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A PRIMER OF  
REAL FUNCTIONS

By

RALPH P. BOAS, JR.

*Professor Emeritus of Mathematics  
Northwestern University*



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are not empty, and which have the property that  $M_k$  is disjoint from  $E_1, E_2, \dots, E_k$ . The common point of all  $M_k$  is a point of the required kind, since it cannot be in any  $E_k$ .

**Exercise 9.3.** If  $E$  is a nonempty bounded subset of  $R_2$ , then  $E$  and the boundary of  $E$  have the same diameter.

## 10. Some applications of Baire's Theorem

(i) **A PROPERTY OF REPEATED INTEGRALS.** Let  $f$  be a continuous real-valued function on a real interval, say  $[0, 1]$ . Let  $f_1$  be any integral of  $f$ ,  $f_2$  any integral of  $f_1$ , and so on. If some  $f_k$  vanishes identically, so does  $f$ : we have only to differentiate  $f_k$  repeatedly. The following proposition generalizes this simple fact: *if for each  $x$  there is an integer  $k$ , possibly differing from one  $x$  to another, such that  $f_k(x) = 0$ , then  $f$  vanishes identically.*

To prove this theorem, let  $E_k$  be the set of points  $x$  for which  $f_k(x) = 0$ ; then our hypothesis says that every  $x$  in  $[0, 1]$  is in some  $E_k$ . By Baire's theorem, not every  $E_k$  is nowhere dense. Hence there is some  $k$  for which the closure of  $E_k$  fills an interval  $I_k$ . For this particular  $k$ , since  $f_k$  is continuous and vanishes on  $E_k$ , we must have  $f_k(x) = 0$  for every  $x$  in  $I_k$ . If  $I_k$  is not all of  $[0, 1]$ , we repeat this argument with any remaining part of  $[0, 1]$ , and so on. In this way we have  $f(x) = 0$  for all points  $x$  of an everywhere dense set; and since  $f$  is continuous, it then follows that  $f(x) = 0$  for every  $x$  in  $[0, 1]$ .

Thus if  $f(x) \neq 0$ , then no matter how the integrals  $f_k$  are selected, there must be some  $x$  (indeed, an everywhere dense set) such that  $f_k(x) \neq 0$  for every  $k$ .

(ii) **A CHARACTERIZATION OF POLYNOMIALS.** Consider again a continuous real-valued function  $f$  on  $[0, 1]$ . If  $f$  has

an  $n$ th derivative that is identically zero, it is easily proved, for example, by repeated application of the law of the mean (and see p. 177) that  $f$  coincides on  $[0, 1]$  with a polynomial (of degree at most  $n - 1$ ). The following theorem generalizes this in the spirit of example (i). *Let  $f$  have derivatives of all orders on  $[0, 1]$ , and suppose that at each point some derivative of  $f$  is zero. That is, for each  $x$  there is an integer  $n(x)$  such that  $f^{(n(x))}(x) = 0$ . Then  $f$  coincides on  $[0, 1]$  with some polynomial.*<sup>7</sup>

We can start the proof just as in (i). Let  $E_n$  be the set of points  $x$  for which  $f^{(n)}(x) = 0$ . By hypothesis every  $x$  is in at least one  $E_n$ . By Baire's theorem there is a closed interval  $I$  in which some  $E_n$  is everywhere dense. Since  $f^{(n)}$  is a continuous function,  $f^{(n)}(x) \equiv 0$  in  $I$  and  $f$  coincides in  $I$  with a polynomial. If  $I$  is not all of  $[0, 1]$ , repeat the reasoning in any remaining part of  $[0, 1]$ , and so on. In this way we see that there is an everywhere dense set of intervals in each of which  $f$  coincides with a polynomial. We still have to show that  $f$  coincides with the same polynomial in all the intervals.

To do this, we are going to apply Baire's theorem again to the nowhere dense set  $H$  that is left when we remove the interiors of our dense set of intervals from  $[0, 1]$ . We first need to show that  $H$  is perfect. In the first place  $H$  is closed, since it is obtained by removing a collection of open intervals from a closed interval. Suppose that  $H$  is not perfect, and not just the pair  $\{0, 1\}$  (otherwise there was only one interval to begin with and there is nothing more to prove). Then  $H$  must have a point  $y$  that is not a limit point. This point is the common endpoint of two intervals in each of which  $f$  coincides with some polynomial. Then if  $n$  exceeds the degree of both polynomials,  $f^{(n)}(x) = 0$  for  $x$  in both intervals, and at the endpoint by the continuity of  $f^{(n)}$ . Therefore  $f$  coincides with a