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An elementary multidimensional fundamental theorem of calculus

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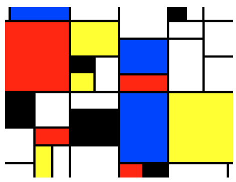
We discuss a version of the fundamental theorem of calculus in several variables and some applications, of potential interest as a teaching material in undergraduate courses.



1 Introduction and main result

In standard undergraduate courses on one variable calculus, differentiation and integration are presented as inverse processes, as stated in the *fundamental theorem of calculus*: if F is differentiable at every point $x \in [a, b]$ and $F' = f$ is integrable in $[a, b]$, then

$$F(x) - F(a) = \int_a^x f(t)\, dt, \quad a \leq x \leq b.$$

This holds both in the context of Riemann and Lebesgue integration (see [4] for a proof in the Lebesgue integration context).

In this note we provide a multidimensional version of this statement. The proof is straightforward and can be included in an undergraduate course of multidimensional calculus.

The first basic concept is that of *interval or cube function* Φ on a domain $U \subset \mathbb{R}^n$. By an interval Q in U we understand a set $Q = \prod_{j=1}^n I_j$ with all the I_j one-dimensional closed intervals of the same length. Notice that all the faces of Q are parallel to the coordinate axis. An interval function is a map defined on all intervals $Q \subset \mathbb{R}^n$ assigning to each Q a real or complex number $\Phi(Q)$ with the property that

$$\Phi(Q) = \sum_i \Phi(Q_i),$$

whenever (Q_i) is a finite partition of Q, that is, $Q = \cup Q_i$ and the Q_i have disjoint interiors. The easiest example is

$$\Phi_f(Q) = \int_Q f\,dx,$$

where f is locally integrable.

We have in mind two other examples. For the first one, assume that $T : U \to V$ is a measurable homeomorphism between two domains such that $m(T(A)) = 0$ if $m(A) = 0$. Here and in the following $m(A)$ denotes the Lebesgue measure of A. Then $\Phi(Q) = m(T(Q))$ is an interval function, because images of the faces have zero measure. For the second one assume that F is a continuous vector field in the plane or in space and set

$$\Phi(Q) = \int_{\partial Q} \langle F, N\rangle\, dm_{n-1},$$

the flow of F through the boundary ∂Q oriented with the outward normal N. Then $\Phi(Q)$ is an interval function. This is because if Q_i, Q_j are two intervals with a face S in common, the outward normals are opposite one each other.

Notice that a Dirac delta at a point $a \in U$, that is, $\Phi(Q) = 1$ if $a \in Q$ and zero otherwise, is not an interval function according to our definition, because if a is a boundary point of both Q_1, Q_2 then $\Phi(Q_1) = \Phi(Q_2) = \Phi(Q) = 1$.

In dimension $n = 1$, with $U = (a, b)$, if g is defined on (a, b), it is immediately seen that

$$\Phi([c, d]) = g(d) - g(c), \tag{1}$$

defines an interval function on (a, b). Indeed, a decomposition of $[c, d]$ into pieces Q_i amounts to a selection of intermediate points (the end-points of the Q_i) $c = t_0 < t_1 < \cdots < t_m = d$, and then

$$\Phi(I) = g(d) - g(c) = \sum_i g(t_{i+1}) - g(t_i) = \sum_i \Phi(Q_i).$$

Conversely, given an interval function defined on (a, b) and $p \in (a, b)$, the function

$$g(x) = \begin{cases} \Phi([p, x]) & p \le x, \\ -\Phi([x, p]) & x \le p, \end{cases}$$

satisfies (1). Thus there is an one-to-one correspondence between interval functions and classical functions.

The second basic concept is that of density. For an interval function Φ we define its *upper density*

$$\overline{D}_\Phi(x) = \limsup_{x\in Q, \delta(Q)\to 0} \frac{\Phi(Q)}{m(Q)} = \inf_\varepsilon \sup_{\delta(Q)\le\varepsilon, x\in Q} \frac{\Phi(Q)}{m(Q)},$$

where $m(Q)$ denotes the measure of Q and $\delta(Q)$ its diameter. Analogously the lower density is defined

$$\underline{D}_\Phi(x) = \liminf_{x\in Q,\delta(Q)\to 0} \frac{\Phi(Q)}{m(Q)} = \sup_\varepsilon \inf_{\delta(Q)\le\varepsilon, x\in Q} \frac{\Phi(Q)}{m(Q)}.$$

In case both are finite and equal we say that Φ has a *finite density* $D_\Phi(x)$ at x.

For instance, if f is continuous in U, the density of Φ_f is f at all points. Indeed, given $\varepsilon > 0$ there is τ such that $|f(y) - f(x)| \le \varepsilon$ if $|x - y| \le \tau$. Then, if $\delta(Q) < \tau$, $x \in Q$ one has $|f(y) - f(x)| \le \varepsilon$ for all $y \in Q$ so

$$\begin{aligned}\left|\frac{\Phi_f(Q)}{m(Q)} - f(x)\right| &= \left|\frac{1}{m(Q)}\int_Q (f(y) - f(x))\,dy\right| \\ &\le \frac{1}{m(Q)}\int_Q |f(y) - f(x)|\,dy \le \varepsilon.\end{aligned}$$

Thus

$$\lim_{\delta(Q)\to 0} \frac{\Phi_f(Q)}{m(Q)} = f(x).$$

A deeper result is Lebesgue's differentiation theorem (see [3]) stating that Φ_f has density $f(x)$ at almost all points $x \in U$ under the sole assumption that f is locally integrable.

For a better understanding of the density consider the following example. Assume $U = (0,1)\times(0,1)$ and let $L = \{(x,x), 0 < x < 1\}$ be the diagonal. Define $\Phi(Q)$ as the length of $L \cap Q$, clearly an interval function. Then $D_\Phi(x) = 0$ for $x \notin L$ while $\underline{D}_\Phi(x) = 0$, $\overline{D}_\Phi(x) = +\infty$ for $x \in L$.

In dimension one, if Φ is given by (1), Φ has a finite density at x if and only g is differentiable at x, because if $x \in [c,d]$

$$\frac{g(d) - g(c)}{d - c} = \frac{g(d) - g(x)}{d - x}\,\frac{d - x}{d - c} + \frac{g(x) - g(c)}{x - c}\,\frac{x - c}{d - c}.$$

Next elementary theorem seems to be unnoticed, to the best of author's knowledge. It holds both in the context of Riemann and Lebesgue's integration.

Theorem 1.1. *If an interval function Φ has a finite upper density $\overline{D}_\Phi$* **at every point** *and $\overline{D}_\Phi$ is locally integrable, then for every cube $Q \subset U$*

$$\Phi(Q) \le \int_Q \overline{D}_\Phi(x)\,dx.$$

Analogously, if Φ has a finite lower density $\underline{D}_\Phi$ **at every point** *and $\underline{D}_\Phi$ is locally integrable, then*

$$\Phi(Q) \ge \int_Q \underline{D}_\Phi(x)\,dx.$$

Thus,

$$\Phi(Q) = \int_Q D_\Phi(x)\,dx,$$

whenever Φ *has a finite integrable density at every point.*

In an informal way, if $\Phi(Q)$ is of the order of $f(x)\,m(Q)$ for infinitesimal cubes $x \in Q$, then $\Phi(Q) = \int_Q f(x)\,dx$ for big cubes.

In dimension one, in view of the remark before the theorem, this is the fundamental theorem of calculus stated in the beginning.

As a corollary we may state:

Corollary. *For an interval function* Φ *and a continuous function* f *on* U *the following two statements are equivalent:*

$$\lim_{x\in Q,\delta(Q)\to 0} \frac{\Phi(Q)}{m(Q)} = f(x), \quad \Phi(Q) = \int_Q f(x)\,dx.$$

We point out some remarks. First, it is essential, as in one variable, that the density is assumed to exist at **every** point. If it exists just a.e. then the theorem does not hold. Secondly, in other type of results the a.e. existence of the density is actually **proved** like in Lebesgue's differentiation theorem quoted before. In fact, the interval functions Φ_f are characterized as those being *absolutely continuous*, meaning that for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\sum_i |\Phi(Q_i)| < \varepsilon$ whenever Q_i are non-overlapping cubes and $\sum_i m(Q_i) < \delta$. So the result can be rephrased by saying that interval functions having finite integrable density at all points are automatically absolutely continuous. A reference for all these results is [3].

2 Proof

Let $Q \subset U$ be a cube and let us break it into 2^n cubes S_i of equal measure. Since $\Phi(Q) = \sum \Phi(S_i)$, one has

$$\Phi(S_i) \geq \frac{\Phi(Q)}{2^n},$$

whence

$$\frac{\Phi(S_i)}{m(S_i)} \geq \frac{\Phi(Q)}{m(Q)},$$

for at least one i. Repeating the argument we find a sequence Q_k of cubes, $Q_k \subset Q$, shrinking to some point $p \in Q$ such that

$$\frac{\Phi(Q_k)}{m(Q_k)} \geq \frac{\Phi(Q)}{m(Q)}.$$

Therefore $\overline{D}_\Phi(p) \geq \frac{\Phi(Q)}{m(Q)}$. So $\Phi(Q) \leq \overline{D}_\Phi(p)\, m(Q)$ for some point $p \in Q$. Similarly, $\Phi(Q) \geq \underline{D}_\Phi(p)\, m(Q)$ for some point. This holds for all cubes. So $\Phi \leq 0$ if $\overline{D}_\Phi \leq 0$. Similarly, $\Phi \geq 0$ if $\underline{D}_\Phi \geq 0$.

We complete now the proof in the Riemann integration context, where just the definition of the Riemann integral is used.

Let (Q_i) be a partition of Q; then

$$\Phi(Q) = \sum_i \Phi(Q_i) \leq \sum_i \overline{D}_\Phi(p_i) m(Q_i).$$

Therefore, if $\overline{D}_\Phi$ is Riemann integrable, it follows that

$$\Phi(Q) \leq \int_Q \overline{D}_\Phi(x)\, dx.$$

In a similar way we see that

$$\Phi(Q) \geq \int_Q \underline{D}_\Phi(x)\, dx,$$

and so the theorem is proved when the density is Riemann integrable.

Assume now that $\overline{D}_\Phi$ is Lebesgue integrable on U. We may assume Φ real-valued and use semi-continuous functions as in [4]. Recall that a function g is called lower semi-continuous at a point p if $\liminf_{x\to p} g(x) \geq g(p)$ and upper semi-continuous if $\liminf_{x\to p} g(x) \leq g(p)$.

Given $\varepsilon > 0$, by the Vitali-Carathedory theorem (see [4]), there is a lower semi-continuous function v such that $\overline{D}_\Phi \leq v$ and $\int_U (v - \overline{D}_\Phi)\, dx < \varepsilon$. Define

$$\Psi(Q) = \int_Q v\, dx - \Phi(Q).$$

Then, v being lower semi-continuous,

$$\begin{aligned}\underline{D}_\Psi(x) = \liminf_{x\in Q} \frac{\Psi(Q)}{m(Q)} &\geq \liminf_{x\in Q} \frac{1}{m(Q)} \int_Q v\, dx - \limsup_{x\in Q} \frac{\Phi(Q)}{m(Q)}) \\ &\geq v(x) - \overline{D}_\Phi(x) \geq 0.\end{aligned}$$

Therefore $\Psi(Q) \geq 0$, whence

$$\Phi(Q) \leq \int_Q v\, dx = \int_Q \overline{D}_\Phi\, dx + \int_Q (v - \overline{D}_\Phi)\, dx < \int_Q \overline{D}_\Phi\, dx + \varepsilon.$$

Since ε is arbitrary, this shows that $\Phi(Q) \leq \int_Q \overline{D}_\Phi\, dx$ and applying the same argument to $-\Phi$ we are done.

3 Applications

1. As a first application we indicate a simplified proof of a version of the change of variables formula, with minimal assumptions and not relying in the one-dimensional version and Fubini's theorem, the one stated in Theorem 7.26 in [4]:

Theorem 3.1. *Let $T : U \to V$ be an homeomorphism between two domains in $\mathbb{R}^n$, differentiable at every point $x \in U$. Assume that $|\det dT(x)|$ is integrable on U. Then for a positive measurable function f in V one has*

$$\int_V f(y)\,dy = \int_U f(T(x))\ |\det dT(x)|\ dx. \tag{2}$$

Note that the assumption $|\det dT(x)| \neq 0$ is not made, so this version includes Sard's theorem.

We modify the proof in [4] replacing the more advanced Radon-Nikodym differentiation theorem for absolutely continuous measures by theorem 1.1.

First, lemma 7.25 in [4] proves that T maps sets of measure zero to sets of measure zero. As a consequence,

$$\Phi(Q) = m(T(Q))$$

is an interval function.

Secondly, theorem 7.24 in [4] proves that Φ has density $|\det dT(p)|$ at every point p. In fact, the proof in [4] uses balls, but it is easily checked that it holds for cubes too. We explain the basic idea for completeness. By hypothesis, we can approximate $T(p+h)$ near p by $L = L(p+h) = T(p) + dT(p)(h)$, and use that $m(L(Q)) = |\det L|\, m(Q)$ for all affine maps.

One has

$$T(p+h) = L + E,\ |E| \leq \tau(|h|)|h|,\ \tau(h) \to 0.$$

with decreasing $\tau(t)$ as $t \to 0$.

Let $v_j = \frac{\partial T}{\partial x_j}(p),\ j = 1, \cdots, n$ be the columns of $dT(p)$. If Q has side δ, L maps Q onto a parallelepiped P with spanning vectors δv_j, whose measure is

$$m(P) = |\det dT(p)|\delta^n = |\det dT(p)| m(Q),$$

so let us compare $T(Q)$ with $P = L(Q)$. Since $|T - L| = |E| \leq \tau(|h|)|h|$, it is clear that $T(Q)$ is included in a parallelepiped P_1 concentric with P with spanning vectors $(\delta + o(\delta))v_j$, whose measure is

$$\delta^n |\det dT(p)| + o(\delta^n).$$

Again by $|T - L| = |E| \leq \tau(|h|)|h|$, the boundary $b(T(Q)) = T(bQ)$ is at distance less than $\tau(\delta)\delta$ from bP. Since T is an homeomorphism, this implies

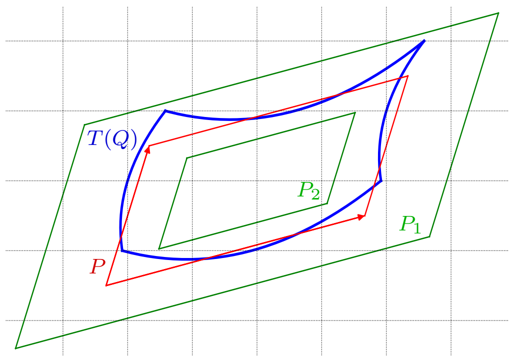


Figure 1

that $T(Q)$ contains a parallelepiped P_2 concentric with P with spanning vectors $(\delta - o(\delta))v_j$ (see figure 1 for $n = 2$; a rigorous proof of this fact relies on Brouwer's fixed point theorem and can be found in lemma 7.23 of [4]), whose measure is

$$\delta^n |\det dT(p)| - o(\delta^n).$$

Altogether, since $m(Q) = \delta^n$,

$$m(Q)\ |\det dT(p)| - o(m(Q)) \leq m(T(Q)) \leq m(Q)\ |\det dT(p)| + o(m(Q)),$$

proving that Φ has density $|\det dT(p)|$ at p.

By theorem 1.1, one has

$$m(T(A)) = \int_A |\det dT(x)|\, dx,$$

when A is a cube, whence when A is a finite union of cubes too. Since every open set is a countable union of cubes, by the monotone convergence theorem this holds when A is an open set and in turn when A is a countable intersection of open sets, a G_δ set. Since every measurable set differs from s G_δ set in a set of zero measure and T preserves those, we conclude that this holds for all measurable sets $A \subset U$, that is, (2) holds for the characteristic function of a measurable set. By linearity it then holds for simple functions, and by the monotone convergence theorem again, for a general measurable function.

Remark. As a first remark for the instructor, in case $|\det dT(x)| \neq 0$ for all $x \in U$, the use of Brouwer's fixed point theorem can be avoided as follows:

By the inclusion $T(Q) \subset P_1$,

$$m(T(Q)) \leq m(Q)\ |\det dT(p)| + o(m(Q)),$$

implying $\overline{D}_\Phi(p) \leq |\det dT(p)|$. Then theorem 1.1 implies

$$m(T(A)) \leq \int_A |\det dT(x)|\,dx,$$

for all cubes, and as before this leads to

$$\int_V f(y)\,dy \leq \int_U f(T(x))\,|\det dT(x)|\,dx.$$

But since the same inequality applies to the inverse f^{-1}, the result follows.

Remark. As a second remark, to be eventually combined with the previous one, a proof in the context of Riemann integration can be further simplified as follows. To prove (2) say for a continuous function f with compact support, introduce

$$\Psi(Q) = \int_{T(Q)} f\,dy.$$

The continuity of f implies

$$D_\Psi(p) = f(T(p))D_\Phi(p),$$

so $D_\Psi(p) = f(T(p))\,|\det dT(p)|$. This leads using theorem 1.1 to

$$\int_{T(Q)} f(y)\,dy = \int_Q f(T(x))\,|\det dT(x)|\,dx,$$

for all cubes. If K is the support of f, the compact $T^{-1}(K)$ can be covered by a finite number of cubes Q, so (2) follows.

2. As a second application we analyze the divergence theorem. Assume that F is a continuous vector field in space and set

$$\Phi(Q) = \int_{\partial Q} \langle F, N\rangle\,dm_{n-1},$$

the flow of F through the boundary ∂Q oriented with the outward normal N. We mentioned before that Φ is indeed an interval function. If its density exists, we call it *the divergence* $\operatorname{div} F$ of F. If integrable, the theorem implies

$$\int_{bQ} \langle F, N\rangle\,dm_{n-1} = \int_Q \operatorname{div} F\,dm_n,$$

and the same holds with Q replaced by a finite union of cubes. From this it follows by approximations that the same holds with Q replaced by a domain with piece-wise regular boundary (details can be found in [1]).

If F is differentiable with components F_i, let us check that the density $\operatorname{div} F$ exists at every point and equals $\langle \nabla, F\rangle = \sum_i D_i F_i$.

First consider an affine field $F(x) = M(X - P)$, where $M = (m_{ij})$ is a constant matrix and X, P are the column vectors x^t, p^t, and let us compute the flux across the boundary ∂Q of a parallelepiped in space spanned by 3 vectors v_1, v_2, v_3

$$Q = \{p' + t_1 v_1 + t_2 v_2 + t_3 v_3,\ 0 \le t_i \le 1\},$$

containing p, oriented by the outward normal. F differs from $M(X - P')$ by a constant field, which obviously has zero flux, so we can replace p by p' and assume $p' = 0$. On the face $t_3 = 1$, the basis v_1, v_2 is positively oriented and the flux is

$$\int_0^1 \int_0^1 \det(M(t_1 v_1 + t_2 v_2 + v_3), v_1, v_2)\, dt_1\, dt_2,$$

while on the face $t_3 = 0$ it is

$$-\int_0^1 \int_0^1 \det(M(t_1 v_1 + t_2 v_2), v_1, v_2)\, dt_1\, dt_2.$$

Therefore they add up to

$$\det(M(v_3), v_1, v_2).$$

If $M(v_3) = \sum_i \lambda_i v_i$, this equals $\lambda_3 \det(v_3, v_1, v_2)$. The same applies to the other two couples of opposite sides, whence the flux is exactly

$$\operatorname{trace}(M) \det(v_1, v_2, v_3),$$

the trace of M times the volume of Q.

Now let F be a differentiable field at p, Q a cube of size δ containing p. As before we expand F around p

$$F(x) = F(p) + dF(p)(X - P) + E, \quad E = o(|x - p|).$$

The contribution to the flux of F across ∂Q of the constant field $F(p)$ is zero, that of the linear field $dF(p)(X - P)$ is the trace of $dF(p)$ times $m(Q)$ while that of E is $o(\delta^n)$, whence the flux equals

$$(D_1 F_1 + \cdots + D_n F_n)(p) m(Q) + o(m(Q)),$$

thus proving that the density is $\langle \nabla, F \rangle$.

Upon replacement of the field $F = (A, B)$ by $JF = (-B, A)$ the divergence theorem in the plane amounts to Green's formula. Using the language of line integrals, if $P\, dx + Q\, dy$ is a differentiable 1-form and $Q_x - P_y$ is integrable one has

$$\int_{bU} P\, dx + Q\, dy = \int_U (Q_x - P_y)\, dA,$$

with no assumption needed separately for Q_x, P_y. A particular case are complex line integrals

$$\Phi(Q) = \int_{bQ} f(z)\, dz,$$

for f continuous in the complex plane $\mathbb{C}$. For differentiable f the density is $\overline{\partial} f$, and so if integrable one has

$$\int_{bU} f(z)\, dz = \int_{U} \overline{\partial} f(z)\, dA(z).$$

Other general versions of Green's theorem with minimal assumptions are known, but the proofs are far from elementary (see [2] and references herein).

3. In a surface S in $\mathbb{R}^3$ oriented by a unit normal field N one can define cubes as those which are so in a local chart. If F is a continuous field, the circulation

$$\Phi(Q) = \int_{bQ} \langle F, T \rangle\, ds,$$

defines an interval function. If F is differentiable, one can show along the same lines that the density is $\langle \nabla \times F, N \rangle$ and one gets Stoke's theorem with minimal assumptions (see the details in the book [1]).

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