides by t:

$$\sum_{i\,:\,\alpha_i>0} \frac{\alpha_i x_i}{t} = \sum_{j\,:\,\alpha_j<0} \frac{(-\alpha_j) x_j}{t}, \qquad \sum_{i\,:\,\alpha_i>0} \frac{\alpha_i}{t} = \sum_{j\,:\,\alpha_j<0} \frac{(-\alpha_j)}{t} = 1.$$

This proves the statement. $\square$

Radon's proof of Helly's theorem. We will prove the theorem by induction on the number of convex sets. Assume that the statement is true for $k < n$ convex sets. Then, for every $n-1$ convex sets, they have a common intersection. Let $\{a_1, \ldots, a_n\}$ be the intersection points.

By 1.2.2, $\{a_1, \ldots, a_n\}$ have a Radon partition S, T. Let p be a point in $\operatorname{conv}(S) \cap \operatorname{conv}(T)$. Then p is contained in every convex set. $\square$

Helly's proof is interesting as well.

Helly's proof of Helly's theorem. We will prove the theorem by induction on the dimension and the number of convex sets. Assume that the statement is true for $l < d$ dimensions and $k < n$ convex sets. Suppose that $K_1 \cap K_2 \cap \ldots \cap K_n = \emptyset$. That is, there is a separating hyperplane H between $C = K_n$ and $M = K_1 \cap \ldots \cap K_{n-1}$. Let $K_i' = K_i \cap H$. By the assumption, each K_i' is nonempty because $K_n \cap K_i$ is nonempty. Then we have a family of $n-1$ convex sets K_i' in H. By the hypothesis, $K_1' \cap \ldots \cap K_{n-1}' \neq \emptyset$, which implies that $K_1 \cap \ldots \cap K_{n-1} \cap H \neq \emptyset$. This contradicts the fact that H is a separating hyperplane. $\square$

1.2.2 Topological generalization

What if the given sets are not convex? The following figure tells that the answer is "no".

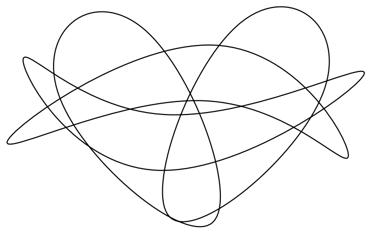

Figure 1.1: Every three sets have a common intersection but four sets do not.

However, Helly's theorem can be generalized to the following topological version.

Theorem 1.2.3 (Topological Helly Theorem)**.** *If $K_1, \ldots, K_n$ have the property that every nonempty intersection is homeomorphic to a ball, then Helly's theorem holds.*

1.2.3 Dimension Helly theorem

One may ask the following question: What happens if we assume that the dimension of the intersection of every $d+1$ sets is larger than or equal to r? Can we deduce that $\dim(K_1 \cap \ldots \cap K_n) \geq r$? Again, the answer is negative. The following figure is a counterexample: A_1, A_2 and A_3, A_4 partition the same rectangle, respectively. And every three sets have a common intersection of dimension 1, but four sets have common intersection of dimension 0.

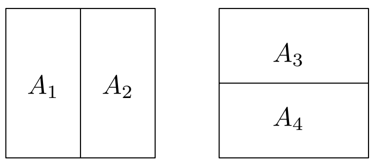

Figure 1.2: Two partitions of the rectangle.

However, Katchalski proved that the statement is true when "many" sets have an r-dimensional intersection.

Theorem 1.2.4 (Katchalski's Dimension Helly Theorem)**.** *If every $h(d,r)$ sets have intersection of dimension $\geq r$, then all sets have intersection of dimension $\geq r$, where $h(d,r) = \max(2(d+1-r), d+1)$.*

Katchalski also proved the following theorem.

Theorem 1.2.5 (Katchalski's Big Theorem)**.** *The dimension of the intersections of all subfamilies of size $\leq d+1$ determines the dimension of the intersections of all subfamilies.*

1.2.4 Fractional Helly theorem

What happens if the assumption "every $d+1$ convex sets have a common intersection", is relaxed to "most $d+1$ convex sets have a common intersection"? Consider a family of convex sets such that

$$K_1 = H_1, \ldots, K_m = H_m, \qquad K_{m+1} = \ldots = K_n = \mathbb{R}^d.$$

Among the $(d+1)$-subfamilies, $\binom{n}{d+1} - \binom{m}{d+1}$ have nonempty intersection. And there is no family of size $n-m+d+1$ with nonempty intersection. The following theorem tells that this is the best possible.

Theorem 1.2.6 (Kalai–Echoff)**.** *If the number of $(d+1)$-subfamilies with nonempty intersection is $> \binom{n}{d+1} - \binom{m}{d+1}$, then there are $n-m+d+1$ sets with nonempty intersection.*

We get the following useful corollary.

Corollary 1.2.7. *If $\alpha\binom{n}{d+1}$ of the $(d+1)$-subfamilies have nonempty intersection, then there are βn sets with nonempty intersection.*

One may ask "how many points are required to *pin* all the convex sets?". Hadwiger and de Brunner proved the following theorem.

Theorem 1.2.8 (Hadwiger–de Brunner)**.** *If $K_1, \ldots, K_n$ are planar convex sets and among every* 5 *of them there are* 4 *with nonempty intersection, then there are two points x, y such that every K_i contains x or y.*

The following two lemmas imply the theorem. Their proof is by plane sweeping. Imagine that a hyperplane H is sweeping the plane. Let K_1, K_2 be the convex sets whose intersection meets H at last. Let x the meet point. Then:

Lemma 1.2.9. *All K_i such that $K_i \cap K_1 \cap K_2 \neq \emptyset$ contain x.*

Proof. Suppose not; that is, there is a set K_i that does not contain x. Then $K_i \cap K_1 \cap K_2$ meets H after H meets x, which implies that $K_i \cap K_1$ or $K_i \cap K_2$ meet H after H meets x. This contradicts the assumption. □

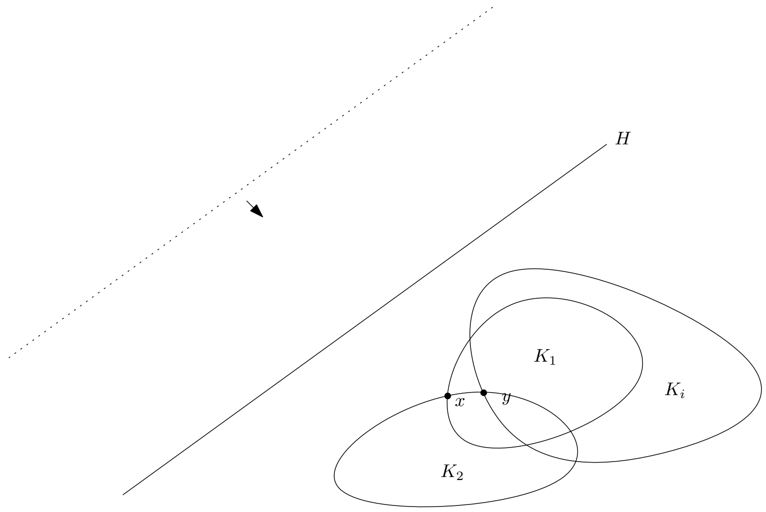

Figure 1.3: Illustration of Lemma 1.2.10.

Lemma 1.2.10. *All the sets K_j which do not intersect $K_1 \cap K_2$ have a point y in common.*

Proof. Pick up K_{j_1}, K_{j_2}, K_{j_3} which do not intersect $K_1 \cap K_2$. Then, by the assumption, $K_{j_1} \cap K_{j_2} \cap K_{j_3} \neq \emptyset$. By Helly's theorem, these sets have a common intersection point y. □

What if we strengthen the condition? That is, how many points are required to pin every convex set if among every 4 convex sets we can find 3 with nonempty intersection? So far, the smallest number of points to pin the convex sets is not known, but Alon and Kleitman proved that the number of points to pin is finite.

1.2.5 Tverberg's theorem

Let us recall Radon's theorem:

Theorem 1.2.11 (Radon's Theorem)**.** *If $x_1, \ldots, x_n \in \mathbb{R}^d$, where $n \geq d+2$, then it is possible to find a partition $S, T \subset \{1, \ldots, n\}$ such that*

$$\mathrm{conv}(x_i,\, i \in S) \cap \mathrm{conv}(x_j,\, j \in T) \neq \emptyset.$$

One may ask a natural question: Can we divide the point set into "more" than two disjoint sets whose convex hulls intersect?

Tverberg answered the question via the following theorem.

Theorem 1.2.12 (Tverberg's Theorem)**.** *Let $x_1, \ldots, x_n \in \mathbb{R}^d$, where $n \geq (d+1)(r-1)+1$. Then we can divide $\{1, \ldots, n\}$ into r parts $S_1, S_2, \ldots, S_r$ such that $\bigcap_{i=1}^r \operatorname{conv}(x_j,\, j \in S_i) \neq \emptyset$.*

This theorem has several proofs. The original proof is very complicated. Tverberg simplified the proof several times by himself. Sarkaria gave a beautiful linear algebraic proof using tensor product and the colorful Carathéodory theorem.

This theorem has a topological generalization.

Theorem 1.2.13 (Topological Tverberg Theorem)**.** *Let f be a continuous function from the n-dimensional simplex σ^n to $\mathbb{R}^d$. If $n \geq (d+1)(r-1)$ and r is a power of a prime, then there are r pairwise disjoint faces of σ^n whose images have a point in common.*

For a composite number r, its proof is not known.

There are several problems around Tverberg's theorem. One is about a relaxed Tverberg condition. Let $t(d, r, k)$ be the smallest integer such that, given n points $x_1, \ldots, x_n$ in $\mathbb{R}^d$, there exists a partition $S_1, \ldots, S_r$ of $\{1, \ldots, m\}$ such that every k among the convex hulls $\operatorname{conv}(x_i,\, i \in S_j)$, $j = 1, \ldots, r$, have a point in common. Reay conjectured that $t(d, r, k) = (d+1)(r-1)+1$, but we do not believe it.

Another problem is about the dimension of Tverberg point sets. Kalai conjectured that, for every finite $A \subset \mathbb{R}^d$, $\sum_{r=1}^{|A|} \dim T_r(A) \geq 0$, where $T_r(A)$ denotes the set of points in $\mathbb{R}^d$ which belong to the convex hull of r pairwise disjoint subsets of A, and $\dim \emptyset = -1$ by convention; thus, the nonexistence of Tverberg r-partitions for large r must be compensated by sufficiently large dimensions of $T_r(A)$ for small r. This conjecture includes Tverberg's theorem as a special case: if $|A| = (r-1)(d+1)+1$, $\dim A = d$, and $T_r(A) = \emptyset$, then the sum in question is at most $(r-1)d + (|A| - r + 1)(-1) = -1$.

Seminar 9

The Erdős–Szekeres Problem

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From now on, we will deal with point sets $\mathcal{P}$ in general position in the Euclidean plane such that $|\mathcal{P}| < \infty$.

Observation 1.0.1. Given 5 points in the plane, there are 4 forming a convex quadrilateral. Note that there are three different possibilities: the convex hull of the 5 points can be a pentagon, a quadrilateral or a triangle.

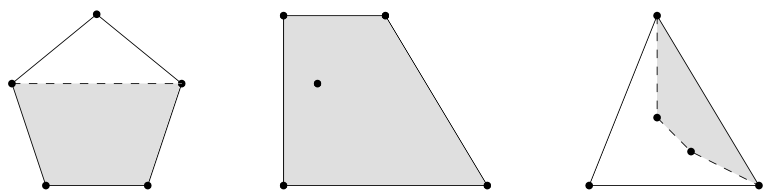

Figure 1.1: Any five points in general position determine a convex quadrilateral.

Theorem 1.0.2 (Erdős–Szekeres Theorem)**.** *For $n \geq 3$ there exists a smallest number $f(n)$ such that, given at least $f(n)$ points in the plane, there are n points in convex position (a convex n-gon).*

For example, $f(4) = 5$.

Theorem 1.0.3 (Ramsey Theorem)**.** *For all $n, m \in \mathbb{N}$ there exists a function $R_2(m, n)$ such that, if the edges of the complete graph with $R_2(m, n)$ vertices* *is coloured by white and blue, then there is either a complete white subgraph of m vertices or a complete blue subgraph of n vertices.*

For example, $R_2(3,3) = 6$.

In general, instead of pairs of vertices we can colour k-tuples, and then we have $R_k(m,n)$.

Application 1. $f(n) \leq R_4(n,5)$: Suppose given a set of at least $R_4(n,5)$ elements. For every subset of 4 vertices, we colour it white if the 4 points are in convex position and blue if the vertices are not in convex position. Then, by Ramsey's theorem, we have two possibilities:

1. There is a complete 4-uniform subhypergraph on n vertices (white).
2. There is a complete 4-uniform subhypergraph on 5 vertices (blue).

But possibility 2 is impossible due to Observation 1.0.1. Then it must be true that our set contains an n-element subset for which all subsets of 4 elements are in convex position, so it is a convex n-gon.

Application 2. $f(n) \leq R_3(n,n)$: Let us colour the triangles white if the number of points inside them is even and blue if it is odd. Then the vertices of a monochromatic complete subhypergraph must be in convex position, because otherwise we would have a contradiction.

Let us now find some upper bounds for $f(n)$.

Proposition 1.0.4. $f(n) \leq \binom{2n-4}{n-2} + 1$.

Proof. Assume that there are no two points with the same x-coordinate. We define a *k-cup* as k points in convex position and an *l-cap* as l points in concave position, as in Figure 1.2.

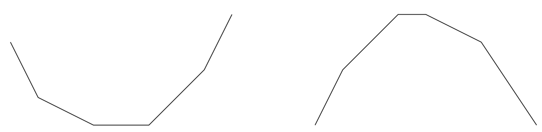

Figure 1.2: Example of a 6-cup and a 6-cap.

Let $f(k,l)$ be the smallest number of points such that there is either a k-cup or an l-cap. It is obvious that $f(n) \leq f(n,n)$.

By construction, we can see that $f(k,l)$ satisfies the following recursion:

$$f(k,l) \leq f(k-1,l) + f(k,l-1) - 1,$$

and, using the base cases $f(k,2) = f(2,l) = 2$ and $f(k,3) = f(3,k) = k$, we can see by induction that

$$f(k,l) \leq \binom{k+l-4}{k-2} + 1,$$

so

$$f(n) \leq f(n,n) \leq \binom{2n-4}{n-2} + 1.$$

□

It is interesting to note that $\binom{2n-4}{n-2} + 1 < 4^n$, which means that $f(n) < 4^n$.

Proposition 1.0.5 (Tóth–Valtr). $f(n) = \binom{2n-5}{n-2} + 2$.

An exponential lower bound is given by Horton's construction. It goes as follows: the empty set $\emptyset$ and the 1-point sets are Horton. Furthermore, H can be partitioned into two sets H^+ and H^- such that

1. both H^+, H^- are Horton;
2. H^+ lies *high above* H^-;
3. the x-coordinates of the points of H^+ and H^- alternate.

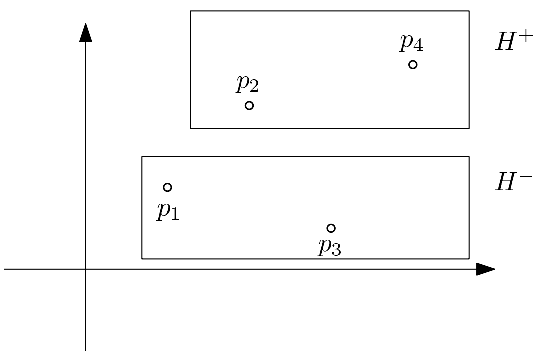

Figure 1.3: Example of Horton's construction.

Let us see an example of a Horton set. If $k = 2^n$, we define $H(k) = H_{2^n}$ as

$$H(k) = \{(x, \mathrm{inv}(x)) \mid x \in \{0, \dots, 2^{n-1}\}\}$$

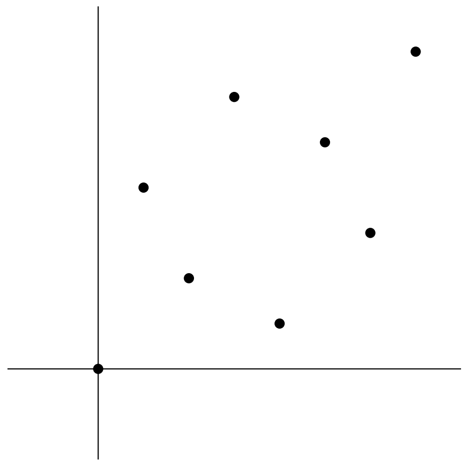

Figure 1.4: $H(8)$.

where $\mathrm{inv}(x)$ is the number whose representation in binary in inverse order is the binary representation of x.

If $H(k) = H_{2^n}$ contains a convex N-gon, then $N \leq 4k - 2$ for $k \geq 3$. In order to see it, it is enough to prove that if $H(k)$ contains an M-cup then $M \leq 2k - 1$ for $k \geq 2$, and we can see it by induction on k.

Finally, Erdős and Szekeres showed that $f(n) \geq 2^{n-2} + 1$ and conjectured that $f(n) = 2^{n-2} + 1$.

Now, if we define a *k-hole* as an empty k-gon, then it is easy to see that Horton's construction does not contain 7-holes. If we have a 7-hole, then, without loss of generality, we can assume that we have a 4-cup in H^-, and, using that H^- is also Horton —as we can see in Figure 1.5— the 7-gon is not empty. So it is not a 7-hole.

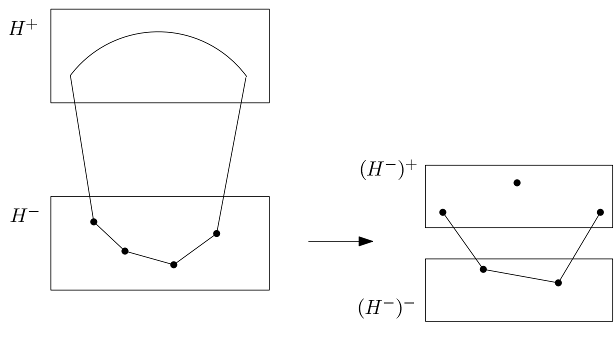

Figure 1.5: A 4-cup in H^- is not empty.

About the 6-holes, it is seen that every sufficiently large set contains 6-holes. A simple proof is given by Valtr in *On empty hexagons*.

The next step is to find k-holes in point sets for $k < 6$. There are many monochromatic empty triangles in two-coloured point sets. It is proved that there are at least $cn^{\frac{4}{3}}$. The question is if there are cn^2. This can be answered by this other question: Are there cn^2 empty 5-gons in a non-coloured point set? It is obvious that for every 5-hole we have at least one monochromatic empty triangle.

Furthermore, it is seen that Horton sets can be coloured so that there is no monochromatic empty 5-gon: First, we colour $\{p_1, \ldots, p_n\}$ repeating the sequence red-blue-green, and observe that there can be no monochromatic triangles because of the construction of Horton sets. Now we recolour the red points and the blue points using white. If we want to find a monochromatic empty 5-gon, it must be a white one. But if we bicolour a monochromatic empty 5-gon with red and blue, again there must be at least one monochromatic empty triangle with the original colouration, which is impossible.

Problem 1.0.6. If $|\mathcal{P}|$ is large enough and bicoloured, is there always an empty monochromatic convex quadrilateral?

Conjecture 1.0.7 (Erdős–Hajnal Conjecture)**.** *Let $R_H(n)$ be the smallest number such that any graph of this many vertices without an induced H contains a homogeneous subgraph on n vertices. Then, for every H, $R_H(n)$ is subexponential.*

Now let us consider the analogue of the Erdős–Szekeres conjecture but with forbidden order types. We need a previous definition. We say that $P = \{p_1, \ldots, p_n\}$ and $Q = \{q_1, \ldots, q_n\}$ are of the *same order type* if for every i, j, k the triangles $p_ip_jp_k$ and $q_iq_jq_k$ have the same orientation. And we say that $\mathcal{P}$ is in *convex position* if $\mathcal{P} = \{p_1, \ldots, p_n\}$ is such that for every $i < j < k$ the triangles $p_ip_jp_k$ have the same orientation.

For simplicity, from now on $\mathcal{T}$ will be a forbidden order type whose convex hull is a triangle. We define $f_{\mathcal{T}(n)}$ as the smallest number such that if $|\mathcal{P}| \geq f_{\mathcal{T}(n)}$ and $\mathcal{P}$ does not contain $\mathcal{T}$, then $\mathcal{P}$ contains $\mathcal{C}_n$ (convex order type of n points).

We define $\mathcal{T} = \mathcal{E}_k$ as the order type in which there are $k-2$ points inside the triangle forming a k-cap with two vertices of the triangle.

Theorem 1.0.8. *If $\mathcal{T} = \mathcal{E}_k$ with $k \geq 1$, then $f_{\mathcal{T}}(n) < C_kn$.*

For example, for $k = 1$, $f_{\mathcal{T}}(n) = n$. C_k is supposed to be about 4^{3k}.

Theorem 1.0.9. *If $\mathcal{T} \neq \mathcal{E}_k$ with $k \geq 0$, then $f_{\mathcal{T}}(n) > cn^2$.*

Proof. We choose m such that $m < \frac{n}{2}$, and construct a point set in a $2m$-gon as in Figure 1.6.

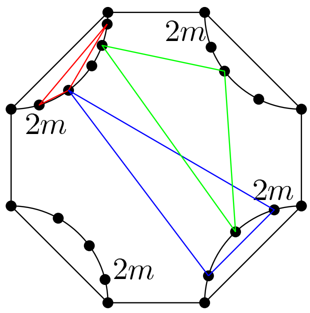

Figure 1.6: Construction of $\mathcal{P}$.

As we can see, all the possible triangles are either empty ($\mathcal{E}_0$) or of the order type $\mathcal{E}_k$. So, we have a point set $\mathcal{P}$ such that $|\mathcal{P}| = 2m^2$ does not contain any point set of the order type $\mathcal{T}$, and does not contain any $\mathcal{C}_n$ because the biggest $\mathcal{C}_k$ in our point set is $\mathcal{C}_{2m}$, and $2m < n$. So $f_{\mathcal{T}}(n) > 2m^2$. But we chose m satisfying $m < \frac{n}{2}$, so $f_{\mathcal{T}}(n) > cn^2$. □

Now we define $\mathcal{T} = \mathcal{F}_k$ as the order type in which there are k points inside the triangle forming a k-cup.

Theorem 1.0.10. *If $\mathcal{T} = \mathcal{F}_k$, then $f_{\mathcal{T}}(n)$ is bounded by a polynomial in n. The degree of the polynomial depends on k.*

Theorem 1.0.11. *If $\mathcal{T} \neq \mathcal{C}_k, \mathcal{F}_l$, then $f_{\mathcal{T}}(n)$ is exponentially large in n.*

Seminar 10

Double Pseudoline Arrangements

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1.1 Introduction

Pseudoline arrangements (or, in higher dimension, pseudohyperplane arrangements) have been extensively studied in the last decades as a useful combinatorial abstraction of projective configurations of points [11, 12, 7, 4, 6, 2, 9, 5]. Recently, Habert and Pocchiola [8] introduced double pseudoline arrangements as a combinatorial abstraction of projective configurations of disjoint convex bodies (motivated by visibility questions [10, 1, 8]). The main structural properties of pseudoline arrangements (embeddability in a geometric projective plane, connectivity of the mutation graph, axiomatic characterization) extend to double pseudoline arrangements.

The goal of the talk is first to give a short overview of the main results on pseudoline arrangements (see Section 1.2), and then to extend these results to double pseudoline arrangements (see Section 1.3).

1.2 Pseudoline arrangements

1.2.1 Definition

Let $\mathcal{P}$ denote the projective plane, i.e., the space of lines of $\mathbb{R}^3$. For all pictures, it is represented as the unit disk where antipodal points in the border are identified. A *pseudoline* of $\mathcal{P}$ is a non-separating simple closed curve of $\mathcal{P}$ (Fig. 1.1). A *pseudoline arrangement* is a finite set of pseudolines such that any two of them have a unique intersection point (Fig. 1.1).

1.2.2 Isomorphism and mutations

Two arrangements A and B are *isomorphic* if there is a homeomorphism of the projective plane that sends A on B (or, equivalently, if there is an isotopy joining A to B). We are naturally interested in isomorphism classes of arrangements.

A particulary simple transformation of the isomorphism class is given by the so-called mutations. A *mutation* is a local transformation of an arrangement L that only inverts a triangular face of L. More precisely, it is a homotopy of arrangements in which only one curve ℓ moves, sweeping a single vertex of the remaining arrangement $L \setminus \{\ell\}$ (Fig. 1.1).

Let us consider the graph G_n whose vertices are the isomorphism classes of arrangements of n pseudolines and whose edges correspond to mutations. This graph is known to be connected:

Theorem 1.2.1. [11] *Any two pseudoline arrangements (with the same number of pseudolines) are homotopic via a finite sequence of mutations.*

According to this result, one can enumerate isomorphism classes of arrangements of n pseudolines by exploring the graph G_n. The following table gives the number p_n of isomorphism classes of simple pseudoline arrangements in the projective plane, for small values of n:

n	1	2	3	4	5	6	7	8	9	10	11
p_n	1	1	1	1	1	4	11	135	4 382	312 356	41 848 591

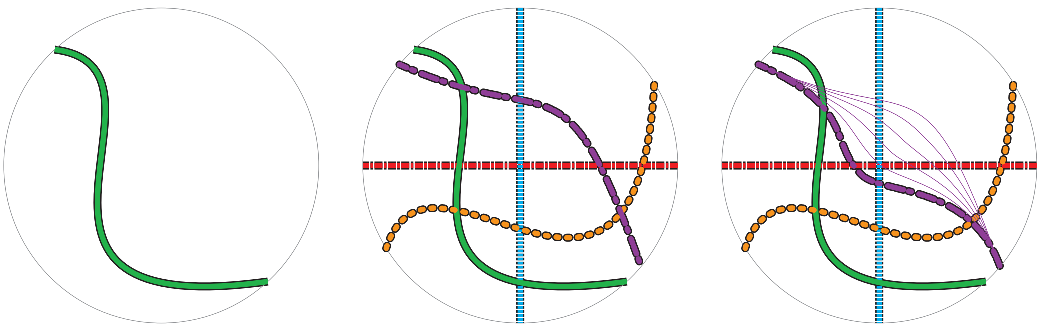

Figure 1.1: A pseudoline; a pseudoline arrangement; and a mutation.

1.2.3 Geometric projective planes and representation

A *geometric projective plane* [14, 13] is given by a set $\mathbb{P}$ of *points* (homeomorphic to the projective plane) together with a set $\mathbb{L}$ of *lines* (each of them being a non-separating simple closed curve of $\mathbb{P}$) such that any two lines meet exactly in one point (which depends continuously on the two lines) and any two points are contained in exactly one common line (which depends continuously on the two points). For example, we obtain a geometric projective plane with $\mathbb{P}$ being the quotient set of the sphere $\mathbb{S}^2$ by the antipodal map $x \mapsto -x$, and $\mathbb{L}$ being the set of projections of great circles of the sphere (Fig. 1.2).

The first representation theorem affirms that any pseudoline arrangement can be extended into a complete geometric projective plane (i.e., can be seen as a subfamily of the line set $\mathbb{L}$):

Theorem 1.2.2. [6, 9] *Any pseudoline arrangement is isomorphic to a finite family of lines of a geometric projective plane.*

Pseudoline arrangements are also related with geometric projective planes via duality. Let $(\mathbb{P}, \mathbb{L})$ be a geometric projective plane, P be a finite set of points of $\mathbb{P}$, and $p \in P$. It turns out that the set p^* of all lines of $\mathbb{L}$ passing through p forms a pseudoline of $\mathbb{L}$, called the *dual pseudoline* of p. Moreover, the set P^* of all dual pseudolines of the points of P forms a pseudoline arrangement of $\mathbb{L}$, called the *dual pseudoline arrangement* of P. The second representation theorem affirms that any pseudoline arrangement can be seen as the dual pseudoline arrangement of a certain point set in a well-chosen geometric projective plane:

Theorem 1.2.3. [6, 9] *Any pseudoline arrangement is isomorphic to the dual of a finite family of points in a geometric projective plane.*

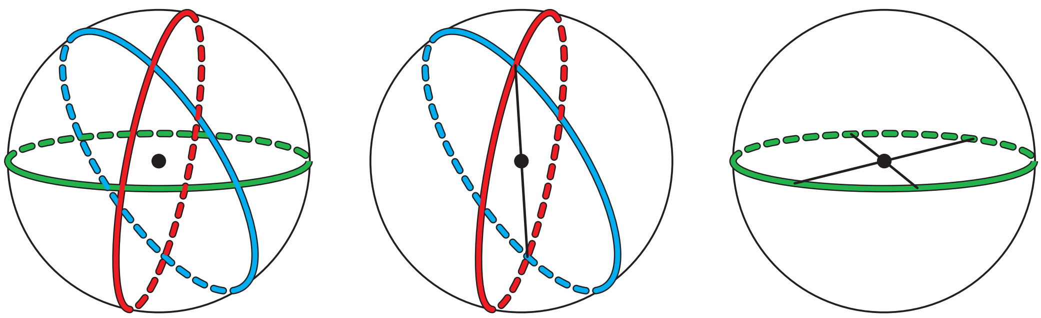

Figure 1.2: A geometric projective plane: points are pairs of antipodal points on the sphere and lines are (projections of) great circles; two lines cross exactly once; two points are contained in a unique common line.

1.2.4 Chirotope

Consider a pseudoline arrangement L that is indexed and oriented: each pseudoline receives a number and an orientation. The *chirotope* of L is the application that assigns to each triple J of indices the isomorphism class of the subarrangement indexed by J. Observe that, since there exist only two non-degenerate isomorphism classes of indexed oriented arrangements of three pseudolines, we can associate a sign $+$ or $-$ to each of them and recover the "usual" definition of chirotope.

The interest of chirotopes is that they encode completely the pseudoline arrangement:

Theorem 1.2.4. [11, 12, 4] *An isomorphism class of indexed oriented arrangements only depends on its chirotope.*

Furthermore, one can characterize what applications correspond to chirotopes of pseudoline arrangements of the projective plane:

Theorem 1.2.5. [11, 12, 4] *Given an application χ that assigns to each triple J of indices an isomorphism class of an oriented pseudoline arrangement indexed by J, the following properties are equivalent:*

1. *χ is the chirotope of an indexed oriented pseudoline arrangement.*
2. *The restriction of χ to the set of triples of any subset of at most five indices is the chirotope of an indexed oriented pseudoline arrangement.*

1.3 Double pseudoline arrangements

1.3.1 Definition

Again, we consider the projective plane $\mathcal{P}$. A separating simple closed curve in $\mathcal{P}$ is called a *double pseudoline*. Its complement has two connected components: a Möbius strip and a topological disk. Observe that any simple closed curve of $\mathcal{P}$ is either a pseudoline (if it is non-separating, or equivalently, non-contractible), or a double pseudoline (if it is separating, or equivalently, contractible).

A *double pseudoline arrangement* is a finite set of double pseudolines such that any two of them have exactly four intersection points, cross transversally at these points, and induce a cell decomposition of $\mathcal{P}$ (Fig. 1.3). Observe that, as for pseudoline arrangements, double pseudoline arrangements are defined only with conditions on subarrangements of size two.

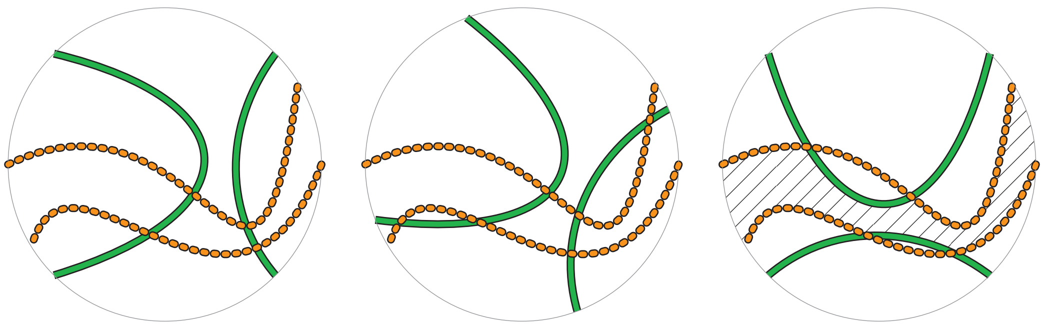

Figure 1.3: Three sets of two double pseudolines: the first one is a double pseudoline arrangement; the second is not since the two double pseudolines intersect 6 times instead of 4; the last is not either since the double pseudolines do not induce a cell decomposition (the hatched face is a Möbius strip).

Again, we are interested in isomorphism classes of simple arrangements. For example, there is only one simple arrangement of two double pseudolines (Fig. 1.3), while already 13 simple arrangements with three double pseudolines (Fig. 1.4).

The end of the talk presents extensions to double pseudoline arrangements of the results on pseudoline arrangements presented before.

1.3.2 Mutations and enumeration

As for pseudoline arrangements, mutations are defined as local transformations inverting a triangular face of the arrangement. The mutation graph is again connected:

Theorem 1.3.1. *Any two double pseudoline arrangements (with the same number of double pseudolines) are homotopic via a finite sequence of mutations.*

Based on this result, an enumeration algorithm has been implemented [3] to count the number q_n of isomorphism classes of simple arrangements of n double pseudolines in the projective plane, for small values of n:

n	1	2	3	4	5
q_n	1	1	13	6570	181 403 533

For example, the 13 simple arrangements of three double pseudolines are represented in Fig. 1.4.

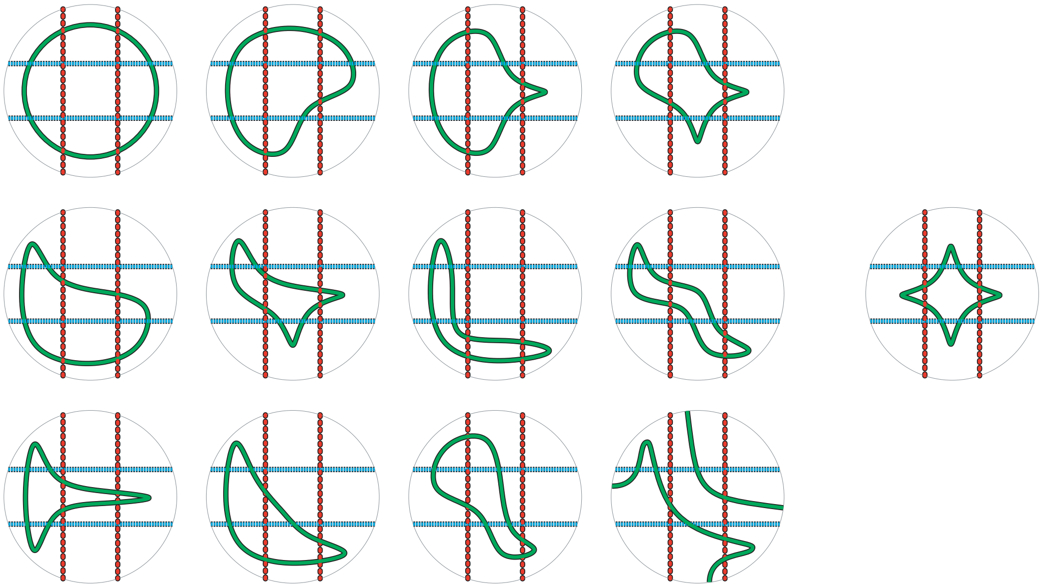

Figure 1.4: The 13 isomorphism classes of arrangements of three double pseudolines.

1.3.3 Duality and representation theorem

A *convex body* of a geometric projective plane is a compact subset C of $\mathcal{P}$ with non-empty interior and whose intersection with any line is an interval of that line. A line is *tangent* to C if this interval is a single point. The set C^* of all tangents to C forms a double pseudoline of $\mathcal{P}$. Furthermore, if Γ is a set of disjoint convex bodies of $\mathcal{P}$, then the set of all their dual double pseudolines forms a double pseudoline arrangement. As for pseudoline arrangements, any double pseudoline arrangement can be seen as the dual of a certain family of disjoint convex bodies in a well-chosen geometric projective plane:

Theorem 1.3.2. *Any double pseudoline arrangement is isomorphic to the dual of a finite family of disjoint convex bodies in a geometric projective plane.*

Two different proofs of this result are possible:

1. By mutations: If L is an arrangement of double pseudolines for which we know a primal representation, and if L' is another double pseudoline arrangement obtained from L by a single mutation, then one can update the primal representation of L to obtain a primal representation of L'. The connectivity of the mutation graph ensures then that any arrangement is representable.

2. By reduction: We prove that any double pseudoline arrangement can be polygonalized, since each double pseudoline can be seen as the convex

hull of finitely many pseudolines. Then we use the Representation Theorem for pseudoline arrangements to derive the similar theorem for double pseudolines.

1.3.4 Chirotope

The *chirotope* of an indexed and oriented double pseudoline arrangement is the application that assigns to each triple J of indices the isomorphism class of the subarrangement indexed by J. As for pseudoline arrangements, the chirotope of a double pseudoline arrangement completely determines the arrangement, and chirotopes of double pseudoline arrangements are characterized by their restrictions to five double pseudolines:

Theorem 1.3.3. *An isomorphism class of indexed oriented double pseudoline arrangements only depends on its chirotope.*

Theorem 1.3.4. *Given an application χ that assigns to each triple J of indices an isomorphism class of an oriented double pseudoline arrangement indexed by J, the following properties are equivalent:*

1. *χ is the chirotope of an indexed oriented double pseudoline arrangement.*
2. *The restriction of χ to the set of triples of any subset of at most five indices is the chirotope of an indexed oriented double pseudoline arrangement.*

In other words, in order to check that a given application is a chirotope of a double pseudoline arrangement, it is again enough to check it only on subconfigurations of size at most five. Observe that this result provides a motivation for the enumeration of arrangements of at most five double pseudolines presented in Subsection 1.3.2.

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Seminar 11

On the Hirsch Conjecture

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This is going to be a survey talk. Most of the theorems mentioned here are at least thirty years old. A *convex polytope* is the convex hull of a finite point set in $\mathbb{R}^d$. A *convex polyhedron* is a finite intersection of closed half-spaces. These two notions are almost the same: Every polytope is a polyhedron, but the converse is not true (there may not be sufficiently many half-spaces). Thus, a polytope is a bounded polyhedron.

Polytopes and polyhedra have faces of various dimensions. A *face* of a polytope or polyhedron is the intersection of a hyperplane with the polytope that does not go through the interior. This is one possible definition: The faces of P are the intersections $P \cap H$, where H is a hyperplane not going through $\text{int}(P)$. Given a polytope of dimension d, there are faces of dimension 0 through d, and some of them have special names.

If P is a d-polytope, then 0-faces are *vertices*, 1-faces are *edges*, and $(d-1)$-faces are called *facets*. Also $(d-2)$-faces are called *ridges*. The Hirsch conjecture was originally posed for the graphs of all polyhedra in [5].

Conjecture 1.0.1 (Hirsch, 1957). *For every polyhedron of dimension d with n facets, the diameter of the graph is $\leq n - d$.*

There are non-bounded counterexamples. Therefore, we nowadays speak of it for polytopes:

Conjecture 1.0.2 (Hirsch, 1957)**.** *For every polytope of dimension d with n facets, the diameter of the graph is $\leq n - d$.*

The case of polyhedra was disproved by Klee and Walkup in 1967. The first question is “Why is this number $n - d$ reasonable?”. The second question is “Why is this number unreasonable?”. Now the conjecture is that the conjecture is actually false.

1.1 Examples and easy observations

Let us see some examples of polytopes and diameters of polytopes. The first easy observation is that there is no loss of generality in assuming that the polytope P is simple. *Simple* means that, at every vertex, exactly d facets meet. This is the same as saying that the hyperplanes that define facets are sufficiently generic (or are in general position).

The reason why we can assume this is that, if we have a polytope that does not have this property, we may consider the polytope as an intersection of half-spaces. By perturbing the half-spaces, we create new vertices which can only increase the diameter of the polytope. Then the resulting graph is d-regular. For example, in an octahedron, we can perturb the facets. Every path in the perturbed graph contracts to a path in the original graph.

One thing I like a lot (since I like triangulations) is that the same question can be asked for the dual. In the dual formulation, if we have a simplicial polytope P, we can ask about the diameter of the ridge-graph $\widehat{G}$: The vertices of $\widehat{G}$ are the facets of P, and the edges of $\widehat{G}$ are the ridges of P. For simplicial polytopes, this is a nice combinatorial object: We are asking what is the minimal number of steps necessary to go from one simplex to the next. So, this way of asking the question makes it more combinatorial, at least for me. This is nicer for constructions.

Now, why is $n - d$ “natural”?

1. One way of saying it is that, for any $n > d$, it is easy to construct a simplicial complex that is topologically a ball and with ridge-graph diameter equal to $n - d$. The number d is one more than the dimension, and n is the number of vertices in this case, not the number of facets.

   The polar of a simple polytope is a simplicial sphere. But the polar of an unbounded polyhedron is a simplicial ball. Not every combinatorial/topological ball can be realized as the polar of a polyhedron, but the ones constructed here can. What this means is that, for any $n \geq d$, it is very easy to construct unbounded polyhedra that are “Hirsch tight”, that is, their diameter is exactly the conjectured upper bound.

If we want to construct *polytopes* that are Hirsch tight, then this is a little more difficult. This is one of the reasons why this number $n - d$ is so natural.

2. For another reason, cubes of any dimension are Hirsch tight, and, actually, products of them too. (Here we are in the simple world; in order to stay simplicial, state this for cross-polytopes, and not cubes.) Why is this true for products of them?

   If P has n_1 facets and dimension d_1 and diameter δ_1, and Q has n_2, d_2 and δ_2, then $P \times Q$ has $n_1 + n_2$ facets, dimension $d_1 + d_2$, and diameter $\delta_1 + \delta_2$. Hence, if P and Q are tight, then $P \times Q$ is tight. This is another reason why this number is natural: it behaves well under products.

   In particular, for every $2d \geq n \geq d + 1$, it is very easy to construct Hirsch-tight polytopes.

The main motivation for the Hirsch conjecture was the simplex method in linear programming. In linear programming, the input is a polyhedron given via the facets for which we want to maximize a certain linear functional. The simplex method starts with any feasible solution (specifically a vertex) and it follows the functional from vertex to vertex. It is the simplest method, and, at the time of the conjecture, it was the only method. The diameter tells us the theoretical lower bound on the number of steps needed. Nowadays, no one knows whether the simplex method is polynomial. To put it better, the simplex method is really not a method, but rather there are several ways to choose the step on each iteration (given by a pivot rule).

There are several pivot rules proposed, and for most of them (if not all of them) there is a polytope that takes an exponential number of steps. One of the reasons why there is no polynomial method known is that the conjecture is still open. Thus, the conjecture is that the bound is linear, although no polynomial upper bound is known. Of course, nowadays, the ellipsoid method and the interior point method give polynomial time bounds.

Question. By polynomial, is it meant polynomial in d and n?

Answer. Yes, or we could just say polynomial in n, since $n > d$. But we want this fixed degree as d grows. This is the interesting fact: If we fix d, then the bound is polynomial because there is a polynomial number of vertices:

$$\lfloor n^{n/2} \rfloor .$$

Note, however, that an upper bound on the diameter does not give an upper bound on the number of iterations needed for the simplex algorithm. Instead, it gives the minimum number of iterations needed for the simplex algorithm under *any* pivot rule, providing that the initial vertex is “far away” from the optimal vertex.

Let me list several positive things that we know on the Hirsch conjecture. Then I will state and show negative things: This means examples that there are almost counterexamples to the Hirsch conjecture.

Also, let me point out that most people believe that the Hirsch conjecture is false. The concern is whether there is a polynomial bound or not. Here, I would say that opinions are split. Actually, the person who was closest to proving a polynomial bound (Gil Kalai [8]) believes that there is no polynomial bound.

1.2 Positive results

Let $H(n, d)$ be the maximum diameter of d-polytopes with n facets.

1. $H(n, 2) = \lfloor \frac{n}{2} \rfloor$. This is very easy to prove. Also, $H(n, 3) = \lfloor \frac{2n}{3} \rfloor - 1$.

2. $H(n, d) \leq n - d$ if $n - d \leq 6$. For $n - d \leq 5$, this was proved by Klee and Walkup in [9]. For $n - d = 6$, this was proved by Bremner and Schewe in [3]. The first is a traditional proof, and the second is more a computer proof with an enumeration of possible polytopes.

3. The Hirsch conjecture holds for 0-1 polytopes, that is, polytopes whose coordinates of all vertices are either 0 or 1. This was proved by Naddef in [11]. There is no polynomial simplex algorithm in this case.

4. $H(n, d) \leq O(n^{\log d + 2})$. This was shown by Kalai and Kleitman in [8]. The proof is very simple. I am going to sketch it. Actually, all the proofs are simple except the one based on computer enumeration. There are subexponential simplex algorithms, but it is still not as good as this diameter bound. The subexponential bound of $e^{O(\sqrt{n \log d})}$ is in [10].

If we are interested in combinatorial polytopes, then we can say more.

- For transportation polytopes, there is a linear bound (see [4]).

- For network-flow polytopes, there is an $O(nm^2)$ bound (see [12]). Network-flow polytopes are polytopes that are defined from a network. In this network, we are going to put flows. Our constraints are that the flows are non-negative and we have fixed flows that come in to or out of vertices (some nodes act as supplies and others as demands). Why is this a polytope? We have an inequality for each edge and for each edge-capacity bound. We have one equation for each node for node balancing. For this polytope, there is a polynomial simplex algorithm given in [12]. Note that transportation polytopes are where our graph is bipartite.

- For the duals of transportation polytopes, the Hirsch bound holds (see [2]).

Let me give the proofs of at least some of these facts.

Lemma 1.2.1. $H(n,3) \leq \lfloor \frac{2n}{3} \rfloor - 1$.

Proof. A simplicial 3-polytope with n facets has $2n-4$ vertices, which follows by Euler's formula. Also, its graph is 3-connected. So, given two vertices u and v, we have three disjoint paths that go from u to v. They use in total $2n-6$ intermediate vertices[1].

This means that the shortest of them uses at most $\frac{2n}{3} - 2$, and if we have that many intermediate vertices, the number of edges is this plus one. □

What fails in dimension 4? What fails is that we now have a quadratic number of vertices.

Actually, this is only half of the proof. Now, we have to construct our polytope.

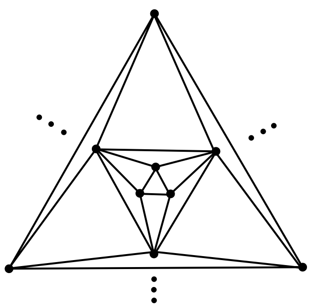

Figure 1.1: A family of 3-dimensional polytopes with $6 + 3k$ vertices for each positive k.

Lemma 1.2.2. $H(n,3) \geq \lfloor \frac{2n}{3} \rfloor - 1$.

Proof. See the picture in Figure 1.1. To move from one layer to another, we need 3 steps. In each layer, we see an octahedron. □

[1]Because this is the number of vertices in the polytope.

This construction works for n a multiple of three, but variations of this are possible.

For the second point, we need a lemma (or theorem) that I should have stated already. Let me add this as part 5:

- If we fix $n - d = k$, then $\max\{H(n,d) \mid n - k = k\} = H(2k,k)$. Put differently, we only need to prove the Hirsch conjecture for the case $n = 2d$. This is called the *d-step conjecture*.

If we want to prove part 2, we only need a finite number of cases. This is why we have a computer proof for [3]. Thus, part 2 is equivalent to stating the following:

Lemma 1.2.3. *$H(8,4) = 4$, $H(10,5) = 5$, and $H(12,6) = 6$.*

I will only prove the first claim.

Proof. If we have a polytope with 8 vertices in dimension 4, let u and v be two vertices. The d-step conjecture allows us to only look at this many facets. Now, if those two vertices share a facet F, then it is a 3-polytope of at most 7 facets, since the facets of F come from the intersection of the other facets of P. But we already have that $H(7,3) = 3$ from Lemma 1.2.1.

If u and v do not share a facet, do a pivot from u to a neighboring vertex u'. By the previous case, the distance from u' to v is at most 3, so the distance from u to v is at most 4. □

For $H(10,5) = 5$, we need some more things. For $H(12,6) = 6$, we need a computer.

1.2.1 The *d*-step conjecture, the Hirsch conjecture, and the "non-revisiting" conjecture

Let me speak a bit about the d-step conjecture. The d-step conjecture is that $H(2d,d) \le d$. The non-revisiting conjecture is the following: For every two vertices u and v of a polytope P, there is an edge path that never revisits a facet that it has previously abandoned.

It is very easy to see that the non-revisiting conjecture implies the Hirsch conjecture. There are at most $n - d$ facets that we can abandon. At every pivot step, we abandon one facet. This, by the way, is another reason why the number $n - d$ is reasonable.

Theorem 1.2.4. *These three conjectures are equivalent.*

Proof. Clearly, the Hirsch conjecture implies the d-step conjecture. The non-revisiting path conjecture implies the Hirsch conjecture: A non-revisiting path can never have length $> n-d$ because, at each step, we abandon a facet and there are d facets at our final vertex.

The difficult part is to prove that the d-step conjecture implies the non-revisiting conjecture. Let me prove first that the d-step conjecture implies the Hirsch conjecture. I will prove that

$$\cdots \leq H(2d-1,d-1) \leq H(2d,d) \geq H(2d+1,d+1) \geq H(2d+2,d+2) \geq \cdots \quad (1.1)$$

In the first half of the above, $n > 2d$, while in the second half $n < 2d$.

Let P be our polytope and let u and v be vertices.

- If $n < 2d$, then u and v have some facet F in common, since each has n facets. Then we do the same trick as before. The facet F has dimension one less (namely $d-1$) and number of facets one less (namely $n-1$).

- If $n > 2d$, we need to find a construction that goes to the right in (1.1). That construction is called the *wedge*. There is a facet F not containing u nor v. Let P' be a "wedge of P on that facet"; see Figure 1.2. We draw two copies in parallel and join an edge together (in dimension three). So P' has one more facet[2], one more dimension, and at least the same diameter.

  Observe that, for polytopes with $2d$ facets, the non-revisiting conjecture and the Hirsch conjecture really say the same. To prove that the d-step conjecture proves the non-revisiting conjecture, use the wedge construction again and prove that the wedge respects the non-revisiting property.

This proves the equivalence of these conjectures. □

Question. But, do we really have our result for $H(n,4)$ from earlier?

Answer. My claim is that $H(9,4) \leq H(10,5)$. We do want to know if $H(10,5) \leq 5$, but this is not the claim.

1.3 Some negative results

Let us speak about a 4-dimensional polytope P with 9 facets that is Hirsch tight, and is basically the only Hirsch-tight polytope known with $n > 2d$.

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[2]Is it clear that we have one more facet? We have two copies of P but not the facet F. Every other facet of P gives a facet of P'.

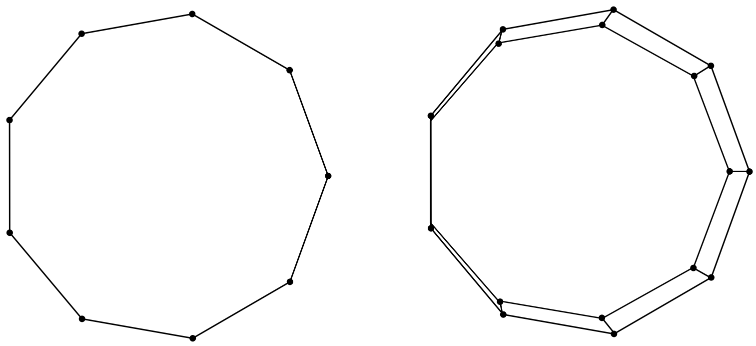

Figure 1.2: A 9-gon and a wedge on its leftmost facet.

1. There is a polytope with $n > 2d$ and diameter $n - d$ ($n = 9$, $d = 4$).
2. There is an unbounded polyhedron with diameter strictly greater than $n - d$.
3. The monotone Hirsch conjecture is false: There is a polytope, a vertex v and a linear functional f such that every f-monotone path from v to the optimum has length strictly greater than $n - d$.
4. Many polytopes meet the Hirsch bound: For every $n \geq d \geq 8$ there is a Hirsch-tight polytope.

The fourth item is a result by Fritzsche and Holt in [6] (which improved the work of [7]). Note: *All of this comes from a single polytope!* What is special about this polytope? It is the Klee–Walkup polytope of [9]. It has dimension 4, with 9 facets and diameter equal to 5. But, also, it is the only combinatorial polytope with those properties. There is more than 1,000 simple 4-polytopes with 9 facets. The 4-polytopes with 9 facets have been enumerated by Altshuler, Bokowski, and Steinberg in [1] in 1980. The polytope was found in [9] in 1967, but the uniqueness of this polytope was proved later.

Figure 1.3 is a picture of a 4-polytope, even if it does not look like it. First, note that we do not draw the simple polytope, but the simplicial polytope. So, what is this picture?

- It "depicts" the polar (simplicial polytope). The polar is a 3-sphere. Drawing a 3-sphere is difficult, but drawing a 3-ball is not that difficult: It is not hard to visualize, because we live in three dimensions.

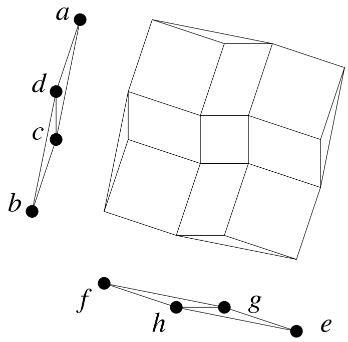

Figure 1.3: A "cross section" of the Mani–Walkup polytope Q_4.

- Removing a vertex from it gives a simplicial ball of dimension 3 with 8 vertices (as $9 - 1 = 8$). If I give the ball, then you know the sphere: The sphere is obtained by coning the boundary with a point at ∞.

- Actually, I am going to construct this simplicial ball as a Delaunay triangulation, which is something that you are more-or-less familiar with. This simplicial ball[3] is the Delaunay triangulation of:

$$\begin{pmatrix} 1 & -1 & 0 & 0 & -4 & -4 & -1 & 1 \\ 4 & -4 & -1 & 1 & -1 & 1 & 0 & 0 \\ 1^+ & 1^+ & 1 & 1 & -1^- & -1^- & -1 & -1 \end{pmatrix},$$

where $1^+ = 1 + \epsilon$ and $1^- = 1 - \epsilon$. I will label these points a, b, c, d, e, f, g, h. There are four points in $z = 1$ and four in $z = -1$, before perturbation.

I want to describe the triangulation. It is enough the describe the triangulation with an intermediate plane. Each tetrahedron has 4 vertices.

If we have two vertices on top and two on bottom, then we see the Minkowski sum of line segments. The triangles represent tetrahedra of 3 and 1 vertices. So, in the picture, we have 13 tetrahedra.

Question. And the mixed subdivision comes from the fact that it is Delaunay?

Answer. No, any subdivision will be. We will need Delaunay just to prove that it is polytopal.

Now, we have little pluses and minuses. We move a and b slightly up and e and f slightly down. The Delaunay triangulation will slightly change. A simplicial 3-ball with 8 vertices, 15 tetrahedra, and which diameter?

[3]I will give 9 points in dimension 3.

Since we have the picture, we can compute the diameter. How do we go from the lower right simplex to the upper right simplex? We need 3. Hence, we get a diameter of 5.

Now the Hirsch conjecture for simplicial balls is essentially the same. The Delaunay triangulation is regular (it can be lifted to a paraboloid $x_4 = x_1^2 + x_2^2 + x_3^2$ in dimension four), hence the lower envelope of what we see is precisely our simplicial ball. And, for example, it should now be easy to derive the polytope: Simply take the lower envelope and a vertical facet at ∞, which is the ninth vertex of our polytope. This is the Klee–Walkup polytope with 9 facets.

The diameter is still at least 5. Why is this true? In this polytope, take the same two tetrahedra as before. There are two ways to go through: One is to not use ∞ (which we had seen before). If we go through infinity, then we need to reintroduce four vertices, so we will need at least four steps after going through infinity, which is still at least five steps.

It is the only 4-polytope of 9 vertices with facet-dimension 5. Hence, we have proved the first two claims (since we actually constructed this unbounded polyhedron).

Before going on to parts 3 and 4, let me point out that we can glue several copies of this thing together. By the way, all I have said is in [9]. Part 3 is due to [13], and part 4 is the work of [7] and the extension in [6]. Before moving on, let me picture this 3-ball as follows: The tetrahedron at $z = 1$ has two triangles on the boundary and two on the interior. The same is true for $z = -1$. So, we can iterate it.

Question. And our adjacent triangulations are okay?

Answer. We might not be able to make them Delaunay, but we can make them regular and convex via projective transformations.

There are 4-polyhedra with $4 + 4k$ facets and diameter $5k$. So it is not only non-Hirsch, but we can beat Hirsch by a linear constant.

Question. We are not introducing the point at ∞, right?

Answer. Right, so we still have a polyhedron. If we introduce the vertex at ∞, then our diameter is essentially $\frac{2}{3}n$.

I am not going to speak about part 3, actually. Basically, it is the same polyhedron. Let me just explain the crucial point. If we look at the polyhedron (without ∞), the polyhedron does not have the non-revisiting property. Monotone paths do not use the facet that corresponds to the vertex at ∞.

Let me now speak about part 4. Again, all these polytopes are derived from our favorite polytope. Parts 2 and 4 follow from the existence of this polytope. Let me show this table from [6]. Since we are interested in $n \geq 2d$, in the table the polytopes are parametrized by d and $n - 2d$. I am going to show what these different arrows mean.

The w's mean wedge, which is something that we have seen already: It transforms an (n, d)-polytope into an $(n + 1, d + 1)$-polytope, and we have already seen that the wedging preserves being Hirsch tight, since wedging can only increase the diameter. And τ is truncation. To truncate means that we cut a vertex. It transforms an (n, d)-polytope into an $(n + 1, d)$-polytope. This second process does *not* preserve Hirsch tightness! If it did, I could fill in the rest of the missing triangle in the table.

The reason why it does not in some cases is the following:

Lemma 1.3.1. *If P is Hirsch tight, then wP, τwP and $\tau^2 wP$ are all Hirsch tight.*

This is not very good notation, because wedging and truncation depend on where we do the operations. The proof is very simple:

Proof. We have two neighboring vertices and another pair so that any path from one of u_1 or u_2 to any one of v_1 or v_2 is Hirsch. So, this polytope is more than Hirsch tight (which is what makes truncation work). Now, if we truncate on u_1 and v_2, and if the distance from u to v is k, then the distance from u_3 to v_3 is at least $k + 2$.

Now we need to prove that we can do all the values of n in dimension 8. Then, by wedging, we will have this in all bigger dimensions[4]. What is this polytope? It is the four-fold wedge of Q_4. We will keep wedging on a new facet. It will all depend on which facet we choose. Let $P \subseteq \mathbb{R}^d$. Then the second wedge is in $\mathbb{R}^{d+2}$. By wedging k times on F,

- F becomes a codimension $k + 1$ face, and
- every vertex of $P \setminus F$ appears $k + 1$ times.

Moreover, if $v_0, v_1, \ldots, v_k$ are the "copies" of v in $w^k P$, then they form a clique in the graph. Put differently, they form a simplicial face. Setting k to four, we obtain a polytope $W = w^4 Q_4$.

Since in Q_4 we had two vertices u and v at distance 5, in W we have two sets $S_1 = \{v_0, v_1, \ldots, v_4\}$ and $T = \{u_0, u_1, \ldots, u_4\}$ of 5 vertices each such that:

- $d(u_i, v_j) \geq 5$ for all i, j;
- the u_i's are neighbors of each other (and the same for the v's).

[4]Here Q_4 is the name of the polytope. For Q_4 we need $n > 2d$.

Let me move to the simplicial picture now. We have W and put another copy of W as in Figure 1.4. We want to understand how to go from the (depicted) left-most facet of the big polytope to the (depicted) right-most facet of the big polytope. Keep in mind that this is the simplicial version: each u_i is a vertex.

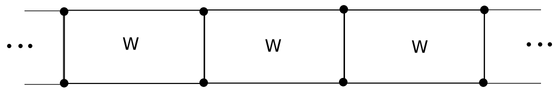

Figure 1.4: Construction of Hirsch-tight polytopes.

How many facets does W have? Every time we wedge, we add one facet. So, W has 13 facets and is 8-dimensional. Here, the diameter is 5, the Hirsch number.

So, in our simplicial picture, each W has 13 vertices and is of dimension 8. What is important is that n increases by 5, because at each meeting of the W's we have 8 vertices. We increase the diameter by at least 4, which is not what we want. We want 5. However, we have not used any of the properties. How can we use the properties that all the u_i's are neighbors of each other? The simplex of dimension 7 has 8 neighbors to the left W and 8 on the right W. If one of these facets is a v_i, we need to increase the bound by one. Using this, we can add 5 and 5 and 5 and then 5. To fill in the intermediate steps, we do some particular things that are simpler than what we just did. This proves property 4. □

What is my guess? If we look at this table, in dimension 3, our polytopes have very small diameter. In dimension 4, we do not know, but we have a clue. The $4, 2$ case has been explored. On the contrary, the other guess is that, when we go to the bottom, we will have extremely non-Hirsch polytopes, including maybe even non-polynomial. One could guess that we can do something for dimension 6. Let me also state that you could try to prove the Hirsch conjecture for dimension 4, if you are in need of open problems.

1.4 Questions

Question. Didn't you say that the Hirsch conjecture d is the same as this d-step?

Answer. No, for codimension d.

Question. Thus, in dimension 4, there is a lot of work to be done.

Answer. In this table, fixed codimension are the SW-to-NE diagonals.

Question. You said that Q_4 is the only polytope with these parameters. Do all the others have a smaller diameter?

Answer. Yes, with dimension 4 and 9 facets. Indeed, the others all have smaller diameter. Some other numbers are known: $H(4,9) = H(4,10) = H(5,10) = 5$. The middle shows non-Hirsch-tight cases. Note that $H(5,11)$ is not known: it is either 6 or 7.

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Seminar 12

Optimization in Geometry: Kissing and Coloring

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1.1 Introduction

We want to consider two optimization problems in geometry:

1. The minimization of the measurable chromatic number.
2. The maximization of the kissing number.

Our strategy will be to study and generalize stable sets of finite graphs and the Lovász theta function based on semidefinite programming (SDP). This will, in a first generalization, lead to new lower bounds for the measurable chromatic number and, in a second generalization, to new upper bounds for the kissing number.

1.2 Two problems in geometry

Definition 1.2.1. The *measurable chromatic number* of $\mathbb{R}^n$, denoted by $\chi(\mathbb{R}^n)$, is defined as the minimal number of colors needed to paint all points in $\mathbb{R}^n$ such that every two points at distance 1 receive different colors.

One easily sees that $\chi(\mathbb{R}^1) = 2$.

Definition 1.2.2. We define $\chi_m(\mathbb{R}^n)$ as the minimal number of colors needed to paint all points in $\mathbb{R}^n$ such that every two points at distance 1 receive different colors and such that points receiving the same color form a Lebesgue measurable set.

By definition, we have $\chi(\mathbb{R}^n) \leq \chi_m(\mathbb{R}^n)$. Furthermore, we know that $4 \leq \chi(\mathbb{R}^2) \leq 7$: The existence of a graph that is not colorable with less than 4 colors yields our lower bound. The tessellation of the plane with hexagons leads to 7 as an upper bound. In 1981 Falconer [2] showed that $\chi_m(\mathbb{R}^2) \geq 5$. But the exact determination of $\chi(\mathbb{R}^2)$ is still an open problem.

Definition 1.2.3. The *kissing number* τ_n in dimension n is defined as the maximum number of unit spheres which can touch a central unit sphere without pairwise overlapping.

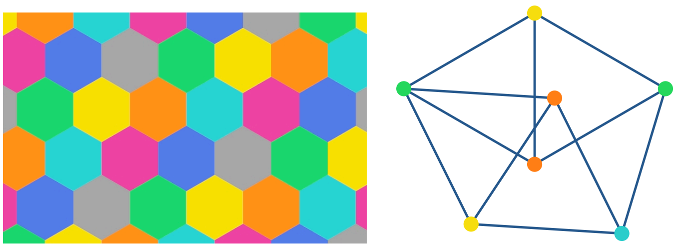
By use of elementary geometry, one can show that the kissing number τ_2 in dimension 2 equals 6. But the exact value of τ_3 was for a long time unknown. Already in 1694, Newton and Gregory were sure about $\tau_3 \geq 12$ but had a disput about the exact value being 12 or 13; see [10]. In 1953 Schütte and van der Waerden [10] showed in a difficult proof that $\tau_3 = 12$. The 8 and 24 dimensional cases were solved in 1979 by Odlyzko, Sloane and Levenshtein [7], who proved by using linear programming that $\tau_8 = 240$ (realized by the E_8 lattice) and $\tau_{24} = 196{,}560$ (Leech lattice Λ_{24}). The value of $\tau_4 = 24$ (D_4 lattice) was determined in 2003 by Musin [6].

We now aim to give a unified proof for all these results using semidefinite programming.

1.3 Strategy

Our strategy for the proof will be to generalize the theta function of finite graphs.

The theta function, which was introduced by Lovász in 1979 [5], gives an upper bound for the stability number of a graph. A big advantage of the



theta function is that it can be efficiently computed, based on the solution of a semidefinite optimization problem. Furthermore, it can be strengthened by a hierarchy of increasingly larger semidefinite optimization problems which converge to the right value (Lovász–Schrijver, 1991).

Definition 1.3.1. Let $G = (V, E)$ be a graph with vertex set V and set of edges E. We call a set $C \subset V$ *stable* (or *independent*) if all pairs in C are not adjacent, i.e., if $\{x, y\} \notin E$ for all $x, y \in C$. If G is finite, we define the *stability number* of G by $\alpha(G) = \max\{|C| : C \text{ stable set}\}$.

Unfortunately, finding (even approximating) the stability number is computationally difficult; see [4].

Definition 1.3.2. *Semidefinite programming* (*SDP*) means the optimization of a linear functional over the intersection of the cone of positive semidefinite matrices with an affine subspace. Thereby, the *primal SDP* is

$$\max\{\mathrm{trace}(CK) : \mathrm{trace}(A_iK) = b_i,\ i = 1, \ldots, m,\ K \succeq 0\},$$

over all semidefinite matrices K for given symmetric matrices C, A_i and $b_i \in \mathbb{R}$. The *dual SDP* is formulated as

$$\min\left\{\sum_{i=1}^{m} b_i y_i \ :\ y_1, \ldots, y_m \in \mathbb{R},\ \sum_{i=1}^{m} y_i A_i - C \succeq 0\right\}.$$

One can think of SDP as a matrix version of linear programming. An advantage of SDP is its expressive power, e.g. for the computation of eigenvalues or the optimization of polynomials. One can use interior point algorithms to approximate optimal solutions, which work in polynomial time. And because of the duality theory, one can use rigorous computer proofs for finding better error bounds. It should be noticed that a solution of the dual problem always gives an upper bound for the primal problem.

Definition 1.3.3. (First formulation.) Let $G = (V, E)$ be a finite graph. Then we define a theta function of G by

$$\theta_1(G) = \max\Big\{\sum_{x\in V}\sum_{y\in V} K(x,y) \mid K \in \mathbb{R}^{V\times V} \text{ positive semidefinite}, \\ \sum_{x\in V} K(x,x) = 1,\ K(x,y) = 0 \text{ if } \{x,y\} \in E\Big\}.$$

Theorem 1.3.4. *Let $G = (V, E)$ be a finite graph. Then $\alpha(G) \leq \theta_1(G)$.*

Proof. Let $C \subset V$ be a stable set and $1_C \colon V \to \{0,1\}$ its characteristic function. We set $K(x,y) = \frac{1}{|C|} 1_C(x) 1_C(y)$. Then $K(x,y)$ satisfies the required conditions and we have $\sum_{x\in V}\sum_{y\in V} K(x,y) = |C|$. Hence, $|C| \leq \theta_1(G)$. $\square$

Dualizing the SDP for θ_1 leads to a second formulation for this theta function:

Definition 1.3.5. (Second formulation.) Let $G = (V, E)$ be a finite graph. Then we define

$$\theta_2(G) = \min\big\{\lambda \mid K \in \mathbb{R}^{V\times V} \text{ positive semidefinite}, \\ K(x,x) = \lambda \text{ for all } x \in V,\ K(x,y) = -1 \text{ if } \{x,y\} \notin E\big\}.$$

Then we have $\theta_1(G) = \theta_2(G)$.

1.4 First generalization

Definition 1.4.1. Let $S^{n-1} = \{x \in \mathbb{R}^n \mid x \cdot x = 1\}$ be the unit sphere. We define the *coloring graph* $G(n,t) = (V, E)$ with $V = S^{n-1}$ and $\{x,y\} \in E$ if and only if $x \cdot y = t$.

One can show that stable sets in $G(n,t)$ can have positive measure. Therefore, we get the stability number of $G(n,t)$ as follows:

$$\alpha(G(n,t)) = \sup\{\omega(C) \mid C \text{ stable and measurable}\},$$

where ω denotes the surface measure of the unit sphere. With this, we have

$$\chi_m(\mathbb{R}^n)\, \alpha(G(n,t)) \geq \omega(S^{n-1}) \quad \text{for } -1 \leq t \leq 1.$$

So, an upper bound for $\alpha(G(n,t))$ will provide a lower bound for $\chi_m(\mathbb{R}^n)$. To generalize the theta function for infinite graphs, we will need a notion of positive semidefinite *infinite matrices*, which leads to (positive, continuous) *Hilbert–Schmidt kernels*:

Definition 1.4.2. A continuous kernel $K \in \mathcal{C}(S^{n-1} \times S^{n-1})$ is called *symmetric* if for all $x, y \in S^{n-1}$ we have $K(x,y) = K(y,x)$, and *positive* if for all N and all $x_1, \ldots, x_N \in S^{n-1}$ we have $\big(K(x_i, x_j)\big)_{1 \le i,j \le N} \succeq 0$.

Thereby, $\mathcal{C}(S^{n-1} \times S^{n-1}) = \{f : S^{n-1} \times S^{n-1} \to \mathbb{R}^n \mid f \text{ continuous}\}$.

We now want to generalize the first formulation of our theta function. For this, the idea is to replace matrices by kernels and sums by integrals:

$$\theta_1(G(n,t)) = \sup\left\{ \int_{S^{n-1}} \int_{S^{n-1}} K(x,y)\, d\omega(x)\, d\omega(y) \mid K \in \mathcal{C}(S^{n-1} \times S^{n-1}), \right.$$
$$\left. K \text{ positive}, \int_{S^{n-1}} K(x,x)\, d\omega(x) = 1,\ K(x,y) = 0 \text{ if } x \cdot y = t \right\}.$$

Theorem 1.4.3. *With the above definitions,* $\alpha(G(n,t)) \le \theta_1(G(n,t))$.

Now the question is how to compute $\theta_1(G(n,t))$. One key observation is that we can restrict our attention to $O(\mathbb{R}^n)$-invariant kernels. Thereby $O(\mathbb{R}^n) = \{A \in \mathbb{R}^{n \times n} \mid A^t A = I_n\}$ denotes the *orthogonal group* (isometry group of S^{n-1}).

Definition 1.4.4. A kernel $K \in \mathcal{C}(S^{n-1} \times S^{n-1})$ is called $O(\mathbb{R}^n)$-*invariant* if for all $A \in O(\mathbb{R}^n)$ and for all $x, y \in S^{n-1}$ we have $K(Ax, Ay) = K(x,y)$.

The restriction to $O(\mathbb{R}^n)$-invariant kernels is possible, because of the following reason: If K is a feasible solution, then the $O(\mathbb{R}^n)$-invariant *group average* $\overline{K}$, defined by

$$\overline{K}(x,y) = \int_{A \in O(\mathbb{R}^n)} K(Ax, Ay),$$

is also feasible and has the same objective value.

To get a "good" characterization of positive, invariant kernels, we are going to use harmonic analysis on compact topological groups. Schoenberg (1942) [8] showed that the $O(\mathbb{R}^n)$-invariant, positive kernels are of the form

$$K(x,y) = \sum_{k=0}^{\infty} f_k P_k^{((n-3)/2,\,(n-2)/2)}(x \cdot y)$$

where $f_k \geq 0$ and $P_k^{(\alpha,\beta)}(u)$ are Jacobi polynomials of degree k.

Using this theory, we can now compute $\theta_1(G(n,t))$ by solving an *infinite linear program* with optimization variables f_k:

$$\begin{aligned} \theta_1(G(n,t)) = \sup\{\omega_n^2 f_0 \mid f_k \geq 0 \text{ for all } k = 0,1,\ldots, \\ \textstyle\sum_{k=0}^{\infty} f_k P_k^{(\alpha,\alpha)}(1) = 1/\omega_n,\ \sum_{k=0}^{\infty} f_k P_k^{(\alpha,\alpha)}(t) \geq 0\}. \end{aligned}$$

The dual problem is

$$\inf\left\{ z_1/\omega_n \mid z_1 + z_t \geq \omega_n^2,\ z_1 + z_t P_k^{(\alpha,\alpha)}(t) \geq 0 \text{ for all } k = 1,2,\ldots \right\}.$$

Theorem 1.4.5. *With the above notations, we have*

$$\lim_{t \to 1} \theta_1(G(n,t)) = \omega_n \left(1 - \frac{j_{\alpha+1}^{\alpha}}{2^{\alpha}\,\Gamma(\alpha+1)\,J_\alpha(j_{\alpha+1})}\right)^{-1}$$

where $\alpha = (n-3)/2$ and J_α denotes the Bessel function of the first kind with parameter α. The first positive zero of $J_{\alpha+1}$ is denoted by $j_{\alpha+1}$.

Corollary 1.4.6. $\chi_m(\mathbb{R}^n) \gtrapprox (1.165)^n$.

In 1981 Frankl and Wilson [3] showed that $\chi(\mathbb{R}^n) \gtrapprox (1.2)^n$, which is slightly better. In a recent development it was shown that the above is also a lower bound for $\chi_m(\mathbb{R}^{n-1})$, which can be improved further with additional tricks.

1.5 Second generalization

Definition 1.5.1. The *packing graph* is defined as $G(n) = (V,E)$ with $V = S^{n-1}$ and $\{x,y\} \in E$ if and only if $x \cdot y \in (\frac{1}{2}, 1)$.

One can show that stable sets in $G(n)$ give *touching points* for kissing configurations. Hence, the stability number of $G(n)$ coincides with the kissing number τ_n:

$$\alpha(G(n)) = \max\{|C| \mid C \subset S^{n-1},\ \forall x,y \in C,\ x \neq y\ :\ x \cdot y \notin (\tfrac{1}{2}, 1)\} = \tau_n.$$

With this, we can generalize the second formulation of our theta function. Again, we replace matrices by kernels and sums by integrals:

$$\begin{gathered}\theta_2(G(n)) = \inf\{\lambda \mid K \in \mathcal{C}(S^{n-1} \times S^{n-1}) \text{ positive}, \\ K(x,x) = \lambda - 1 \text{ for all } x \in S^{n-1},\ K(x,y) = -1 \text{ if } x \cdot y \in [-1, \tfrac{1}{2}]\}.\end{gathered}$$

Theorem 1.5.2. *With the above notation, we have* $\alpha(G(n)) \leq \theta_2(G(n))$.

By applying Schoenberg's characterization, we can compute $\theta_2(G(n))$:

$$\begin{gathered}\theta_2(G(n)) = \inf\{\lambda \mid f_0 \geq 0, f_1 \geq 0, \ldots, \\ \textstyle\sum_{k=0}^{\infty} f_k P_k^{(\alpha,\alpha)}(1) = \lambda - 1,\ \sum_{k=0}^{\infty} f_k P_k^{(\alpha,\alpha)}(u) = -1 \text{ for all } u \in [-1, \tfrac{1}{2}]\}.\end{gathered}$$

Relaxing it by requiring $\sum_{k=0}^{\infty} f_k P_k^{\alpha,\alpha}(u) \leq -1$ gives the linear programming bound θ_2', established by Delsarte, Goethals and Seidel in 1979 [1]. Some results are: $\theta_2'(G(8)) = 240 = \tau_8$, $\theta_2'(G(24)) = 196{,}560 = \tau_{24}$, although $\theta_2'(G(3)) > 13$, $\theta_2'(G(4)) > 25$.

To get new SDP bounds for kissing numbers, we have to use some more techniques. Here, we present only a rough idea: The LP bound exploits obstructions coming from pairs of points and the orthogonal group. To get more obstructions, we want to use triples of points and the stabilizer subgroup of a point instead. This approach was motivated by Schrijver's work on binary codes in 2005 [9]. The main technical step is to prove a generalization of Schoenberg's characterization for positive $O(\mathbb{R}^{n-1})$-invariant kernels. Instead of the univariate Jacobi polynomials $P_k^{(\alpha,\alpha)}$ and non-negative real coefficients, one has to use multivariate orthogonal polynomials and positive semidefinite matrix coefficients. A mathematical rigorous computer proof uses Putinar's representation of positive polynomials as sums of squares and Borcher's csdp and a check of error bounds using SDP duality.

n	best lower bound known	best upper bound previously known		SDP bound
3	12	12	(Schütte, v. d. Waerden, 1953)	12
4	24	24	(Musin, 2003)	24
5	40	46	(Odlyzko, Sloane, 1979)	45
6	72	82	(O., S.)	78
7	126	140	(O., S.)	134
8	240	240	(O., S., Levenshtein, 1979)	240
9	306	379	(Rzhevskii, Vsemirnov, 2002)	364
10	500	594	(Pfender, 2007)	556
11	582	915	(O., S.)	873
12	840	1 416	(O., S.)	1 362
13	1 130	2 233	(O., S.)	2 080
14	1 582	3 492	(O., S.)	3 202
15	2 564	5 431	(O., S.)	4 893
16	4 320	8 312	(P.)	7 432
17	5 346	12 210	(P.)	11 333
18	7 398	17 877	(O., S.)	17 034
19	10 668	25 900	(Boyvalenkov, 1994)	25 636
20	17 400	37 974	(O., S.)	37 844
21	27 720	56 851	(B.)	56 079
22	49 896	86 537	(O., S.)	84 922
23	93 150	128 095	(B.)	127 323

1.6 Conclusion

1. One succeeded with a symbiosis of human and computer reasoning.
2. Similarly to NP-proofs, the presented proof was easy to understand but in this case difficult to find.
3. A combination of SDP and harmonic analysis is widely applicable, e.g. in continuous combinatorial optimization, correlation theory of stochastic processes, energy minimization or shape optimization.

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Student Seminar Abstracts

Ina Voigt
Voronoi cells of discrete point sets

Abstract. Reviewing the literature on Voronoi cells, one notices that the considered point sets are in most cases either finite or Delaunay sets. In case of a Delaunay set, all Voronoi cells are polytopes. And for a finite point set we still get polyhedral cells. Thereby the question emerges if all Voronoi cells of an arbitrary discrete point set are polyhedral. We will show by an example that this is not the case and characterize those discrete sets for which all Voronoi cells are polytopal/polyhedral.

Edward Kim
On the graphs of transportation polytopes

Abstract. Transportation polytopes are classical objects in optimization. In this talk, we will consider the status of theorems and conjectures on the graphs (or 1-skeleton) of two-way (that is, classical) and three-way (both axial and planar) transportation polytopes. Our primary concern is the diameter, and the core results are on an upper and a lower bound for the diameters of classical transportation polytopes.

The talk is completely self-contained and based on joint work with Jesús A. De Loera (University of California at Davis), Shmuel Onn (Technion, Israel Institute of Technology), and Francisco Santos (Universidad de Cantabria).

Marek Krčál
Embedding of simplicial complexes into Euclidean spaces and the deleted product obstruction

Abstract. We will review the known complexity of the problem of embeddability of a given simplicial complex of a dimension k into $\mathbb{R}^d$ (denoted as $\mathrm{EMBED}_{k\to d}$, for some numbers k and d) which can be seen as a higher-dimensional generalization of the well known graph planarity problem.

There is a class of remaining cases (where k is not "too big") left open that could be possibly tractable. For these cases, the so-called deleted product obstruction gives an equivalent condition, namely the existence of a "$\mathbb{Z}_2$-equivariant" map of a simplicial complex into a $(d-1)$-dimensional sphere. Tools of algebraic topology seem to be able to deal with this question.

Is there a polynomial algorithm to test this? Besides embedding, the algorithm would have applications for graph coloring and homomorphisms.

Anna Gundert
Embedding large complexes

Abstract. It is easy to prove that any k-dimensional simplicial complex embeds into $\mathbb{R}^{2k+1}$, so the maximal number of k-simplices one can get when embedding into $\mathbb{R}^{2k+1}$ is $\binom{n}{k+1} = \Theta(n^{k+1})$, where n is the number of vertices.

For the case $k = 2$, this yields $\Theta(n^3)$ for embeddability into $\mathbb{R}^5$. One can also show that a 2-complex which embeds into $\mathbb{R}^3$ can have at most $n(n-3) = \Theta(n^2)$ triangles. What happens in $\mathbb{R}^4$ is an open question.

We will address the general question of the maximal number of k-simplices for a complex which is embeddable into $\mathbb{R}^d$ for some $k \leq d \leq 2k$. Lower bounds are given for the cyclic polytope. To find upper bounds for the case $d = 2k$, we look for forbidden subcomplexes. A generalisation of the theorem of van Kampen and Flores yields those. Then the problem can be tackled with the methods of extremal hypergraph theory. This gives $O(n^s)$, $k < s < k+1$.

Aaron Dall
Enumeration of integral tensions

Abstract. Let $G = (V, E)$ be an oriented graph with n edges. An integral tension on G is a function $\tau\colon E \to \mathbb{Z}$ with the property that the sum of the edge labels (with respect to the orientation) around any cycle is zero. A nowhere-zero tension is an integral tension such that $\tau(e) \neq 0$ for all edges e. First we use Ehrhart theory to show that the number of integral tensions on G taking values in $[-k, k]$ is a polynomial in k. Then we show that the number of nowhere-zero tensions taking values between $[-k, k]$ is also a polynomial. To do this, we introduce the notion of an inside-out polytope which leads to a generalization of Ehrhart's theorem.

Juanjo Rué
Enumerating simplicial decompositions of surfaces with boundaries

Abstract. It is well-known that the triangulations of the disc with $n+2$ vertices on its boundary are counted by the nth Catalan number $C(n) = \frac{1}{n+1}\binom{2n}{n}$. This paper deals with the generalisation of this problem to any arbitrary compact surface S with boundaries. We obtain the asymptotic number of simplicial decompositions of the surface S with n vertices on its

boundary. More generally, we determine the asymptotic number of dissections of S when the faces are δ-gons with δ belonging to a set of admissible degrees $\Delta \subseteq \{3, 4, 5, \dots\}$. We also give the limit laws of certain parameters of such dissections.

This is a joint work with Olivier Bernardi, from Paris. An online version is available at arXiv: `http://lanl.arxiv.org/abs/0901.1608`

Canek Peláez Valdés

Allowable sequences

Abstract. Given a family of points in $\mathbb{R}^2$, we can define the crossing number, the number of k-sets, and the number of halving lines of the set using its geometric graph. However, we can also be interested in the case when we connect the points not with straight lines, but with simple curves. In this case, we can use arrangements of pseudolines (which are a concrete geometric model for oriented matroids of rank 3) and generalized configurations of points to model our problems, and then attack them using allowable sequences. We will show how can we construct allowable sequences, and an algorithm that we can use to find the number of halving pseudolines of a generalized configuration of points, which in turn helps us to find the pseudolinear crossing number.

Vincent Pilaud

Multi-pseudotriangulations

Abstract. We introduce multi-pseudotriangulations: the definition, given in terms of pseudoline arrangements, naturally generalizes both pseudotriangulations and multitriangulations. We first present various structural properties of multi-pseudotriangulations: number of edges, decomposition into stars, existence of flips. Then, we propose an enumeration algorithm for multi-pseudotriangulations based on certain greedy multi-pseudotriangulations that are closely related with sorting networks. This algorithm requires a polynomial running time per multi-pseudotriangulation and its working space is polynomial.

This is joint work with Michel Pocchiola (École Normale Supérieure, Paris).

Benjamin Matschke
The Square Peg Problem and beyond

ABSTRACT. The famous still open Square Peg Problem (Toeplitz, 1911) asks: "Does every simple closed curve in the plane contain four points spanning a square?". We will give some background, a proof for the case when the curves are smooth, and related things: New results and two again-open easy-looking conjectures.

Frederik von Heymann
Counting points with Euclid

ABSTRACT. We will give an easy answer to the question how to describe the lattice points in a triangle. Surprisingly (or not), this will deliver us new insights on the problem of enumerating lattice points in polytopes, and (if there is time) a partly new proof of a theorem that classifies all tetrahedra in which the vertices are the only integral points.

Noa Nitzan
A planar 3*-convex set is indeed a union of six convex sets*

ABSTRACT. Suppose S is a planar set. Two points a, b in S "see each other" via S if $[a, b]$ is included in S. F. Valentine proved in 1957 that if S is closed, and if for every three points of S at least two see each other via S, then S is a union of three convex sets. The pentagonal star shows that the number three is best possible. We discard the condition that S is closed and show that S is a union of (at most) six convex sets. The number six is best possible.

Daria Schymura
Measuring the similarity of geometric graphs

ABSTRACT. What does it mean for geometric graphs to be similar? We propose a distance for geometric graphs that is a metric, and that can be computed by solving an integer linear program.

Mareike Massow

Linear extension diameter of Boolean lattices

ABSTRACT. Given a finite poset P, we consider pairs of linear extensions of P with maximum distance. The distance of two linear extensions L_1, L_2 is the number of pairs of elements of P appearing in different orders in L_1 and L_2. The maximum possible distance is the linear extension diameter of P. We prove a formula for the linear extension diameter of Boolean lattices which was conjectured by Felsner and Reuter in 1999.

Birgit Vogtenhuber

Large bichromatic point sets admit empty monochromatic 4*-gons*

ABSTRACT. We consider a variation of a problem stated by Erdős and Szekeres in 1935 about the existence of a number $f(k)$ such that any set S of at least $f(k)$ points in general position in the plane has a subset of k points that are the vertices of a convex k-gon. In our setting, the points of S are colored, and we say that a (not necessarily convex) spanned polygon is monochromatic if all its vertices have the same color. Moreover, a polygon is called empty if it does not contain any points of S in its interior. We show that any sufficiently large bichromatic set of points in $\mathbb{R}^2$ in general position determines at least one empty, monochromatic quadrilateral (and thus linearly many).

Matthias Henze

Small steps towards a reverse Blichfeldt–van der Corput inequality

ABSTRACT. In the early 20th century, Blichfeldt and van der Corput gave a sharp upper bound on the volume of zero symmetric lattice polytopes in terms of the number of interior lattice points. A corresponding lower bound is conjectured by Bey, Henk and Wills, but still remains to be proven. A short introduction to the theory and some results for lattice zonotopes are presented.

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