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A PRIMER OF REAL FUNCTIONS

By

RALPH P. BOAS, JR.

Professor Emeritus of Mathematics

Northwestern University

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are not empty, and which have the property that M_k is disjoint from $E_1, E_2, \ldots, E_k$. The common point of all M_k is a point of the required kind, since it cannot be in any E_k.

Exercise 9.3. If E is a nonempty bounded subset of R_2, then E and the boundary of E have the same diameter.

10. Some applications of Baire's Theorem

(i) A PROPERTY OF REPEATED INTEGRALS. Let f be a continuous real-valued function on a real interval, say $[0, 1]$. Let f_1 be any integral of f, f_2 any integral of f_1, and so on. If some f_k vanishes identically, so does f: we have only to differentiate f_k repeatedly. The following proposition generalizes this simple fact: *if for each x there is an integer k, possibly differing from one x to another, such that $f_k(x) = 0$, then f vanishes identically.*

To prove this theorem, let E_k be the set of points x for which $f_k(x) = 0$; then our hypothesis says that every x in $[0, 1]$ is in some E_k. By Baire's theorem, not every E_k is nowhere dense. Hence there is some k for which the closure of E_k fills an interval I_k. For this particular k, since f_k is continuous and vanishes on E_k, we must have $f_k(x) = 0$ for every x in I_k. If I_k is not all of $[0, 1]$, we repeat this argument with any remaining part of $[0, 1]$, and so on. In this way we have $f(x) = 0$ for all points x of an everywhere dense set; and since f is continuous, it then follows that $f(x) = 0$ for every x in $[0, 1]$.

Thus if $f(x) \not\equiv 0$, then no matter how the integrals f_k are selected, there must be some x (indeed, an everywhere dense set) such that $f_k(x) \neq 0$ for every k.

(ii) A CHARACTERIZATION OF POLYNOMIALS. Consider again a continuous real-valued function f on $[0, 1]$. If f has

an nth derivative that is identically zero, it is easily proved, for example, by repeated application of the law of the mean (and see p. 177) that f coincides on $[0, 1]$ with a polynomial (of degree at most $n - 1$). The following theorem generalizes this in the spirit of example (i). *Let f have derivatives of all orders on $[0, 1]$, and suppose that at each point some derivative of f is zero. That is, for each x there is an integer $n(x)$ such that $f^{(n(x))}(x) = 0$. Then f coincides on $[0, 1]$ with some polynomial.*[7]

We can start the proof just as in (i). Let E_n be the set of points x for which $f^{(n)}(x) = 0$. By hypothesis every x is in at least one E_n. By Baire's theorem there is a closed interval I in which some E_n is everywhere dense. Since $f^{(n)}$ is a continuous function, $f^{(n)}(x) \equiv 0$ in I and f coincides in I with a polynomial. If I is not all of $[0, 1]$, repeat the reasoning in any remaining part of $[0, 1]$, and so on. In this way we see that there is an everywhere dense set of intervals in each of which f coincides with a polynomial. We still have to show that f coincides with the same polynomial in all the intervals.

To do this, we are going to apply Baire's theorem again to the nowhere dense set H that is left when we remove the interiors of our dense set of intervals from $[0, 1]$. We first need to show that H is perfect. In the first place H is closed, since it is obtained by removing a collection of open intervals from a closed interval. Suppose that H is not perfect, and not just the pair $\{0, 1\}$ (otherwise there was only one interval to begin with and there is nothing more to prove). Then H must have a point y that is not a limit point. This point is the common endpoint of two intervals in each of which f coincides with some polynomial. Then if n exceeds the degree of both polynomials, $f^{(n)}(x) = 0$ for x in both intervals, and at the endpoint by the continuity of $f^{(n)}$. Therefore f coincides with a