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Examples and counterexamples for Markus-Yamabe and LaSalle global asymptotic stability problems

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Abstract

We revisit the known counterexamples and the state of the art of the Markus-Yamabe and LaSalle's problems on global asymptotic stability of discrete dynamical systems. We also provide new counterexamples, associated to difference equations, for some of these problems.

1 Introduction

Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a $\mathcal{C}^1$ map and consider the discrete dynamical system

$$\mathbf{x}_{k+1} = F(\mathbf{x}_k). \tag{1}$$

Let $A = (a_{ij})$ be a real $n \times n$ matrix. We denote by $\sigma(A)$ the spectrum of A, i.e., the set of eigenvalues of A and by $|A| = (|a_{ij}|)$. We also denote by $DF(\mathbf{x}) = \left(\frac{\partial F_i(\mathbf{x})}{\partial x_j}\right)$ the Jacobian matrix of F at $\mathbf{x} \in \mathbb{R}^n$. When $F(\mathbf{0}) = \mathbf{0}$, we can write $F(\mathbf{x})$ in the form $F(\mathbf{x}) = A(\mathbf{x})\mathbf{x}$, where $A(\mathbf{x})$ is an $n \times n$ matrix function. Note that this $A(\mathbf{x})$ is not unique.

LaSalle in [12] gave some possible generalizations of the sufficient conditions for global asymptotic stability (GAS) for $n = 1$. Concretely, the conditions are the following:

(I) $|\lambda| < 1$ for all $\lambda \in \sigma(A(\mathbf{x}))$ and for all $\mathbf{x} \in \mathbb{R}^n$,

(II) $|\lambda| < 1$ for all $\lambda \in \sigma(|A(\mathbf{x})|)$ and for all $\mathbf{x} \in \mathbb{R}^n$,

(III) $|\lambda| < 1$ for all $\lambda \in \sigma(DF(\mathbf{x}))$ and for all $\mathbf{x} \in \mathbb{R}^n$,

(IV) $|\lambda| < 1$ for all $\lambda \in \sigma(|DF(\mathbf{x})|)$ and for all $\mathbf{x} \in \mathbb{R}^n$.

In [6] it is proved that none of the conditions I and II implies GAS, even for $n = 2$. In particular in both cases there are polynomial maps satisfying them and such that the origin of (1) is not GAS.

Conditions III and IV are also known as Markus-Yamabe type conditions because they are similar to a condition proposed for ordinary differential equations, see [4, 10] and the references therein. In [5] it is proved that condition III implies GAS for planar polynomial maps and that there are planar rational maps satisfying it having other periodic points. In [4, 8] there are also examples of polynomial maps defined in $\mathbb{R}^n$, $n \geq 3$, satisfying the condition and having unbounded orbits. Moreover in [5] it is also proved that when F is polynomial condition IV implies GAS.

Taking into account all these examples and the known results it turns out that it only remains to study the following problem:

Open Question. *Set $n \geq 2$. Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a $\mathcal{C}^1$ map satisfying $F(0) = 0$ and such that condition IV holds. Is the origin GAS for the discrete dynamical system* (1)*?*

In the forthcoming paper [7] we give a general result on GAS for maps of the form

$$F(x_1, x_2, \ldots, x_n) = (x_2, x_3, \ldots, f(x_1, x_2, \ldots, x_n)). \tag{2}$$

In particular it implies that when $F(0) = 0$ and condition IV is satisfied then the origin is GAS for the dynamical system generated by (2). Notice that precisely, difference equations of order n can be studied through dynamical systems generated by maps of the form (2).

In this note, we revisit the known counterexamples and the state of the art of the Markus-Yamabe and LaSalle's problems. We also provide difference equations counterexamples to conditions I and III.

2 Examples and counterexamples

2.1 Condition I.

The map given in [6],

$$F(x, y) = A(x, y) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 + xy & 1 \\ -(x^2 + xy)^2 & -(x^2 + xy) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

satisfies condition (I) because

$$\det\left(A(x, y) - \lambda\, \mathrm{Id}\right) = \lambda^2$$

and so, $\sigma(A(x, y)) = \{0\}$. On the other hand it is easy to check that $F^4(1, -1) = (1, -1)$. Therefore the origin is not GAS.

Remark 2.1. It is clear that the above map can be extended to $\mathbb{R}^n$, $n > 2$, as

$$\widetilde{F}(x_1, x_2, \ldots, x_n) = \left(F(x_1, x_2), \frac{x_3}{2}, \frac{x_4}{2}, \ldots, \frac{x_n}{2}\right),$$

providing an example satisfying condition (I), with $\sigma(A(x)) = \{0, 1/2\}$ and not having the origin as a global attractor. A similar trick can be used in all the counterexamples presented in this section.

2.2 Condition II.

Following again [6], consider the map

$$F(x, y) = A(x, y) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{3}{2}xy^3 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{3}$$

For it, $\sigma(|A(x,y)|) = \{1/2\}$. Moreover the hyperbola $\{(x,y) : xy = 1\}$ is invariant for (1), because

$$F_1(x,y)F_2(x,y) = \frac{x}{2}\left(\frac{(3x^2y^2+1)y}{2}\right)\bigg|_{xy=1} = 1.$$

Hence we have proved that condition (II) does not imply that the origin is GAS.

Remark 2.2. As usual, given a matrix A we denote its spectral radius by $\rho(A) = \max_{\{\lambda\in\sigma(A)\}} |\lambda|$. It is well known that $\rho(|A|) \le \rho(A)$, see [6, 9]. Therefore condition II is more restrictive that condition I. In particular the map (3) gives a counterexample for both conditions I and II.

2.3 Condition III.

For $n = 2$, consider the rational map introduced by Szlenk in the appendix of [5],

$$F(x,y) = \left(-\frac{ky^3}{1+x^2+y^2}, \frac{kx^3}{1+x^2+y^2}\right), \quad k \in (1, 2/\sqrt{3}). \tag{4}$$

It can be seen that

$$\sigma(DF(\mathbf{x})) \subset \{z \in \mathbb{C} : |z| < \sqrt{3}k/2\}$$

and

$$F^4\left(\frac{1}{\sqrt{k-1}}, 0\right) = \left(\frac{1}{\sqrt{k-1}}, 0\right).$$

In [5] it was also proved that when $n = 2$ and F is a polynomial map it is true that condition (III) implies GAS, see next subsection. Nevertheless, for $n \ge 3$, polynomial counterexamples can also be constructed, see [4]. If we take the map

$$F(x,y,z) = \left(\frac{x}{2} + y(x+yz)^2, \frac{y}{2} - (x+yz)^2, \frac{z}{2}\right) \tag{5}$$

it can be seen that when $(x_0, y_0, z_0) = (174/32, -63/32, 1)$,

$$F^m(x_0,y_0,z_0) = \left(\frac{174}{32}\,2^m, -\frac{63}{32}\,2^{2m}, 2^{-m}\right),$$

and so the origin can not be GAS. This example satisfies that $\sigma(DF(\mathbf{x})) = \{1/2\}$ because $DF(\mathbf{x}) = \mathbf{x}/2 + N(\mathbf{x})$, where $N(\mathbf{x})$ is a nilpotent matrix. These maps belong to a bigger class of counterexamples constructed in [8].

The example (4) has also been modified in [5] to get a counterexample given by a diffeomorphism. Later, other conditions have been added to condition III, like the one of having the infinite as a repeller, for trying to obtain GAS. Nevertheless, assuming also these additional conditions it turns out that it is possible to obtain dynamical systems for which the origin is not GAS, see [2]. Recently, in [1] a new family of counterexamples satisfying condition III together with these more restrictive conditions is introduced. The maps

$$F_{a,b,c}(x,y) = (ae^{-x^2} - by, cx), \tag{6}$$

for some concrete values of the parameters a, b and c, provide an explicit family of counterexamples. The nice point with these new counterexamples is that their dynamics are very complicated, because they can be seen as perturbed twist maps. In Section 2.5 we will use the above family to construct a difference equation counterexample to Condition III.

2.4 Conditions III and IV in the polynomial case.

For the particular case of F being a polynomial map there are some positive results. For instance condition III implies GAS for $n = 2$, see [5], and condition IV also implies GAS for any n, see [6]. We do not give here the proofs but we want to comment a key difference between the polynomial and the non-polynomial cases.

Let $B(\mathbf{x})$ be an $n \times n$ matrix with polynomial entries and such that the set $\cup_{\mathbf{x}\in\mathbb{R}^n}\sigma(B(\mathbf{x}))$ is contained in a compact set. Let us prove that its characteristic polynomial

$$p_{\mathbf{x}}(\lambda) = \det(B(\mathbf{x}) - \lambda\,\mathrm{Id})$$

is indeed independent of $\mathbf{x}$. Observe that the coefficients of $p_{\mathbf{x}}(\lambda)$ are polynomials on $\mathbf{x}$. Therefore the result follows if we prove that these coefficients are bounded functions. This is a straightforward consequence of the Vieta's formulas that give the coefficients of a monic polynomial as symmetric polynomial functions of its roots, because it is clear that by hypothesis, for all $\mathbf{x}$, all the roots of $p_{\mathbf{x}}$ are bounded.

Therefore when one of the conditions III or IV holds and F is a polynomial map the characteristic polynomial of $DF(\mathbf{x})$ is independent of $\mathbf{x}$. As can be seen in the proofs of the above mentioned cases this fact forces some kind of triangular structures in F that allow to prove that the origin is GAS. Nevertheless, recall that for $n = 3$ there is a polynomial counterexample (5) satisfying condition III.

2.5 A difference equation counterexample for condition III

Motivated by family (6) we will construct a counterexample of condition III given by a map associated to a difference equation.

Consider the family of difference equations

$$x_{m+2} = 2e^{-x_{m+1}^2} - bx_m. \tag{7}$$

To study its behavior we can consider the dynamical system generated by the map

$$\widetilde{F}_b(x,y) = \left(y, 2e^{-y^2} - bx\right),$$

that, for $b \neq -1$, has a unique fixed point (x_0, x_0), where $x_0 = x_0(b)$ is the only solution of $2e^{-x_0^2} - (b+1)x_0 = 0$. Therefore, with a translation, we can conjugate $\widetilde{F}_b$ with

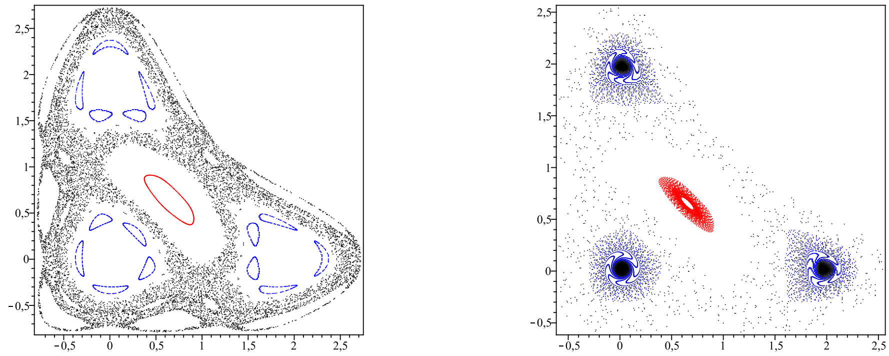
$$F_b(x,y) = (y, 2e^{-(y+x_0)^2} - (b+1)x_0 - bx).$$

Notice that $F_b(0,0) = (0,0)$. Moreover, since

$$DF_b(x,y) = \begin{pmatrix} 0 & 1 \\ -b & -4(y+x_0)e^{-(y+x_0)^2} \end{pmatrix}$$

and $\max_{w\in\mathbb{R}} |4we^{-w^2}| = 2\sqrt{2/e}$ we know that for $b > 2/e$ the eigenvalues of the above matrix are complex conjugated with modulus less than or equal to $\sqrt{b}$. Therefore for each $b \in (2/e, 1)$, the map F_b satisfies condition III.

Following [1] we consider first the map F_1. It is an area preserving map which numerically seems to present all the complicated dynamics associated to the perturbed twist maps. In Figure 1



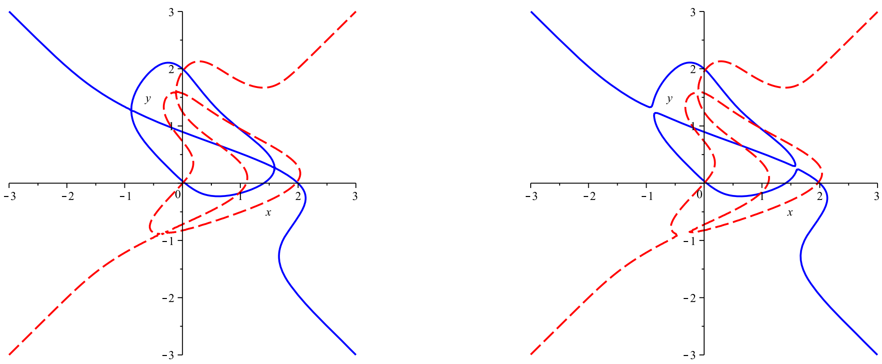
$b = 1$ $\qquad$ $b = 0.999$

Figure 1: Three obits of $\widetilde{F}_b$

we show several thousands of points of three orbits of $\widetilde{F}_1$ and $\widetilde{F}_{0.999}$. Recall that these maps are conjugated to the corresponding F_b.

Although it is not easy to prove, from the above pictures it seems natural to believe that F_1 has hyperbolic periodic orbits and associated to them transversal heteroclinic points. Therefore, for $b \lesssim 1$ many of these hyperbolic points remain, providing a counterexample that satisfies condition III and has complicated dynamics.

In any case, for $b = 1$ it is not difficult to see that the map $\widetilde{F}_1$ has two orbits of three periodic points suggested by Figure 1. One of them is of elliptic type and the other one of saddle type. Moreover, they remain for $b \lesssim 1$, see again Figure 1. In fact in Figure 2 we present the two curves corresponding to the first and the second components of $\widetilde{F}_b^3(x, y) - (x, y) = (0, 0)$ for $b \in \{1, 0.999\}$. The second component corresponds to the dashed line. Notice that these curves intersect transversally at seven ponts which are the fixed point and the two 3-periodic orbits. Therefore the origin of the corresponding F_b is not GAS.



$b = 1$ $\qquad$ $b = 0.999$

Figure 2: Fixed and three periodic points of $\widetilde{F}_b$

2.6 A difference equation counterexample for condition I

In the previous section we have seen that there are counterexamples of GAS, satisfying condition III and which are of the form

$$F(x, y) = (y, f(y) - bx),$$

with f smooth and $f(0) = 0$. Let us see that they also satisfy condition I. We write

$$F(x, y) = A(x, y) \begin{pmatrix} x \\ y \end{pmatrix} := \begin{pmatrix} 0 & 1 \\ -b & \frac{f(y)}{y} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

By the mean value Theorem $f(y)/y = f'(z)$ for some z between 0 and x. Therefore $A(x, y) = DF(x, z)$ for some z. Then $\sigma(A(x, y)) \subset \sigma(DF(x, y))$ and the result follows.

3 On the existence of the fixed point

One could think that the hypothesis that $F(\mathbf{0}) = \mathbf{0}$ is not essential when one considers the problem of GAS under any of the conditions I-IV. Soon, one realizes that even when $n = 1$ it has to be taken into account. For instance, if one considers the dynamical system generated by the map

$$F(x) = \log(1 + e^x),$$

it is clear that $|DF(x)| = |F'(x)| < 1$, satisfying condition III, but having no fixed point.

In fact it was proved in [5] an interesting relation between the existence of a fixed point for polynomial maps under condition III and the celebrated Jacobian Conjecture. We reproduce here this result.

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a polynomial map satisfying condition III. Until now we were interested in knowing whether the dynamical system associated to F had a GAS fixed point. Now we formulate a weaker problem:

Fixed Point Conjecture. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a polynomial map satisfying condition III. Then F has a unique fixed point.

Considering the real and the imaginary part of the components of a polynomial map $F : \mathbb{C}^n \to \mathbb{C}^n$ and using standard arguments of linear algebra it is easy to see that this conjecture can be formulated in the following equivalent form:

Fixed Point Conjecture. Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map such that $DF(x)$ has all its eigenvalues with modulus less than one at each $x \in \mathbb{C}^n$. Then F has a unique fixed point.

Theorem 3.1 shows that this problem is equivalent to the Jacobian Conjecture (JC), formulated in 1939 by Keller [11], and which can be established as follows.

Jacobian Conjecture. Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map with $\det(DF(x)) \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ at each $x \in \mathbb{C}^n$. Then F is invertible.

Theorem 3.1. *([5]) The Jacobian Conjecture is equivalent to the Fixed Point Conjecture*

Proof. Assume that the JC holds and let F be satisfying the hypothesis of Fixed Point Conjecture (FPC) for some n. Consider $G(\mathbf{x}) = F(\mathbf{x}) - \mathbf{x}$. Then the eigenvalues of $DG(\mathbf{x})$ are the eigenvalues of $DF(\mathbf{x})$ minus one. Hence by the results of Section 2.4 we know that $\det DG(\mathbf{x})$ is constant. Moreover, from the hypothesis on F we have that this constant is not zero. So, G is invertible and it exists a unique zero of G, which is the unique fixed point of $F(\mathbf{x})$.

Now assume that the JC fails for some m. From the Reduction Theorem ([3]) this means that there exist $n \in \mathbb{N}$ and $G : \mathbb{C}^n \to \mathbb{C}^n$ polynomial and non invertible of the form

$$G(\mathbf{x}) = \mathbf{x} + H(\mathbf{x}),$$

with $DH(\mathbf{x})$ a nilpotent matrix at each $\mathbf{x} \in \mathbb{C}^n$. Now set $g(\mathbf{x}) = \frac{1}{2} G(\mathbf{x})$ and let $\mathbf{y}, \mathbf{z} \in \mathbb{C}^n$, $\mathbf{y} \neq \mathbf{z}$ with $g(\mathbf{y}) = g(\mathbf{z}) = \mathbf{p}$. Denoting by $h(\mathbf{x}) = \mathbf{x} + \mathbf{p} - g(\mathbf{x})$ we have that $h(\mathbf{y}) = \mathbf{y}$ and $h(\mathbf{z}) = \mathbf{z}$. On the other hand, since $DH(\mathbf{x})$ is a nilpotent matrix at each $\mathbf{x} \in \mathbb{C}^n$, all its eigenvalues are zero. From the definition of $h(\mathbf{x})$ we obtain

$$Dh(\mathbf{x}) = \mathrm{Id} - Dg(\mathbf{x}) = \mathrm{Id} - \left(\frac{1}{2}\,\mathrm{Id} + \frac{1}{2}\,DH(\mathbf{x})\right) = \frac{1}{2}\,\mathrm{Id} - \frac{1}{2}\,DH(\mathbf{x})$$

which implies that $\sigma(Dh(\mathbf{x})) = \{1/2\}$. Hence, $h(\mathbf{x})$ is under the hypothesis of the FPC and it has two different fixed points. □

Recall that in [5] it is proved that the FPC is true in $\mathbb{R}^2$. From this fact, and the proof of Theorem C, we can only deduce that the JC is true for some special subcases, but it can not be deduced that it is true for $n = 2$.

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