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Tutorials by

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Foreword

The Young Set Theory Workshop 2009 was held April 14–18 at the CRM in Bellaterra (Catalonia, Spain). The purpose of this workshop was to give talented young researchers in set theory an opportunity to learn from experts and from each other in a friendly co-operative environment. Four tutorial speakers, Mirna Džamonja, Moti Gitik, Ernest Schimmerling, and Boban Veličković, were invited and gave tutorials of four hours each. Moreover, the following postdoc speakers were invited to give a one-hour talk each: Andrew Brooke-Taylor, Bernhard König, Jordi López-Abad, Luis Pereira, Hiroshi Sakai, Dima Sinapova, and Asger Törnquist. Apart from the talks, plenty of time was devoted to discussion sessions, where many ideas were discussed, many questions were asked, and some problems were solved.

This volume contains two papers related to the tutorials held at the workshop. Mirna Džamonja contributed a paper on an open problem related to the subject of her tutorial, and Laura Fontanella and Boban Veličković contributed a paper on the tutorial of the second author. We would like to thank the authors for their contributions and the CRM for their organizational and financial support, for hosting this conference at their premises, and for publishing this volume. We would also like to thank the following institutions for their financial support: Generalitat de Catalunya (AGAUR), Universitat de Barcelona (Grup de Recerca en Teoria de Conjunts de Barcelona, Comissió de Recerca de l'Agrupació en Humanitats, and Facultat de Filosofia), and the Association for Symbolic Logic.

June 2011

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Ramsey Methods and the Problem DU

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Abstract. We consider Fremlin's notion of 1/2-density and the related notion of Fremlin cardinals. A well-known related question is if every 1/2-dense hereditary family on an uncountable cardinal must have an infinite homogeneous family. These notions do not seem to lend themselves to Ramseyan methods. In particular, it is not known if a Fremlin cardinal must be a large cardinal. We introduce a related notion of 1/2-dense cardinals which is easier to handle using Ramsey methods. We show that a 1/2-dense cardinal must be at least strongly inaccessible. On the other hand, David Asperó showed that an ω-Erdős cardinal must be 1/2-dense.[1]

Preface

I was a tutorial speaker at the Young Set Theory Workshop 2009 in Barcelona. The topic of my lectures were Ramsey principles. I talked both about many successes of the applications of the Ramseyan methods in set theory, topology and analysis, and about one Ramsey-like problem that is still unsolved many years after it was posed. It is the problem of 1/2-density, which we explain below. Rather than writing an article about successes of Ramseyan methods, which are well documented in the literature (see for example [2], [9]), I have decided to explain in detail the problem of 1/2-density, bringing a Ramseyan perspective into it. There are several new results in this article, but the answer to the main questions 1.1.2 is still not known.

[1]My thanks go to EPSRC for their support through the grant EP/G068720 and to the organisers of the Young Set Theory Workshop 2009 in Barcelona for their invitation to give a tutorial.

1.1 Introduction

We start with some definitions.

Definition 1.1.1. (i) A family $\mathcal{D}$ of finite subsets of a cardinal κ is called $1/2$-*dense* if for all finite $F \subseteq \kappa$ there is $F_0 \subseteq F$ with $F_0 \in \mathcal{D}$ and $|F_0| \geq 1/2 \cdot |F|$. We say that $\mathcal{D}$ is *hereditary* if it is closed under subsets. Hereditary $1/2$-dense families are called $1/2$-*filling*.

(ii) Suppose that $\mathcal{D}$ is a family of finite subsets of a cardinal κ and $H \subseteq \kappa$. Then H is *homogeneous* for $\mathcal{D}$ if $[H]^{<\aleph_0} \subseteq \mathcal{D}$.

The most interesting cardinals in the context of $1/2$-dense families are ω_1 and $\mathfrak{c} = 2^{\aleph_0}$. The following questions appear stated as Problem DU on Fremlin's list (see [6]):

Questions 1.1.2. Suppose that $\mathcal{D}$ is a $1/2$-filling family on ω_1.

(i) (Argyros) Must there be an infinite set homogeneous for $\mathcal{D}$?

(ii) Under MA + $\neg$CH, is it true that $\mathcal{D}$ must have an uncountable homogeneous set?

It is known that under $\mathrm{cov}(\mathcal{N}) = \aleph_1$ in place of MA + $\neg$CH, the answer to (ii) is negative; see [1] or [6] for a folklore proof. It is not known if the positive answer is consistent. A meaningful concept is obtained if Definition 1.1.1 is made with an arbitrary $\alpha \in (0, 1)$ in place of $1/2$; however, it is known that this change does not add any generality. Namely, Fremlin [6] showed that the truth of the statement "every α-filling family $\mathcal{D}$ on κ has a homogeneous set of size λ" does not depend on $\alpha \in (0, 1)$. The paper [4] gives a combinatorial characterisation of $1/2$-filling families on ω_1 which have an uncountable homogeneous set under MA + $\neg$CH.

We use the notation $P(\kappa, \lambda)$ to state that every $1/2$-filling family $\mathcal{D}$ on κ has a homogeneous set of size λ. This notation was introduced by Fremlin. We use the word 'homogeneous' and notation from the theory of partition relations to emphasise the intuition we expressed in [1], that $P(\kappa, \lambda)$ is a large-cardinal statement. Along these lines, Fremlin proved in [6] that if κ is a real-valued measurable cardinal then $P(\kappa, \omega)$ holds; hence it is consistent modulo a measurable cardinal that $P(\mathfrak{c}, \omega)$ holds. On the other hand, it is an observation of Apter and Džamonja in [1] that, if κ is λ-Erdős, then $P(\kappa, \lambda)$ holds. The statement $\neg P(\mathfrak{c}, \omega)$ is of interest in analysis, as it can be used to construct interesting examples of spaces and functions. From this point of view, the former of the two large-cardinal results is more interesting, as an

Erdős cardinal is necessarily strongly inaccessible. On the other hand, the consistency strength of Erdős cardinals is weaker than that of real-valued measurable cardinals. Namely, the consistency strength of the existence of a real-valued measurable cardinal is that of a measurable cardinal, and the consistency strength of the existence of an Erdős cardinal is that of the assertion that the existence of $0^{\#}$ implies that in L there is an α-Erdős cardinal for every $\alpha < \omega_1$ (while from the existence of an ω_1-Erdős cardinal one can derive the existence of $0^{\#}$).

One difficulty in treating the problem has been that 1/2-density is a density notion which does not fit into the classical treatment of partition relations. In this paper we explore the influence of Ramsey theory on this notion. For example, we show that there is a notion closely connected to 1/2-filling families and including 1/2-density which can be treated by classical Ramsey theory. Specifically, in §1.2 we show just in ZFC that there is a 1/2-dense family $\mathcal{D}$ of finite subsets of $\mathfrak{c}$ such that there is no infinite $X \subseteq \mathfrak{c}$ homogeneous for $\mathcal{D}$, and in fact that there is such a family on every cardinal below the first strongly inaccessible cardinal. The family is not hereditary. In view of the Fremlin's result mentioned above, the result is optimal. However, as pointed out by Kojman, if we completely give up on the requirement of hereditariness, it is easy to give a trivial example of a 1/2-dense family $\mathcal{F}$ of any cardinal such that there is no infinite X homogeneous for $\mathcal{F}$; namely, just taking the finite subsets with even cardinality will do. This family $\mathcal{F}$ has the property that there is no non-empty set homogeneous for $\mathcal{F}$. The family $\mathcal{D}$ constructed in §1.2 will have homogeneous sets of arbitrary finite size within every infinite set. In §1.3 we remark how these results relate to another known weakening of 1/2-fillingness.

In §1.4 we consider the problem of 1/2-density when restricted to sets of fixed finite size.

Following the notation from [1], we say that κ is a λ-*Fremlin cardinal* if $P(\kappa, \lambda)$ holds, and when λ is ω we just speak of *Fremlin cardinals*. It is still not known if the first Fremlin cardinal must be a large cardinal. In §1.2 we introduce a related type of large cardinals, 1/2-dense cardinals, and we prove that such a cardinal must be strongly inaccessible. In a previous version of this article we asked if 1/2-dense cardinals exist. Asperó answered this by observing that, in fact, an ω-Erdős cardinal must be 1/2-dense. We give an argument for this at the end of §1.2.

1.2 1/2-dense cardinals

In this section, 1/2-dense cardinals will be defined as cardinals that satisfy a stronger version of Fremlin's property $P(\kappa, \omega)$. We have been interested in $P(\kappa, \omega)$ rather than $P(\kappa, \lambda)$ for $\lambda > \omega$, but many arguments in this section apply to $\lambda > \omega$ as well.

Definition 1.2.1. Let $\kappa \geq \aleph_0$ be a cardinal. A family $\mathcal{D}$ of finite subsets of κ is said to satisfy $\varphi(\kappa)$ if the following properties hold:

(i) all singletons are in $\mathcal{D}$,

(ii) $\mathcal{D}$ is a 1/2-dense family which has no infinite homogeneous set, and

(iii) (*Spread Property*) for any infinite $A \subseteq \kappa$ there are subsets of A of arbitrarily large finite size which are homogeneous for $\mathcal{D}$.

A cardinal κ such that $\varphi(\kappa)$ is not satisfied by any family of finite subsets of κ is said to be a 1/2-*dense* cardinal.

In other words, a cardinal κ is 1/2-dense if every 1/2-dense family of finite subsets of κ with the spread property and containing the singletons has an infinite homogeneous set. Clearly every 1/2-dense hereditary family has the spread property and contains the singletons, and therefore we have:

Observation 1.2.2. A 1/2-dense cardinal is necessarily Fremlin.

We may also observe that if a cardinal is 1/2-dense then so are all the larger cardinals.

Lemma 1.2.3. *Suppose that λ is an infinite cardinal $\leq \kappa$ and there is a family $\mathcal{D}_\kappa$ satisfying $\varphi(\kappa)$. Then there is a family $\mathcal{D}_\lambda$ satisfying $\varphi(\lambda)$.*

Proof. Let $\mathcal{D}_\lambda = \mathcal{D}_\kappa \cap [\lambda]^{<\aleph_0}$. It is clear that $\mathcal{D}_\lambda$ is a 1/2-dense family of finite subsets of λ which has no infinite homogeneous set and which contains all singletons. If $A \subseteq \lambda$ is infinite, then there are subsets of A of arbitrarily large finite size which are homogeneous for $\mathcal{D}_\kappa$, and hence also for $\mathcal{D}_\lambda$. $\square_{1.2.3}$

We now prove that the first 1/2-dense cardinal is a large cardinal.

Theorem 1.2.4. *The first 1/2-dense cardinal, if it exists, is strongly inaccessible.*

Proof. Suppose that λ^* is the first 1/2-dense cardinal. By the example of the Schreier family,[2] we know that $\lambda^* > \aleph_0$.

Now we shall show that if $\kappa < \lambda^*$ then also $2^\kappa < \lambda^*$.

Lemma 1.2.5. *Suppose that there is a family $\mathcal{D}_\kappa$ satisfying $\varphi(\kappa)$. Then there is a family $\mathcal{D}_{2^\kappa}$ satisfying $\varphi(2^\kappa)$.*

Proof of the Lemma. Let $<^*$ be a fixed well-order of ${}^\kappa 2$ in order type 2^κ. We identify the cardinal 2^κ with the tree ${}^\kappa 2$ ordered by $<^*$. Let $K \subseteq {}^{\omega>}({}^\kappa 2)$ be the set of all $u = \langle x_0 <^* x_1 <^* \cdots <^* x_{r-1}\rangle$ where $r \geq 2$ is such that u is either $<_{\text{lex}}$-increasing or $<_{\text{lex}}$-decreasing.

If $x \neq y$ in ${}^\kappa 2$ we let $\Delta(x,y) = \min\{\alpha : x(\alpha) \neq y(\alpha)\}$. For u as above, we let $\delta(u) = \langle \Delta(x_0,x_1), \Delta(x_1,x_2), \ldots, \Delta(x_{r-2},x_{r-1})\rangle$. Note that $\delta(u)$ is a finite sequence of ordinals $< \kappa$.

We let P_0 consist of all $u \in K$ such that $\delta(u)$ is strictly increasing, and define P_1 as the set of all those $u \in K$ for which $\delta(u)$ is strictly decreasing. Let $P = P_0 \cup P_1$.

Let $\mathcal{D} = \mathcal{D}_{2^\kappa}$ be given by

$$\mathcal{D} = \{\{f\} : f \in 2^\kappa\} \cup \{u \in P : \operatorname{ran}(\delta(u)) \in \mathcal{D}_\kappa\} \cup ([{}^\kappa 2]^{<\aleph_0} \setminus P).$$

Clearly $\mathcal{D}$ contains all singletons. To show that $\mathcal{D}$ is 1/2-dense in ${}^\kappa 2$, it suffices to consider $u \in P$. Let us first suppose that $u \in P_0$. If $r = |u| = 2$ then $|\delta(u)| \leq 1$, so $\operatorname{ran}(\delta(u)) \in \mathcal{D}_\kappa$. Otherwise, $\operatorname{ran}(\delta(u))$ is in any case a finite subset of κ and therefore there is $F \subseteq \operatorname{ran}(\delta(u))$ with $F \in \mathcal{D}_\kappa$ and $|F| \geq |\operatorname{ran}(\delta(u))|/2 = (|u|-1)/2$. Now F is the range of a sequence of the form $\langle \Delta(x_{i_0}, x_{i_0+1}), \ldots, \Delta(x_{i_k}, x_{i_k+1})\rangle$ for some $i_0 < i_1 < \cdots < i_k$ and $k \leq r-2$ and therefore F is not immediately seen to be of the form $\operatorname{ran}(\delta(v))$ for any $v \subseteq u$. However, since $u \in P_0$, we have that $\Delta(x_i, x_{i+1})$ increases with i. Therefore for every $s < k$ we have $\Delta(x_{i_s}, x_{i_{s+1}}) = \Delta(x_{i_s}, x_{i_s+1})$ and hence $F = \operatorname{ran}(\delta(v))$ for $v = \langle x_{i_0}, \ldots, x_{i_{k+1}}\rangle$. Since $|v| = |F|+1 \geq (|u|-1)/2+1 \geq |u|/2$, we have found $v \in \mathcal{D}$ as desired. The argument for $u \in P_1$ is similar.

Now suppose that X is infinite and homogeneous for $\mathcal{D}$, and assume simply that X has order type ω under $<^*$. Define a colouring c by colouring pairs $\{x,y\}$ in X with colour 0 if the $<^*$ and $<_{\text{lex}}$ order agree on $\{x,y\}$, and colour 1 otherwise. By Ramsey's theorem we can assume that all pairs are coloured the same colour and therefore X is $<_{\text{lex}}$-increasing or $<_{\text{lex}}$-decreasing. Suppose for simplicity that it is $<_{\text{lex}}$-increasing —the argument in the other case is similar.

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[2] The Schreier family consists of finite subsets F of ω which satisfy $\min(F) \geq |F|+1$, and the singleton $\{0\}$. This family is a well-known example of a 1/2-dense hereditary family of subsets of ω for which there is no infinite homogeneous set.

By induction on $n < \omega$, we choose $x_n \in X$ and a final segment B_n of X so that all B_n are non-empty and decreasing, and $\Delta(x_n, x_m)$ for $n < m$ only depends on n. Let x_0 be the $<^*$-minimal element of X and

$$\xi_0 = \min\{\Delta(x_0, x_n) : n > 0\}.$$

Let $B_0 = \{x_n : \Delta(x_0, x_n) = \xi_0\}$, so clearly B_0 is non-empty. Suppose that $x_n \in B_0$ and $n < m$. Then $x_n <_{\rm lex} x_m$ by the assumptions above and $x_0 <_{\rm lex} x_n$. By the choice of ξ_0 we can only have $\Delta(x_0, x_m) \geq \xi_0$ and therefore it must be that $\Delta(x_0, x_m) = \xi_0$ and $m \in B_0$. Hence B_0 is a final segment of X. Now we let x_1 be the $<^*$-minimal element of B_0 and continue. Note that the sequence $\bar{\xi} = \langle \xi_n : n < \omega \rangle$ is strictly increasing.

At the end, by renaming, we can assume that $X = \{x_n : n < m\}$. Then note that $[X]^{<\aleph_0} \subseteq P_0$, exactly because $\bar{\xi}$ is strictly increasing. Hence for any $u \in [X]^{\geq 2}$ we have that $\mathrm{ran}(\delta(u)) \in \mathcal{D}_\kappa$. This means that $\{\Delta(x_n, x_{n+1}) : n < \omega\}$ is infinite homogeneous for $\mathcal{D}_\kappa$, a contradiction.

To prove the final claim, suppose that A is an infinite subset of 2^κ, where $2 \leq n < \omega$, and we shall find a $\mathcal{D}$-homogeneous subset of A of size $\geq n$. By an application of Ramsey's theorem we can assume as above that the order type of A under $<^*$ is ω, that $A = \{y_k : k < \omega\}$ is an $<^*$-increasing enumeration and that either A is $<_{\rm lex}$-increasing or $<_{\rm lex}$-decreasing. Note that since the y_ks are binary sequences we must have that if $k_0 < k_1 < k_2$ then $\Delta(y_{k_0}, y_{k_1}) \neq \Delta(y_{k_1}, y_{k_2})$. Since there is no infinite decreasing sequence of ordinals we can thin A further if necessary to obtain that $\Delta(y_k, y_{k+1}) < \Delta(y_{k+1}, y_{k+2})$ for any k. Therefore, any finite subset of A gives rise to a sequence in P. Let $B = \{\Delta(y_k, y_{k+1}) : k < \omega\}$. By the inductive hypothesis, there is a subset of B of size n which is homogeneous for $\mathcal{D}_\kappa$. This implies, as in the argument for 1/2-density, that $\{y_k : \Delta(y_k, y_{k+1}) \in B\}$ is homogeneous for $\mathcal{D}_{2^\kappa}$. $\square_{1.2.5}$

Our next task is to show that λ^* cannot be singular. For this, we recast the property $\neg\varphi(\kappa)$ in terms of a classically-looking partition relation:

Definition 1.2.6. We say that $\kappa \to_0 (\omega, \lceil n/2 \rceil, \hat{\omega})^{<\omega}$ if for every function $f\colon [\kappa]^{<\omega} \to 2$ satisfying $f(\{\alpha\}) = 0$ for all α,

(i) *either* there is an infinite $A \subseteq \kappa$ such that $f \restriction [A]^{<\omega}$ is the constant 0 function,

(ii) *or* there is a finite $B \subseteq \kappa$ such that $f \restriction [B]^{\lceil |B|/2 \rceil}$ is the constant 1 function,

(iii) *or* there is an infinite $A \subseteq \kappa$ such that, for some $n < \omega$, every $B \in [A]^{\geq n}$ has a subset C with $f(C) = 1$.

Lemma 1.2.7. *A cardinal κ satisfies the property $\neg\varphi(\kappa)$ if and only if $\kappa \to_0 (\omega, \lceil n/2 \rceil, \hat{\omega})^{<\omega}$ holds.*

Proof of the Lemma. In the forward direction, given $f\colon [\kappa]^{<\omega} \to 2$ satisfying $f(\{\alpha\}) = 0$ for all α, define $\mathcal{D} = \{F : f(F) = 0\}$. If (ii) does not hold, then $\mathcal{D}$ is 1/2-dense. If (iii) does not hold, then for every infinite $A \subseteq \kappa$ and for every $n > \omega$ there is $B \subseteq A$ of size at least n all whose subsets are in $\mathcal{D}$. Since $\mathcal{D}$ cannot witness $\varphi(\kappa)$, there must be an infinite $\mathcal{D}$-homogeneous set, so (i) holds.

In the backward direction the proof is similar: if we are given a 1/2-dense family $\mathcal{D}$ of finite subsets of κ which contains all singletons and has the property that within every infinite subset of κ there is an arbitrarily large finite $\mathcal{D}$-homogeneous set, then we can define $f\colon [\kappa]^{<\omega} \to 2$ by $f(F) = 0$ if and only if $F \in \mathcal{D}$. Then 1/2-density of $\mathcal{D}$ implies that condition (ii) in $\kappa \to_0 (\omega, \lceil n/2 \rceil, \hat{\omega})^{<\omega}$ cannot hold and the property that within every infinite subset of κ there is an arbitrarily large finite $\mathcal{D}$-homogeneous set shows that (iii) cannot hold. Hence, (i) holds, and any infinite A witnessing it gives an infinite $\mathcal{D}$-homogeneous set. $\square_{1.2.7}$

Lemma 1.2.8. *λ^* is not singular.*

Proof of the Lemma. By Lemma 1.2.7, this amounts to showing that the first κ satisfying $\kappa \to_0 (\omega, \lceil n/2 \rceil, \hat{\omega})^{<\omega}$ cannot be singular. Suppose for contradiction that this is the case. Let $\kappa > \mathrm{cf}(\kappa)$ and let $\langle \kappa_i : i < \mathrm{cf}(\kappa) \rangle$ be an increasing continuous sequence of cardinals converging to κ, with $\kappa_0 = 0$ and $\kappa_1 \geq \omega$. For $\alpha < \kappa$, define $h(\alpha) = i$ if and only if $\alpha \in [\kappa_i, \kappa_{i+1})$. Let $f\colon [\mathrm{cf}(\kappa)]^{<\omega} \to 2$ exemplify that $\mathrm{cf}(\kappa) \not\to_0 (\omega, \lceil n/2 \rceil, \hat{\omega})^{<\omega}$, and let $f_i\colon [\kappa_{i+1}]^{<\omega} \to 2$ exemplify the same for κ_{i+1}.

Define $g\colon [\kappa]^{<\omega} \to 2$ as follows: for an increasing sequence $(\xi_1, \ldots, \xi_n)$ in $[\kappa]^{<\omega}$, let

$$g(\xi_1, \ldots, \xi_n) = \begin{cases} 0 & \text{if } n = 1; \\ f_{i+1}(\xi_1, \ldots, \xi_n) & \text{if } h(\xi_1) = \cdots = h(\xi_n) = i; \\ f(h(\xi_1), \ldots, h(\xi_n)) & \text{if } h(\xi_1) < \cdots < h(\xi_n); \\ 0 & \text{otherwise.} \end{cases}$$

In our notation we use $g(\xi_1, \ldots, \xi_n)$ in place of $g(\{\xi_1, \ldots, \xi_n\})$, for clarity. We claim that g exemplifies that the required partition relation does not hold at κ. Clearly g maps all singletons to 0. Suppose that (i) holds, as shown by an infinite $A \subseteq \kappa$. Suppose that $h``A$ is infinite. By thinning $A$ if necessary, we can assume that for $\xi < \zeta$ in $A$ we have $h(\xi) < h(\zeta)$. Therefore $\{h(\zeta) : \zeta \in A\}$ gives an infinite subset of $\mathrm{cf}(\kappa)$ which shows that condition (i) holds for $f$, a contradiction. Otherwise $h``A$ is finite and by thinning A if necessary we can assume that $h(\alpha)$ for $\alpha \in A$ is constantly i. Then A shows that (i) holds for f_{i+1}, a contradiction. A similar contradiction is obtained by assuming that (iii) holds for g. Finally, suppose that $2 \leq n < \omega$ is given and $B \subseteq A$ has size n. If neither the second or the third clause of the definition of g apply to B, then $g(B) = 0$ and so B does not exemplify (ii). If either the second or the third clause applies to B, then so does it to any of its subsets, and hence there must be a subset C of B of size $\geq n/2$ which satisfies $g(C) = f_{i+1}(C) = 0$, or $g(C) = f(h^{-1}(C)) = 0$. So (ii) does not hold for g either, and hence we have a contradiction. $\square_{1.2.8}$

We have thus shown that λ^* must be strongly inaccessible, hence the theorem is proved. $\square_{1.2.4}$

Remark 1.2.9. To see specifically that $\mathcal{D}$ from the proof of Lemma 1.2.5 is not closed under subsets, say on 2^ω, notice for example that there are sequences in the complement of P with a subsequence in P_0 which is not in $\mathcal{D}$. The proof of Lemma 1.2.5 shows that all infinite subsets of ${}^\omega 2$ 'concentrate' on P_0 and that $\mathcal{D} \cap P_0$ is closed under subsets. However, there is no uncountable subset of ${}^\omega 2$ all whose finite subsets are in P_0. To see this, suppose that A were such a set and let $\{x_\alpha : \alpha < \omega_1\}$ be a $<^*$-increasing subset of A. Let $n_\alpha = \Delta(x_\alpha, x_{\alpha+1})$. Then if $\alpha < \beta$ we have that $\{x_\alpha, x_{\alpha+1}, x_\beta, x_{\beta+1}\} \in P_0$, so $n_\alpha < \Delta(x_\alpha, x_\beta) < n_\beta$, letting us obtain an increasing ω_1-sequence in ω.

Theorem 1.2.4 in conjunction with Fremlin's result that a real-valued measurable cardinal is Fremlin shows that 1/2-dense cardinals are strictly stronger than Fremlin cardinals. Asperó proved that any ω-Erdős cardinal is 1/2-dense, as we now show.

Theorem 1.2.10 (Asperó)**.** *An Erdős cardinal is necessarily* 1/2*-dense.*

Proof. Recall that a cardinal κ is Erdős if and only if $\kappa \to (\omega)^{<\omega}$, which means that for every $f\colon [\kappa]^{<\omega} \to 2$ there is $H \in [\kappa]^\omega$ which is 'homogeneous' for f. In this context, homogenous means that either there are unboundedly many $n < \omega$ such that $f''[H]^n = \{1\}$ or there is $n_0 < \omega$ such that, for all $n \geq n_0$, we have $f''[H]^n = \{0\}$.

Suppose for contradiction that κ is an Erdős cardinal and that $\mathcal{D}$ is a 1/2-dense family of subsets of κ satisfying the property $\varphi(\kappa)$ from Definition 1.2.1. Let f be the following coloring of $[\kappa]^{<\aleph_0}$ into 2: $f(F) = 1$ if and only if all subsets of F are in $\mathcal{D}$. Let H be an infinite set homogeneous for f. By the spread property, there cannot be $n_0 < \omega$ such that for all $n \geq n_0$ we have $f''[H]^n = \{0\}$. Therefore, there are unboundedly many $n < \omega$ such that $f''[H]^n = \{1\}$. Let $m < \omega$ be arbitrary and let $n \geq m$ be such that $f''[H]^n = \{1\}$. Therefore $[H]^{\leq n} \subseteq \mathcal{D}$ and hence $[H]^m \subseteq \mathcal{D}$. In conclusion, H is homogeneous for $\mathcal{D}$. $\square_{1.2.10}$

1.3 Remarks on Fremlin cardinals

As mentioned before, Fremlin proved that a real-valued measurable cardinal must be, in our notation, a Fremlin cardinal. Modulo the existence of a measurable cardinal, it is consistent that $2^{\aleph_0}$ is a real-valued measurable cardinal, hence the analogue of Theorem 1.2.4 cannot be true for Fremlin cardinals. Remark 1.2.9 shows exactly where the proof would fail. However, some of the techniques of the proof do apply. For example, we can easily prove the following theorem of Fremlin, the statement of which was communicated to us by Henryk Michalewski:

Theorem 1.3.1 (Fremlin). *There is a family of finite subsets $\mathcal{F}_\mathfrak{c}$ of $\mathfrak{c}$ such that $\mathcal{F}_\mathfrak{c}$ is closed under subsets, has no infinite homogeneous set, but for all $\alpha < \mathfrak{c}$ and $n < \omega$ there is $F \in \mathcal{F}_\mathfrak{c}$ with $|F| \geq n$ and $F \cap \alpha = \emptyset$.*

Proof. We use the notation of the proof of Theorem 1.2.4 with $\kappa = \omega$. Let $\mathcal{F}_\mathfrak{c}$ be the family of all sets $\{f_{\alpha_0}, \dots, f_{\alpha_{n-1}}\}$ in ${}^{\omega}2$ such that $\{\Delta(f_{\alpha_i}, f_{\alpha_j}) : i \neq j < n,\ \alpha_i < \alpha_j\}$ is in the Schreier family. This family of functions is clearly closed under subsets, and any infinite homogeneous set would give us an infinite homogeneous subset of the Schreier family. If $\alpha < \mathfrak{c}$ and $n < \omega$ are given, we can find a finite subset of $\{f_\beta : \beta > \alpha\}$ of the form $\{f_{\beta_0}, \dots, f_{\beta_{2n}}\}$ with β_i increasing with i and $\Delta(f_{\beta_i}, f_{\beta_{i+1}})$ also increasing with i. Then the set of such values has size $2n$ and it has a subset F of size n which is in the Schreier family. From F we can recover a subset H of $2n+1$ such that $\{f_{\beta_i} : i \in H\}$ satisfies $\{\Delta(f_{\alpha_i}, f_{\alpha_j}) : i < j,\ i, j \in H\} = F$ and then $\{f_{\beta_i} : i \in H\}$ is in $\mathcal{F}_\mathfrak{c}$. $\square_{1.3.1}$

A version of this theorem was used by Avilés, Plebanek and Rodríguez in [3] to prove that there exists a weakly compactly generated Banach space X and a scalarly null function $f\colon [0,1] \to X$ which is not Mc Shane integrable. This answered several open questions in the theory of Mc Shane integration.

1.4 Homogenous sets of fixed exponent

It is natural to ask to what extent the problem of 1/2-density is linked to considering all finite subsets of a given set, rather than just finite sets of some bounded cardinality. We concentrate on ω_1 and observe that restricting to fixed cardinalities gives rise to infinite homogeneous sets of order type $\omega + 1$ for any 1/2-dense open family on ω_1, as follows.

Theorem 1.4.1. *Suppose that $n < \omega$ and let $\mathcal{D}$ be a family of subsets of $[\omega_1]^{\le n}$ closed under subsets and having the property that every element F of $[\omega_1]^{<\omega}$ has a subset F_0 of size at least $1/2 \cdot |F|$ such that $[F_0]^{\le n} \subseteq \mathcal{D}$. Then there is an $A \subseteq \omega_1$ of order type $\omega + 1$ with $[A]^{\le n} \subseteq \mathcal{D}$.*

If $n = 2$, then there is an uncountable such A and, in fact, for any infinite κ, if $\mathcal{D}$ is a family of subsets of $[\kappa]^{\le 2}$ closed under subsets and having the property that every element F of $[\kappa]^{<\omega}$ has a subset F_0 of size at least $1/2 \cdot |F|$ such that $[F_0]^{\le 2} \subseteq \mathcal{D}$, then there is $A \in [\kappa]^{\ge\omega}$ with $[A]^{\le 2} \subseteq \mathcal{D}$.

Proof. Let $n < \omega$ be given. The following theorem seems to be folklore in partition calculus for ω_1: for all $n < \omega$,

$$\omega_1 \longrightarrow (\omega + 1, \omega + 1)^n.$$

If $\mathcal{D}$ is a family as in the assumptions, then we can define a colouring $f\colon [\omega_1]^n \to 2$ by letting $f(F) = 0$ if and only if $F \in \mathcal{D}$. The property of $1/2$-density prevents any 1-homogeneous set of size $2n$, so there must be a 0-homogeneous set of order type $\omega + 1$.

The Erdős–Dushnik–Miller theorem (see [5]) states that $\kappa \to (\kappa, \omega)^2$ for any infinite κ, so the conclusion follows as in the previous argument. $\square_{1.4.1}$

Corollary 1.4.2. *Suppose that $\mathcal{D}$ is a $1/2$-dense open family on ω_1. Then for every $n < \omega$ there is an $A \subseteq \omega_1$ of order type $\omega + 1$ with $[A]^{\le n} \subseteq \mathcal{D}$, and if $n = 2$ then there is an uncountable $A \subseteq \omega_1$ with $[A]^{\le 2} \subseteq \mathcal{D}$.*

Proof. Suppose that $\mathcal{D}$ is a $1/2$-dense open family on ω_1 and $n < \omega$. Let $\mathcal{D}_0 = \mathcal{D} \cap [\omega_1]^{\le n}$. Then $\mathcal{D}_0$ satisfies the assumptions of Theorem 1.4.1, so the conclusions follow from the relevant parts of the Theorem. $\square_{1.4.2}$

A natural way to build a $1/2$-dense open family on ω_1 is to build, for some fixed n, a family $\mathcal{D}_0$ satisfying the assumptions of Theorem 1.4.1, and then to take $\mathcal{D} = \{F \in [\omega_1]^{<\omega} : [F]^{\le n} \subseteq \mathcal{D}_0\}$. Corollary 1.4.2 says that such a family will always have a homogeneous subset of order type $\omega + 1$, and, if $n = 2$, then it will have an uncountable homogeneous subset.

We note that improvements are available for larger order types in the second coordinate of the Erdős–Dushnik–Miller theorem; for example, if κ is regular then the original theorem has it as $\omega + 1$, and for κ singular one can consult [8]. For $n \ge 4$, it is a well-known fact in partition calculus that $(\omega_1) \not\to (\omega + 2, 5)^n$, as is $(\omega_1) \not\to (\omega + 2, \omega)^3$. Schipperus proved recently ([7]) that $\omega_1 \to (\omega^2 + 1, 4)^3$.

Corollary 1.4.2 does not say anything about uncountable homogeneous sets with $n > 2$. The following is a well-known folklore fact mentioned in the Introduction:

Fact 1.4.3. Suppose that there are $\aleph_1$ many measure 0 sets whose union is $[0, 1]$ (i.e., $\mathrm{cov}(\mathcal{N}) = \aleph_1$). Then there is a 1/2-open dense family on ω_1 with no uncountable homogeneous sets.

The proof (see [1] or [6]) uses compactness. In particular, it does not provide either an answer to the question about uncountable homogeneous sets with $n > 2$ in the situation of Theorem 1.4.1. We state the question explicitly:

Question 1.4.4. Suppose that $n \geq 3$ and $\mathcal{D}$ is a 1/2-dense open family on ω_1. Must there be a set $H \in [\omega_1]^{\aleph_1}$ with $[H]^n \subseteq \mathcal{D}$?

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Oscillations and Their Applications in Partition Calculus

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Abstract. Oscillations are a powerful tool for building examples of colorings witnessing negative partition relations. We survey several results illustrating the general technique and present a number of applications.

Introduction

We start by recalling some well-known notation. Given three cardinals κ, λ, μ and $n < \omega$, the notation

$$\kappa \to (\lambda)^n_\mu$$

means that for all functions $f\colon [\kappa]^n \to \mu$ there exists $H \subseteq \kappa$ with $|H| = \lambda$ and such that $f \restriction [H]^n$ is constant. We say that f is a *coloring of* $[\kappa]^n$ *in* μ *colors* and H is a *homogeneous set*. Given κ, λ, μ, σ and n as before, we write

$$\kappa \to [\lambda]^n_\mu$$

if for every coloring $f\colon [\kappa]^n \to \mu$ there exists $H \subseteq \kappa$ of cardinality λ such that $f''[H]^n \neq \mu$. We write

$$\kappa \to [\lambda]^n_{\mu,\sigma}$$

if for every coloring $f\colon [\kappa]^n \to \mu$ there exists $H \subseteq \kappa$ of cardinality λ such that $|f''[H]^n| \leq \sigma$.

One can extend the above notation to sets with additional structures, such as linear or partial orderings, graphs, trees, topological or vector spaces, etc. For instance, if two topological spaces X and Y are given, then

$$X \to (\text{top } Y)^n_\mu$$

means that for all $f\colon [X]^n \to \mu$ there exists a subset H of X homeomorphic to Y such that $f \restriction [H]^n$ is constant. Similarly, we can define statements such as $X \to [\text{top } Y]^n_\mu$, $X \to [\text{top } Y]^n_{\mu,\sigma}$, etc.

We denote by $[\mathbb{N}]^{<\omega}$ the set of all finite subsets of $\mathbb{N}$ and by $[\mathbb{N}]^\omega$ the set of all infinite subsets of $\mathbb{N}$. We often identify a set s in $[\mathbb{N}]^{<\omega}$ (or $[\mathbb{N}]^\omega$) with its increasing enumeration. When we do this, we will write $s(i)$ for the i-th element of s, assuming it exists. In this way, we identify $[\mathbb{N}]^\omega$ with $(\omega)^\omega$, the set of strictly increasing sequences from ω to ω, which is a G_δ subset of the Baire space ω^ω, and thus is itself a Polish space. For $s, t \in [\mathbb{N}]^{<\omega}$, we write $s \sqsubseteq t$ to say that s is an initial segment of t. In this way, we can view $([\mathbb{N}]^{<\omega}, \sqsubseteq)$ as a tree. For a given $s \in [\mathbb{N}]^{<\omega}$, we denote by N_s the set of all infinite increasing sequences of integers which extend s. In general, if T is a subtree of $[\mathbb{N}]^{<\omega}$ then T_s will denote the set of all sequences of T extending s. We will need some basic properties of the Baire space (or rather $[\mathbb{N}]^\omega$) and the Cantor space $\{0,1\}^\omega$ with the usual product topologies. For these facts and all undefined notions, we refer the reader to [5].

The paper is organized as follows. In §2.1 we discuss partitions of the rationals as a topological space. The basic tool is oscillations of finite sets of integers. In §2.2 we consider infinite oscillations of tuples of real numbers and discuss several applications to the study of inner models of set theory. In §2.3 we discuss finite oscillations of tuples of reals of a slightly different type. Finally, in §2.4 we present oscillations of pairs of countable ordinals and, in particular, outline Moore's ZFC construction of an L-space. We point out that none of the results of this paper are new and we will give a reference to the original paper for each of the results we mention. Our goal is not to give a comprehensive survey of all applications of oscillations in combinatorial set theory, but rather to present several representative results which illustrate the general method.

These are lecture notes of a tutorial given by the second author at the 2nd Young Set Theory Workshop held at the CRM in Bellaterra, April 14–18, 2009. The notes were taken by the first author.

2.1 Negative partition relations on the rationals

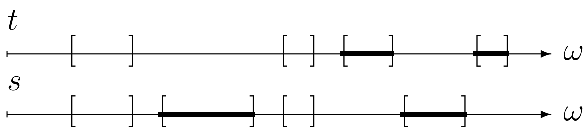
We start with a simple case of oscillations. Given $s, t \in [\mathbb{N}]^{<\omega}$, we define an equivalence relation $\sim$ on $s \mathbin{\triangle} t$ by:

$$n \sim m \iff ([n,m] \subseteq (s \setminus t) \vee [n,m] \subseteq (t \setminus s))$$

for all $n \leq m$ in $s \mathbin{\triangle} t$. We now define a function $\mathrm{osc}\colon ([\mathbb{N}]^{<\omega})^2 \to \mathbb{N}$ by

$$\mathrm{osc}(s,t) = |(s \mathbin{\triangle} t)/_\sim|$$

for all $s, t \in [\mathbb{N}]^{<\omega}$. If, for example, s and t are the two sets represented in the following picture, then $\mathrm{osc}(s,t) = 4$.



The following theorem is due to Baumgartner (see [1]):

Theorem 2.1.1 ([1])**.** $\mathbb{Q} \nrightarrow [\mathrm{top}\ \mathbb{Q}]^2_\omega$.

This means, with our notation, that there exists a coloring $c\colon [\mathbb{Q}]^2 \to \omega$ such that $c''[A]^2 = \omega$ for all $A \subseteq \mathbb{Q}$ with $A \approx \mathbb{Q}$.

Consider $[\mathbb{N}]^{<\omega}$ with the topology of pointwise convergence. Let $X \subseteq [\mathbb{N}]^{<\omega}$ and $s \in [\mathbb{N}]^{<\omega}$. Then $s \in \overline{X}$ if and only if for every $n > \sup(s)$ there is $t \in X$ such that $t \cap n = s$. Given $s, t \in [\mathbb{N}]^{<\omega}$, we write $s < t$ if $\max s < \min t$.

Remark 2.1.2. It is well known that $\mathbb{Q} \simeq [\mathbb{N}]^{<\omega}$, so we can view osc as a coloring of $[\mathbb{Q}]^2$.

In order to prove Theorem 2.1.1, we recall the definition of the Cantor–Bendixson derivative:

$$\delta(X) = \{x \in X : x \in \overline{X \setminus \{x\}}\},$$
$$\delta^0(X) = X,$$
$$\delta^{k+1}(X) = \delta(\delta^k(X)).$$

We need the following lemma.

Lemma 2.1.3. *Suppose that* $X \subseteq [\mathbb{N}]^{<\omega}$ *and let* $k > 0$ *be an integer such that* $\delta^k(X) \neq \emptyset$*. Then* $\mathrm{osc}''[X]^2 \supseteq \{1, 2, \ldots, 2k-1\}$*.*

Proof. The proof is by induction on k. First assume $k = 1$; then let $s \in \delta(X)$. This means that we can find $t, u \in X$ such that $s \sqsubset t, u$ and $t \setminus s < u \setminus s$. It follows that $\mathrm{osc}(t, u) = 1$. Assume that the property holds for all $l < k$; we show that $\mathrm{osc}''[X]^2$ takes values $2k-2$, $2k-1$. Fix $s \in \delta^k(X)$. Recursively pick $u_i, t_i \in \delta^{k-i}(X)$ for all $i \leq k$ such that the following hold:

1. $t_0 = u_0 = s$;
2. $s \sqsubset t_1 \sqsubset t_2 \sqsubset \cdots \sqsubset t_k$;
3. $s \sqsubset u_1 \sqsubset u_2 \sqsubset \cdots \sqsubset u_k$;
4. $t_i \setminus t_{i-1} < u_i \setminus u_{i-1} < t_{i+1} \setminus t_i$ for all $i \in \{1, 2, \ldots, k\}$.

Then $\mathrm{osc}(t_{k-1}, u_{k-1}) = 2k - 2$ and $\mathrm{osc}(t_k, u_{k-1}) = 2k - 1$. □

Proof of Theorem 2.1.1. By Remark 2.1.2, it is sufficient to check that, for all $A \subseteq [\mathbb{N}]^{<\omega}$ homeomorphic to $\mathbb{Q}$, $\mathrm{osc}''[A]^2 = \omega$. Since $A \approx \mathbb{Q}$, we have $\delta^k(A) \neq \emptyset$ for all integers k. Hence we can apply Lemma 2.1.3 and this completes the proof. □

An unpublished result of Galvin states that

$$\eta \to [\eta]^2_{n,2}$$

when η is the order type of the rational numbers and n is any integer. Therefore, the order-theoretic version of Theorem 2.1.1 does not hold. Also, the coloring we build to prove Baumgartner's theorem is not continuous. In fact, if we only consider continuous colorings, then we have

$$\mathbb{Q} \to_{\mathrm{cont}} [\mathrm{top}\ \mathbb{Q}]^2_2.$$

If we want a continuous coloring, we need to work in $[\mathbb{Q}]^3$. The following result is due to Todorčević ([10]).

Theorem 2.1.4 ([10])**.** *There is a continuous coloring* $c\colon [\mathbb{Q}]^3 \to \omega$ *such that* $c''[A]^3 = \omega$ *for all* $A \subseteq \mathbb{Q}$ *with* $A \approx \mathbb{Q}$*.*

Proof. Given $s, t, u \in [\mathbb{N}]^{<\omega}$, we define

$$\triangle(s,t) = \min(s \triangle t)$$
$$\triangle(s,t,u) = \max\{\triangle(s,t),\ \triangle(t,u),\ \triangle(s,u)\}.$$

The value of $\triangle(s,t,u)$ is equal to the least $n \in \mathbb{N}$ such that

$$|\{s \cap (n+1), t \cap (n+1), u \cap (n+1)\}| = 3.$$

So, in particular, for such an integer n we have $|\{s \cap n, t \cap n, u \cap n\}| = 2$. Let $\{v, w\} = \{s \cap n, t \cap n, u \cap n\}$. Then we define

$$\mathrm{osc}_3(s,t,u) = \mathrm{osc}(v,w).$$

The coloring osc_3 is obviously continuous. The proof that this coloring works is similar to the one given for Theorem 2.1.1. We can prove, analogously, that if $X \subseteq [\mathbb{N}]^{<\omega}$ and $\delta^k(X) \neq \emptyset$ for some integer $k > 0$, then $\mathrm{osc}_3''[X]^2 \supseteq \{1, 2, \ldots, 2k-1\}$. Let us just see the case $k = 1$. Fixing $s \in \delta(X)$, we can find $t, u \in X$ such that $s \sqsubset t, u$ and $t \setminus s < u \setminus s$. Then $\mathrm{osc}_3(s,t,u) = 1$. Finally, one can apply this result to all subsets of $[\mathbb{N}]^{<\omega}$ that are homeomorphic to $\mathbb{Q}$, and this completes the proof. □

2.2 Oscillations of real numbers – Part 1

We now discuss infinite oscillations and their applications.

For $x \subseteq \mathbb{N}$, we define an equivalence relation $\sim_x$ on $\mathbb{N} \setminus x$ as

$$n \sim_x m \iff [n, m] \cap x = \emptyset$$

for all $n \leq m$ in $\mathbb{N} \setminus x$. Thus, the equivalence classes of $\sim_x$ are the intervals between consecutive elements of x. Given $x, y, z \subseteq \mathbb{N}$, suppose that $(I_k)_{k \leq t}$ for $t \leq \omega$ is the natural enumeration of those equivalence classes of x which meet both y and z. We define a function $o(x,y,z)\colon t \to \{0,1\}$ as follows:

$$o(x,y,z)(k) = 0 \iff \min(I_k \cap y) \leq \min(I_k \cap z).$$

Notice that o is a continuous function from

$$\{(x,y,z) \in [[\mathbb{N}]^\omega]^3 : |(\mathbb{N} \setminus x)/_{\sim_x}| = \aleph_0\}$$

to $2^{\leq\omega}$. Note also that $[\mathbb{N}]^{<\omega}$ ordered by $\sqsubseteq$ is a tree. A subset T of $[\mathbb{N}]^{<\omega}$ is a *subtree* if it is closed under initial segments.

Definition 2.2.1. Let T be a subtree of $[\mathbb{N}]^{<\omega}$. We say that $t \in T$ is ∞-*splitting* if for all k there exists $u \in T$ such that $t \sqsubseteq u$ and $u(|t|) > k$.

Definition 2.2.2. A subtree T of $[\mathbb{N}]^{<\omega}$ is *superperfect* if for all $s \in T$ there exists $t \in T$ such that $s \sqsubseteq t$ and t is ∞-splitting in T.

Definition 2.2.3. We say that $X \subseteq [\mathbb{N}]^\omega$ is *superperfect* if there is a superperfect tree $T \subseteq [\mathbb{N}]^{<\omega}$ such that $X = [T] = \{A \in [\mathbb{N}]^\omega : A \cap k \in T \text{ for all } k\}$.

The following theorem is due to Veličković and Woodin ([11]).

Theorem 2.2.4 ([11])**.** *Let $X, Y, Z \subseteq [\mathbb{N}]^\omega$ be superperfect sets. Then*

$$o''[X \times Y \times Z] \supseteq 2^\omega.$$

Proof. Let $T_1, T_2, T_3 \subseteq [\mathbb{N}]^{<\omega}$ be superperfect trees such that $X = [T_1]$, $Y = [T_2]$ and $Z = [T_3]$. Given an $\alpha \in 2^\omega$, we build sequences $\langle s_k \rangle_k$, $\langle t_k \rangle_k$, $\langle u_k \rangle_k$ of nodes of T_1, T_2 and T_3, respectively, such that the following properties hold:

1. s_0, t_0, u_0 are the least ∞-splitting nodes of T_1, T_2 and T_3, respectively;
2. $s_0 \sqsubset s_1 \sqsubset s_2 \sqsubset \cdots \sqsubset s_k \sqsubset \cdots$;
3. $t_0 \sqsubset t_1 \sqsubset t_2 \sqsubset \cdots \sqsubset t_k \sqsubset \cdots$;
4. $u_0 \sqsubset u_1 \sqsubset u_2 \sqsubset \cdots \sqsubset u_k \sqsubset \cdots$;
5. $t_i \setminus t_{i-1}$, $u_i \setminus u_{i-1} < s_i \setminus s_{i-1}$;
6. $t_i \setminus t_{i-1} < u_i \setminus u_{i-1}$ if $\alpha(i) = 0$ and $u_i \setminus u_{i-1} < t_i \setminus t_{i-1}$ if $\alpha(i) = 1$.

If $x = \bigcup_{k<\omega} s_k$, $y = \bigcup_{k<\omega} t_k$ and $z = \bigcup_{k<\omega} u_k$, then $o(x, y, z) = \alpha$, and this completes the proof. □

Corollary 2.2.5 ([11])**.** *If $X \subseteq [\mathbb{N}]^\omega$ is superperfect, then $o''[X]^3 \supseteq 2^\omega$.* □

We now apply the previous theorem to prove some results about reals of inner models of set theory.

Theorem 2.2.6 ([11])**.** *Let V, W be models of set theory such that $W \subseteq V$. If there is a superperfect set X in V such that $X \subseteq W$, then $\mathbb{R}^W = \mathbb{R}^V$.*

Proof. This is trivial by applying Corollary 2.2.5. □

Question 2.2.7. Can we replace superperfect by perfect in the previous theorem?

Surprisingly, the answer depends on whether CH holds in the model W, as asserted in the following theorem due to Groszek and Slaman (see [4]).

Theorem 2.2.8 ([4])**.** *Suppose that W and V are two models of set theory such that $W \subseteq V$. Assume that there is a perfect set P in V such that $P \subseteq W$. If* CH *holds in W, then $\mathbb{R}^W = \mathbb{R}^V$.*

In order to prove this theorem, let us introduce the following notion.

Definition 2.2.9. Given two models of set theory W and V with $W \subseteq V$, we say that (W, V) satisfies the *countable covering property for the reals* if, for all X in V such that $X \subseteq \mathbb{R}^W$ and X is countable in V, there is an Y in W such that $X \subseteq Y$ and Y is countable in W.

We prove first the following theorem.

Theorem 2.2.10. *Given two models of set theory W and V with $W \subseteq V$, suppose that there is a perfect set P in V such that $P \subseteq W$. If (W, V) satisfies the countable covering property for the reals, then $\mathbb{R}^W = \mathbb{R}^V$.*

Proof. Work in V and fix a perfect subset P of $(2^\omega)^W$. Let X be a countable dense subset of P. By the countable covering property for the reals, we can cover X by some set $D \in W$ such that D is countable in W, it is a dense subset of 2^ω, and $D \cap P$ is dense in P. In W, fix an enumeration $\{d_n : n < \omega\}$ of D. For $x, y \in 2^\omega$ with $x \neq y$, let

$$\triangle(x, y) = \min\{n : x(n) \neq y(n)\}.$$

Given $x \in 2^\omega \setminus D$, first define a sequence $\langle k_x(n) : n < \omega \rangle$ by induction as follows:

$$k_x(n) = \min\{k : \triangle(x, d_k) > \triangle(x, d_{k_x(i)}) \text{ for all } i < n\}.$$

Note that $k_x(0) = 0$. Since D is dense in P and $x \in P \setminus D$, $k_x(n)$ is defined for all n. Now define $f \colon P \setminus D \to [\mathbb{N}]^\omega$ by setting

$$f(x)(n) = \triangle(x, d_{k_x(n)}).$$

Clearly, f is continuous and $f(x)$ is a strictly increasing function for all $x \in 2^\omega \setminus D$. Since $D \in W$, f is coded in W. We can now prove that $f''[P \setminus D]$ is superperfect. Let $T = \{f(x) \restriction n : x \in P \setminus D \wedge n \in \omega\}$. First note that $f''[P \setminus D]$ is closed, i.e., it is equal to $[T]$. To see this, note that, if $b \in [T]$, then for every i there is $x_i \in P \setminus D$ such that $b \restriction i = f(x_i) \restriction i$. Since P is compact, it follows that the sequence $(x_i)_i$ converges to some $x \in P$. Note then that $k_x(n) = k_{x_m}(n)$ for all $m > n$; in particular, $k_x(n)$ is defined for all n. It follows that $x \notin D$. Since $f(x) = b$, it follows that $b \in f''[P \setminus D]$, as desired.

Next, we show that every node of T is ∞-splitting. Let $s \in T$ and suppose that $n = |s|$. Then there is some $x \in P \setminus D$ such that $s \sqsubseteq f(x)$. Therefore, $s(i) = \triangle(x, d_{k_x(i)})$ for all $i < n$. Let $l = k_x(n)$. Since P is perfect, we can find, for every $j \geq \triangle(x, d_l)$, some $x_j \in P \setminus D$ such that $\triangle(x_j, d_l) \geq j$. It follows that $f(x_j) \restriction n = s$ and $f(x_j)(n) \geq j$. This shows that s is ∞-splitting.

Since $P \subseteq W$ and f is coded in W, we have $f''[P \setminus D] \subseteq W$, that is, W contains a superperfect set. By Corollary 2.2.5, we have $\mathbb{R}^W = \mathbb{R}^V$ and this completes the proof. □

Proof of Theorem 2.2.8. By the previous theorem, it is enough to prove that (W, V) satisfies the countable covering property for the reals. By assumption, W satisfies CH, so we can fix in W a well-ordering on $(\mathbb{R})^W$ of height $(\omega_1)^W$. Since every perfect set is uncountable and $P \subseteq W$, we have ${\omega_1}^W = {\omega_1}^V$. Therefore, any $X \subseteq (\mathbb{R})^W$ which is countable in V is contained in a proper initial segment Y of the well-ordering. Then $Y \in W$ and Y is countable in W. This completes the proof. □

In particular, we can state the following corollary.

Corollary 2.2.11 ([4]). *If there is a perfect set of constructible reals, then* $\mathbb{R} \subseteq L$. □

Is the countable covering condition necessary to obtain this result? Theorem 2.2.12 below (see [11]) gives a partial answer to this question.

Theorem 2.2.12 ([11]). *There is a pair (W, V) of generic extensions of L with $W \subseteq V$ such that $\aleph_1^W = \aleph_1^V$ and V contains a perfect set of W-reals, but $\mathbb{R}^W \neq \mathbb{R}^V$.*

On the other hand, in [11] we also have the following theorem.

Theorem 2.2.13 ([11]). *Suppose that M is an inner model of set theory and $\mathbb{R}^M$ is analytic. Then either $\aleph_1^M$ is countable or all reals are in M.*

In order to prove Theorem 2.2.13, let us introduce a generalization of the notion of a superperfect set.

Definition 2.2.14. Suppose that λ is a limit ordinal and T is a subtree of $[\lambda]^{<\omega}$. We say that $t \in T$ is λ-*splitting* if for all $\xi < \lambda$ there exists $u \in T$ such that $t \sqsubseteq u$ and $u(|t|) > \xi$.

Definition 2.2.15. Suppose that λ is a limit ordinal and let T be a subtree of $[\lambda]^{<\omega}$. We say that T is λ-*superperfect* if for all $s \in T$ there exists $t \in T$ such that $s \sqsubseteq t$ and t is λ-splitting.

Definition 2.2.16. A set $P \subseteq [\lambda]^\omega$ is λ-*superperfect* if there is a λ-superperfect tree $T \subseteq [\lambda]^{<\omega}$ such that $P = \{x \in [\lambda]^\omega : \forall n < \omega\, (x \restriction n \in T)\}$. Here $x \restriction n$ denotes the set of the first n elements of x in the natural order.

The definition of $o\colon ([\mathbb{N}]^\omega)^3 \to \{0,1\}^\omega$ can be trivially generalized to a coloring

$$o_\lambda\colon ([\lambda]^\omega)^3 \longrightarrow \{0,1\}^\omega.$$

As for o, one can easily check that, for all λ-superperfect P, we have $o''_\lambda[P^3] \supseteq \{0,1\}^\omega$ (the proof is the same as for Theorem 2.2.4). Moreover, we have $o_\lambda(x,y,z) \in L[x,y,z]$ for all $x,y,z \in [\lambda]^\omega$. To complete the proof of Theorem 2.2.13, it suffices to prove the following lemma.

Lemma 2.2.17. *Suppose that A is an analytic set such that*

$$\sup\{\omega_1^{\mathrm{CK},x} : x \in A\} = \omega_1.$$

Then every real is hyperarithmetic in a quadruple of elements of A.

Proof. Let $T \subset (\omega\times\omega)^{<\omega}$ be a tree such that $A = p[T]$. Note that the statement $\sup\{\omega_1^{\mathrm{CK},x} : x \in p[T]\} = \omega_1$ is $\Pi^1_2(T)$ and thus absolute.

For an ordinal α, let $\mathrm{Coll}(\aleph_0,\alpha)$ be the usual collapse of α to $\aleph_0$ using finite conditions. Let $\mathcal{P}$ denote $\mathrm{Coll}(\aleph_0,\aleph_1)$. If G is V-generic over $\mathcal{P}$, then, by Shoenfield's absoluteness theorem, in $V[G]$ there is $x \in p[T]$ such that $\omega_1^{\mathrm{CK},x} > \omega_1^V$. In V fix a name $\dot{x}$ for x and a name σ for a cofinal ω-sequence in ω_1^V such that the maximal condition in $\mathcal{P}$ forces that $\dot{x} \in p[T]$ and $\sigma \in L[\dot{x}]$.

Claim 2.2.18. *For every $p \in \mathcal{P}$ there is $k < \omega$ such that for every $\alpha < \omega_1$ there is $q \le p$ such that $q \Vdash \sigma(k) > \alpha$.*

Proof. Assume otherwise and fix p for which the claim is false. Then for every k there is $\alpha_k < \omega_1$ such that $p \Vdash \sigma(k) < \alpha_k$. Let $\alpha = \sup\{\alpha_k : k < \omega\}$. Then $p \Vdash \mathrm{ran}(\sigma) \subset \alpha$, contradicting the fact that σ is forced to be cofinal in ω_1^V. □

Let $\mathcal{Q}$ denote $\mathrm{Coll}(\aleph_0,\aleph_2)$ as defined in V. Suppose that H is V-generic over $\mathcal{Q}$. Work for a moment in $V[H]$. If G is a V-generic filter over $\mathcal{P}$, let σ_G denote the interpretation of σ in $V[G]$. Let B be the set of all σ_G where G ranges over all V-generic filters over $\mathcal{P}$.

Claim 2.2.19. *B contains an ω_1^V-superperfect set in $(\omega_1^V)^\omega$.*

Proof. Let $\{D_n : n < \omega\}$ be an enumeration of all dense subsets of $\mathcal{P}$ which belong to the ground model. For each $t \in (\omega_1^V)^{<\omega}$, we define a condition p_t in the regular open algebra of $\mathcal{P}$ as computed in V, and $s_t \in (\omega_1^V)^{<\omega}$ inductively on the length of t, such that:

1. $p_t \in D_{lh(t)}$;
2. $p_t \Vdash s_t \subset \sigma$;

3. if $t \subseteq r$ then $p_r \leq p_t$ and $s_t \subset s_r$;
4. if t and r are incomparable then s_t and s_r are incomparable;
5. for every t the set $\{\alpha < \omega_1^V : \text{there is } q \leq p \text{ such that } q \Vdash s_t{}^\frown\alpha \subset \sigma\}$ is unbounded in ω_1^V.

Suppose that p_t and s_t have been defined. Using 5, choose in V a 1-1 order-preserving function $f_t\colon \omega_1^V \to \omega_1^V$ and for every α $q_{t,\alpha} \leq p_t$ such that $q_{t,\alpha} \Vdash s_t{}^\frown f_t(\alpha) \subset \sigma$. By extending $q_{t,\alpha}$ if necessary, we may assume that it belongs to $D_{lh(t)+1}$. Now, by applying Claim 2.2.18, we can find a condition $p \leq q_{t,\alpha}$ and $k > lh(s_t) + 1$ such that for some $s \in (\omega_1^V)^k$ $p \Vdash s \subset \sigma$ and for every $\gamma < \omega_1^V$ there is $q \leq p$ such that $q \Vdash \sigma(k) > \gamma$. Let then $s_{t^\frown\alpha} = s$ and $p_{t^\frown\alpha} = p$. This completes the inductive construction.

Now if $b \in (\omega_1^V)^\omega$ then $\{p_{b\restriction n} : n < \omega\}$ generates a filter G_b which is V-generic over $\mathcal{P}$. The interpretation of σ under G_b is $s_b = \bigcup_{n<\omega} s_{b\restriction n}$. Since the set $R = \{s_b : b \in (\omega_1^V)^\omega\}$ is ω_1^V-superperfect, this proves Claim 2.2.19. □

Using the remark following Definition 2.2.16, for any real $r \in \{0,1\}^\omega$ we can find $b_1, b_2, b_3 \in (\omega_1^V)^\omega$ such that $r \in L[s_{b_1}, s_{b_2}, s_{b_3}]$. Let x_i be the interpretation of $\dot{x}$ under G_{b_i}. Then it follows that $x_i \in p[T]$ and $s_{b_i} \in L[x_i]$ for $i = 1, 2, 3$. Thus $r \in L[x_1, x_2, x_3]$. Pick a countable ordinal δ such that $r \in L_\delta[x_1, x_2, x_3]$. Using the fact that $\sup\{\omega_1^{\mathrm{CK},x} : x \in p[T]\} = \omega_1$ in $V[H]$, we can find $y \in p[T]$ such that $\omega_1^{\mathrm{CK},y} > \delta$. Then we have that r is $\Delta^1_1(x_1, x_2, x_3, y)$. Note that the statement that there are $x_1, x_2, x_3, y \in p[T]$ such that $r \in \Delta^1_1(x_1, x_2, x_3, y)$ is $\Sigma^1_2(r, T)$. Thus for any real $r \in V$, by Shoenfield absoluteness again, it must be true in V. This proves Lemma 2.2.17. □

We complete this section by stating some related results.

Theorem 2.2.20 ([11])**.** *There is a pair $W \subseteq V$ of generic extensions of L such that $\mathbb{R}^W$ is an uncountable F_δ set in V and $\mathbb{R}^W \neq \mathbb{R}^V$.*

Theorem 2.2.21 ([3])**.** *Suppose that $W \subseteq V$ are two models of set theory, $\kappa > \omega_1^V$, and there exists $C \subseteq [\kappa]^\omega$ which is a club in V such that $C \subseteq W$. Then $\mathbb{R}^W = \mathbb{R}^V$.*

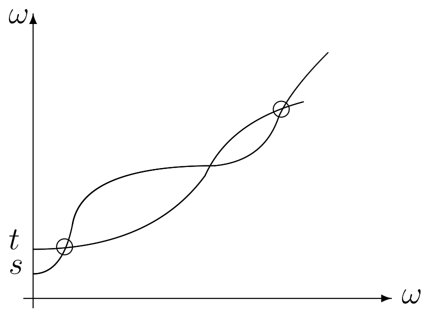
Theorem 2.2.22 ([2])**.** *Let $W \subseteq V$ be two models of set theory such that $V, W \models \mathrm{PFA}$ and $\aleph_2^W = \aleph_2^V$. Then $\mathbb{R}^W = \mathbb{R}^V$; in fact, $\mathscr{P}(\omega_1)^W = \mathscr{P}(\omega_1)^V$.*

2.3 Oscillations of real numbers – Part 2

The results of this section are taken from [9]. We look at increasing sequences of integers and slightly change the definition of oscillation. For $s, t \in (\omega)^{\leq\omega}$, we define

$$\operatorname{osc}(s,t) = |\{n < \omega : s(n) \leq t(n) \wedge s(n+1) > t(n+1)\}|.$$

In the next picture, s and t are two functions in $(\omega)^{<\omega}$ with $\operatorname{osc}(s,t) = 2$.



We now define two orders $\leq_m$ and $\leq_*$ on $(\omega)^\omega$:

$$f \leq_m g \iff \forall n \geq m\,(f(n) \leq g(n));$$
$$f \leq_* g \iff \exists m\,(f \leq_m g).$$

Given $X \subseteq (\omega)^\omega$ and $s \in (\omega)^{<\omega}$, we let $X_s = \{f \in X : s \sqsubseteq f\}$ and

$$T_X = \{s \in (\omega)^{<\omega} : X_s \text{ is unbounded under } \leq_*\}.$$

Lemma 2.3.1. *Suppose that $X \subseteq (\omega)^\omega$ is unbounded under $\leq_*$ and that $X = \bigcup_{n<\omega} A_n$. Then there exists n such that A_n is unbounded.*

Proof. Suppose that every A_n is bounded and, for all n, let g_n be such that $f \leq_* g_n$ for all $f \in A_n$. If we define $g(n) = \sup\{g_k(n) : k \leq n\}$, then X is bounded by g with respect to $\leq_*$. This leads to a contradiction. □

Lemma 2.3.2. *Suppose that $X \subseteq (\omega)^\omega$ is unbounded under $\leq_*$. Then T_X is superperfect.*

Proof. Suppose, by way of contradiction, that there is a node $s \in T_X$ with no ∞-splitting extensions in T_X. We define a function $g_s\colon \omega \to \omega$ as follows:

$$g_s(n) = \sup\{t(n) : t \in (T_X)_s \wedge n \in \operatorname{dom}(t)\},$$

where $(T_X)_s = \{f \in T_X : s \sqsubseteq f\}$. First note that $g_s(n) < \omega$ for all $n < \omega$. Let $Q = \{t \in (\omega)^{<\omega} : X_t$ is bounded under $\leq_*\}$. By Lemma 2.3.1, the set $\bigcup\{X_t : t \in Q\}$ is bounded under $\leq_*$ by some function g. Now let $h = \max(g, g_s)$. It follows that X_s is $\leq_*$-bounded by h, a contradiction. $\square$

We first consider oscillations of elements of $(\omega)^{<\omega}$. Our first goal is to prove that if T is a superperfect subtree of $(\omega)^{<\omega}$ then $\mathrm{osc}''[T]^2 = \omega$. In fact, we prove a slightly stronger lemma.

Lemma 2.3.3. *Let S and T be two superperfect subtrees of $(\omega)^{<\omega}$ and let s and t be ∞-splitting nodes of S and T respectively. Then for all n there are ∞-splitting nodes s' in S and t' in T such that $s \sqsubseteq s'$, $t \sqsubseteq t'$ and*

$$\mathrm{osc}(s', t') = \mathrm{osc}(s, t) + n.$$

Proof. We may assume without loss of generality that $|s| < |t|$ and $s(|s|-1) \leq t(|s|-1)$. We can recursively pick some ∞-splitting extensions $s_i \in S$ and $t_i \in T$, for $i \leq n$, such that:

- $s_0 = s$ and $t_0 = t$;
- $s_0 \sqsubset s_1 \sqsubset \cdots \sqsubset s_n$;
- $t_0 \sqsubset t_1 \sqsubset \cdots \sqsubset t_n$;
- $\mathrm{osc}(s_i, t_i) = \mathrm{osc}(s, t) + i$ for all i;
- $|s_i| < |t_i|$ and $s_i(|s_i| - 1) \leq t_i(|s_i| - 1)$.

Given s_i and t_i, since S is superperfect and s_i is ∞-splitting in S, we can find some ∞-splitting extension u of s_i in S such that $u(|s_i|) > t_i(|t_i| - 1)$ and such that $|u| > |t_i| + 1$. In the same way, we can take an ∞-splitting extension v of t_i in T such that $v(|t_i|) > u(|u| - 1)$ and $|v| > |u| + 1$. Since u and v are strictly increasing, we have $\mathrm{osc}(u, v) = \mathrm{osc}(s_i, t_i) + 1$, so we can define $s_{i+1} = u$ and $t_{i+1} = v$.

Finally, $\mathrm{osc}(s_n, t_n) = \mathrm{osc}(s, t) + n$ and this completes the proof. $\square$

Corollary 2.3.4. *If T is a superperfect subtree of $(\omega)^{<\omega}$ then the equality $\mathrm{osc}''[T]^2 = \omega$ holds.* $\square$

We now turn to oscillations of elements of $(\omega)^{\omega}$. We will need the following definition.

Definition 2.3.5. A subset X of $(\omega)^{\omega}$ is *σ-directed under* $\leq_*$ if, and only if, for all countable $D \subseteq X$ there is $f \in X$ such that $d \leq_* f$ for all $d \in D$.

Lemma 2.3.6. *Suppose that $X \subseteq (\omega)^\omega$ is σ-directed and unbounded under $\leq_*$ and $Y \subseteq (\omega)^\omega$ is such that for every $a \in X$ there is $b \in Y$ such that $a \leq_* b$. Then there is an integer n_0 such that for all $k < \omega$ there are $f \in X$ and $g \in Y$ such that* $\operatorname{osc}(f, g) = n_0 + k$.

Proof. Fix a countable dense subset D of X. Since X is σ-directed, there is a function $a \in X$ such that $d \leq_* a$ for all $d \in D$. Then $Y' = \{g \in Y : a \leq_* g\}$ is unbounded under $\leq_*$. We define $Y_m = \{g \in Y' : a \leq_m g\}$ for all $m < \omega$. By Lemma 2.3.1 and the fact that $Y' = \bigcup\{Y_m : m < \omega\}$, there exists $m_0 < \omega$ such that Y_{m_0} is also $\leq_*$-unbounded. Let $s_0 \in T_X$ and $t_0 \in T_{Y_{m_0}}$ be the two least ∞-splitting nodes of T_X and $T_{Y_{m_0}}$ respectively. Let $n_0 = \operatorname{osc}(s_0, t_0)$. Now, fix $k < \omega$. By Lemma 2.3.3, there are two ∞-splitting $s \in T_X$ and $t \in T_{Y_{m_0}}$ such that $\operatorname{osc}(s, t) = n_0 + k$. We may assume without loss of generality that $|t| \leq |s|$ and $t(|t| - 1) > s(|t| - 1)$. Since D is dense, there is $f \in D$ such that $s \sqsubseteq f \leq_* a$. Fix $m \geq m_0$ such that $f \leq_m a$. Since t is ∞-splitting in $T_{Y_{m_0}}$, we can pick $i > f(m)$ and $g \in Y_{m_0}$ such that $t^\frown i \sqsubseteq g$. We know that for all $k \geq m_0$, $a(k) \leq g(k)$, so $f(k) \leq g(k)$ for all $k \geq m$. Moreover, f and g are increasing and $t^\frown i \sqsubseteq g$, so for all k between $|t|$ and m we have $g(k) > f(k)$. It follows that $\operatorname{osc}(f, g) = \operatorname{osc}(s, t) = n_0 + k$ and this completes the proof. □

The following theorem is due to Todorčević (see [9]).

Theorem 2.3.7 ([9]). *Suppose that $X \subseteq (\omega)^\omega$ is unbounded under $\leq_*$ and σ-directed. Then $osc''[X]^2 = \omega$.*

Proof. The proof is the same as for Lemma 2.3.6, by assuming $Y = X$. Thus $s_0 = t_0$ and, consequently, $n_0 = 0$ in the previous proof. Hence, for all $k < \omega$ there are $f, g \in X$ such that $\operatorname{osc}(f, g) = k$. This completes the proof. □

We recall that $\mathfrak{b}$ is the least cardinal of an $\leq_*$-unbounded subset of $(\omega)^\omega$. Fix an unbounded $\mathscr{F} \subseteq (\omega)^\omega$ of cardinality $\mathfrak{b}$. We may assume that $\mathscr{F}$ is well ordered under $\leq_*$ and $(\mathscr{F}, \leq_*)$ has order type $\mathfrak{b}$.

Remark 2.3.8. Every unbounded subset of $\mathscr{F}$ is σ-directed and cofinal in $\mathscr{F}$ under $\leq_*$.

Corollary 2.3.9. *Let $X, Y \subseteq \mathscr{F}$ be unbounded under $\leq_*$. There exists $n_0 < \omega$ such that for all $k < \omega$ there exist $f \in X$ and $g \in Y$ such that* $\operatorname{osc}(f, g) = n_0 + k$.

Proof. Trivial by Remark 2.3.8 and Lemma 2.3.6. □

In [9], Todorčević proved a more general result:

Theorem 2.3.10 ([9]). *Suppose that $\mathscr{F}$ is $\leq_*$-unbounded and well ordered by $\leq_*$ in order type $\mathfrak{b}$. Suppose that $\mathfrak{A} \subseteq [\mathscr{F}]^n$, $|\mathfrak{A}| = \mathfrak{b}$, and $\mathfrak{A}$ consists of pairwise disjoint n-tuples. Then there exists $h\colon n \times n \to \omega$ such that for all $k < \omega$ there exist $A, B \in \mathfrak{A}$ such that $A \neq B$ and $\operatorname{osc}(A(i), B(j)) = h(i,j) + k$ for all $i, j < n$. Here $A(i)$ denotes the i-th element of A in increasing order, and similarly $B(j)$ denotes the j-th element of B.*

Proof. For any $A, B \in [\mathscr{F}]^n$, we will write $A <_m B$ if, and only if, $a <_m b$ for all $a \in A$ and $b \in B$. Similarly, with $A \leq_* B$ we mean that $a \leq_* b$ for all $a \in A$ and $b \in B$. Finally, if $A \in \mathfrak{A}$ and $m < \omega$, we denote by $A \restriction m$ the sequence $\langle A(i) \restriction m \rangle_{i<n}$.

We may assume that $\mathfrak{A}$ is *everywhere unbounded*, that is, for all $m < \omega$ and $A \in \mathfrak{A}$, the set $\{B \in \mathfrak{A} : B \restriction m = A \restriction m\}$ is also unbounded in $((\omega)^\omega)^n$ under $\leq_*$. Take a countable dense $\mathfrak{D} \subseteq \mathfrak{A}$. There is $A \in \mathfrak{A}$ such that $D \leq_* A$ for all $D \in \mathfrak{D}$. For all $m < \omega$, let $\mathfrak{A}_m = \{B \in \mathfrak{A} : A <_m B\}$. As before, there is $m_0 < \omega$ such that $\mathfrak{A}_{m_0}$ is everywhere unbounded.

Given any $\vec{t} \in (\omega^{<\omega})^n$, we denote by t_i the i-th element of $\vec{t}$ in increasing order. If $B \in (\omega^\omega)^n$, then $\vec{t} \sqsubseteq B$ means $t_i \sqsubseteq B(i)$ for all $i < n$. Now, we define

$$T_{\mathfrak{A}_{m_0}} = \{\vec{t} \in (\omega^{<\omega})^n : \forall i < n\,(|t_i| < |t_{i+1}|) \wedge \exists B \in \mathfrak{A}_{m_0}(\vec{t} \sqsubseteq B)\}.$$

For any sequence $\vec{s} \in T_{\mathfrak{A}_{m_0}}$, we say that $\vec{s}$ *is ∞-splitting* if for all $l < \omega$ there is $\vec{t} \in T_{\mathfrak{A}_{m_0}}$ such that $\vec{s} \sqsubseteq \vec{t}$ and $t_i(|s_i|) > l$ for all $i < n$.

Claim 2.3.11. *$T_{\mathfrak{A}_{m_0}}$ is superperfect, that is, for all $\vec{s} \in T_{\mathfrak{A}_{m_0}}$ there is an ∞-splitting sequence $\vec{t} \in T_{\mathfrak{A}_{m_0}}$ such that $\vec{s} \sqsubseteq \vec{t}$.*

Proof. Given $\vec{s} \in T_{\mathfrak{A}_{m_0}}$, define t_0 as the least ∞-splitting extension of s_0 in $T_{Z(0)}$, where $Z(0) = \{B(0) : B \in \mathfrak{A}_{m_0}\}$. Assume that $\vec{t} \restriction i$ is defined. The set $Z(i) = \{B(i) : B \in \mathfrak{A}_{m_0}$ and $B \restriction i = \vec{t} \restriction i\}$ is unbounded (because $\mathfrak{A}_{m_0}$ is everywhere unbounded). Let t_i be any ∞-splitting extension of s_i in $T_{Z(i)}$ such that $|t_i| > |t_{i-1}|$. The sequence $\vec{t}$, so defined, is ∞-splitting in $T_{\mathfrak{A}_{m_0}}$. This completes the proof of the claim. □

Let $\vec{r} \in T_{\mathfrak{A}_{m_0}}$ be the least ∞-splitting sequence. We define, for all $i, j < n$,

$$h(i,j) = \operatorname{osc}(r_i, r_j).$$

Claim 2.3.12. *For all $k < \omega$, there are two ∞-splitting sequences $\vec{s}, \vec{t} \in T_{\mathfrak{A}_{m_0}}$ such that $\vec{r} \sqsubseteq \vec{s}, \vec{t}$ and $\operatorname{osc}(s_i, t_j) = \operatorname{osc}(r_i, r_j) + k$ for all $i, j < n$.*

Proof. We prove this by induction on $k < \omega$. The case $k = 0$ is trivial. Let $\vec{s}, \vec{t} \in T_{\mathfrak{A}_{m_0}}$ be ∞-splitting, such that $\vec{r} \sqsubseteq \vec{s}, \vec{t}$ and $\operatorname{osc}(s_i, t_j) = \operatorname{osc}(r_i, r_j) + k$ for all i, j. Assume without loss of generality that $|s_i| < |t_j|$ and $s_i(|s_i| - 1) \leq t_j(|s_i| - 1)$ for all i, j. Since $\vec{s}$ is ∞-splitting, there is an ∞-splitting sequence $\vec{u} \in T_{\mathfrak{A}_{m_0}}$ such that $\vec{s} \sqsubseteq \vec{u}$ and $u_i(|s_i|) > t_j(|t_j| - 1)$ for all i, j. We also ask that $|u_i| > |t_j| + 1$ for all i, j. Similarly, we can find an ∞-splitting sequence $\vec{v} \in T_{\mathfrak{A}_{m_0}}$ such that $\vec{t} \sqsubseteq \vec{v}$ and $v_i(|t_i|) > u_j(|u_j| - 1)$ for all i, j. It follows that $\operatorname{osc}(u_i, v_j) = \operatorname{osc}(s_i, t_j) + 1$ for all i, j. This completes the proof of the claim. □

Fix $\vec{s}$ and $\vec{t}$ as in Claim 2.3.12. Assume without loss of generality that $|s_i| \leq |t_j|$ and $s_i(|s_i| - 1) > t_j(|s_i| - 1)$ for all i, j. Consider now the families $X = \{B \in \mathfrak{A} : \vec{t} \sqsubseteq B\}$ and $\mathfrak{D}' = \mathfrak{D} \cap X$. We have that X is everywhere unbounded and $\mathfrak{D}'$ is dense in X. Take any $D \in \mathfrak{D}'$. Then $\vec{t} \sqsubseteq D <_m A$ for some $m > m_0$. Since $\vec{s}$ is ∞-splitting, there is $l \geq D(n-1)(m)$ and $B \in \mathfrak{A}_{m_0}$ such that $\vec{s}^\frown l := \langle s_i{}^\frown l \rangle_{i<n} \sqsubseteq B$. By construction, $\operatorname{osc}(D(i), B(j)) = \operatorname{osc}(s_i, t_j) = h(i, j) + k$ for all $i, j < n$. This completes the proof. □

Sometimes we need to improve osc to get an even better coloring. First we want to get rid of the function h of Theorem 2.3.10. We fix a bijection $e \colon \omega \to \omega \times \omega$ and define a new partial function o on pairs of elements of $(\omega)^{<\omega}$ or $(\omega)^\omega$ as follows. Let $\operatorname{osc}(f, g) = 2^{i_0} + 2^{i_1} + \cdots + 2^{i_k}$ for $i_0 > i_1 > \cdots > i_k$ be the binary expansion of $\operatorname{osc}(f, g)$. We define $o(f, g) = \pi_0 \circ e(i_0)$, where π_0 is the projection onto the first component.

Lemma 2.3.13. *Suppose that $\mathscr{F}$ and $\mathfrak{A} \subseteq [\mathscr{F}]^n$ are as in Theorem 2.3.10. For all $k < \omega$ there exist $A, B \in \mathfrak{A}$ such that $A \neq B$ and $o(A(i), B(j)) = k$ for all $i, j < n$.*

Proof. Given k, consider the function $h \colon n \times n \to \omega$ of Theorem 2.3.10. For all $i, j < n$, let $l_{i,j}$ be the largest integer such that $2^{l_{i,j}} \leq h(i, j)$, and let $l = \max\{l_{i,j} : i, j < n\}$. The set $\{m : \exists p\, (e(m) = (k, p))\}$ is infinite, so we can find $m > l$ such that $\pi_0 \circ e(m) = k$. By definition of h, there exist two different $A, B \in \mathfrak{A}$ such that $\operatorname{osc}(A(i), B(j)) = h(i, j) + 2^m$ for all $i, j < n$. It follows that $o(A(i), B(j)) = \pi_0 \circ e(m) = k$ for all $i, j < n$. This completes the proof. □

Finally, we want to be able to choose the color of $\{A(i), B(j)\}$ independently for all i, j. First we need the following lemma.

Lemma 2.3.14. *Given an unbounded family $\mathfrak{A} \subseteq [\mathscr{F}]^n$ of pairwise disjoint sets, there are $k < \omega$ and $\mathfrak{A}^* \subseteq \mathfrak{A}$ unbounded such that, for every $i < n$, there exists $a_i \in (\omega)^k$ such that $A(i) \restriction k = a_i$ for all $A \in \mathfrak{A}^*$ and $a_i \neq a_j$ for all $i \neq j < n$.*

Proof. We prove it by induction on $n < \omega$. It is trivial for $n = 1$. Assume that the statement is true for n. We prove it for $n+1$. Given $\mathfrak{A} \subseteq [\mathscr{F}]^{n+1}$, let $k < \omega$, $\mathfrak{A}' \subseteq \mathfrak{A} \restriction n$, and $\{a_i\}_{i<n}$ be as in the conclusion of the lemma for $\mathfrak{A} \restriction n$. The set $\mathfrak{B} = \{A \in \mathfrak{A} : A \restriction n \in \mathfrak{A}'\}$ is unbounded, hence $X = \{A(n) : A \in \mathfrak{B}\}$ is also unbounded. By Lemma 2.3.2, we have that T_X is superperfect, so let b be the least ∞-splitting node of T_X. We can assume without loss of generality that $|b| < k$. Take any $a_n \sqsupseteq b$ in T_X such that $|a_n| = k$ and $a_n(k-1) > \max\{a_i(k-1) : i < n\}$. Then $a_n \neq a_i$ for all $i < n$. Recall that $T_X = \{s \in (\omega)^{<\omega} : \{f \in X : s \sqsubseteq f\} \text{ is unbounded}\}$. Thus $\mathfrak{A}^* = \{B \in \mathfrak{B} : a_n \sqsubseteq B(n)\}$ is unbounded. This completes the proof. □

Consider all finite functions $t\colon D \times E \to \omega$ where $D, E \subseteq (\omega)^k$ and k is an integer. Let $\{(t_n, D_n, E_n, k_n)\}_{n<\omega}$ be any enumeration of such functions. We define $c\colon [\mathscr{F}]^2 \to \omega$ as follows: given $f, g \in \mathscr{F}$ and letting $n = o(f,g)$, we set

$$c(f,g) = \begin{cases} t_n(f \restriction k_n, g \restriction k_n) & \text{if } f \restriction k_n \in D_n \text{ and } g \restriction k_n \in E_n; \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2.3.15 ([9])**.** *Given an unbounded family $\mathfrak{A} \subseteq [\mathscr{F}]^n$ of pairwise disjoint sets and an arbitrary $u\colon n \times n \to \omega$, there are two different $A, B \in \mathfrak{A}$ such that $c(A(i), B(j)) = u(i,j)$ for all $i, j < n$.*

Proof. Take $k < \omega$, $\mathfrak{A}^*$, and $\{a_i\}_{i<n}$ as in the conclusion of Lemma 2.3.14, and let $D = \{a_i : i < n\}$. Consider the function $t\colon D \times D \to \omega$ defined by $t(a_i, a_j) = u(i,j)$ for all $i, j < n$. Assume that (t_m, D_m, E_m, k_m) is the corresponding sequence in the previous enumeration. By Lemma 2.3.13, there exist different $A, B \in \mathfrak{A}^*$ such that $o(A(i), B(j)) = m$ for all $i, j < n$. It follows that $u(i,j) = t(a_i, a_j) = t_m(A(i) \restriction k_m, B(j) \restriction k_m) = c(A(i), B(j))$. This completes the proof. □

Corollary 2.3.16. *There exists a $\mathfrak{b}$-c.c. partial order whose square is not $\mathfrak{b}$-c.c.* □

The following question is still open.

Question 2.3.17. Can we do the same for some other cardinal invariant such as $\mathfrak{t}$ or $\mathfrak{p}$?

2.4 Partitions of countable ordinals

Oscillations provide the main tool for constructing partitions of pairs of countable ordinals with very strong properties. The goal of this section is to present the construction of an L-space due to Moore [7] which uses oscillations in an ingenious way. In order to motivate this construction we start with a simple example.

For each limit $\alpha < \omega_1$, fix $c_\alpha \subseteq \alpha$ cofinal of order type ω. As before, we will view c_α both as a set and as an ω-sequence which enumerates it in increasing order. Thus, we will write $c_\alpha(n)$ for the n-th element of c_α. The sequence $\langle c_\alpha : \alpha < \omega_1, \lim(\alpha)\rangle$ is called a $\vec{c}$-sequence.

We can generalize the definition of osc as follows: for $f, g \in (\omega_1)^{\leq\omega}$,

$$\operatorname{osc}(f, g) = |\{n < \omega : f(n) \leq g(n) \wedge f(n+1) > g(n+1)\}|.$$

Given a subset S of ω_1 consisting of limit ordinals, let

$$U_S = \{s \in [\omega_1]^{<\omega} : \{\alpha \in S : s \sqsubseteq c_\alpha\} \text{ is stationary}\}.$$

Lemma 2.4.1. *Assume that $S \subseteq \omega_1$ is stationary. Then U_S is an ω_1-superperfect tree.*

Proof. Given $s \in U_S$ let $(U_S)_s = \{t \in U_S : s \sqsubseteq t\}$ and let $\alpha_{s,n} = \sup\{t(n) : t \in (U_S)_s\}$. Then there is n such that $\alpha_{s,n} = \omega_1$. To see this, assume otherwise and let $\alpha = \sup\{\alpha_{s,n} : n < \omega\}$. Then $\alpha < \omega_1$. For each $\delta \in S \setminus (\alpha + 1)$ such that $s \sqsubseteq c_\delta$, let n_δ be the least integer such that $c_\delta(n_\delta) > \alpha$. By the Pressing Down Lemma, there is $t \in [\omega_1]^{<\omega}$ such that $s \sqsubseteq t$ and the set $\{\delta \in S : c_\delta \restriction (n_\delta + 1) = t\}$ is stationary. It follows that $s \sqsubseteq t \in U_S$ and $\max(t) > \alpha$, a contradiction. □

Lemma 2.4.2. *Given two stationary sets $S, T \subseteq \omega_1$, there is $n_0 < \omega$ such that for all $k < \omega$ there exist $\alpha \in S$ and $\beta \in T$ such that* $\operatorname{osc}(c_\alpha, c_\beta) = n_0 + k$.

Proof. By Lemma 2.4.1 both U_S and U_T are ω_1-superperfect. Let s and t be the least ω_1-splitting nodes of U_S and U_T respectively. We may assume that $|s| \leq |t|$ and $s(|s|-1) \leq t(|s|-1)$. Let $n_0 = \operatorname{osc}(s, t) + 1$. Now, as in the proof of Lemma 2.3.3, given an integer k we can find ω_1-splitting nodes s' and t' of U_S and U_T respectively, such that $s \sqsubseteq s'$, $t \sqsubseteq t'$ and $\operatorname{osc}(s', t') = n_0 + k - 1$. Moreover, we can arrange that $|s'| \leq |t'|$ and $s'(|s'| - 1) \leq t'(|s'| - 1)$. Now, pick any $\beta \in T$ such that $t' \sqsubseteq c_\beta$. Since s' is an ω_1-splitting node of U_S, there is $\gamma > \beta$ such that $s'^\frown\gamma \in U_S$. Pick $\alpha \in S$ such that $s'^\frown\gamma \sqsubseteq c_\alpha$. It follows that $\operatorname{osc}(c_\alpha, c_\beta) = \operatorname{osc}(s', t') + 1 = n_0 + k$, as desired. □

We can then improve osc as before to get some better coloring. We know that our coloring cannot be as strong as in the case of $\mathfrak{b}$, since $\mathrm{MA}_{\aleph_1}$ implies that the countable chain condition is productive, so we have to give up some of the properties of our coloring.

We now present a construction of Moore [7] of a coloring of pairs of countable ordinals witnessing $\omega_1 \not\twoheadrightarrow [\omega_1; \omega_1]^2_\omega$ and use it to construct an L-space. As before, we fix a sequence $\langle C_\alpha : \alpha < \omega_1 \rangle$ such that

- if $\alpha = \xi + 1$, then $C_\alpha = \{\xi\}$;
- if α is limit, then $C_\alpha \subseteq \alpha$ is cofinal and of order type ω.

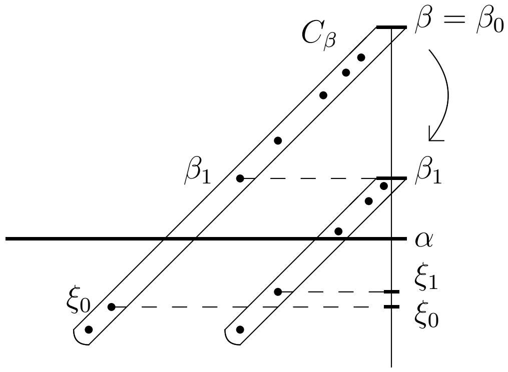
Given $\alpha < \beta$, we define the *walk* from β to α. We first define a sequence $\beta_0 > \beta_1 > \cdots > \beta_l = \alpha$ as follows:

- $\beta_0 = \beta$;
- $\beta_{i+1} = \min(C_{\beta_i} \setminus \alpha)$.

Then we define $\xi_0 \leq \xi_1 \leq \cdots \leq \xi_{l-1}$ by setting

$$\xi_k = \max \bigcup_{j=0}^{k} (C_{\beta_j} \cap \alpha)$$

for all $k \leq l-1$. We call $\mathrm{Tr}(\alpha, \beta) = \{\beta_0, \ldots, \beta_l\}$ the *upper trace* and $L(\alpha, \beta) = \{\xi_0, \ldots, \xi_{l-1}\}$ the *lower trace* of the walk from β to α.



Lemma 2.4.3. *Suppose that* $\alpha \leq \beta \leq \gamma$ *and* $\max(L(\beta, \gamma)) < \min(L(\alpha, \beta))$. *Then* $L(\alpha, \gamma) = L(\alpha, \beta) \cup L(\beta, \gamma)$.

Proof. Since $\max(L(\beta,\gamma)) < \min(L(\alpha,\beta))$, we have $C_\xi \cap \alpha = C_\xi \cap \beta$ whenever ξ is in $\mathrm{Tr}(\beta,\gamma)$ and $\xi \neq \beta$. It follows that $\beta \in \mathrm{Tr}(\alpha,\gamma)$ and

$$\mathrm{Tr}(\alpha,\gamma) = \mathrm{Tr}(\alpha,\beta) \cup \mathrm{Tr}(\beta,\gamma).$$

Assume that $\mathrm{Tr}(\alpha,\gamma) = \{\gamma_0,\dots,\gamma_l\}$ and $L(\alpha,\gamma) = \{\xi_0,\dots,\xi_{l-1}\}$. Then there is $l_0 \leq l$ such that $\gamma_{l_0} = \beta$. Therefore, $\{\xi_k\}_{k\leq l_0-1} = L(\beta,\gamma)$. On the other hand, $\max(C_{\gamma_{l_0}} \cap \alpha) > \xi_{l_0-1}$ because $\xi_{l_0-1} \in L(\beta,\gamma)$ and $\max C_{\gamma_{l_0}} \in L(\alpha,\beta)$. Hence, if $k \geq l_0$, then

$$\xi_k = \max \bigcup_{j=0}^{k} (C_{\gamma_j} \cap \alpha) = \max \bigcup_{j=l_0}^{k} (C_{\gamma_j} \cap \alpha),$$

and so $L(\alpha,\beta) = \{\xi_k\}_{k=l_0}^{l-1}$. □

Lemma 2.4.4. *If $\alpha < \beta$, then $L(\alpha,\beta)$ is a non-empty finite set and, for every limit ordinal β, $\lim\limits_{\alpha\to\beta} \min(L(\alpha,\beta)) = \beta$.*

Proof. The first statement is trivial. Let us prove that $\lim\limits_{\alpha\to\beta} \min(L(\alpha,\beta)) = \beta$ for every limit ordinal β. Given $\alpha < \beta$, one can take $\alpha' \in C_\beta \setminus (\alpha+1)$. Then $\alpha < \alpha' = \max(C_\beta \cap (\alpha'+1)) = \min L(\alpha'+1,\beta) \leq \lim\limits_{\alpha\to\beta} \min(L(\alpha,\beta))$. It follows that $\beta \leq \lim\limits_{\alpha\to\beta} \min(L(\alpha,\beta)) \leq \beta$, and this completes the proof. □

Fix a sequence $\langle e_\alpha : \alpha < \omega_1 \rangle$ satisfying the following conditions:

1. $e_\alpha : \alpha \to \omega$ is finite-to-one;

2. $\alpha < \beta$ implies $e_\beta \upharpoonright \alpha =_* e_\alpha$, i.e., $\{\xi < \alpha : e_\beta(\xi) \neq e_\alpha(\xi)\}$ is finite.

Given $\alpha < \beta < \omega_1$, let $\triangle(\alpha,\beta)$ be the least $\xi < \alpha$ such that $e_\alpha(\xi) \neq e_\beta(\xi)$, if it exists, and α otherwise. We define $\mathrm{osc}(\alpha,\beta)$ as follows:

$$\mathrm{osc}(\alpha,\beta) = |\{i \leq l-1 : e_\alpha(\xi_i) \leq e_\beta(\xi_i) \wedge e_\alpha(\xi_{i+1}) > e_\beta(\xi_{i+1})\}|,$$

where $L(\alpha,\beta) = \{\xi_0 < \cdots < \xi_{l-1}\}$.

It will be convenient to also use the notation $\mathrm{Osc}(e_\alpha, e_\beta, L(\alpha,\beta))$ for the set $\{\xi_i \in L(\alpha,\beta) : e_\alpha(\xi_i) \leq e_\beta(\xi_i) \wedge e_\alpha(\xi_{i+1}) > e_\beta(\xi_{i+1})\}$.

Our aim is to prove the following theorem due to Moore (see [7]).

Theorem 2.4.5 ([7])**.** *Let $A, B \subseteq \omega_1$ be uncountable. Then for all $n < \omega$ there exist $\alpha \in A$, $\beta_0, \beta_1, \dots, \beta_{n-1} \in B$, and k_0 such that $\alpha < \beta_0, \dots, \beta_{n-1}$ and $\mathrm{osc}(\alpha, \beta_m) = k_0 + m$ for all $m < n$.*

This means that we can get arbitrarily long intervals of oscillations with a fixed lower point $\alpha \in A$. We can generalize this to get even more:

Theorem 2.4.6 ([7]). *Given* $\mathfrak{A} \subseteq [\omega_1]^k$ *and* $\mathfrak{B} \subseteq [\omega_1]^l$ *uncountable and pairwise disjoint, and given* $n < \omega$, *we can find* $A \in \mathfrak{A}$ *and* $B_0, \ldots, B_{n-1} \in \mathfrak{B}$ *such that* $\max A < \min B_i$ *for all* $i < n$, *and*

$$\text{osc}(A(i), B_m(j)) = \text{osc}(A(i), B_0(j)) + m$$

for all $i < k$, $j < l$ *and* $m < n$.

In order to prove Theorem 2.4.5, we demonstrate the following lemma.

Lemma 2.4.7. *Let* $A, B \subseteq \omega_1$ *be uncountable. There exists a club* $C \subseteq \omega_1$ *such that if* $\delta \in C$, $\alpha \in A \setminus \delta$, $\beta \in B \setminus \delta$, *and* $R \in \{=, >\}$, *then there are* $\alpha' \in A \setminus \delta$ *and* $\beta' \in B \setminus \delta$ *satisfying the following properties:*

1. $\max L(\alpha, \beta) < \triangle(\alpha, \alpha'), \triangle(\beta, \beta')$;
2. $L(\delta, \beta) \sqsubseteq L(\delta, \beta')$;
3. *for all* $\xi \in L^+ = L(\delta, \beta') \setminus L(\delta, \beta)$, *we have* $e_{\alpha'}(\xi) \; R \; e_{\beta'}(\xi)$.

Proof. Fix a sufficiently large regular cardinal θ. We are going to show that, if $M \prec H_\theta$ is a countable elementary substructure containing all the relevant objects, then $\delta = M \cap \omega_1$ satisfies the conclusion of the lemma. Since the set of such δ contains a club in ω_1, this will be sufficient. Thus, fix M and δ as above and let α and β be as in the hypothesis of the lemma. We first suppose that R is $=$. Since δ is a limit ordinal, we can take $\gamma_0 < \delta$ such that

1. $\max(L(\delta, \beta)) < \gamma_0$, and
2. for all $\xi \in (\gamma_0, \delta)$, $e_\alpha(\xi) = e_\beta(\xi)$.

By Lemma 2.4.4, we can also fix $\gamma < \delta$ such that $\gamma_0 < \min L(\xi, \delta)$ for all $\xi \in (\gamma, \delta)$. Let D be the set of all $\delta' < \omega_1$ such that for some $\alpha' \in A \setminus \delta'$ and $\beta' \in B \setminus \delta'$ the following properties are satisfied:

(a) $e_{\alpha'} \restriction \gamma_0 = e_\alpha \restriction \gamma_0$, $e_{\beta'} \restriction \gamma_0 = e_\beta \restriction \gamma_0$;

(b) $L(\delta', \beta') = L(\delta, \beta)$;

(c) for all $\xi \in (\gamma, \delta')$, $\gamma_0 < \min L(\xi, \delta')$;

(d) for all $\xi \in (\gamma_0, \delta')$, $e_{\alpha'}(\xi) = e_{\beta'}(\xi)$.

For all $\xi \geq \gamma_0$, $e_\xi \restriction \gamma_0$ is in M, since, by definition, $e_\xi \restriction \gamma_0 =_* e_{\gamma_0}$. This means that D is definable in M; hence $D \in M$. Moreover $D \not\subseteq M$ (since $\delta \in D$), and therefore D is uncountable. Choose $\delta' > \delta$ in D with $\alpha' \in A \setminus \delta'$ and $\beta' \in B \setminus \delta'$ witnessing $\delta' \in D$. By condition (a) of the definition of D,

$$\gamma_0 \leq \triangle(\alpha, \alpha'), \triangle(\beta, \beta').$$

Put $L^+ = L(\delta, \delta')$. Then $\max L(\delta, \beta) = \max L(\delta', \beta') < \min L^+$; hence

$$L(\delta, \beta') = L(\delta', \beta') \cup L^+ = L(\delta, \beta) \cup L^+.$$

Given $\xi \in L^+$, by condition (c) we have $\gamma_0 < \min L^+ \leq \xi$. It follows that $\xi \in (\gamma_0, \delta')$, so (d) implies that $e_{\alpha'}(\xi) = e_{\beta'}(\xi)$.

Now assume that R is $>$. Let E be the set of all limits $\nu < \omega_1$ such that, for all $\alpha_0 \in A \setminus \nu$, $\nu_0 < \nu$, $\varepsilon < \omega_1$, $n < \omega$ and finite $L^+ \subseteq \omega_1 \setminus \nu$, there exists $\alpha_1 \in A \setminus \varepsilon$ with $\nu_0 \leq \triangle(\alpha_0, \alpha_1)$ and $e_{\alpha_1}(\xi) > n$ for all $\xi \in L^+$. Since E is definable from parameters in M, it follows that $E \in M$ as well.

Claim 2.4.8. *The ordinal δ is in E. In particular, E is uncountable.*

Proof. Let α_0, ν_0, ε, n, L^+ be given as in the definition of E for $\nu = \delta$. Since e_{α_0} is finite-to-one, we can assume without loss of generality that $\nu_0 > \sup\{\xi < \delta : e_{\alpha_0}(\xi) \leq n\}$. By the elementarity of M, there exists δ' bigger than ε, δ and $\max L^+$, and $\alpha_1 \in A \setminus \delta'$, such that the following conditions hold:

- $e_{\alpha_0} \restriction \nu_0 = e_{\alpha_1} \restriction \nu_0$;
- for all ξ in (ν_0, δ'), we have $e_{\alpha_1}(\xi) > n$.

Since $L^+ \subseteq \delta' \setminus \delta$, this completes the proof of the claim. □

Now apply the elementarity of M and the fact that E is uncountable to find $\gamma_0 \in E$ such that $L(\delta, \beta) < \gamma_0 < \delta$. By Lemma 2.4.4, we can find $\gamma < \delta$ such that if $\xi \in (\gamma, \delta)$ then $\gamma_0 < L(\xi, \delta)$. Again by the elementarity of M, we may select $\delta' > \delta$ and $\beta' \in B \setminus \delta'$ such that the following conditions hold:

- $e_{\beta'} \restriction \gamma_0 = e_\beta \restriction \gamma_0$;
- $L(\delta', \beta') = L(\delta, \beta)$;
- $\gamma < \xi < \delta'$ implies $\gamma_0 < L(\xi, \delta')$.

If $L^+ = L(\delta, \delta')$, then $L^+ \subseteq \omega_1 \setminus \gamma_0$. Since $\gamma_0 \in E$, we can apply the definition of E with $\nu_0 = \max L(\delta, \beta) + 1$, $n = \max\{e_{\beta'}(\xi) : \xi \in L^+\}$ to find $\alpha' \in A \setminus \delta$ such that, for all $\xi \in L^+$, $\max L(\delta, \beta) < \triangle(\alpha, \alpha')$ and $e_{\alpha'}(\xi) > e_{\beta'}(\xi)$. This completes the proof of Lemma 2.4.7. □

We can finally prove Theorem 2.4.5.

Proof of Theorem 2.4.5. Let $A, B \subseteq \omega_1$ be uncountable sets and let $M \prec H_{\aleph_2}$ be a countable substructure containing everything relevant with $\delta = M \cap \omega_1$. Since M contains A and B, the club C provided by Lemma 2.4.7 is in M. Use Lemma 2.4.7 to select $\alpha_0, \alpha_1, \ldots, \alpha_n, \ldots$ in $A \setminus \delta$, $\beta_0, \beta_1, \ldots, \beta_n, \ldots$ in $B \setminus \delta$ and $\xi_0, \xi_1, \ldots, \xi_n, \ldots$ in δ such that for all $n < \omega$ the following conditions are satisfied:

1. $L(\delta, \beta_n) \sqsubset L(\delta, \beta_{n+1})$;
2. $\xi_n \in L(\delta, \beta_{n+1}) \setminus L(\delta, \beta_n)$;
3. $\mathrm{Osc}(e_{\alpha_{n+1}}, e_{\beta_{n+1}}, L(\delta, \beta_{n+1})) = \mathrm{Osc}(e_{\alpha_n}, e_{\beta_n}, L(\delta, \beta_n)) \cup \{\xi_n\}$;
4. if $m > n$, then $\xi_n < \triangle(\alpha_m, \alpha_{m+1}), \triangle(\beta_m, \beta_{m+1})$;
5. $e_{\alpha_n}(\max L(\delta, \beta_n)) > e_{\beta_n}(\max L(\delta, \beta_n))$.

Suppose that α_n and β_n have been defined. We obtain α_{n+1} and β_{n+1} by applying Lemma 2.4.7 twice: first with R being $=$, second with R being $>$. If α' and β' are the two ordinals obtained by applying the lemma the first time, then $\xi_n = \min(L(\delta, \beta_{n+1}) \setminus L(\delta, \beta'))$.

Now let n be given, and pick $\gamma_0 < \delta$ such that

$$\gamma_0 > \max L(\delta, \beta_n),\ \max\{\xi < \delta : \exists m, m' \leq n\, (e_{\beta_m}(\xi) \neq e_{\beta_{m'}}(\xi))\}.$$

Using the elementarity of M and Lemma 2.4.4, select $\alpha \in A \cap \delta$ such that

$$\max L(\delta, \beta_n) < \triangle(\alpha, \alpha_n) \text{ and } \gamma_0 < \min L(\alpha, \delta).$$

Now let $m < n$ be fixed. It follows from Lemma 2.4.3 that

$$L(\alpha, \beta_m) = L(\alpha, \delta) \cup L(\delta, \beta_m).$$

Finally, $e_{\beta_m} \restriction L(\alpha, \delta)$ does not depend on m, since

$$\min L(\alpha, \delta) > \gamma_0 > \max\{\xi < \delta : \exists m, m' \leq n\, (e_{\beta_m}(\xi) \neq e_{\beta_{m'}}(\xi))\}.$$

Therefore,

$$\mathrm{Osc}(e_\alpha, e_{\beta_0}, L(\alpha, \delta)) = \mathrm{Osc}(e_\alpha, e_{\beta_m}, L(\alpha, \delta)).$$

By 5, $\mathrm{Osc}(e_\alpha, e_{\beta_m}, L(\alpha, \beta_m)) = \mathrm{Osc}(e_\alpha, e_{\beta_m}, L(\alpha, \delta)) \cup \mathrm{Osc}(e_\alpha, e_{\beta_m}, L(\delta, \beta_m))$, so, by 3, $\mathrm{Osc}(e_\alpha, e_{\beta_m}, L(\alpha, \beta_m)) = \mathrm{Osc}(e_\alpha, e_{\beta_0}, L(\alpha, \beta_0)) \cup \{\xi_{m'}; m' < m\}$. Hence $\mathrm{osc}(\alpha, \beta_m) = \mathrm{osc}(\alpha, \beta_0) + m$ and this completes the proof. □

By using the previous results we can, finally, prove the existence of an L-space, that is, a regular Hausdorff space which is hereditarily Lindelöf but not hereditarily separable. We will work in $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. We fix a sequence $\langle z_\alpha : \alpha < \omega_1 \rangle$ of rationally independent elements of $\mathbb{T}$. It is easy to find such a sequence, since, given any countable rationally independent subset I of $\mathbb{T}$, there are only countable many z for which $I \cup \{z\}$ is rationally dependent. Consider now the function defined as

$$o(\alpha, \beta) = z_\alpha^{\operatorname{osc}(\alpha,\beta)+1}$$

for all $\alpha < \beta < \omega_1$.

We will use *Kronecker's Theorem* (see [6] or [8]), which is the following statement:

Theorem 2.4.9. *Suppose that $\langle z_i \rangle_{i<k}$ is a sequence of elements of $\mathbb{T}$ which are rationally independent. For every $\epsilon > 0$, there is $n_\epsilon \in \mathbb{N}$ such that, if $u, v \in \mathbb{T}^k$, then there is $m < n_\epsilon$ such that*

$$|u_i z_i^m - v_i| < \epsilon$$

for all $i < k$.

Now we can define the L-space. For every $\beta < \omega_1$, we define a function $w_\beta \colon \omega_1 \to \mathbb{T}$ as follows:

$$w_\beta(\xi) = \begin{cases} o(\xi, \beta) & \text{if } \xi < \beta; \\ 1 & \text{otherwise.} \end{cases}$$

Let $\mathscr{L} = \{w_\beta : \beta < \omega_1\}$, viewed as a subspace of $\mathbb{T}^{\omega_1}$.

Remark 2.4.10. $\mathscr{L}$ is not separable.

For all $X \subseteq \omega_1$, let $\mathscr{L}_X = \{w_\beta \restriction X : \beta \in X\}$, viewed as a subspace of $\mathbb{T}^X$. We will simply write w_β for $w_\beta \restriction X$ when referring to elements of $\mathscr{L}_X$. Our aim is to prove that $\mathscr{L}_X$ is an L-space for every X uncountable.

Lemma 2.4.11. *Let $\mathscr{A} \subseteq [\omega_1]^k$ and $\mathscr{B} \subseteq [\omega_1]^l$ be uncountable families of pairwise disjoint sets. For every sequence $\langle U_i \rangle_{i<k}$ of open neighborhoods in $\mathbb{T}$ and every $\phi \colon k \to l$, there are $a \in \mathscr{A}$ and $b \in \mathscr{B}$ such that* $\max(a) < \min(b)$ *and, for all $i < k$,*

$$o(a(i), b(\phi(i))) \in U_i.$$

Proof. We may assume without loss of generality that every U_i is an ϵ-ball about a point v_i, for some fixed $\epsilon > 0$. We can also assume that the integer n_ϵ of Kronecker's Theorem for the sequence $\langle z_{a(i)} \rangle_{i<k}$ is uniform for $a \in \mathscr{A}$. Apply Theorem 2.4.6 to find $a \in \mathscr{A}$ and a sequence $\langle b_m \rangle_{m<n_\epsilon}$ of elements of $\mathscr{B}$ such that

$$\max(a) < \min(b_m),$$
$$\operatorname{osc}(a(i), b_m(j)) = \operatorname{osc}(a(i), b_0(j)) + m,$$

for all $i < k$, $j < l$ and $m < n_\epsilon$. For each $i < k$, put $u_i = o(a(i), b_0(\phi(i)))$. There is an $m < n_\epsilon$ such that

$$|u_i z_{a(i)}^m - v_i| < \epsilon$$

for all $i < k$ or, equivalently, $o(a(i), b_m(\phi(i))) \in U_i$. This fact completes the argument. □

Lemma 2.4.12. *If $X, Y \subseteq \omega_1$ have countable intersection, then there is no continuous injection from any uncountable subspace of $\mathscr{L}_X$ into $\mathscr{L}_Y$.*

Proof. Suppose, by way of contradiction, that such an injection g does exist. Then there are an uncountable set $X_0 \subseteq X$ and an injection $f \colon X_0 \to Y$ such that $g(w_\beta) = w_{f(\beta)}$. We may assume without loss of generality that X_0 is disjoint from Y. For each $\xi < \omega_1$, let $\beta_\xi \in X_0$ and $\zeta_\xi \in Y$ be such that $f(\beta_\xi) > \zeta_\xi$ and, if $\xi < \xi'$, then $\beta_\xi < \zeta_{\xi'}$. Let $\Xi \subseteq \omega_1$ be uncountable and such that for every $\xi \in \Xi$ there is an open neighborhood V in $\mathbb{T}$ such that $g(w_{\beta_\xi})(\zeta_\xi) \notin \bar{V}$. Let $W_\xi = \{w \in \mathscr{L}_Y : w(\zeta_\xi) \notin \bar{V}\}$, for all $\xi < \omega_1$. Since g is continuous at w_{β_ξ}, there is a basic open neighborhood U_ξ of w_{β_ξ} such that $U_\xi \subseteq g^{-1} W_\xi$. Using the Δ-system lemma and the second countability of $\mathbb{T}$, we can find an uncountable $\Xi' \subseteq \Xi$, a sequence of open neighborhoods $\langle U_i \rangle_{i<k}$ in $\mathbb{T}$, and $a_\xi \in [X]^k$ such that, for all $\xi \in \Xi'$, the following conditions hold:

- $\{a_\xi\}_{\xi \in \Xi'}$ is a Δ-system with root a;
- $w_{\beta_\xi} \in \{w \in \mathscr{L}_X : \forall i < k\, (w(a_\xi(i)) \in U_i)\} \subseteq U_\xi$;
- the inequality $\beta_\xi < f(\beta_\xi)$ does not depend on ξ;
- $|\zeta_\xi \cap a_\xi|$ does not depend on ξ.

Let $\mathscr{A} = \{a_\xi \cup \{\xi\} \setminus a\}_{\xi \in \Xi'}$ and $\mathscr{B} = \{\beta_\xi, f(\beta_\xi)\}_{\xi \in \Xi'}$. By applying Lemma 2.4.11 we can find $\xi < \xi'$ in Ξ' such that, for all $i < k$,

$$\begin{aligned} \max(a_\xi \cup \{\zeta_\xi\}) &< \min(\beta_{\xi'}, f(\beta_{\xi'})), \\ w_{\beta_{\xi'}}(a_\xi(i)) &= o(a_\xi(i), \beta_{\xi'}) \in U_i, \\ g(w_{\beta_{\xi'}}) &= w_{f(\beta_{\xi'})}(\zeta_\xi) = o(\zeta_\xi, f(\beta_{\xi'})) \in V. \end{aligned}$$

Thus we have that $w_{\beta_{\xi'}} \in U_\xi$ and $g(w_{\beta_{\xi'}}) \notin W_\xi$, a contradiction. □

Theorem 2.4.13 ([7]). *For every X, $\mathscr{L}_X$ is hereditarily Lindelöf.*

Proof. If not, then $\mathscr{L}_X$ would contain an uncountable discrete subspace. Moreover it would be possible to find disjoint $Y, Z \subseteq X$ such that $\mathscr{L}_Y$ and $\mathscr{L}_Z$ contain uncountable discrete subspaces. It is well known that any function from a discrete space to another discrete space is continuous, and this contradicts Lemma 2.4.12. □

Corollary 2.4.14 ([7]). *There exists an L-space, i.e., a hereditarily Lindelöf non-separable T_3 topological space.* □

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