Survey on $\mathbb{Z}_2\mathbb{Z}_4$-additive codes [1]

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Abstract

A code $\mathcal{C}$ is $\mathbb{Z}_2\mathbb{Z}_4$-additive if the set of coordinates can be partitioned into two subsets X and Y such that the punctured code of $\mathcal{C}$ by deleting the coordinates outside X (respectively, Y) is a binary linear code (respectively, a quaternary linear code). The corresponding binary codes of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes under an extended Gray map are called $\mathbb{Z}_2\mathbb{Z}_4$-linear codes, which seem to be a very distinguished class of binary group codes.

As for binary and quaternary linear codes, for these codes the fundamental parameters are shown and standard forms for generator and parity-check matrices are given, defining the appropriate concept of duality. The main results on $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual and $\mathbb{Z}_2\mathbb{Z}_4$-additive formally self-dual codes are also presented, as well as, the results on the invariants rank and dimension of the kernel for these codes are given. Several families of important binary codes fall in the class of $\mathbb{Z}_2\mathbb{Z}_4$-linear codes. In this survey, we review characterizations, properties and constructions of perfect and extended perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear codes, Hadamard $\mathbb{Z}_2\mathbb{Z}_4$-linear codes, Reed-Muller $\mathbb{Z}_2\mathbb{Z}_4$-linear codes, maximum distance separable $\mathbb{Z}_2\mathbb{Z}_4$-linear codes, and Preparata-like and Kerdock-like $\mathbb{Z}_2\mathbb{Z}_4$-linear codes. Finally, applications of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes to steganography are also presented.

1. Introduction

Let $\mathbb{Z}_2$ and $\mathbb{Z}_4$ be the ring of integers modulo 2 and 4 respectively. Let $\mathbb{Z}_2^n$ denote the set of all binary vectors of length n and let $\mathbb{Z}_4^n$ be the set of all n-tuples over the ring $\mathbb{Z}_4$. In this paper, the elements of $\mathbb{Z}_4^n$ will also be called quaternary vectors of length n. We denote by $\mathbf{0}^\ell$ and $\mathbf{1}^\ell$ the all-zero and the all-one vectors, respectively, of length ℓ. If the length of such vectors is clear from the context we omit the parameter ℓ.

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The *Hamming distance* $d_H(u,v)$ between two vectors $u, v \in \mathbb{Z}_2^n$ is the number of coordinates in which u and v differ, and the *Hamming weight* of a vector $u \in \mathbb{Z}_2^n$, denoted by $w_H(u)$, is the number of nonzero coordinates of u, so $d_H(u,v) = w_H(u-v)$. On the other hand, the *Lee weights* over the elements in $\mathbb{Z}_4$ are defined as: $w_L(0) = 0$, $w_L(1) = w_L(3) = 1$, $w_L(2) = 2$. Then, the *Lee weight* of a vector $u \in \mathbb{Z}_4^n$, denoted by $w_L(u)$, is the addition of the weights of its coordinates, whereas the *Lee distance* $d_L(u,v)$ between two vectors $u, v \in \mathbb{Z}_4^n$ is $d_L(u,v) = w_L(u-v)$.

Any nonempty subset C of $\mathbb{Z}_2^n$ is a binary code and a subgroup of $\mathbb{Z}_2^n$ is called a *binary linear code* or a $\mathbb{Z}_2$*-linear code*. Equivalently, any nonempty subset $\mathcal{C}$ of $\mathbb{Z}_4^n$ is a quaternary code and a subgroup of $\mathbb{Z}_4^n$ is called a *quaternary linear code*. The elements of a code are called *codewords*. If C is a binary linear code, it is isomorphic to an additive group $\mathbb{Z}_2^k$, so C has dimension k and it has 2^k codewords. Equivalently, if $\mathcal{C}$ is a quaternary linear code, since it is a subgroup of $\mathbb{Z}_4^n$, it is isomorphic to an abelian structure $\mathbb{Z}_2^\gamma \times \mathbb{Z}_4^\delta$. Therefore, we have that $\mathcal{C}$ is of type $2^\gamma 4^\delta$ as a group, and it has $|\mathcal{C}| = 2^{\gamma+2\delta}$ codewords.

Quaternary codes can be viewed as binary codes under the usual Gray map defined as $\phi(0) = (0,0)$, $\phi(1) = (0,1)$, $\phi(2) = (1,1)$, $\phi(3) = (1,0)$ in each coordinate. If $\mathcal{C}$ is a quaternary linear code, then the binary code $C = \phi(\mathcal{C})$ is called a $\mathbb{Z}_4$*-linear* code. The dual of a quaternary linear code $\mathcal{C}$, denoted by $\mathcal{C}^\perp$, is called the *quaternary dual code* and is defined in the standard way [41] in terms of the usual inner product for quaternary vectors [33]. The binary code $C_\perp = \phi(\mathcal{C}^\perp)$ is called the $\mathbb{Z}_4$*-dual code* of $C = \phi(\mathcal{C})$.

The *minimum Hamming distance* $d_H(C)$ of a binary code C is the minimum value of $d_H(u,v)$ for $u, v \in C$ satisfying $u \neq v$. The *minimum Hamming weight* of a binary code C, denoted by $w_H(C)$, is the minimum value of $w_H(u)$ for $u \in C \backslash \{\mathbf{0}\}$. It is well known that if C is a binary linear code, $d_H(C) = w_H(C)$ [41]. Equivalently, the *minimum Lee distance* $d_L(\mathcal{C})$ of a quaternary code $\mathcal{C}$ is the minimum value of $d_L(u,v)$ for $u, v \in \mathcal{C}$ satisfying $u \neq v$. The *minimum Lee weight* of a quaternary code $\mathcal{C}$, denoted by $w_L(\mathcal{C})$, is the minimum value of $w_L(u)$ for $u \in \mathcal{C} \backslash \{\mathbf{0}\}$. Again, if $\mathcal{C}$ is a quaternary linear code, $d_L(\mathcal{C}) = w_L(\mathcal{C})$. Note that the Gray map ϕ is an isometry which transforms Lee distances over $\mathbb{Z}_4^n$ into Hamming distances over $\mathbb{Z}_2^{2n}$. Therefore, the minimum Lee weight of a quaternary code $\mathcal{C}$ coincides with the minimum Hamming weight of $C = \phi(\mathcal{C})$, that is $w_L(\mathcal{C}) = w_H(\phi(\mathcal{C}))$.

Since 1994, quaternary linear codes have been studied and became significant since, after applying the Gray map, we obtain binary nonlinear codes better than any known binary linear code with the same parameters. More specifically, Hammons et. al. [33, 64] show how to construct well known binary nonlinear codes like the Nordstrom-Robinson code, Kerdock codes and Delsarte-Goethals codes as $\mathbb{Z}_4$-linear codes, that is, as the Gray map image of quaternary linear codes. Furthermore, they solve an old open problem on coding theory about that the Hamming weight enumerators of the nonlinear Kerdock and Preparata codes satisfy the MacWilliams identities. Actually, they prove that the Kerdock codes and some Preparata-like codes are $\mathbb{Z}_4$-linear codes and, moreover, the $\mathbb{Z}_4$-dual code of the Kerdock code is a Preparata-like code. Later, several other $\mathbb{Z}_4$-linear codes with the same parameters as some well known families of binary linear codes (for example, extended Hamming, Hadamard, and Reed-Muller codes) have been studied and classified [8, 9, 15, 14, 38, 43, 44, 48, 50, 63].

Additive codes were first defined by Delsarte in 1973 in terms of association schemes [22, 23]. According to this definition, an additive code is a subgroup of the underlying abelian group in a translation association scheme. On the other hand, translation invariant propelinear

codes were first defined in 1997 [55, 58], where it is proved that they are group-isomorphic to subgroups of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta \times \mathbb{Q}_8^\sigma$, being $\mathbb{Q}_8$ the non-abelian quaternion group on eight elements. In the special case of a binary Hamming scheme, that is, when the underlying abelian group is of order 2^n, the additive codes coincide with the abelian translation invariant propelinear codes. Hence, as it is pointed out in [23, 54], the only structures for the abelian group are those of the form $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$, with $\alpha+2\beta = n$. Therefore, the codes that are subgroups of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ are the only additive codes in the binary Hamming scheme. In order to distinguish them from additive codes over finite fields [4], from now on, we will call them $\mathbb{Z}_2\mathbb{Z}_4$-*additive codes*. Note that one could think of other families of codes with an algebraic structure that also include the $\mathbb{Z}_2\mathbb{Z}_4$-additive codes, such as mixed group codes [13, 34, 40].

Since $\mathbb{Z}_2\mathbb{Z}_4$-additive codes are subgroups of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$, they can be seen as a generalization of binary (when $\beta = 0$) and quaternary (when $\alpha = 0$) linear codes. As for quaternary linear codes, after applying the Gray map to the $\mathbb{Z}_4$ coordinates of a $\mathbb{Z}_2\mathbb{Z}_4$-additive code, we obtain binary codes called $\mathbb{Z}_2\mathbb{Z}_4$-*linear codes*. There are $\mathbb{Z}_2\mathbb{Z}_4$-linear codes in several important classes of binary codes. For example, $\mathbb{Z}_2\mathbb{Z}_4$-linear perfect single error-correcting codes (or 1-perfect codes) are found in [55] and fully characterized in [16]. Also, in subsequent papers [14, 38, 47, 48], $\mathbb{Z}_2\mathbb{Z}_4$-linear extended perfect and Hadamard codes are studied and classified. Note that $\mathbb{Z}_2\mathbb{Z}_4$-additive codes have allowed to classify more binary nonlinear codes, giving them a structure as $\mathbb{Z}_2\mathbb{Z}_4$-additive codes. Although it is not easy to determine whether a code has a $\mathbb{Z}_2\mathbb{Z}_4$-additive structure, and whether it is unique or not, it seems that there are many more $\mathbb{Z}_2\mathbb{Z}_4$-linear codes than linear. In this sense, recently, a preliminary proposal about counting $\mathbb{Z}_2\mathbb{Z}_4$-additive codes can be found in [26]. Finally, mention that a permutation decoding method for $\mathbb{Z}_2\mathbb{Z}_4$-linear codes is described in [3].

Part of the research developed by the Combinatorics, Coding and Security Group (CCSG) deals with quaternary linear codes, as well as $\mathbb{Z}_2\mathbb{Z}_4$-additive codes. Since there is not any symbolic software to work with $\mathbb{Z}_2\mathbb{Z}_4$-additive codes, the members of CCSG have been developing a new package [12, 46, 52] in MAGMA [19] that supports the basic facilities for these codes. Specifically, this new MAGMA package generalizes most of the functions for quaternary linear codes to $\mathbb{Z}_2\mathbb{Z}_4$-additive codes, and includes new functions specific for these kind of codes. A beta version of this package and the manual with the description of all functions can be downloaded from the web page `http://ccsg.uab.cat` (for any comment or further information about this package, you can send an e-mail to *support-ccsg@deic.uab.cat*).

The aim of this paper is to give a complete survey on $\mathbb{Z}_2\mathbb{Z}_4$-additive codes and their corresponding $\mathbb{Z}_2\mathbb{Z}_4$-linear codes. It is organized as follows. In Section 2, we recall the definition of a $\mathbb{Z}_2\mathbb{Z}_4$-additive and $\mathbb{Z}_2\mathbb{Z}_4$-linear code, and we describe their fundamental parameters and a standard form for the generator matrices of these codes. Section 3 is devoted to the duality concept for $\mathbb{Z}_2\mathbb{Z}_4$-additive codes defining the appropriate inner product, showing how the generator and parity-check matrices are related, as well as how the parameters of the dual code can be computed from the parameters of the code. In Section 4, we discuss about $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual codes. Separability, antipodality and Type of these codes are studied. Moreover, we give different constructions of such codes. Briefly, we also discuss about formally self-dual additive codes. In Section 5, we present the possible values of two invariants: the rank and dimension of the kernel, for $\mathbb{Z}_2\mathbb{Z}_4$-additive codes. Such techniques are further applied to specific families of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes. In Section 6, constructions, classification and properties like rank and

dimension of the kernel are established for several families of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes. An interesting application of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes to steganography is presented in Section 7. Finally, in Section 8, we give some conclusions and discuss about further research on these codes.

2. $\mathbb{Z}_2\mathbb{Z}_4$-additive codes

In this section, we recall some definitions and concepts related to $\mathbb{Z}_2\mathbb{Z}_4$-additive codes. We also describe their fundamental parameters and a standard form for the generator matrices of these codes. The material of this section is a summary of the results presented in [10, 11].

From now on, we focus on $\mathbb{Z}_2\mathbb{Z}_4$-additive codes $\mathcal{C}$, which are subgroups of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$. The $\mathbb{Z}_2\mathbb{Z}_4$-additive codes can also be seen as binary codes, called $\mathbb{Z}_2\mathbb{Z}_4$-*linear codes*, by considering the extension of the usual Gray map: $\Phi : \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta \longrightarrow \mathbb{Z}_2^n$, where $n = \alpha + 2\beta$, given by

$$\begin{aligned}\Phi(x, y) &= (x, \phi(y_1), \ldots, \phi(y_\beta)) \\ &\forall x \in \mathbb{Z}_2^\alpha,\ \forall y = (y_1, \ldots, y_\beta) \in \mathbb{Z}_4^\beta;\end{aligned}$$

where $\phi : \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2^2$ is the usual Gray map, that is, $\phi(0) = (0, 0),\ \phi(1) = (0, 1),\ \phi(2) = (1, 1),\ \phi(3) = (1, 0)$. For a vector $v = (v_1, v_2) \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$, we define the weight of v, denoted by $w(v)$, as $w_H(v_1) + w_L(v_2)$. Note that since $w(v) = w_H(\Phi(v))$, the Gray map Φ is an isometry which transforms distances defined in a $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ over $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ to Hamming distances defined in the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-linear code $C = \Phi(\mathcal{C})$. Note that the length of $C = \Phi(\mathcal{C})$ is $n = \alpha + 2\beta$.

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code. Since $\mathcal{C}$ is a subgroup of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$, it is isomorphic to an abelian structure $\mathbb{Z}_2^\gamma \times \mathbb{Z}_4^\delta$. Therefore, $\mathcal{C}$ is of type $2^\gamma 4^\delta$ as a group, and it has $|\mathcal{C}| = 2^{\gamma+2\delta}$ codewords. Let X (respectively Y) be the set of $\mathbb{Z}_2$ (respectively $\mathbb{Z}_4$) coordinate positions, so $|X| = \alpha$ and $|Y| = \beta$. Unless otherwise stated, the set X corresponds to the first α coordinates and Y corresponds to the last β coordinates. Call $\mathcal{C}_X$ (respectively $\mathcal{C}_Y$) the punctured code of $\mathcal{C}$ by deleting the coordinates outside X (respectively Y). Let $\mathcal{C}_b$ be the subcode of $\mathcal{C}$ which contains all order two codewords and let κ be the dimension of $(\mathcal{C}_b)_X$, which is a binary linear code. For the case $\alpha = 0$, we write $\kappa = 0$. Considering all these parameters, we say that the $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ (or equivalently the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-linear code $C = \Phi(\mathcal{C})$) is of type $(\alpha, \beta; \gamma, \delta; \kappa)$.

Note that $\mathcal{C}_Y$ is a quaternary linear code of type $(0, \beta; \gamma_Y, \delta; 0)$, where $0 \leq \gamma_Y \leq \gamma$, and $\mathcal{C}_X$ is a binary linear code of type $(\alpha, 0; \gamma_X, 0; \gamma_X)$, where $\kappa \leq \gamma_X \leq \kappa + \delta$. Note also that $\mathbb{Z}_2\mathbb{Z}_4$-linear codes are a generalization of binary linear codes and $\mathbb{Z}_4$-linear codes. When $\beta = 0$, the binary code $C = \mathcal{C}$ corresponds to a binary linear code. On the other hand, when $\alpha = 0$, the $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ is a quaternary linear code and the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-linear code $C = \Phi(\mathcal{C})$ is a $\mathbb{Z}_4$-linear code.

Two $\mathbb{Z}_2\mathbb{Z}_4$-additive codes $\mathcal{C}_1$ and $\mathcal{C}_2$ both of type $(\alpha, \beta; \gamma, \delta; \kappa)$ are said to be *monomially equivalent*, if one can be obtained from the other by permutating the coordinates and (if necessary) changing the signs of certain $\mathbb{Z}_4$ coordinates. Two $\mathbb{Z}_2\mathbb{Z}_4$-additive codes are said to be *permutation equivalent* if they differ only by a permutation of coordinates. If two $\mathbb{Z}_2\mathbb{Z}_4$-additive

codes $\mathcal{C}_1$ and $\mathcal{C}_2$ are monomially equivalent, then, after the Gray map, the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-linear codes $C_1 = \Phi(\mathcal{C}_1)$ and $C_2 = \Phi(\mathcal{C}_2)$ are isomorphic as binary codes. Note that the inverse statement is not always true. The *monomial automorphism group* of a $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$, denoted by $\mathrm{MAut}(\mathcal{C})$, is the group generated by all permutations and sign-changes of the $\mathbb{Z}_4$ coordinates that preserves the set of codewords of $\mathcal{C}$, while the *permutation automorphism group* of $\mathcal{C}$, denoted by $\mathrm{PAut}(\mathcal{C})$, is the group generated by all permutations that preserves the set of codewords of $\mathcal{C}$ [36].

Although $\mathcal{C}$ is not a free module, every codeword is uniquely expressible in the form

$$\sum_{i=1}^{\gamma} \lambda_i u^{(i)} + \sum_{j=\gamma+1}^{\gamma+\delta} \mu_j v^{(j)},$$

where $\lambda_i \in \mathbb{Z}_2$ for $1 \leq i \leq \gamma$, $\mu_j \in \mathbb{Z}_4$ for $\gamma + 1 \leq j \leq \gamma + \delta$ and $u^{(i)}, v^{(j)}$ are vectors in $\mathbb{Z}_2^{\alpha} \times \mathbb{Z}_4^{\beta}$ of order two and order four, respectively. Moreover, the vectors $u^{(i)}, v^{(j)}$ give us a generator matrix $\mathcal{G}$ of size $(\gamma + \delta) \times (\alpha + \beta)$ for the code $\mathcal{C}$. This generator matrix $\mathcal{G}$ can be written as

$$\mathcal{G} = \left(\begin{array}{c|c} B_1 & 2B_3 \\ \hline B_2 & Q \end{array}\right), \tag{1}$$

where B_1, B_2 are matrices over $\mathbb{Z}_2$ of size $\gamma \times \alpha$ and $\delta \times \alpha$, respectively; B_3 is a matrix over $\mathbb{Z}_4$ of size $\gamma \times \beta$ with all entries in $\{0, 1\} \subset \mathbb{Z}_4$; and Q is a matrix over $\mathbb{Z}_4$ of size $\delta \times \beta$ with quaternary row vectors of order four.

Let I_k be the identity matrix of size $k \times k$. In [33, 64], it was shown that any quaternary linear code of type $(0, \beta; \gamma, \delta; 0)$ is permutation equivalent to a quaternary linear code with a generator matrix of the form

$$\mathcal{G}_S = \left(\begin{array}{ccc} 2T & 2I_{\gamma} & \mathbf{0} \\ \hline S & R & I_{\delta} \end{array}\right), \tag{2}$$

where R, T are matrices over $\mathbb{Z}_4$ with all entries in $\{0, 1\} \subset \mathbb{Z}_4$, of size $\delta \times \gamma$ and $\gamma \times (\beta - \gamma - \delta)$, respectively; and S is a matrix over $\mathbb{Z}_4$ of size $\delta \times (\beta - \gamma - \delta)$. In [10, 11], this result was generalized for $\mathbb{Z}_2\mathbb{Z}_4$-additive codes as follows:

Theorem 2.1 *[10, 11] Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$. Then, $\mathcal{C}$ is permutation equivalent to a $\mathbb{Z}_2\mathbb{Z}_4$-additive code with canonical generator matrix of the form*

$$\mathcal{G}_S = \left(\begin{array}{cc|ccc} I_{\kappa} & T_b & 2T_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 2T_1 & 2I_{\gamma-\kappa} & \mathbf{0} \\ \hline \mathbf{0} & S_b & S_q & R & I_{\delta} \end{array}\right), \tag{3}$$

where T_b, S_b are matrices over $\mathbb{Z}_2$; T_1, T_2, R are matrices over $\mathbb{Z}_4$ with all entries in $\{0, 1\} \subset \mathbb{Z}_4$; and S_q is a matrix over $\mathbb{Z}_4$.

Example 2.2 *Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(3, 4; 3, 1; 3)$ with generator matrix*

$$\mathcal{G} = \left(\begin{array}{ccc|cccc} 1 & 0 & 0 & 2 & 2 & 0 & 0 \\ 1 & 1 & 1 & 2 & 2 & 2 & 2 \\ 1 & 1 & 0 & 2 & 2 & 0 & 0 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right).$$

By Theorem 2.1, $\mathcal{C}$ is permutation equivalent to a $\mathbb{Z}_2\mathbb{Z}_4$-additive code with canonical generator matrix

$$\mathcal{G}_S = \left(\begin{array}{ccc|cccc} 1 & 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right). \tag{4}$$

3. Duality of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes

For linear codes over finite fields or finite rings, there exists the well known concept of duality. In this section, we explain the duality for $\mathbb{Z}_2\mathbb{Z}_4$-additive codes, taking advantage of their abelian group structure. We also show how to compute the type of an additive dual code, and how to construct a parity-check matrix of a $\mathbb{Z}_2\mathbb{Z}_4$-additive code, or equivalently a generator matrix of its additive dual code, when the code is generated by a canonical generator matrix as in (3). The material of this section is a summary of the results presented in [10, 11].

The appropriate inner product of two vectors $u = (u_1, u_2, \ldots, u_{\alpha+\beta})$, $v = (v_1, v_2, \ldots, v_{\alpha+\beta})$ in $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ is given by [10, 11]

$$\langle u, v\rangle = 2(\sum_{i=1}^{\alpha} u_i v_i) + \sum_{j=\alpha+1}^{\alpha+\beta} u_j v_j \ \in \ \mathbb{Z}_4,$$

as usual, assuming that the first α coordinates are binary and the last β coordinates are quaternary. Note that the computations are made considering the zeros and ones in the α binary coordinates as quaternary zeros and ones, respectively. We refer to this product as the *standard inner product*, that can also be written as

$$\langle u, v\rangle = u{\cdot}J_n{\cdot}v^t,$$

where $J_n = \left(\begin{array}{c|c} 2I_\alpha & \mathbf{0} \\ \hline \mathbf{0} & I_\beta \end{array}\right)$ is a diagonal matrix over $\mathbb{Z}_4$. Note that when $\alpha = 0$ the inner product is the usual one for quaternary vectors, and when $\beta = 0$ it is twice the usual one for binary vectors.

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ and let $C = \Phi(\mathcal{C})$ be the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-linear code. The *additive orthogonal code* of $\mathcal{C}$, denoted by $\mathcal{C}^\perp$, is defined in the standard way as

$$\mathcal{C}^\perp = \{v \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta \mid \langle u, v\rangle = 0 \text{ for all } u \in \mathcal{C}\}.$$

We also call $\mathcal{C}^\perp$ the *additive dual code* of $\mathcal{C}$. The corresponding binary code $\Phi(\mathcal{C}^\perp)$ is denoted by $C_\perp$ and called $\mathbb{Z}_2\mathbb{Z}_4$-*dual code* of C. In the case that $\alpha = 0$, that is, when $\mathcal{C}$ is a quaternary linear code, $\mathcal{C}^\perp$ is also called the *quaternary dual code* of $\mathcal{C}$ and $C_\perp$ the $\mathbb{Z}_4$-*dual code* of C. The additive dual code $\mathcal{C}^\perp$ is also a $\mathbb{Z}_2\mathbb{Z}_4$-additive code, that is, a subgroup of $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$.

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$. The *weight enumerator* of $\mathcal{C}$ is

$$W_{\mathcal{C}}(x, y) = \sum_{c\in\mathcal{C}} x^{n-w(c)} y^{w(c)}, \tag{5}$$

where $n = \alpha + 2\beta$ and $w(c)$ stands for the Lee weight of a codeword $c \in \mathcal{C}$. We know from [10, 11, 23, 55] that for the weight enumerator defined in (5) we have that

$$W_{\mathcal{C}^\perp}(x, y) = \frac{1}{|\mathcal{C}|} W_{\mathcal{C}}(x + y, x - y). \tag{6}$$

Therefore, taking $x = y$, we obtain that $|\mathcal{C}||\mathcal{C}^\perp| = 2^n$. Note that the weight distribution of a $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ refers to the Lee weight, which coincides with the Hamming weight of the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-linear code $C = \Phi(\mathcal{C})$. The codes C and $C_\perp$ are not necessarily linear, so they are not dual in the binary linear sense, but the weight enumerator of $C_\perp$ is the MacWilliams transform of the weight enumerator of C. This remarkable relationship was first established for the specific case of $\mathbb{Z}_4$-linear codes in [33, 64], where it is pointed out that the Kerdock code is the quaternary dual of some Preparata-like code.

Note that one could think on $\mathbb{Z}_2\mathbb{Z}_4$-additive codes (or $\mathbb{Z}_2\mathbb{Z}_4$-linear codes) just as quaternary linear codes (or $\mathbb{Z}_4$-linear codes), replacing the ones with twos in the binary coordinates. However, in this case, the corresponding quaternary linear codes of an $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ and its additive dual code $\mathcal{C}^\perp$ are not necessarily quaternary dual codes. Take, for example, $\alpha = \beta = 1$ and the vectors $v = (1, 3)$ and $w = (1, 2)$. It is easy to check that $\langle v, w \rangle = 0$, so v and w are orthogonal. However, if we replace the ones with twos in the binary coordinates of these vectors, we obtain $v' = (2, 3)$ and $w' = (2, 2)$, which are not orthogonal in the quaternary sense.

The next results were established in [10, 11], where the computation of the type of the additive dual code $\mathcal{C}^\perp$ of a given $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$, as well as the construction of the generator matrix of $\mathcal{C}^\perp$ in terms of the generator matrix of $\mathcal{C}$ are given. Previously, in [33, 64], it was shown that if $\mathcal{C}$ is a quaternary linear code of type $(0, \beta; \gamma, \delta; 0)$, then the quaternary dual code $\mathcal{C}^\perp$ is of type $(0, \beta; \gamma, \beta - \gamma - \delta; 0)$. Moreover, if $\mathcal{C}$ has canonical generator matrix (2), then the generator matrix of $\mathcal{C}^\perp$ is

$$\mathcal{H}_S = \left(\begin{array}{ccc|ccc} \mathbf{0} & 2I_\gamma & 2R^t \\ \hline I_{\beta-\gamma-\delta} & T^t & -(S + RT)^t \end{array} \right), \tag{7}$$

where R, T are matrices over $\mathbb{Z}_4$ with all entries in $\{0, 1\} \subset \mathbb{Z}_4$ of size $\delta \times \gamma$ and $\gamma \times (\beta - \gamma - \delta)$, respectively; and S is a matrix over $\mathbb{Z}_4$ of size $\delta \times (\beta - \gamma - \delta)$. In [10, 11], these two results were generalized for $\mathbb{Z}_2\mathbb{Z}_4$-additive codes as follows:

Theorem 3.1 *[10, 11] Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$. The additive dual code $\mathcal{C}^\perp$ is then of type $(\alpha, \beta; \bar{\gamma}, \bar{\delta}; \bar{\kappa})$, where*

$$\begin{aligned} \bar{\gamma} &= \alpha + \gamma - 2\kappa, \\ \bar{\delta} &= \beta - \gamma - \delta + \kappa, \\ \bar{\kappa} &= \alpha - \kappa. \end{aligned}$$

Theorem 3.2 *[10, 11] Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ with canonical generator matrix (3). Then, the generator matrix of $\mathcal{C}^\perp$ is*

$$\mathcal{H}_S = \left(\begin{array}{cc|ccc} T_b^t & I_{\alpha-\kappa} & \mathbf{0} & \mathbf{0} & 2S_b^t \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 2I_{\gamma-\kappa} & 2R^t \\ \hline T_2^t & \mathbf{0} & I_{\beta+\kappa-\gamma-\delta} & T_1^t & -\big(S_q + RT_1\big)^t \end{array} \right), \tag{8}$$

where T_b, T_2 are matrices over $\mathbb{Z}_2$; T_1, R, S_b are matrices over $\mathbb{Z}_4$ with all entries in $\{0,1\} \subset \mathbb{Z}_4$; and S_q is a matrix over $\mathbb{Z}_4$.

Note that by Theorems 3.1 and 3.2, if $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ with canonical generator matrix (3), then $\mathcal{C}^\perp$ is permutation equivalent to a $\mathbb{Z}_2\mathbb{Z}_4$-additive code with canonical generator matrix

$$\left(\begin{array}{cc|ccc} I_{\bar{\kappa}} & T_b^t & 2S_b^t & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 2R^t & 2I_{\bar{\gamma}-\bar{\kappa}} & \mathbf{0} \\ \hline \mathbf{0} & T_2^t & -\left(S_q + RT_1\right)^t & T_1^t & I_{\bar{\delta}} \end{array}\right), \tag{9}$$

where T_b, T_2 are matrices over $\mathbb{Z}_2$; T_1, R, S_b are matrices over $\mathbb{Z}_4$ with all entries in $\{0,1\} \subset \mathbb{Z}_4$; and S_q is a matrix over $\mathbb{Z}_4$. Moreover, $\bar{\gamma} = \alpha + \gamma - 2\kappa$, $\bar{\delta} = \beta - \gamma - \delta + \kappa$ and $\bar{\kappa} = \alpha - \kappa$.

Finally, it is also worth mentioning that a generator matrix of $\mathcal{C}^\perp$ can be seen as a parity-check matrix for $\mathcal{C}$. Analogously, by linearity, we can use a generator matrix of $\mathcal{C}$ as a parity-check matrix for $\mathcal{C}^\perp$. Therefore, Theorem 3.2 also shows how to construct the parity-check matrix of a $\mathbb{Z}_2\mathbb{Z}_4$-additive code generated by a canonical generator matrix as in (3).

Example 3.3 *Let $\mathcal{C}_S$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(3,4;3,1;1)$ with canonical generator matrix (4) given in Example 2.2. By Theorems 3.1 and 3.2, the additive dual code $\mathcal{C}_S^\perp$ is of type $(3,4;0,3;0)$ and has generator matrix*

$$\mathcal{H}_S = \left(\begin{array}{ccc|cccc} 1 & 0 & 1 & 1 & 0 & 0 & 3 \\ 1 & 0 & 1 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right).$$

4. $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual codes

A $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ is *self-orthogonal* if $\mathcal{C}^\perp \subseteq \mathcal{C}$, and it is *self-dual* if $\mathcal{C} = \mathcal{C}^\perp$. Self-duality for binary and quaternary linear codes has been extensively studied. For the quaternary case, the Gray map images of quaternary self-dual codes are also very interesting since they are *formally self-dual*, that is, their Hamming weight enumerators are invariant under the MacWilliams transform [33]. Therefore, a next logical step is to study $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual codes and their Gray map images. The material in this chapter is a summary of the results presented in [7, 25].

4.1. Classification of $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual codes

In this subsection, we show that $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual codes can be characterized in terms of some properties such us separability, antipodality and Type. Moreover, we determine all the possible values for the parameters α and β of a $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual code, as well as the weight enumerator of such codes.

4.1.1. SEPARABILITY AND ANTIPODALITY

The following proposition determine the type of a $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual code taking into account the relation of the parameters given in Theorem 3.1.

Proposition 4.1 *[7] If $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual code, then $\mathcal{C}$ is of type $(2\kappa, \beta; \beta + \kappa - 2\delta, \delta; \kappa)$, $|\mathcal{C}| = 2^{\kappa+\beta}$, $|\mathcal{C}_b| = 2^{\kappa+\beta-\delta}$, and $(\mathcal{C}_b)_X$ is a binary self-dual code.*

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code. If $\mathcal{C} = \mathcal{C}_X \times \mathcal{C}_Y$, then $\mathcal{C}$ is called *separable*. If $\mathcal{C}$ is separable, then the generator matrix of $\mathcal{C}$ in standard form is

$$\mathcal{G}_S = \left(\begin{array}{cc|ccc} I_\kappa & T_b & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 2T_1 & 2I_{\gamma-\kappa} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & S_q & R & I_\delta \end{array}\right).$$

The following theorem show some properties of separable $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual codes.

Theorem 4.2 *[7] Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual code of type $(2\kappa, \beta; \beta + \kappa - 2\delta, \delta; \kappa)$. The following statements are equivalent:*

(i) $\mathcal{C}_X$ *is a binary self-orthogonal code.*

(ii) $\mathcal{C}_X$ *is a binary self-dual code.*

(iii) $|\mathcal{C}_X| = 2^\kappa$.

(iv) $\mathcal{C}_Y$ *is a quaternary self-orthogonal code.*

(v) $\mathcal{C}_Y$ *is a quaternary self-dual code.*

(vi) $|\mathcal{C}_Y| = 2^\beta$.

(vii) $\mathcal{C}$ *is separable.*

From the above theorem, if $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual code, then $\mathcal{C}_X$ is binary self-dual if and only if $\mathcal{C}_Y$ is quaternary self-dual. Moreover, if $\mathcal{C}$ is a separable $\mathbb{Z}_2\mathbb{Z}_4$-additive code, $\mathcal{C}_X$ is binary self-dual and $\mathcal{C}_Y$ is quaternary self-dual, then $\mathcal{C}$ is also self-dual, as it is stated in the following theorem.

Theorem 4.3 *[7] If C is a binary self-dual code of length α and $\mathcal{D}$ is a quaternary self-dual code of length β, then $C \times \mathcal{D}$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual code of length $\alpha + \beta$.*

We say that a binary code C is *antipodal* if, for any codeword $z \in C$, we have that $z+\mathbf{1} \in C$. If $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive code, we say that $\mathcal{C}$ is antipodal if $\Phi(\mathcal{C})$ is antipodal. Clearly, a $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C} \subseteq \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ is antipodal if and only if $(\mathbf{1}^\alpha, \mathbf{2}^\beta) \in \mathcal{C}$.

We define the Type of a $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual code $\mathcal{C}$ in terms of the weights of its codewords. If $\mathcal{C}$ has odd weights, then it is said to be *Type 0*. If it has only even weights, then the code is said to be *Type I*. If all the codewords have doubly-even weight, then it is said to be *Type II*. In general, if all the codewords of $\mathcal{C}$ have even weights, then $\mathcal{C}$ is an even code;

otherwise, $\mathcal{C}$ is an odd code. We remark that applying Theorem 4.3 to a binary self-dual code and a quaternary self-dual code gives a Type I code, and applying Theorem 4.3 to a binary doubly-even self-dual code and a quaternary doubly-even code gives a Type II code.

Now, we give some relations among Type, separability and antipodality.

Proposition 4.4 *[7] Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual code.*

(i) $\mathcal{C}$ is antipodal if and only if $\mathcal{C}$ is of Type I or Type II.

(ii) If $\mathcal{C}$ is separable, then $\mathcal{C}$ is antipodal.

(iii) If $\mathcal{C}$ is of Type 0, then $\mathcal{C}$ is non-separable and non-antipodal.

The following examples show the existence of all possible cases we have described.

Example 4.5 (Type 0) *Let $\mathcal{D} = \{(00|00), (00|22), (11|02), (11|20)\}$. Then, the code $\mathcal{C}_1 = \mathcal{D} \cup (\mathcal{D} + (01|11))$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual code of type $(2, 2; 1, 1; 1)$ and has generator matrix*

$$\mathcal{G}_1 = \left(\begin{array}{cc|cc} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right).$$

The weight enumerator of this code is $W_{\mathcal{C}}(x, y) = x^6 + 4x^3y^3 + 3x^2y^4$. Note that it has codewords of odd weight, hence it is a Type 0 code and by Proposition 4.4 it is non-separable.

Example 4.6 (Type I, separable) *A $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual code with $\alpha, \beta \geq 1$ should have $\alpha \geq 2$, since α must be even. A $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual code with minimum length has $\alpha = 2$, $\beta = 1$ and $2^{\kappa+\beta} = 2^{1+1} = 4$ codewords. For example, $\mathcal{C}_2 = \{(00|0), (00|2), (11|0), (11|2)\}$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual code of type $(2, 1; 2, 0; 1)$ and has generator matrix*

$$\mathcal{G}_2 = \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 2 \end{array}\right).$$

Notice that for $\alpha = 2$ and $\beta = 1$, it is not possible to have odd weight codewords. Thus, the code must be of Type I and antipodal. Also, we have that the code restricted to the quaternary coordinates is $\{0, 2\}$, which is self-dual and hence, by Theorem 4.2, $\mathcal{C}_2$ is separable.

Example 4.7 (Type I, non-separable) *The codes $\mathcal{C}_3$ and $\mathcal{C}_4$, generated by $\mathcal{G}_3$ and $\mathcal{G}_4$, respectively, are Type I non-separable, where*

$$\mathcal{G}_3 = \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 2 & 0 \\ \hline 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right) \quad \mathcal{G}_4 = \left(\begin{array}{cccc|cccccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 2 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 \\ \hline 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \end{array}\right).$$

Example 4.8 (Type II, separable) *Let C be the extended binary Hamming code of length 8 and let $\mathcal{D}$ be the quaternary linear code generated by*

$$\begin{pmatrix} 2 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Then, $|\mathcal{D}| = 2^2 4^1 = 2^4$, which is the correct size to be self-dual. Clearly, $\mathcal{D}$ is quaternary self-orthogonal and hence self-dual. On the other hand, C is a binary self-dual code. Since both codes have only doubly-even weights, we conclude that $\mathcal{C}_5 = C \times \mathcal{D}$ is Type II separable.

Example 4.9 (Type II, non-separable) *The code $\mathcal{C}_6$ generated by $\mathcal{G}_6$ is self-orthogonal, where*

$$\mathcal{G}_6 = \left(\begin{array}{cccccccc|cccc} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 2 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right).$$

Since $|\mathcal{C}_6| = 2^8$, the code $\mathcal{C}_6$ is self-dual. Clearly it is non-separable, since $(\mathcal{C}_6)_X$ is not self-orthogonal. On the other hand, it can be checked that all weights are doubly-even.

4.1.2. ALLOWABLE ALPHA AND BETA VALUES

The following lemma is easily proven.

Lemma 4.10 *[7] If $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ and $\mathcal{D}$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual code of type $(\alpha', \beta'; \gamma', \delta'; \kappa')$, then $\mathcal{C} \times \mathcal{D}$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual code of type $(\alpha + \alpha', \beta + \beta'; \gamma + \gamma', \delta + \delta'; \kappa + \kappa')$.*

The following statements show some conditions for the parameters of a $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual code depending on its Type.

Theorem 4.11 *[7] Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual code of type $(\alpha, \beta; \gamma, \delta; \kappa)$, with $\alpha, \beta > 0$.*

(i) If $\mathcal{C}$ is Type 0, then $\alpha \geq 2$, $\beta \geq 2$.

(ii) If $\mathcal{C}$ is Type I and separable, then $\alpha \geq 2$, $\beta \geq 1$.

(iii) If $\mathcal{C}$ is Type I and non-separable, then $\alpha \geq 4$, $\beta \geq 4$.

(iv) If $\mathcal{C}$ is Type II, then $\alpha \geq 8$, $\beta \geq 4$.

Let α_{min}, β_{min} be the minimum values of α and β given in Theorem 4.11 for each case (i) to (iv). Note that the codes $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_5$ and $\mathcal{C}_6$ are examples where α and β are the minimum values α_{min} and β_{min}.

Theorem 4.12 *[7] Let α_{min} and β_{min} be as defined above.*

(i) There exist a Type 0 or Type I code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ if and only if $\alpha = \alpha_{min} + 2a$, $a \geq 0$, $\beta \geq \beta_{min}$.

(ii) There exist a Type II code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ if and only if $\alpha = \alpha_{min} + 8a$, $\beta = \beta_{min} + 4b$, $a, b \geq 0$.

Finally, we remark a special case where the Gray map image is also a self-dual code.

Theorem 4.13 *[7] If $\mathcal{C}$ is a Type II code and $\Phi(\mathcal{C})$ is linear, then $\Phi(\mathcal{C})$ is a binary doubly-even self-dual code.*

4.1.3. WEIGHT ENUMERATORS

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual code of type $(\alpha, \beta; \gamma, \delta; \kappa)$, and let $C = \Phi(\mathcal{C})$. Since the weight enumerator of $\mathcal{C}$ satisfies (6), the code $\mathcal{C}$ is held invariant by the action of the matrix

$$M = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}, \tag{10}$$

which satisfies $M^2 = I_2$. We also know that the length of C, $n = \alpha + 2\beta$, is even, so $W_C(-x, -y) = W_C(x, y)$ and so the weight enumerator is held invariant by the matrix

$$B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{11}$$

Theorem 4.14 *[7] Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual code. Then,*

$$\begin{cases} W_{\mathcal{C}}(x, y) \in \mathbb{C}[x^2 + y^2, y(x - y)], & \text{if } \mathcal{C} \text{ is Type 0,} \\ W_{\mathcal{C}}(x, y) \in \mathbb{C}[x^2 + y^2, x^2y^2(x^2 - y^2)^2], & \text{if } \mathcal{C} \text{ is Type I,} \\ W_{\mathcal{C}}(x, y) \in \mathbb{C}[x^8 + 14x^4y^4 + y^8, x^4y^4(x^4 - y^4)^4], & \text{if } \mathcal{C} \text{ is Type II.} \end{cases} \tag{12}$$

The results on the characterization of $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual codes of Type 0, Type I and Type II are summarized in the following table:

	Type 0	Type I	Type II
separability	non-separable	separable or non-separable	separable or non-separable
antipodality	non-antipodal	antipodal	antipodal
separable $\alpha, \beta; a, b \geq 0$	- -	$\alpha = 2 + 2a$ $\beta = 1 + b$	$\alpha = 8 + 8a$ $\beta = 4 + 4b$
non-separable $\alpha, \beta; a, b \geq 0$	$\alpha = 2 + 2a$ $\beta = 2 + b$	$\alpha = 4 + 2a$ $\beta = 4 + b$	$\alpha = 8 + 8a$ $\beta = 4 + 4b$
$W_{\mathcal{C}}(x, y)$	$\mathbb{C}[x^2 + y^2,$ $y(x - y)]$	$\mathbb{C}[x^2 + y^2,$ $x^2y^2(x^2 - y^2)^2]$	$\mathbb{C}[x^8 + 14x^4y^4 + y^8,$ $x^4y^4(x^4 - y^4)^4]$

4.2. Construction technique of $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual codes

4.2.1. USING THE SHADOW OF THE CODE

Consider the code given in Example 4.5. The codewords that have even weight are precisely $\{(00|00), (11|20), (11|02), (00|22)\}$. These form a linear subcode consisting of exactly half the codewords. This fact holds in general, that is, if $\mathcal{C}$ is a Type 0 code, then the subcode $\mathcal{C}_0 = \{v \mid v \in \mathcal{C}, w(v) \equiv 0 \pmod 2\}$ is a linear subcode with $|\mathcal{C}| = 2|\mathcal{C}_0|$, and $W_{\mathcal{C}_0}(x, y) = \frac{1}{2}(W_{\mathcal{C}}(x, -y) + W_{\mathcal{C}}(x, y))$. Note that for binary linear self-dual codes of Type I, a similar property is satisfied except that $\mathcal{C}_0$ consists of doubly-even codewords. This notion cannot be extended here to $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual codes of Type I, since the sum of two codewords with doubly-even weight may not have doubly-even weight.

We define the shadow of a $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ to be $S = \mathcal{C}_0^{\perp}\backslash\mathcal{C}$. The shadow is a nonlinear code with $|S| = |\mathcal{C}|$. Recall that the matrix M given in (10) represents the action of the MacWilliams relations. Shadows of binary codes first appeared in [65] but were first specifically labeled as a code in [20]. The shadow has been generalized to numerous alphabets, see [53] for a complete description. For a $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual code of Type 0, the weight enumerator of the shadow S is $W_S(x, y) = \frac{1}{|\mathcal{C}|}M \cdot W_{\mathcal{C}}(x, -y) = W_{\mathcal{C}}\left(\frac{x+y}{\sqrt{2}}, \frac{-(x-y)}{\sqrt{2}}\right)$ [7]. The difference with the usual binary case is that these weight enumerators are not necessarily possible weight enumerators for binary self-dual codes, since in the case of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes of Type 0 it may be odd weight vectors. Given a possible weight enumerator for Type 0 codes, one can compute the weight enumerator of the shadow.

Example 4.15 *Let $\mathcal{C}$ be the $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual code given in Example 4.5. We can compute*

$$\begin{aligned} W_{\mathcal{C}}(x, y) &= x^6 + 4x^3y^3 + 3x^2y^4, \\ W_{\mathcal{C}_0}(x, y) &= x^6 + 3x^2y^4, \\ W_S(x, y) &= 3x^4y^2 + 4x^3y^3 + y^6. \end{aligned}$$

Note that $\mathcal{C}_0 = \mathcal{D}$, and the shadow is $S = \mathcal{C}_0^{\perp}\backslash\mathcal{C} = \{(11|00), (01|11), (10|13), (00|20), (01|33), (00|02), (11|22), (10|31)\}$.

The code $\mathcal{C}_0$ has 4 cosets in $\mathcal{C}_0^{\perp}$. Let $\mathcal{C}_{0,0} = \mathcal{C}_0$ and $\mathcal{C}_{1,0} = \mathcal{C}\backslash\mathcal{C}_{0,0}$. Let $\mathcal{C}_0^{\perp} = \mathcal{C} \cup \mathcal{C}_{0,1} \cup \mathcal{C}_{1,1}$, i.e. $S = \mathcal{C}_{0,1} \cup \mathcal{C}_{1,1}$. There are vectors s and t such that $\mathcal{C} = \langle \mathcal{C}_0, t\rangle$ and $\mathcal{C}_0^{\perp} = \langle \mathcal{C}, s\rangle$. Then, we have that $\mathcal{C}_{i,j} = \mathcal{C}_0 + it + js$. Taking $s = (\mathbf{1}^{\alpha}, \mathbf{2}^{\beta})$, $\mathcal{C}_{0,0}$ and $\mathcal{C}_{0,1}$ consist of even weight vectors and $\mathcal{C}_{1,0}$ and $\mathcal{C}_{1,1}$ consist of odd weight vectors. Hence we have the orthogonality given in Table 1.

	$\mathcal{C}_{0,0}$	$\mathcal{C}_{1,0}$	$\mathcal{C}_{0,1}$	$\mathcal{C}_{1,1}$
$\mathcal{C}_{0,0}$	0	0	0	0
$\mathcal{C}_{1,0}$	0	0	2	2
$\mathcal{C}_{0,1}$	0	2	0	2
$\mathcal{C}_{1,1}$	0	2	2	0

Table 1: Orthogonality Relations

Proposition 4.16 *[7] Let $\mathcal{C}$ be a Type 0 code. Then, the codes $\mathcal{C}_{0,0} \cup \mathcal{C}_{0,1} = \langle \mathcal{C}, s \rangle$ and $\mathcal{C}_{0,0} \cup \mathcal{C}_{1,1} = \langle \mathcal{C}, s + t \rangle$ are self-dual codes that are not Type 0.*

We can now generalize the construction first described in [17] but greatly expanded in [28].

Theorem 4.17 *[7] Let $\mathcal{C}$ and $\mathcal{D}$ be Type 0 codes in $\mathbb{Z}_2^{\alpha} \times \mathbb{Z}_4^{\beta}$ and $\mathbb{Z}_2^{\alpha'} \times \mathbb{Z}_4^{\beta'}$, respectively. If $\langle C_{i,j}, C_{i,j} \rangle = \langle D_{i,j}, D_{i,j} \rangle$, for all $i, j \in \{0, 1\}$, then the code*

$$(\mathcal{C}_{0,0}, \mathcal{D}_{0,0}) \cup (\mathcal{C}_{0,1}, \mathcal{D}_{0,1}) \cup (\mathcal{C}_{1,0}, \mathcal{D}_{1,0}) \cup (\mathcal{C}_{1,1}, \mathcal{D}_{1,1})$$

is a self-dual code in $\mathbb{Z}_2^{\alpha+\alpha'} \times \mathbb{Z}_4^{\beta+\beta'}$. If $\langle C_{i,j}, C_{i,j} \rangle \neq \langle D_{i,j}, D_{i,j} \rangle$, for some $i, j \in \{0, 1\}$, then the code

$$(\mathcal{C}_{0,0}, \mathcal{D}_{0,0}) \cup (\mathcal{C}_{0,1}, \mathcal{D}_{1,1}) \cup (\mathcal{C}_{1,0}, \mathcal{D}_{1,0}) \cup (\mathcal{C}_{1,1}, \mathcal{D}_{0,1})$$

is a self-dual code in $\mathbb{Z}_2^{\alpha+\alpha'} \times \mathbb{Z}_4^{\beta+\beta'}$.

4.2.2. EXTENDING THE LENGTH

Let $\mathcal{C}$ be a self-dual code in $\mathbb{Z}_2^{\alpha} \times \mathbb{Z}_4^{\beta}$ and let $v \in \mathbb{Z}_2^{\alpha} \times \mathbb{Z}_4^{\beta}$ such that $v \notin \mathcal{C}$. We define $\mathcal{C}_v = \{u \in \mathcal{C} \mid \langle u, v \rangle = 0\}$. It is immediate that $\mathcal{C}_v$ is a subgroup of $\mathcal{C}$ and that the index $[\mathcal{C} : \mathcal{C}_v]$ is either 2 or 4. In either case, we have that $[\mathcal{C} : \mathcal{C}_v] = [\mathcal{C}_v^{\perp} : \mathcal{C}]$ and that $\mathcal{C}_v^{\perp} = \langle \mathcal{C}, v \rangle$. Let w be the vector such that $\mathcal{C} = \langle \mathcal{C}_v, w \rangle$. We can then write $\mathcal{C}_v^{\perp} = \langle \mathcal{C}_v, w, v \rangle$.

Example 4.18 *Let $\mathcal{C}_1$ be the $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual code given in Example 4.5, generated by $\mathcal{G}_1$, and let $v = (00|20)$. Then, $\mathcal{C}_v = \{(00|00), (11, |20), (00|22), (11|02)\}$, generated by*

$$\mathcal{G}_v = \left(\begin{array}{cc|cc} 1 & 1 & 2 & 0 \\ 0 & 0 & 2 & 2 \end{array}\right),$$

and $\mathcal{C} = \langle \mathcal{C}_v, w \rangle$, where $w = (01|11)$. Therefore, the code $\mathcal{C}_v^{\perp}$ is generated by

$$\mathcal{H}_v = \left(\begin{array}{cc|cc} 1 & 1 & 2 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 0 \\ \hline 0 & 1 & 1 & 1 \end{array}\right).$$

We can form a code $\bar{C}$ by extending the code $C = \mathcal{C}_v^{\perp}$ in the following manner. For $u \in \mathcal{C}_v^{\perp}$ we let $\bar{u} = (u'_X, u_X, u_Y, u'_Y)$, where u'_X is an extension of the binary part and u'_Y is an extension of the quaternary part. Then let $\bar{C} = \langle \{\bar{u} \mid u \in \mathcal{C}_v^{\perp}\} \rangle$. We choose u'_X and u'_Y so that $\bar{C}$ is a self-orthogonal code. We denote by α' the length of u'_X and by β' the length of u'_Y. If $\bar{C}$ is not self-dual, then we may need to add additional vectors to the code. In all cases, we let u'_X and u'_Y be $\mathbf{0}$ if $u \in \mathcal{C}_v$, and we denote by $\bar{\mathcal{C}}_v$ the extension of $\mathcal{C}_v$. Since $\mathcal{C} = \langle \mathcal{C}_v, w \rangle$, we denote $\bar{\mathcal{C}} = \langle \bar{\mathcal{C}}_v, \bar{w} \rangle$. We separate the construction into three cases, namely Case 1 is when $\beta' = 0$, Case 2 is when $\alpha' = 0$, and Case 3 is when neither are 0.

Theorem 4.19 *[7] Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ and $v \notin \mathcal{C}$. Let w, $\mathcal{C}_v$ be as before and $C = \mathcal{C}_v^{\perp} = \langle \mathcal{C}_v, w, v \rangle$. There exists a $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual code $D = \langle \bar{C}, V \rangle$ of type $(\alpha + \alpha', \beta + \beta'; \gamma', \delta'; \kappa')$, for some set of vectors V with the following conditions:*

(i) $\alpha' \neq 0$ *and* $\beta' = 0$ *only if* $\langle v, w \rangle = 2$ *and* $\langle v, v \rangle \in \{0, 2\}$,

(ii) $\alpha' = 0$ *and* $\beta' \neq 0$ *only if* $\langle v, w \rangle = 2$ *or* $\langle v, w \rangle \in \{1, 3\}$ *and* $\langle v, v \rangle \in \{1, 3\}$,

(iii) $\alpha' \neq 0$ *and* $\beta' \neq 0$.

The steps to obtain an extended $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\bar{C}$ from a $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual code $\mathcal{C}$ are the following. First, select $v \notin \mathcal{C}$, $w \in \mathcal{C} \backslash \mathcal{C}_v$ such that $\mathcal{C} = \langle \mathcal{C}_v, w \rangle$, with the conditions in $\langle v, v \rangle, \langle v, w \rangle$ described in Theorem 4.19. After that, determine the values of $v'_X, v'_Y, w'_X, w'_Y, V$ from Tables 2-6. Finally, if $\mathcal{G}_v$ is the generator matrix of $\mathcal{C}_v$, then the generator matrix of $\bar{C}$ is

$$\mathcal{G}_D = \begin{pmatrix} \mathbf{0} & \mathcal{G}_v & \mathbf{0} \\ v'_X & v & v'_Y \\ w'_X & w & w'_Y \\ & V & \end{pmatrix}.$$

$\langle v, v \rangle$	v'_X	w'_X	V
0	$(0,0,1,1)$	$(0,1,0,1)$	$\{(1,1,1,1,\mathbf{0})\}$
2	$(0,1)$	$(1,1)$	$\emptyset$

Table 2: Case $\alpha' \neq 0, \beta' = 0$.

$\langle v, v \rangle$	v'_Y	w'_Y	V
0	$(1,1,1,1)$	$(2,0,0,0)$	$\{(\mathbf{0},0,2,2,0), (\mathbf{0},0,0,2,2)\}$
1	$(1,1,1)$	$(2,0,0)$	$\{(\mathbf{0},0,2,2)\}$
2	$(1,1)$	$(2,0)$	$\emptyset$
3	(1)	(2)	$\emptyset$

Table 3: Case $\alpha' = 0, \beta' \neq 0$, $\langle v, w \rangle = 2$.

$\langle v, v \rangle$	v'_Y	w'_Y	V
1	$(1,1,1,0)$	$(1,1,1,1)$	$\{(\mathbf{0},0,2,2,0), (\mathbf{0},2,2,0,0)\}$
3	$(3,0,0,0)$	$(1,1,1,1)$	$\{(\mathbf{0},0,2,2,0), (\mathbf{0},0,0,2,2)\}$

Table 4: Case $\alpha' = 0, \beta' \neq 0$, $\langle v, w \rangle = 1$.

$\langle v, v\rangle$	v'_X	v'_Y	w'_X	w'_Y	V
0	$(1,0)$	$(1,0,1)$	$(1,0)$	$(1,1,0)$	$\{(1,1,\mathbf{0},2,0,0),(1,1,\mathbf{0},0,2,0)\}$
1	$(1,0)$	$(1,0)$	$(1,0)$	$(1,1)$	$\{(1,1,\mathbf{0},2,0)\}$
2	$(1,1)$	$(0,1,1)$	$(1,0)$	$(1,1,0)$	$\{(1,1,\mathbf{0},2,0,0),(1,1,\mathbf{0},0,2,2)\}$
3	$(1,1)$	$(1,0)$	$(1,0)$	$(1,1)$	$\{(1,1,\mathbf{0},0,2)\}$

Table 5: Case $\alpha' \neq 0, \beta' \neq 0$, $\langle v, w\rangle = 1$.

$\langle v, v\rangle$	v'_X	v'_Y	w'_X	w'_Y	V
0	$(1,0)$	$(1,1)$	$(1,1)$	$(2,2)$	$\{(1,1,\mathbf{0},2,0)\}$
1	$(1,0)$	$(1,0)$	$(1,1)$	$(0,2)$	$\{(1,1,\mathbf{0},2,0)\}$
2	$(1,1)$	$(1,3)$	$(1,0)$	$(1,1)$	$\emptyset$
3	$(0,0)$	$(0,1)$	$(1,1)$	$(0,2)$	$\{(1,1,\mathbf{0},2,0)\}$

Table 6: Case $\alpha' \neq 0, \beta' \neq 0$, $\langle v, w\rangle = 2$.

4.2.3. NEIGHBOR CONSTRUCTION

Let $\mathcal{C}$ be a self-dual code in $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ and let v be a self-orthogonal vector such that $v \notin \mathcal{C}$. As before, we denote by $\mathcal{C}_v$ the subcode $\mathcal{C}_v = \{u \in \mathcal{C} \mid \langle u, v\rangle = 0\}$. Let $N(\mathcal{C}, v) = \langle \mathcal{C}_v, v\rangle$. The following construction technique is a generalization of the technique used for the neighbor construction for codes over finite fields.

Theorem 4.20 *[7] Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual code and let v be a self-orthogonal vector such that $v \notin \mathcal{C}$. Then, $N(\mathcal{C}, v)$ is a self-dual code.*

Theorem 4.21 *[7] Every $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual code can be found by repeated application of the neighbor code from any self-dual code of that length.*

4.3. $\mathbb{Z}_2\mathbb{Z}_4$-additive formally self-dual codes

In general, a code C over any ring is said to be *formally self-dual* if its weight enumerator is the same as the weight enumerator of its orthogonal. For example, any self-dual code is necessarily formally self-dual but, of course, there are formally self-dual codes that are not self-dual. For quaternary codes, a code can be formally self-dual with respect to the Lee weight enumerator but not with respect to the Hamming weight enumerator and vice versa.

A $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ is *formally self-dual* if $W_{\mathcal{C}^\perp}(x, y) = W_{\mathcal{C}}(x, y)$, with respect to the weight enumerator given in (5). In this subsection, we summarizes the results given in [25].

Example 4.22 *Let $\mathcal{C}$ and $\mathcal{C}$ be the codes generated by the following matrices*

$$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & 0 \end{array}\right), \quad \left(\begin{array}{cc|c} 0 & 0 & 1 \end{array}\right),$$

respectively. It is clear that $\mathcal{C}^\perp = \mathcal{D}$ and that the weight enumerator of both is $W_{\mathcal{C}}(x, y) = W_{\mathcal{C}^\perp}(x, y) = x^4 + 2x^3y + x^2y^2$. Hence, these codes are $\mathbb{Z}_2\mathbb{Z}_4$-additive formally self-dual. The

code $\mathcal{C}$ has parameters $(2,1;2,0;2)$ whereas the code $\mathcal{D}$ has parameters $(2,1;0,1;0)$. Note that $\mathbb{Z}_2\mathbb{Z}_4$-additive formally self-dual codes do not have necessarily the same parameters.

Example 4.23 *Consider the following generator matrices*

$$\mathcal{G} = \left(\begin{array}{cc|cc} 0 & 1 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right), \quad \mathcal{G}' = \left(\begin{array}{cc|cc} 1 & 0 & 2 & 0 \\ 1 & 1 & 3 & 1 \end{array}\right).$$

Let $\mathcal{C}$ and $\mathcal{C}'$ be the codes generated by $\mathcal{G}$ and $\mathcal{G}'$, respectively. Theses codes are odd $\mathbb{Z}_2\mathbb{Z}_4$-additive formally self-dual of length 6 that are orthogonal of each other with weight enumerator $W_{\mathcal{C}}(x,y) = W_{\mathcal{C}^\perp}(x,y) = x^6 + 4x^3y^3 + 3x^2y^4$. Note that its Gray map image has minimum weight 3, which is higher than any self-dual code at that length.

Let C be a binary (possibly nonlinear) code. Then, we say that C is a *formally self-dual code* if the weight enumerator of C is held invariant by the action of the MacWilliams relations. Note that in this case we are not assuming that C is a linear code. If C is a nonlinear binary code, then we do not have that $W_C(x,y) = W_{C^\perp}(x,y)$ since $C^\perp = \langle C \rangle^\perp$ and $\langle C \rangle$ is larger than C when C is nonlinear. In general, what we are seeking are binary codes that are images of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes under the Gray map, since they correspond to the structures defined by Delsarte, and have weight enumerators held invariant by the MacWilliams relations.

The following theorem is immediate, given that the Gray map Φ is an isometry.

Theorem 4.24 *[25] If $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive formally self-dual code, then $\Phi(\mathcal{C})$ is a binary formally self-dual code.*

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive formally self-dual code. The weight enumerator is held invariant by the action of the MacWilliams relations, and hence the invariant theory for binary self-dual codes described in [41, Chapter 19] also applies to $\mathcal{C}$ in order to study the possible weight enumerators.

As in the case of $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual codes, if $\mathcal{C}$ and $\mathcal{C}'$ are $\mathbb{Z}_2\mathbb{Z}_4$-additive formally self-dual codes, then $\mathcal{C} \times \mathcal{C}'$ is a formally self-dual code and $W_{\mathcal{C}\times\mathcal{C}'}(x,y) = W_{\mathcal{C}}(x,y)W'_{\mathcal{C}}(x,y)$. Therefore, if C is a binary formally self-dual code and $\mathcal{D}$ is a quaternary formally self-dual code, then $C \times \mathcal{D}$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive formally self-dual code. Unlike for $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual codes, a $\mathbb{Z}_2\mathbb{Z}_4$-additive formally self-dual code of Type 0 can be separable. For example, consider the code $\langle (10|) \rangle \times \langle |2 \rangle \subseteq \mathbb{Z}_2^2 \times \mathbb{Z}_4$. Its orthogonal code is $\langle (01|) \rangle \times \langle |2 \rangle$. These codes have vectors of weight 1 and hence are odd. Therefore, the code is an odd separable formally self-dual code. Moreover, it is separable but not antipodal; i.e., $(11|2)$ is not in the code.

Similarly to $\mathbb{Z}_2\mathbb{Z}_4$-additive self-dual codes, in [25], we can find the possible weight enumerators, the allowable parameters α and β and the relation among Type, separability and antipodality. We sumarize the results in the following table:

	Type 0	Type I	Type II
separability	separable or non-separable	separable or non-separable	separable or non-separable
antipodality	non-antipodal	antipodal	antipodal
allowable α, β $a, b, c \geq 0$	$\alpha = 2 + 2a$ and $\beta = b$; or $\alpha = 0$ and $\beta = 2 + b$	$\alpha = 2a$ $\beta = b$ $a + b > 0$	$\alpha = 8a + 4b$ $\beta = 2b + 4c$ $a + b + c > 0$
$W_{\mathcal{C}}(x, y)$	$\mathbb{C}[x^2 + y^2,$ $y(x - y)]$	$\mathbb{C}[x^2 + y^2,$ $x^8 + 14x^4y^4 + y^8]$	$\mathbb{C}[x^8 + 14x^4y^4 + y^8,$ $x^4y^4(x^4 - y^4)^4]$

In [25], we also show different constructions of $\mathbb{Z}_2\mathbb{Z}_4$-additive formally self-dual codes starting from a $\mathbb{Z}_2\mathbb{Z}_4$-additive formally self-dual code $\mathcal{C}$; namely the neighbor and the building up constructions. For the neighbor construction, if $\mathcal{C}$ is Type 0 $\mathbb{Z}_2\mathbb{Z}_4$-additive formally self-dual code, then a $\mathbb{Z}_2\mathbb{Z}_4$-additive formally self-dual even code is obtained with the same length as $\mathcal{C}$. On the other hand, if $C = \Phi(\mathcal{C})$ is linear, then we can consider the code $\bar{\mathcal{C}}$ constructed as $\bar{\mathcal{C}} = \Phi^{-1}(\bar{C})$, where $\bar{C}$ is obtained from C by applying the binary building up construction, as in [24]. However, usually and most interestingly, $\Phi(\mathcal{C})$ is nonlinear. In that case, the building up construction is generalized for $\mathbb{Z}_2\mathbb{Z}_4$-additive formally self-dual codes whose Gray map image is nonlinear.

5. Rank and kernel of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes

Two structural properties of nonlinear binary codes are the rank and dimension of the kernel. The *rank* of a binary code C, $rank(C)$, is simply the dimension of $\langle C \rangle$, which is the linear span of the codewords of C. The *kernel* of a binary code C, $K(C)$, is the set of vectors that leave C invariant under translation, i.e. $K(C) = \{x \in \mathbb{Z}_2^n \mid C + x = C\}$. If C contains the all-zero vector, then $K(C)$ is a binary linear subcode of C. In general, C can be written as the union of cosets of $K(C)$, and $K(C)$ is the largest such linear code for which this is true [2]. We denote the dimension of the kernel of C by $ker(C)$.

The rank and dimension of the kernel have been studied for some families of $\mathbb{Z}_2\mathbb{Z}_4$-linear codes [8, 9, 15, 14, 38, 43, 44, 48]. These two parameters do not always give a full classification of $\mathbb{Z}_2\mathbb{Z}_4$-linear codes, since two nonisomorphic $\mathbb{Z}_2\mathbb{Z}_4$-linear codes could have the same rank and dimension of the kernel. In spite of that, they can help in classification, since if two $\mathbb{Z}_2\mathbb{Z}_4$-linear codes have different ranks or dimensions of the kernel, they are nonisomorphic. Moreover, in this case the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-additive codes are not monomially equivalent, so these two parameters can also help to distinguish between $\mathbb{Z}_2\mathbb{Z}_4$-additive codes that are not monomially equivalent.

In this section, we give some results on the rank and dimension of the kernel of $\mathbb{Z}_2\mathbb{Z}_4$-linear codes, including properties related to the linearity of $\mathbb{Z}_2\mathbb{Z}_4$-linear codes. We show the possible ranks and dimensions of the kernel for these codes, and how to construct a $\mathbb{Z}_2\mathbb{Z}_4$-linear code with rank r and dimension of the kernel k for any possible pair of values (r, k). The material of this section is a summary of the results presented in [31, 29].

Regarding the linearity of a $\mathbb{Z}_2\mathbb{Z}_4$-linear code, in [29], the same results proved for quaternary vectors and quaternary linear codes in [33, 64] were generalized as follows. Let $u * v$ denote the component-wise product for any $u, v \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$. We have that for all $u, v \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$, $\Phi(u+v) = \Phi(u) + \Phi(v) + \Phi(2u * v)$. Note that if u or v are vectors in $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ of order two, then $\Phi(u+v) = \Phi(u) + \Phi(v)$. Therefore, if $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive code, then the $\mathbb{Z}_2\mathbb{Z}_4$-linear code $C = \Phi(\mathcal{C})$ is a binary linear code if and only if $2u * v \in \mathcal{C}$ for all $u, v \in \mathcal{C}$. Note that if $\mathcal{G}$ is a generator matrix of a $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ as in (1) and $\{u_i\}_{i=1}^\gamma$ and $\{v_j\}_{j=1}^\delta$ are the row vectors of order two and four in $\mathcal{G}$, respectively, then the $\mathbb{Z}_2\mathbb{Z}_4$-linear code $C = \Phi(\mathcal{C})$ is a binary linear code if and only if $2v_j * v_k \in \mathcal{C}$, for all j, k satisfying $1 \le j < k \le \delta$, since the component-wise product is bilinear.

Proposition 5.1 *[29] There exists a $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ of type $(\alpha, \beta; \gamma, \delta; \kappa)$ if and only if*

$$\begin{array}{c} \alpha, \beta, \gamma, \delta, \kappa \ge 0, \quad \alpha + \beta > 0, \\ 0 < \delta + \gamma \le \beta + \kappa \quad \textit{and} \quad \kappa \le \min(\alpha, \gamma). \end{array} \tag{13}$$

5.1. Rank of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes

In this subsection, we focus on the rank of $\mathbb{Z}_2\mathbb{Z}_4$-linear codes and show that there exists a $\mathbb{Z}_2\mathbb{Z}_4$-linear code C of type $(\alpha, \beta; \gamma, \delta; \kappa)$ with $r = rank(C)$ for any possible value of r.

Proposition 5.2 *[29] Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ and let $C = \Phi(\mathcal{C})$. Let $\mathcal{G}$ be a generator matrix of $\mathcal{C}$ as in (1) and let $\{u_i\}_{i=1}^\gamma$ be the rows of order two and $\{v_j\}_{j=1}^\delta$ the rows of order four in $\mathcal{G}$. Then, $\langle C \rangle$ is generated by $\{\Phi(u_i)\}_{i=1}^\gamma$, $\{\Phi(v_j), \Phi(2v_j)\}_{j=1}^\delta$ and $\{\Phi(2v_j * v_k)\}_{1 \le j < k \le \delta}$. Moreover, $\langle C \rangle$ is both binary linear and $\mathbb{Z}_2\mathbb{Z}_4$-linear.*

Proposition 5.3 *[29] Let C be a $\mathbb{Z}_2\mathbb{Z}_4$-linear code of binary length $n = \alpha + 2\beta$ and type $(\alpha, \beta; \gamma, \delta; \kappa)$. Then, $rank(C) \in \{\gamma + 2\delta, \ldots, \min(\beta + \delta + \kappa,\ \gamma + 2\delta + \binom{\delta}{2})\}$.*

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ and $C = \Phi(\mathcal{C})$ with $rank(C) = \gamma + 2\delta + \bar{r}$, where $\bar{r} \in \{0, \ldots, \min(\beta - (\gamma - \kappa) - \delta, \binom{\delta}{2})\}$. Let $\mathcal{G}$ be a generator matrix of $\mathcal{C}$ as in (1), where $\{u_i\}_{i=1}^\gamma$ are the rows of order two and $\{v_j\}_{j=1}^\delta$ the rows of order four. The $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{S}_\mathcal{C}$ generated by $\{u_i\}_{i=1}^\gamma$, $\{v_j\}_{j=1}^\delta$ and $\{2v_j * v_k\}_{1 \le j < k \le \delta}$ is of type $(\alpha, \beta; \gamma + \bar{r}, \delta; \kappa)$, and it is easy to check that $\Phi(\mathcal{S}_\mathcal{C}) = \langle C \rangle$, by Proposition 5.2.

For the parameters $\alpha, \beta, \gamma, \delta, \kappa$ given by some families of $\mathbb{Z}_2\mathbb{Z}_4$-linear codes such as, for example, extended perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear codes ([14, 47] or Example 5.5), the upper bound above is tight. We also know $\mathbb{Z}_2\mathbb{Z}_4$-linear codes such that the rank is in between these two bounds such as, for example, the Hadamard $\mathbb{Z}_4$-linear codes ([48] or Example 5.4).

Example 5.4 *For any integer $t \ge 3$ and each $\delta \in \{1, \ldots, \lfloor (t+1)/2 \rfloor\}$ there exists a unique (up to isomorphism) extended perfect $\mathbb{Z}_4$-linear code C of binary length $n = 2^t$, such that the $\mathbb{Z}_4$-dual code of C is of type $(0, \beta; \gamma, \delta)$, where $\beta = 2^{t-1}$ and $\gamma = t + 1 - 2\delta$ [38]. The Hadamard $\mathbb{Z}_4$-linear codes H are the $\mathbb{Z}_4$-dual of the extended perfect $\mathbb{Z}_4$-linear codes.*

The rank of Hadamard $\mathbb{Z}_4$-linear codes was computed in [48] and the rank of extended perfect $\mathbb{Z}_4$-linear codes in [14, 38]. Specifically,

$$rank(H) = \begin{cases} \gamma + 2\delta + \binom{\delta-1}{2} & \text{if} \quad \delta \geq 3 \\ \gamma + 2\delta & \text{if} \quad \delta = 1, 2 \end{cases}$$

and $rank(C) = \bar{\gamma} + 2\bar{\delta} + \delta = \beta + \bar{\delta}$ (except when $t = 4$ and $\delta = 1$), where $\bar{\gamma} = \gamma$ and $\bar{\delta} = \beta - \gamma - \delta$ by Theorem 3.1 taking $\alpha = 0 = \kappa$. Note that the rank of the extended perfect $\mathbb{Z}_4$-linear codes satisfies the upper bound.

Example 5.5 *For any integer $t \geq 3$ and each $\delta \in \{0, \ldots, \lfloor t/2 \rfloor\}$ there exists a unique (up to isomorphism) extended perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear code C of binary length $n = 2^t$, such that the $\mathbb{Z}_2\mathbb{Z}_4$-dual code of C is of type $(\alpha, \beta; \gamma, \delta)$ with $\alpha \neq 0$, where $\alpha = 2^{t-\delta}, \beta = 2^{t-1} - 2^{t-\delta-1}$ and $\gamma = t + 1 - 2\delta$ [16]. The Hadamard $\mathbb{Z}_2\mathbb{Z}_4$-linear codes H are the $\mathbb{Z}_2\mathbb{Z}_4$-dual of the extended perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear codes.*

The rank of Hadamard $\mathbb{Z}_2\mathbb{Z}_4$-linear codes was computed in [48] and the rank of extended perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear codes in [14]. Specifically,

$$rank(H) = \begin{cases} \gamma + 2\delta + \binom{\delta}{2} & \text{if} \quad \delta \geq 2 \\ \gamma + 2\delta & \text{if} \quad \delta = 0, 1 \end{cases}$$

and $rank(C) = \bar{\gamma} + 2\bar{\delta} + \delta = \beta + \bar{\delta} + \bar{\gamma}$, where $\bar{\gamma} = \alpha - \gamma$ and $\bar{\delta} = \beta - \delta$ by Theorem 3.1 taking $\gamma = \kappa$. Note that the rank of these two families of $\mathbb{Z}_2\mathbb{Z}_4$-linear codes satisfies the upper bound.

Example 5.6 *Let $\overline{QRM}(r, m)$ be the class of $\mathbb{Z}_4$-linear Reed-Muller codes defined in [8]. These are $\mathbb{Z}_2\mathbb{Z}_4$-linear codes of type $(0, 2^m; 0, \delta; 0)$, where $\delta = \sum_{i=0}^{r} \binom{m}{i}$. An important property is that any $\mathbb{Z}_4$-linear Kerdock-like code of binary length 4^m is in the class $\overline{QRM}(1, 2m-1)$ and any extended $\mathbb{Z}_4$-linear Preparata-like code of binary length 4^m is in the class $\overline{QRM}(2m-3, 2m-1)$.*

The rank of any code $C \in \overline{QRM}(r, m)$ is

$$rank(C) = \sum_{i=0}^{r} \binom{m}{i} + \sum_{i=0}^{t} \binom{m}{i},$$

where $t = \min(2r, m)$, [8]. Hence, if $2r \geq m$, then $rank(C) = \delta + \beta$, i.e. the maximum possible. A $\mathbb{Z}_4$-linear Kerdock-like code K of binary length $4^m \geq 16$ has $rank(P) = 2m^2 + m + 1$ and an extended $\mathbb{Z}_4$-linear Preparata-like code P of binary length $4^m \geq 64$ has $rank(P) = 2^{2m} - 2m$ [15], attaining the upper bound of Proposition 5.3.

Theorem 5.7 *[29] Let $\alpha, \beta, \gamma, \delta, \kappa$ be integers satisfying (13). Then, there exists a $\mathbb{Z}_2\mathbb{Z}_4$-linear code C of type $(\alpha, \beta; \gamma, \delta; \kappa)$ with $rank(C) = r$ if and only if*

$$r \in \{\gamma + 2\delta, \ldots, \min(\beta + \delta + \kappa,\ \gamma + 2\delta + \binom{\delta}{2})\}.$$

Finally, we give a construction of codes for any possible rank, for some given parameters β, γ, δ and κ. Let e_k, $1 \leq k \leq \delta$, denote the column vector of length δ, with a one in the kth coordinate and zeroes elsewhere. For each $\bar{r} \in \{0, \ldots, \min(\beta - (\gamma - \kappa) - \delta, \binom{\delta}{2})\}$, we can construct S_r as a quaternary matrix where in $\bar{r}$ columns there are $\bar{r}$ different column vectors $e_k + e_l$ of length δ, $1 \leq k < l \leq \delta$, and in the remaining columns there is the all-zero column vector. Then, the $\mathbb{Z}_2\mathbb{Z}_4$-linear code $C = \Phi(\mathcal{C})$ of type $(\alpha, \beta; \gamma, \delta; \kappa)$ with $\mathcal{C}$ generated by

$$\mathcal{G} = \left(\begin{array}{cc|ccc} I_\kappa & T' & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 2T_1 & 2I_{\gamma-\kappa} & \mathbf{0} \\ \hline \mathbf{0} & S' & S_r & \mathbf{0} & I_\delta \end{array}\right)$$

has rank $r = \gamma + 2\delta + \bar{r}$. Note that the matrices T', T_1 and S' can be chosen arbitrarily.

Example 5.8 *By Theorem 5.7, we know that the possible ranks for $\mathbb{Z}_2\mathbb{Z}_4$-linear codes of type $(\alpha, 9; 2, 5; 1)$ are $r \in \{12, 13, 14, 15\}$. For each possible r, we can construct a $\mathbb{Z}_2\mathbb{Z}_4$-linear code C with $rank(C) = r$, taking the following generator matrix of $\mathcal{C} = \Phi^{-1}(C)$:*

$$\mathcal{G}_S = \left(\begin{array}{cc|ccc} 1 & T' & \mathbf{0} & 0 & \mathbf{0} \\ 0 & \mathbf{0} & 2T_1 & 2 & \mathbf{0} \\ \hline \mathbf{0} & S' & S_r & \mathbf{0} & I_5 \end{array}\right),$$

where $S_{12} = (\mathbf{0})$ and S_{13}, S_{14}, and S_{15} are constructed as follows:

$$S_{13} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_{14} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_{15} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

5.2. Kernel dimension of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes

In this subsection, we focus on the dimension of the kernel of $\mathbb{Z}_2\mathbb{Z}_4$-linear codes and also show that there exists a $\mathbb{Z}_2\mathbb{Z}_4$-linear code C of type $(\alpha, \beta; \gamma, \delta; \kappa)$ with $k = ker(C)$ for any possible value of k.

Proposition 5.9 *[29] Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code and let $C = \Phi(\mathcal{C})$. Then, $K(C) = \{\Phi(u) \mid u \in \mathcal{C}$ and $2u * v \in \mathcal{C}, \forall v \in \mathcal{C}\}$.*

Note that if $\mathcal{G}$ is a generator matrix of a $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ and $C = \Phi(\mathcal{C})$, $\Phi(u) \in K(C)$ if and only if $u \in \mathcal{C}$ and $2u * v \in \mathcal{C}$ for all $v \in \mathcal{G}$. Moreover, all codewords of order two in $\mathcal{C}$ belong to $K(C)$. Also given $x, y \in \mathcal{C}$, $\Phi(x) + \Phi(y) \in K(C)$ if and only if $\Phi(x + y) \in K(C)$.

Proposition 5.10 *[29] Let C be a $\mathbb{Z}_2\mathbb{Z}_4$-linear code of binary length $n = \alpha + 2\beta$ and type $(\alpha, \beta; \gamma, \delta; \kappa)$. Then, $ker(C) \in \{\gamma + \delta, \gamma + \delta + 1, \ldots, \gamma + 2\delta - 2, \gamma + 2\delta\}$.*

Given an integer $m > 0$, a set of vectors $\{v_1, v_2, \ldots, v_m\}$ in $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ and a subset $I = \{i_1, \ldots, i_l\} \subseteq \{1, \ldots, m\}$, we denote by v_I the vector $v_{i_1} + \cdots + v_{i_l}$. If $I = \emptyset$, then $v_I = \mathbf{0}$.

Proposition 5.11 *[29] Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$, with generator matrix $\mathcal{G}$, and let $C = \Phi(\mathcal{C})$ be the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-linear code with $ker(C) = \gamma + 2\delta - \bar{k}$, where $\bar{k} \in \{2, \ldots, \delta\}$. Then, there exists a set $\{v_1, v_2, \ldots, v_{\bar{k}}\}$ of row vectors of order four in $\mathcal{G}$, such that*

$$C = \bigcup_{I \subseteq \{1,\ldots,\bar{k}\}} (K(C) + \Phi(v_I))$$

It is important to note that if C is a $\mathbb{Z}_2\mathbb{Z}_4$-linear code, then $K(C)$ is a $\mathbb{Z}_2\mathbb{Z}_4$-linear subcode of C. The *kernel* of a $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ of type $(\alpha, \beta; \gamma, \delta; \kappa)$, denoted by $\mathcal{K}(\mathcal{C})$, can be defined as $\mathcal{K}(\mathcal{C}) = \Phi^{-1}(K(C))$, where $C = \Phi(\mathcal{C})$ is the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-linear code. It is easy to see that $\mathcal{K}(\mathcal{C}) = \{u \in \mathcal{C} \mid 2u * v \in \mathcal{C}, \forall v \in \mathcal{C}\}$ and that $\mathcal{K}(\mathcal{C})$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive subcode of $\mathcal{C}$ of type $(\alpha, \beta; \gamma + \bar{k}, \delta - \bar{k}; \kappa)$.

Note also that replacing the ones with twos in the first α coordinates, we can see $\mathbb{Z}_2\mathbb{Z}_4$-additive codes as quaternary linear codes. Let χ be the map from $\mathbb{Z}_2$ to $\mathbb{Z}_4$, which is the usual inclusion from the additive structure in $\mathbb{Z}_2$ to $\mathbb{Z}_4$: $\chi(0) = 0, \chi(1) = 2$. This map can be extended to the map $(\chi, Id) : \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta \to \mathbb{Z}_4^{\alpha+\beta}$, which are also denoted by χ. If $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ with generator matrix $\mathcal{G}$, then $\chi(\mathcal{C})$ is a quaternary linear code of length $\alpha+\beta$ and type $2^\gamma 4^\delta$ with generator matrix $\mathcal{G}_{\chi(\mathcal{C})} = \chi(\mathcal{G})$. Note that $\mathcal{K}(\mathcal{C}) = \chi^{-1}\mathcal{K}(\chi(\mathcal{C}))$ and $\mathcal{K}(\chi(\mathcal{C}))^\perp$ is the quaternary linear code generated by the matrix

$$\begin{pmatrix} \mathcal{H}_{\chi(\mathcal{C})} \\ 2\mathcal{G}_{\chi(\mathcal{C})} * \mathcal{H}_{\chi(\mathcal{C})} \end{pmatrix},$$

where $\mathcal{H}_{\chi(\mathcal{C})}$ is the generator matrix of the quaternary dual code of $\chi(\mathcal{C})$ and $2\mathcal{G}_{\chi(\mathcal{C})} * \mathcal{H}_{\chi(\mathcal{C})}$ is the matrix obtained computing the component-wise product $2u * v$ for all $u \in \mathcal{G}_{\chi(\mathcal{C})}$, $v \in \mathcal{H}_{\chi(\mathcal{C})}$.

By Proposition 5.11, given a $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ with generator matrix $\mathcal{G}$, there exists a set $\{v_1, v_2, \ldots, v_{\bar{k}}\}$ of row vectors of order four in $\mathcal{G}$, such that $\mathcal{C} = \bigcup_{I \subseteq \{1,\ldots,\bar{k}\}} (\mathcal{K}(\mathcal{C}) + v_I)$.

Example 5.12 *Continuing with Example 5.4, the dimension of the kernel for a Hadamard $\mathbb{Z}_4$-linear code H was computed in [38, 48] and the dimension of the kernel for an extended perfect $\mathbb{Z}_4$-linear code C in [14]. Specifically,*

$$ker(H) = \begin{cases} \gamma + \delta + 1 & \text{if} \quad \delta \geq 3 \\ \gamma + 2\delta & \text{if} \quad \delta = 1, 2 \end{cases} \qquad \text{and} \qquad ker(C) = \begin{cases} \bar{\gamma} + \bar{\delta} + 1 & \text{if} \quad \delta \geq 3 \\ \bar{\gamma} + \bar{\delta} + 2 & \text{if} \quad \delta = 2 \\ \bar{\gamma} + \bar{\delta} + t & \text{if} \quad \delta = 1. \end{cases}$$

Example 5.13 *Continuing with Example 5.5, the dimension of the kernel for a Hadamard $\mathbb{Z}_2\mathbb{Z}_4$-linear code H was computed in [48] and the dimension of the kernel for an extended perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear code C in [14]. Specifically,*

$$ker(H) = \begin{cases} \gamma + \delta & \text{if} \quad \delta \geq 2 \\ \gamma + 2\delta & \text{if} \quad \delta = 0, 1 \end{cases} \qquad \text{and} \qquad ker(C) = \begin{cases} \bar{\gamma} + \bar{\delta} + 1 & \text{if} \quad \delta \geq 1 \\ \bar{\gamma} + 2\bar{\delta} & \text{if} \quad \delta = 0. \end{cases}$$

Note that the kernel dimension of the Hadamard $\mathbb{Z}_2\mathbb{Z}_4$-linear codes satisfies the lower bound.

Example 5.14 *Let $\overline{QRM}(r,m)$ be the class of $\mathbb{Z}_4$-linear Reed-Muller codes defined in [8], as in Example 5.6. The dimension of the kernel of any code $C \in \overline{QRM}(r,m)$ is*

$$ker(C) = \sum_{i=0}^{r} \binom{m}{i} + 1 = \delta + 1,$$

except for $r = m$ (in this case, $C = \mathbb{Z}_2^{2^{m+1}}$), [8].

Therefore, $\mathbb{Z}_4$-linear Kerdock-like codes and extended $\mathbb{Z}_4$-linear Preparata-like codes of binary length 4^m have dimension of the kernel $ker(K) = 2m+1$ and $ker(P) = 2^{2m-1}-2m+1$, respectively [8, 15].

Theorem 5.15 *[29] Let $\alpha, \beta, \gamma, \delta, \kappa$ be integers satisfying (13). Then, there exists a $\mathbb{Z}_2\mathbb{Z}_4$-linear code C of type $(\alpha, \beta; \gamma, \delta; \kappa)$ with $ker(C) = \gamma + 2\delta - \bar{k}$ if and only if*

$$\begin{cases} \bar{k} = 0, & \text{if} \quad s = 0, \\ \bar{k} \in \{0\} \cup \{2, \ldots, \delta\} \text{ and } \bar{k} \text{ even}, & \text{if} \quad s = 1, \\ \bar{k} \in \{0\} \cup \{2, \ldots, \delta\}, & \text{if} \quad s \geq 2, \end{cases}$$

where $s = \beta - (\gamma - \kappa) - \delta$.

Finally, we give a construction of codes for any possible dimension of the kernel, for some given parameters β, γ, δ and κ. First, when $s = 1$, for each $\bar{k} \in \{2, \ldots, \delta\}$ and even, we can construct a matrix S_k over $\mathbb{Z}_4$ with an even number of ones, $\bar{k}$, and zeroes elsewhere. On the other hand, when $s \geq 2$, for each $\bar{k} \in \{2, \ldots, \delta\}$, we can construct a matrix S_k, such that only in the last $\delta - \bar{k}$ row vectors all components are zero and, moreover, in the first $\bar{k}$ coordinates of each column vector there are an even number of ones and zeros elsewhere. Then, the $\mathbb{Z}_2\mathbb{Z}_4$-linear code $C = \Phi(\mathcal{C})$ of type $(\alpha, \beta; \gamma, \delta; \kappa)$ with $\mathcal{C}$ generated by

$$\mathcal{G} = \left(\begin{array}{cc|ccc} I_\kappa & T' & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & 2I_{\gamma-\kappa} & \mathbf{0} \\ \hline \mathbf{0} & S' & S_k & \mathbf{0} & I_\delta \end{array} \right)$$

has dimension of the kernel $k = \gamma + 2\delta - \bar{k}$. Note that T' and S' can be chosen arbitrarily.

Example 5.16 *By Theorem 5.15, we know that the possible dimensions of the kernel for $\mathbb{Z}_2\mathbb{Z}_4$-linear codes of type $(\alpha, 9; 2, 5; 1)$ are $k \in \{12, 10, 9, 8, 7\}$. For each possible k, we can construct a $\mathbb{Z}_2\mathbb{Z}_4$-linear code C with $ker(C) = k$, taking the following generator matrix of $\mathcal{C} = \Phi^{-1}(C)$:*

$$\mathcal{G}_S = \left(\begin{array}{cc|ccc} 1 & T' & \mathbf{0} & 0 & \mathbf{0} \\ 0 & \mathbf{0} & \mathbf{0} & 2 & \mathbf{0} \\ \hline \mathbf{0} & S' & S_k & \mathbf{0} & I_5 \end{array} \right),$$

where $S_{12} = (\mathbf{0})$ and S_{10}, S_9, S_8 and S_7 are constructed as follows:

$$S_{10} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, S_9 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, S_8 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, S_7 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

5.3. Pairs of rank and kernel dimension of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes

In this subsection, we give the lower and upper bounds on the rank, once the dimension of the kernel is fixed. Moreover, we show that there exists a $\mathbb{Z}_2\mathbb{Z}_4$-linear code C of type $(\alpha, \beta; \gamma, \delta; \kappa)$ with $r = rank(C)$ and $k = ker(C)$ for any possible pair of values (r, k).

Proposition 5.17 *[29] Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ and let $C = \Phi(\mathcal{C})$. If $rank(C) = \gamma + 2\delta + \bar{r}$ and $ker(C) = \gamma + 2\delta - \bar{k}$, with $\bar{k} \geq 2$, then $1 \leq \bar{r} \leq \binom{\bar{k}}{2}$. Moreover, $\langle C \rangle$ is generated by $\{\Phi(u_i)\}_{i=1}^{\gamma}, \{\Phi(v_j), \Phi(2v_j)\}_{j=1}^{\delta}$ and $\{\Phi(2v_t * v_s)\}_{1 \leq s < t \leq \bar{k}}$.*

Let C be a $\mathbb{Z}_2\mathbb{Z}_4$-linear code with $ker(C) = \gamma + 2\delta - \bar{k}$ and $rank(C) = \gamma + 2\delta + \bar{r}$. Note that if $\bar{r} = 0$ then, necessarily, $\bar{k} = 0$ (and vice versa) and C is a linear code. The next theorem determine all possible pairs of rank and dimension of the kernel for nonlinear $\mathbb{Z}_2\mathbb{Z}_4$-linear codes.

Theorem 5.18 *[29] Let $\alpha, \beta, \gamma, \delta, \kappa$ be integers satisfying (13). Then, there exists a nonlinear $\mathbb{Z}_2\mathbb{Z}_4$-linear code C of type $(\alpha, \beta; \gamma, \delta; \kappa)$ with $ker(C) = \gamma + 2\delta - \bar{k}$ and $rank(C) = \gamma + 2\delta + \bar{r}$ if and only $\bar{k} \in \{2, \ldots, \delta\}$ and*

$$\begin{cases} \bar{r} \in \{2, \ldots, \min(\beta - (\gamma - \kappa) - \delta, \ \binom{\bar{k}}{2})\}, & \text{if} \quad \bar{k} \text{ is odd,} \\ \bar{r} \in \{1, \ldots, \min(\beta - (\gamma - \kappa) - \delta, \ \binom{\bar{k}}{2})\}, & \text{if} \quad \bar{k} \text{ is even.} \end{cases}$$

Combining both constructions, for a given rank and a given dimension of the kernel, it is easy to construct of a $\mathbb{Z}_2\mathbb{Z}_4$-linear code for all possible pairs of rank and dimension of the kernel.

Example 5.19 *By Theorem 5.18, we know that the possible pairs of rank and dimension of the kernel of $\mathbb{Z}_2\mathbb{Z}_4$-linear codes, C, of type $(\alpha, 9; 2, 5; 1)$ are given in the following table:*

$k \backslash r$	12	13	14	15
12	*			
10		*		
9			*	*
8		*	*	*
7			*	*

For each possible pair (r, k), we can construct a $\mathbb{Z}_2\mathbb{Z}_4$-linear code C with $rank(C) = r$ and $ker(C) = k$, taking the following generator matrix of $\mathcal{C} = \Phi^{-1}(C)$:

$$\mathcal{G}_S = \left(\begin{array}{cc|ccc} 1 & T' & \mathbf{0} & 0 & \mathbf{0} \\ 0 & \mathbf{0} & \mathbf{0} & 2 & \mathbf{0} \\ \hline \mathbf{0} & S' & S_{r,k} & \mathbf{0} & I_5 \end{array} \right),$$

where $S_{12,12} = (\mathbf{0})$ and the other possible $S_{r,k}$ are constructed as follows:

$$S_{13,10} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_{13,8} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$S_{14,9} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_{14,8} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_{14,7} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$S_{15,9} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_{15,8} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S_{15,7} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

6. Families of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes

In this section, we show the classification and construction of some well known families of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes: $\mathbb{Z}_2\mathbb{Z}_4$-additive (extended) perfect codes, $\mathbb{Z}_2\mathbb{Z}_4$-additive Hadamard codes, $\mathbb{Z}_2\mathbb{Z}_4$-additive Reed-Muller codes, maximum distance separable $\mathbb{Z}_2\mathbb{Z}_4$-additive codes, and $\mathbb{Z}_4$-linear Preparata-like and Kerdock-like codes.

6.1. $\mathbb{Z}_2\mathbb{Z}_4$-additive (extended) perfect codes

A *binary perfect 1-error correcting code* (briefly in this paper, *binary perfect code*) C of length n is a binary code, with $d_H(C) = 3$, such that all the vectors in $\mathbb{Z}_2^n$ are within distance one from a codeword. For any $t > 1$ there exists exactly one binary linear perfect code of length $2^t - 1$, up to equivalence, which is the well known *Hamming code*. An *extended code* of the code C is a code resulting from adding an overall parity-check digit to each codeword of C.

The $\mathbb{Z}_2\mathbb{Z}_4$-additive codes such that, under the Gray map, give a binary perfect code are called $\mathbb{Z}_2\mathbb{Z}_4$*-additive perfect codes*. Equivalently, the $\mathbb{Z}_2\mathbb{Z}_4$-additive codes such that, under the Gray map, give a code with the same parameters as an extended binary perfect code are called $\mathbb{Z}_2\mathbb{Z}_4$*-additive extended perfect codes*. Given a $\mathbb{Z}_2\mathbb{Z}_4$-additive (extended) perfect code, after applying the Gray map, the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-linear code is called *(extended) perfect* $\mathbb{Z}_2\mathbb{Z}_4$*-linear code*.

Apart from the linear binary case (when $\beta = 0$), there are two different kinds of $\mathbb{Z}_2\mathbb{Z}_4$-additive extended perfect codes, those with $\alpha = 0$ and those with $\alpha \neq 0$. We distinguish between these two cases because the construction of the codes is different. In Subsection 6.1.1, we focus on $\mathbb{Z}_2\mathbb{Z}_4$-additive extended perfect codes with $\alpha = 0$, and in Subsection 6.1.2, on $\mathbb{Z}_2\mathbb{Z}_4$-additive (extended) perfect codes with $\alpha \neq 0$. Finally, in Subsection 6.1.3, a recursive construction of parity-check matrices for all these codes is shown.

The $\mathbb{Z}_2\mathbb{Z}_4$-additive extended perfect codes can also be classified using either the rank or the dimension of the kernel, as it is proven in [14, 38], where these parameters are computed (see also Examples 5.4, 5.5, 5.12 and 5.13). The intersection problem for these codes, i.e., which are the possibilities for the number of codewords in the intersection of two $\mathbb{Z}_2\mathbb{Z}_4$-additive codes of

the same length is investigated in [59]. Finally, also mention that the permutation automorphism group of the corresponding extended perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear codes, which include the extended perfect $\mathbb{Z}_4$-linear codes, has been studied in [39, 47].

6.1.1. $\mathbb{Z}_2\mathbb{Z}_4$-ADDITIVE EXTENDED PERFECT CODES WITH $\alpha = 0$

The $\mathbb{Z}_2\mathbb{Z}_4$-additive extended perfect codes with $\alpha = 0$ are also called *quaternary linear extended perfect codes*, since they are also quaternary linear codes. For the same reason, after applying the Gray map to these codes, their corresponding $\mathbb{Z}_2\mathbb{Z}_4$-linear codes are extended perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear codes (with $\alpha = 0$), which are also called *extended perfect $\mathbb{Z}_4$-linear codes*.

Theorem 6.1 *[38] For any integer $t \geq 4$ and each $\delta \in \{1, \ldots, \lfloor (t+1)/2 \rfloor\}$, there exists a unique (up to isomorphism) extended perfect $\mathbb{Z}_4$-linear code C^* of binary length $2^t \geq 16$, such that the $\mathbb{Z}_4$-dual code of C^* is of type $(0, \beta; \gamma, \delta)$, where $\beta = 2^{t-1}$ and $\gamma = t + 1 - 2\delta$.*

In view of this theorem, we can write the following table:

t	δ	$(\alpha, \beta; \gamma, \delta)$
2	1	$(0, 2; 1, 1)$
3	1,2	$(0, 4; 2, 1)$, $(0, 4; 0, 2)$
4	1,2	$(0, 8; 3, 1)$, $(0, 8; 1, 2)$
5	1,2,3	$(0, 16; 4, 1)$, $(0, 16; 2, 2)$, $(0, 16; 0, 3)$
6	1,2,3	$(0, 32; 5, 1)$, $(0, 32; 3, 2)$, $(0, 32; 1, 3)$
7	1,2,3,4	$(0, 64; 6, 1)$, $(0, 64; 4, 2)$, $(0, 64; 2, 3)$, $(0, 64; 0, 4)$
...	...	...

Note that for $t = 3$ (binary length 8), both extended perfect $\mathbb{Z}_4$-linear codes are isomorphic, and for $t \geq 4$, all such codes are nonisomorphic and unique. Therefore, by Theorem 6.1, the number of nonisomorphic extended perfect $\mathbb{Z}_4$-linear codes of binary length 2^t is $\lfloor (t+1)/2 \rfloor$ for all $t \geq 4$, and it is 1 for $t = 2$ and $t = 3$. Note that the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-additive codes $\mathcal{C}^* = \Phi^{-1}(C^*)$ of these extended perfect $\mathbb{Z}_4$-linear codes C^* are $\mathbb{Z}_2\mathbb{Z}_4$-additive extended perfect codes with $\alpha = 0$, or equivalently quaternary linear extended perfect codes.

We remark that that if C^* is an extended perfect $\mathbb{Z}_4$-linear code of binary length $2^t \geq 16$, then the punctured code can not be a perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear code, up to the extended Hamming code of length 16. A proof of this fact can be found in [30].

In order to construct the corresponding quaternary linear extended perfect code $\mathcal{C}^*$ of an extended perfect $\mathbb{Z}_4$-linear code $C^* = \Phi(\mathcal{C}^*)$, such that its $\mathbb{Z}_4$-dual code is of type $(0, \beta; \gamma, \delta)$, we can consider the matrix consisting of all column vectors of the form $\mathbb{Z}_2^\gamma \times \{1 \in \mathbb{Z}_4\} \times \mathbb{Z}_4^{\delta-1}$,

$$\left(\frac{B}{Q}\right),$$

where B is a matrix over $\mathbb{Z}_2$ of size $\gamma \times \beta$, and Q is a matrix over $\mathbb{Z}_4$ of size $\delta \times \beta$. Then, the matrix $\mathcal{H}^*$ is a parity-check matrix of $\mathcal{C}^*$, where

$$\mathcal{H}^* = \left(\frac{2B}{Q}\right).$$

Example 6.2 *For $t = 3$, there is a unique (up to isomorphism) extended perfect $\mathbb{Z}_4$-linear code C^* of binary length 8. Using the above construction, it is possible to construct two different quaternary linear extended perfect codes, $\mathcal{C}_1$ and $\mathcal{C}_2$ (taking $\delta = 1$ and $\delta = 2$), with parity-check matrices, $\mathcal{H}_1^*$ and $\mathcal{H}_2^*$, respectively, such that $C_1 = \Phi(\mathcal{C}_1)$ and $C_2 = \Phi(\mathcal{C}_2)$ are both isomorphic to C^*, where*

$$\mathcal{H}_1^* = \left(\begin{array}{cccc} 0 & 0 & 2 & 2 \\ 0 & 2 & 0 & 2 \\ \hline 1 & 1 & 1 & 1 \end{array}\right) \quad \mathcal{H}_2^* = \left(\overline{\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{array}}\right).$$

Example 6.3 *For $t = 4$, there are two nonisomorphic extended perfect $\mathbb{Z}_4$-linear codes of binary length 16, since we have two possible parameters: $\delta = 1$ and $\delta = 2$. The following matrices are parity-check matrices of the corresponding two quaternary linear extended perfect codes:*

$$\mathcal{H}_1^* = \left(\begin{array}{cccccccc} 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 \\ 0 & 0 & 2 & 2 & 0 & 0 & 2 & 2 \\ 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right) \quad \mathcal{H}_2^* = \left(\begin{array}{cccccccc} 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 \end{array}\right).$$

6.1.2. $\mathbb{Z}_2\mathbb{Z}_4$-ADDITIVE (EXTENDED) PERFECT CODES WITH $\alpha \neq 0$

Theorem 6.4 *[16] For any integer $t \geq 4$ and each $\delta \in \{0, \ldots, \lfloor t/2 \rfloor\}$, there exists a unique (up to isomorphism) extended perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear code C^* of binary length $2^t \geq 16$, such that the $\mathbb{Z}_2\mathbb{Z}_4$-dual code of C^* is of type $(\alpha, \beta; \gamma, \delta)$, where $\alpha = 2^{t-\delta}$, $\beta = 2^{t-1} - 2^{t-\delta-1}$, and $\gamma = t + 1 - 2\delta$.*

In view of this theorem, we can write the following table:

t	δ	$(\alpha, \beta; \gamma, \delta)$
2	0,1	$(2, 1; 1, 1)$, $(4, 0; 3, 0)$
3	0,1	$(4, 2; 2, 1)$, $(8, 0; 4, 0)$
4	0,1,2	$(4, 6; 1, 2)$, $(8, 4; 3, 1)$, $(16, 0; 5, 0)$
5	0,1,2	$(8, 12; 2, 2)$, $(16, 8; 4, 1)$, $(32, 0; 6, 0)$
6	0,1,2,3	$(8, 28; 1, 3)$, $(16, 24; 3, 2)$, $(32, 16; 5, 1)$, $(64, 0; 7, 0)$
7	0,1,2,3	$(16, 56; 2, 3)$, $(32, 48; 4, 2)$, $(64, 32; 6, 1)$, $(128, 0; 8, 0)$
...	...	...

Note that for $t = 2$ and $t = 3$ (binary length 4 and 8, respectively), both extended perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear codes are isomorphic, and for $t \geq 4$, all such codes are nonisomorphic and unique. Therefore, by Theorem 6.4, the number of nonisomorphic extended perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear codes of binary length 2^t is $\lfloor (t+2)/2 \rfloor$ for all $t \geq 4$, and it is 1 for $t = 2$ and $t = 3$. Note that the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-additive codes $\mathcal{C}^* = \Phi^{-1}(C^*)$ of these extended perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear codes C^* are $\mathbb{Z}_2\mathbb{Z}_4$-additive extended perfect codes with $\alpha \neq 0$.

Corollary 6.5 *[16] For any integer $t \geq 4$ and each $\delta \in \{0, \ldots, \lfloor t/2 \rfloor\}$, there exists a unique (up to isomorphism) perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear code C of binary length $n = 2^t - 1 \geq 15$, such that the $\mathbb{Z}_2\mathbb{Z}_4$-dual code of C is of type $(\alpha, \beta; \gamma, \delta)$, where $\alpha = 2^{t-\delta} - 1$, $\beta = 2^{t-1} - 2^{t-\delta-1}$, and $\gamma = t - 2\delta$.*

In view of this corollary, we can write the following table:

t	δ	$(\alpha, \beta; \gamma, \delta)$
2	0,1	$(1, 1; 0, 1)$, $(3, 0; 2, 0)$
3	0,1	$(3, 2; 1, 1)$, $(7, 0; 3, 0)$
4	0,1,2	$(3, 6; 0, 2)$, $(7, 4; 2, 1)$, $(15, 0; 4, 0)$
5	0,1,2	$(7, 12; 1, 2)$, $(15, 8; 3, 1)$, $(31, 0; 5, 0)$
6	0,1,2,3	$(7, 28; 0, 3)$, $(15, 24; 2, 2)$, $(31, 16; 4, 1)$, $(63, 0; 6, 0)$
7	0,1,2,3	$(15, 56; 1, 3)$, $(31, 48; 3, 2)$, $(63, 32; 5, 1)$, $(127, 0; 7, 0)$
...	...	...

By Corollary 6.5, we also have that the number of nonisomorphic perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear codes of binary length $n = 2^t - 1$ is $\lfloor (t+2)/2 \rfloor$ for all $t \geq 4$, and it is 1 for $t = 2$ and $t = 3$. Note also that the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-additive codes $\mathcal{C} = \Phi^{-1}(C)$ of these perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear codes C are $\mathbb{Z}_2\mathbb{Z}_4$-additive perfect codes.

Again, in order to construct the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-additive extended perfect code $\mathcal{C}^*$ of an extended perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear code $C^* = \Phi(\mathcal{C}^*)$, such that its $\mathbb{Z}_2\mathbb{Z}_4$-dual code is of type $(\alpha, \beta; \gamma, \delta)$ with $\alpha \neq 0$, we can consider the matrix consisting of all column vectors of the form $\{1 \in \mathbb{Z}_2\} \times \mathbb{Z}_2^{\gamma-1} \times \mathbb{Z}_4^{\delta}$ (up to sign). There are exactly $\alpha = 2^{t-\delta}$ nonzero column vectors of order two, which can be placed in the first α columns. Therefore, this matrix can be expressed as

$$\left(\begin{array}{c|c} A' & B' \\ \hline 2D' & Q \end{array}\right),$$

where A', B', D' are matrices over $\mathbb{Z}_2$ of size $\gamma \times \alpha$, $\gamma \times \beta$ and $\delta \times \alpha$, respectively; and Q is a matrix over $\mathbb{Z}_4$ of size $\delta \times \beta$ with all column vectors of order four. Then, the matrix $\mathcal{H}^*$ is a parity-check matrix of $\mathcal{C}^*$, where

$$\mathcal{H}^* = \left(\begin{array}{c|c} A' & 2B' \\ \hline D' & Q \end{array}\right). \tag{14}$$

A parity-check matrix $\mathcal{H}$ of the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-additive perfect code $\mathcal{C}$ can be constructed from $\mathcal{H}^*$ deleting the first row (that is, the row vector $(\mathbf{1}|\mathbf{2})$) and the first column (that is, the all-zero column vector). Since we can write the matrix $\mathcal{H}^*$ in (14) as a matrix of the form

$$\mathcal{H}^* = \left(\begin{array}{cc|c} 1 & \mathbf{1} & \mathbf{2} \\ \mathbf{0} & A & 2B \\ \hline \mathbf{0} & D & Q \end{array}\right),$$

where $A' = \begin{pmatrix} 1 & \mathbf{1} \\ \mathbf{0} & A \end{pmatrix}$, $B' = \begin{pmatrix} \mathbf{1} \\ B \end{pmatrix}$, and $D' = \begin{pmatrix} \mathbf{0} & D \end{pmatrix}$, we have that

$$\mathcal{H} = \left(\begin{array}{c|c} A & 2B \\ \hline D & Q \end{array}\right), \tag{15}$$

where A, B, D are matrices over $\mathbb{Z}_2$ of size $(\gamma-1)\times(\alpha-1)$, $(\gamma-1)\times\beta$ and $\delta\times(\alpha-1)$, respectively; and Q is a matrix over $\mathbb{Z}_4$ of size $\delta\times\beta$.

Example 6.6 *For $t=2$, the parity-check matrices of the $\mathbb{Z}_2\mathbb{Z}_4$-additive extended perfect codes with $\delta=0$ and $\delta=1$, are the same as for the extended Hamming code and $\mathcal{H}_1'$ given below, respectively. And for $t=3$, the matrices for $\delta=0$ and $\delta=1$ are again the same as for the extended Hamming code and $\mathcal{H}_1''$ given below, respectively, where*

$$\mathcal{H}_1'=\left(\begin{array}{cc|c}1&1&2\\\hline 0&1&1\end{array}\right)\quad \mathcal{H}_1''=\left(\begin{array}{cccc|cc}0&0&1&1&0&2\\1&1&1&1&2&2\\\hline 0&1&0&1&1&1\end{array}\right).$$

Example 6.7 *For $t=4$, there are three nonisomorphic extended perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear codes of binary length 16, which are not $\mathbb{Z}_4$-linear codes, so with $\alpha\neq 0$. They are given by the following parameters: $\delta=0$, $\delta=1$ and $\delta=2$.*

For the case $\delta=1$, let us construct the matrix consisting of all column vectors of the form $\{1\in\mathbb{Z}_2\}\times\mathbb{Z}_2^2\times\mathbb{Z}_4^1$, up to sign. Note that there are exactly 8 column vectors of order two. Rearranging columns in order to have the column vectors of order two in the first columns, we obtain the matrix

$$\left(\begin{array}{cccccccc|cccc}1&1&1&1&1&1&1&1&1&1&1&1\\0&0&1&1&0&0&1&1&0&0&1&1\\0&1&0&1&0&1&0&1&0&1&0&1\\\hline 0&0&0&0&2&2&2&2&1&1&1&1\end{array}\right).$$

Therefore, the following matrix $\mathcal{H}^*$ is a parity-check matrix of the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-additive extended perfect code $\mathcal{C}^*$ with $\delta=1$, so such that its additive dual code $\mathcal{C}^{*\perp}$ is of type $(8,4;3,1)$:

$$\mathcal{H}^*=\left(\begin{array}{cccccccc|cccc}1&1&1&1&1&1&1&1&2&2&2&2\\0&0&1&1&0&0&1&1&0&0&2&2\\0&1&0&1&0&1&0&1&0&2&0&2\\\hline 0&0&0&0&1&1&1&1&1&1&1&1\end{array}\right)$$

Note that deleting the first row and column of $\mathcal{H}^*$, we obtain the following matrix $\mathcal{H}$ which is a parity-check matrix of the $\mathbb{Z}_2\mathbb{Z}_4$-additive perfect code $\mathcal{C}$ with $\delta=1$, so such that its additive dual code $\mathcal{C}^{\perp}$ is of type $(7,4;2,1)$:

$$\mathcal{H}=\left(\begin{array}{ccccccc|cccc}0&1&1&0&0&1&1&0&2&0&2\\0&0&0&1&1&1&1&0&0&2&2\\\hline 1&1&0&1&0&1&0&1&1&1&1\end{array}\right).$$

The study of $\mathbb{Z}_2\mathbb{Z}_4$-additive extended perfect codes is absolutely different if they come from the extended code of a perfect $\mathbb{Z}_4$-linear code (with $\alpha=0$) as in Theorem 6.1 or from an extended perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear codes (with $\alpha\neq 0$) as in Theorem 6.4. Note that, in the first case, the quaternary all-one vector is always in both codes, the $\mathbb{Z}_2\mathbb{Z}_4$-additive extended perfect code and its additive dual code. However, in the second case, the vector with binary ones in the binary part and quaternary twos in the quaternary part is always in these two codes, the $\mathbb{Z}_2\mathbb{Z}_4$-additive extended perfect code and its additive dual code.

It is easy to see that for the $\mathbb{Z}_2\mathbb{Z}_4$-additive (extended) perfect codes (or the corresponding (extended) perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear codes) with $\alpha \neq 0$, we do not need to specify the parameter κ, because $\kappa = \gamma$ by construction of these codes. Therefore, we just write $(\alpha, \beta; \gamma, \delta)$ instead of $(\alpha, \beta; \gamma, \delta; \kappa)$. In this case, the additive dual code of a $\mathbb{Z}_2\mathbb{Z}_4$-additive perfect code of type $(\alpha, \beta; \gamma, \delta)$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \alpha - \gamma, \beta - \delta)$.

6.1.3. RECURSIVE CONSTRUCTION

A parity-check matrix for all $\mathbb{Z}_2\mathbb{Z}_4$-additive extended perfect codes can also be constructed in a recursive way. First, the construction for codes with $\alpha = 0$ is given [38] and then, the generalization for the ones with $\alpha \neq 0$ is shown [59].

Let $\mathcal{H}$ be a parity-check matrix of a $\mathbb{Z}_2\mathbb{Z}_4$-additive extended perfect code, such that its additive dual code is of type $(0, \beta; \gamma, \delta)$, where $\beta = 2^{t-1}$, $\gamma = t+1-2\delta$ and $\delta \in \{1, \ldots, \lfloor (t+1)/2 \rfloor\}$. A parity-check matrix for the $\mathbb{Z}_2\mathbb{Z}_4$-additive extended perfect code, such that its additive dual code is of type $(0, \beta'; \gamma + 1, \delta)$, where $\beta' = 2\beta = 2^t$, can be constructed as follows:

$$\mathcal{H}' = \begin{pmatrix} \mathbf{0} & \mathbf{2} \\ \mathcal{H} & \mathcal{H} \end{pmatrix}. \tag{16}$$

Therefore, from codes of binary length $2\beta = 2^t$, we obtain codes of binary length $2\beta' = 2^{t+1}$. Specifically, using this construction, from all the nonisomorphic codes of binary length 2^t, we obtain all the nonisomorphic codes of binary length 2^{t+1}, except when t is odd and $\gamma = 0$. Note also that this construction give us a new matrix $\mathcal{H}'$ with the same number of row vector of order four, δ, and one more row vector of order two, $\gamma + 1$.

When t is odd and $\gamma = 0$, we need to use another recursive construction. This construction gives us a new matrix $\mathcal{H}'$ with the same number of row vector of order two, γ, and one more row vector of order four, $\delta + 1$. A parity-check matrix for the $\mathbb{Z}_2\mathbb{Z}_4$-additive extended perfect code, such that its additive dual code is of type $(0, \beta'; \gamma, \delta + 1)$, where $\beta' = 4\beta$, can be constructed as follows:

$$\mathcal{H}' = \begin{pmatrix} \mathcal{H} & \mathcal{H} & \mathcal{H} & \mathcal{H} \\ \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} \end{pmatrix}. \tag{17}$$

Therefore, starting with a parity-check matrix of the $\mathbb{Z}_2\mathbb{Z}_4$-additive extended perfect code, such that its additive dual code is of type $(0, 4; 0, 2)$, that is, with $t = 3$ and $\gamma = 0$, it is possible to construct all the cases for t odd and $\gamma = 0$.

Example 6.8 *Applying matrix (16) to the matrices $\mathcal{H}_1^*$ and $\mathcal{H}_2^*$ of type $(0, 4; 2, 1)$ and $(0, 4; 0, 2)$, respectively, given in Example 6.2, we can obtain the parity-check matrices $\mathcal{H}_1^*$ and $\mathcal{H}_2^*$ for the quaternary linear extended perfect codes with quaternary dual of dual type $(0, 8; 3, 1)$ and $(0, 8; 1, 2)$, respectively, given in Example 6.3.*

Example 6.9 *For $t = 5$, there are three nonisomorphic extended perfect $\mathbb{Z}_4$-linear codes of binary length 32, since we have three possible parameters: $\delta = 1$, $\delta = 2$ and $\delta = 3$.*

We can construct parity-check matrices of the corresponding quaternary linear extended perfect codes with $\delta = 1$ and $\delta = 2$, using (16) and matrices $\mathcal{H}_1^*$ and $\mathcal{H}_2^*$ given in Example 6.3.

On the other hand, using (17) and matrix $\mathcal{H}_2^*$ of type $(0,4;0,2)$, given in Example 6.2 we can obtain the parity-check matrix for the quaternary linear extended perfect code with $\delta = 3$.

Now, let $\mathcal{H} = (\mathcal{H}_\alpha|\mathcal{H}_\beta)$ be a parity-check matrix of a $\mathbb{Z}_2\mathbb{Z}_4$-additive extended perfect code, such that its additive dual code is of type $(\alpha, \beta; \gamma, \delta)$, where $\alpha = 2^{t-\delta}$, $\beta = 2^{t-1} - 2^{t-\delta-1}$, $\gamma = t + 1 - 2\delta$ and $\delta \in \{0, \ldots, \lfloor t/2 \rfloor\}$. A parity-check matrix for the $\mathbb{Z}_2\mathbb{Z}_4$-additive extended perfect code, such that its additive dual code is of type $(\alpha', \beta'; \gamma+1, \delta)$, where $\alpha' = 2\alpha = 2^{t-\delta+1}$ and $\beta' = 2\beta = 2^t - 2^{t-\delta}$, can be constructed as follows:

$$\mathcal{H}' = \left(\begin{array}{cc|cc} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{2} \\ \mathcal{H}_\alpha & \mathcal{H}_\alpha & \mathcal{H}_\beta & \mathcal{H}_\beta \end{array} \right). \tag{18}$$

Therefore, from codes of binary length $\alpha + 2\beta = 2^t$ and $\alpha = 2^{t-\delta}$, we obtain codes of binary length $\alpha' + 2\beta' = 2^{t+1}$ and $\alpha' = 2^{t-\delta+1}$. This means that using this construction, from all the nonisomorphic codes of binary length 2^t, we obtain all the codes of binary length 2^{t+1}, except when t is even and $\gamma = 1$.

Again, when t is even and $\gamma = 1$, we need to use another recursive construction. A parity-check matrix for the $\mathbb{Z}_2\mathbb{Z}_4$-additive extended perfect code, such that its additive dual code is of type $(\alpha', \beta'; \gamma, \delta + 1)$, where $\alpha' = 2\alpha = 2^{t-\delta+1}$ and $\beta' = \alpha + 4\beta = 2^{t-\delta} + 2^{t+1} - 2^{t-\delta+1} = 2^{t+1} - 2^{t-\delta}$, can be constructed as follows:

$$\mathcal{H}' = \left(\begin{array}{cc|ccccc} \mathcal{H}_\alpha & \mathcal{H}_\alpha & 2\mathcal{H}_\alpha & \mathcal{H}_\beta & \mathcal{H}_\beta & \mathcal{H}_\beta & \mathcal{H}_\beta \\ \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} \end{array} \right). \tag{19}$$

Therefore, starting with a parity-check matrix of the $\mathbb{Z}_2\mathbb{Z}_4$-additive extended perfect code, such that its additive code is of type $(4, 6; 1, 2)$, that is, with $t = 4$ and $\gamma = 1$, it is possible to construct all the cases for t even and $\gamma = 1$.

Example 6.10 *For $t = 4$, there are three nonisomorphic extended perfect $\mathbb{Z}_2\mathbb{Z}_4$-linear codes of binary length 16, since we have three possible parameters: $\delta = 0$, $\delta = 1$ and $\delta = 2$.*

We can construct parity-check matrices of the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-additive extended perfect codes with $\delta = 0$ and $\delta = 1$, using (18) and the matrix $\mathcal{H}_1''$ given in Example 6.6. On the other hand, using (19) and the matrix $\mathcal{H}_1'$ of type $(2, 1; 1, 1)$, given in Example 6.6, we can obtain the parity-check matrix for the $\mathbb{Z}_2\mathbb{Z}_4$-additive extended perfect code with $\delta = 2$.

6.2. $\mathbb{Z}_2\mathbb{Z}_4$-additive Hadamard codes

A *Hadamard matrix* H of order n is an $n \times n$ matrix of $+1$'s and -1's such that $HH^T = nI$, where I is the $n \times n$ identity matrix. It is well known that if a Hadamard matrix H of order n exists, then n is 1, 2 or a multiple of 4 [1, 41]. Two Hadamard matrices are *equivalent* if one matrix can be obtained from the other by permuting rows and (or) columns and multiplying rows and (or) columns by -1. We can change the first row and column of H into $+1$'s and we obtain an equivalent Hadamard matrix H', which is called *normalized*.

If $+1$'s are replaced by 0's and -1's by 1's, H' is changed into a *binary Hadamard matrix* $c(H')$. Since the rows of H' are orthogonal, any two rows of $c(H')$ agree in $n/2$ places and

differ in $n/2$ places, and so are at Hamming distance $n/2$ apart. The binary code consisting of the rows of $c(H')$ and their complements is called a *(binary) Hadamard code* [1, 41] and we use H to denote it. A Hadamard code of length n has $2n$ codewords and minimum distance $n/2$.

The $\mathbb{Z}_2\mathbb{Z}_4$-additive codes such that, under the Gray map, give a Hadamard code are called *$\mathbb{Z}_2\mathbb{Z}_4$-additive Hadamard codes*. Given a $\mathbb{Z}_2\mathbb{Z}_4$-additive Hadamard code, after applying the Gray map, the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-linear code is called *Hadamard $\mathbb{Z}_2\mathbb{Z}_4$-linear code*. The $\mathbb{Z}_2\mathbb{Z}_4$-additive Hadamard codes with $\alpha = 0$ are also called *quaternary linear Hadamard codes*, and their corresponding $\mathbb{Z}_2\mathbb{Z}_4$-linear codes are also called *Hadamard $\mathbb{Z}_4$-linear codes*. Actually, the $\mathbb{Z}_2\mathbb{Z}_4$-additive Hadamard codes can also be seen as the additive dual codes of the $\mathbb{Z}_2\mathbb{Z}_4$-additive extended perfect codes.

The Hadamard $\mathbb{Z}_4$-linear codes H are the $\mathbb{Z}_4$-dual of the extended perfect $\mathbb{Z}_4$-linear codes C^*, that is, $H = C^*_\perp$. For any integer $t \geq 4$ and each $\delta \in \{1, \ldots, \lfloor (t+1)/2 \rfloor\}$, there exists a unique $\mathbb{Z}_4$-dual code H of the extended perfect $\mathbb{Z}_4$-linear code and all these codes H are pairwise nonequivalent, except for $\delta = 1$ and $\delta = 2$, where the codes H are isomorphic to the binary dual of the extended Hamming code [38]. Therefore, by Theorem 6.1, the number of nonisomorphic Hadamard $\mathbb{Z}_4$-linear codes of binary length 2^t is $\lfloor \frac{t-1}{2} \rfloor$ for all $t \geq 3$, and it is 1 for $t = 2$, $t = 3$ and $t = 4$. On the other hand, we have analogous results for the Hadamard $\mathbb{Z}_2\mathbb{Z}_4$-linear codes with the exception for $\delta = 0$ and $\delta = 1$, where the dual codes are isomorphic to the binary dual of the extended Hamming code [14]. Therefore, by Theorem 6.4, the number of nonisomorphic Hadamard $\mathbb{Z}_2\mathbb{Z}_4$-linear codes with $\alpha \neq 0$ of binary length 2^t is $\lfloor t/2 \rfloor$ for all $t \geq 2$, and it is 1 for $t = 2$ and $t = 3$. Finally, note that the parity-check matrices of the $\mathbb{Z}_2\mathbb{Z}_4$-additive extended perfect codes, described in Subsections 6.1.1 and 6.1.2 can be taken as generator matrices for the $\mathbb{Z}_2\mathbb{Z}_4$-additive Hadamard codes.

As for $\mathbb{Z}_2\mathbb{Z}_4$-additive extended perfect codes, the $\mathbb{Z}_2\mathbb{Z}_4$-additive Hadamard codes can also be classified using either the rank or the dimension of the kernel, as it is proven in [38, 48], where these parameters are computed (see also Examples 5.4, 5.5, 5.12 and 5.13). The intersection problem for these codes, i.e., which are the possibilities for the number of codewords in the intersection of two $\mathbb{Z}_2\mathbb{Z}_4$-additive codes of the same length is investigated in [60]. Finally, also mention that the permutation automorphism group of these $\mathbb{Z}_2\mathbb{Z}_4$-additive extended perfect and the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-linear codes has been studied in [39, 45].

6.3. $\mathbb{Z}_2\mathbb{Z}_4$-additive Reed-Muller codes

Linear Reed-Muller codes have been extensively studied and have good combinatorics properties [41]. There are several nonbinary generalizations of such codes [8, 9, 63]. In this subsection, we present the generalization given in [50, 51]. The codes in these families of $\mathbb{Z}_2\mathbb{Z}_4$-additive Reed-Muller codes satisfy that, after the Gray map, the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-linear codes have the same parameters and properties as the codes of the binary linear Reed-Muller family.

For $m \geq 1$ and $0 \leq r \leq m$, there is a binary linear Reed-Muller code $RM(r, m)$ such that,

(i) Length $n = 2^m$, minimum distance $d = 2^{m-r}$, and dimension $k = \sum_{i=0}^{r} \binom{m}{i}$.

(ii) $RM(r-1, m)$ is a subcode of $RM(r, m)$ for $r > 0$.

(iii) $RM(r, m)$ is the dual code of $RM(m-1-r, m)$ for $r < m$.

Moreover, $RM(1,m)$ is equivalent to the linear Hadamard code and $RM(m-2,m)$ is equivalent to the extended Hamming code of the same length. In [33], it is proved that $RM(r,m)$ is $\mathbb{Z}_4$-linear for $r = 0, 1, 2, m-1$ and not $\mathbb{Z}_4$-linear for $r = m-2$ ($m \geq 5$). In a subsequent work [35], it is proved that $RM(r,m)$ is not $\mathbb{Z}_4$-linear for $3 \leq r \leq m-2$.

Binary linear Reed-Muller codes can be build using Plotkin construction [41], so it is a natural question to use some sort of Plotkin construction to generate $\mathbb{Z}_2\mathbb{Z}_4$-additive Reed-Muller codes such that for $r = 1$ we have a Hadamard $\mathbb{Z}_2\mathbb{Z}_4$-linear code and, for $r = m-2$ we have a $\mathbb{Z}_2\mathbb{Z}_4$-additive extended perfect code.

Plotkin Construction: Let $\mathcal{X}$ and $\mathcal{Y}$ be any two $\mathbb{Z}_2\mathbb{Z}_4$-additive codes of types $(\alpha, \beta; \gamma_{\mathcal{X}}, \delta_{\mathcal{X}})$, $(\alpha, \beta; \gamma_{\mathcal{Y}}, \delta_{\mathcal{Y}})$ and minimum distances $d_{\mathcal{X}}$, $d_{\mathcal{Y}}$, respectively. If $\mathcal{G}_{\mathcal{X}}$ and $\mathcal{G}_{\mathcal{Y}}$ are the generator matrices of $\mathcal{X}$ and $\mathcal{Y}$, then the matrix

$$\mathcal{G}_P = \begin{pmatrix} \mathcal{G}_{\mathcal{X}} & \mathcal{G}_{\mathcal{X}} \\ 0 & \mathcal{G}_{\mathcal{Y}} \end{pmatrix}$$

is the generator matrix of a $\mathbb{Z}_2\mathbb{Z}_4$-additive code, denoted by $\mathcal{P}(\mathcal{X}, \mathcal{Y})$.

Proposition 6.11 *[50] The code $\mathcal{P}(\mathcal{X}, \mathcal{Y})$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(2\alpha, 2\beta; \gamma, \delta)$, where $\gamma = \gamma_{\mathcal{X}} + \gamma_{\mathcal{Y}}$, $\delta = \delta_{\mathcal{X}} + \delta_{\mathcal{Y}}$, binary length $n = 2\alpha + 4\beta$, size $2^{\gamma+2\delta}$ and minimum distance $d = \min\{2d_{\mathcal{X}}, d_{\mathcal{Y}}\}$.*

Applying twice the Plotkin construction, one after another, but slightly changing the submatrices in the generator matrix, we obtain a new construction with interesting properties with regard to the minimum distance of the generated code. We call this new construction *BQ-Plotkin* construction for $\alpha = 0$ and *BA-Plotkin* for $\alpha \neq 0$.

BQ-Plotkin Construction: Let $\mathcal{G}_{\mathcal{A}}$, $\mathcal{G}_{\mathcal{B}}$, and $\mathcal{G}_{\mathcal{C}}$ be generators matrices of the quaternary linear codes $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, respectively. We define the code $\mathcal{BQ}(\mathcal{A}, \mathcal{B}, \mathcal{C})$ as the quaternary linear code generated by

$$\mathcal{G}_{BQ} = \begin{pmatrix} \mathcal{G}_{\mathcal{A}} & \mathcal{G}_{\mathcal{A}} & \mathcal{G}_{\mathcal{A}} & \mathcal{G}_{\mathcal{A}} \\ 0 & \mathcal{G}'_{\mathcal{B}} & 2\mathcal{G}'_{\mathcal{B}} & 3\mathcal{G}'_{\mathcal{B}} \\ 0 & 0 & \hat{\mathcal{G}}_{\mathcal{B}} & \hat{\mathcal{G}}_{\mathcal{B}} \\ 0 & 0 & 0 & \mathcal{G}_{\mathcal{C}} \end{pmatrix},$$

where $\mathcal{G}'_{\mathcal{B}}$ is the matrix obtained from $\mathcal{G}_{\mathcal{B}}$ after switching twos and ones in their $\gamma_{\mathcal{B}}$ rows of order two and $\hat{\mathcal{G}}_{\mathcal{B}}$ is the matrix obtained from $\mathcal{G}_{\mathcal{B}}$ after removing their $\gamma_{\mathcal{B}}$ rows of order two.

Proposition 6.12 *[50] The code $\mathcal{BQ}(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is a quaternary linear code of type $(4\beta; \gamma, \delta)$, where $\gamma = \gamma_{\mathcal{A}} + \gamma_{\mathcal{C}}$, $\delta = \delta_{\mathcal{A}} + \gamma_{\mathcal{B}} + 2\delta_{\mathcal{B}} + \delta_{\mathcal{C}}$, binary length $n = 8\beta$, size $2^{\gamma+2\delta}$ and minimum distance $d = \min\{4d_{\mathcal{A}}, 2d_{\mathcal{B}}, d_{\mathcal{C}}\}$.*

As in the binary case, given $m \geq 2$ and a family of quaternary linear Reed-Muller codes, $\mathcal{RM}_s(r, m-1)$, we can apply the Plotkin construction to obtain a new family $\mathcal{RM}_s(r, m)$. And, given $m \geq 3$, from $\mathcal{RM}_{s-1}(r, m-2)$, $\mathcal{RM}_{s-1}(r-1, m-2)$, $\mathcal{RM}_{s-1}(r-2, m-2)$, $0 < s \leq \lceil \frac{m-1}{2} \rceil$, we can apply the BQ-Plotkin construction to obtain a new family $\mathcal{RM}_s(r, m)$.

Theorem 6.13 *[50] For $m \geq 1$, the $\lceil \frac{m-1}{2} \rceil$ families of quaternary linear codes $\mathcal{RM}_s(r, m)$, $0 < s \leq \lceil \frac{m-1}{2} \rceil$, $0 \leq r \leq m$, have the following properties:*

(i) Binary length 2^m, minimum Lee weight $d = 2^{m-r}$, and size $2^{\sum_{i=0}^{r}\binom{m}{i}}$.

(ii) $\mathcal{RM}_s(r-1,m)$ is a subcode of $\mathcal{RM}_s(r,m)$, $r > 0$. $\mathcal{RM}_s(0,m)$ is the repetition code, $\mathcal{RM}_s(m,m)$ is the whole space $\mathbb{Z}_4^{2^{m-1}}$ and $\mathcal{RM}_s(m-1,m)$ is the even code.

(iii) $\mathcal{RM}_s(1,m)$ is a quaternary linear Hadamard code and $\mathcal{RM}_s(m-2,m)$ is a quaternary linear extended perfect code.

(iv) $\mathcal{RM}_s(r,m)$ is the dual code of $\mathcal{RM}_s(m-1-r,m)$, $-1 \leq r \leq m$.

These families of quaternary linear Reed-Muller codes, $\mathcal{RM}_s(r,m)$, $0 < s \leq \lceil \frac{m-1}{2} \rceil$, $0 \leq r \leq m$, were completely classified in [44], computing the dimension of the kernel of the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-linear codes and generalizing the known results about the dimension of the kernel for $\mathbb{Z}_2\mathbb{Z}_4$-linear Hadamard and $\mathbb{Z}_2\mathbb{Z}_4$-linear extended perfect codes.

BA-Plotkin Construction: Let $\mathcal{G}_\mathcal{X}$, $\mathcal{G}_\mathcal{Y}$, and $\mathcal{G}_\mathcal{Z}$ be generators matrices of the $\mathbb{Z}_2\mathbb{Z}_4$-additive codes $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$, respectively. We define the code $\mathcal{BA}(\mathcal{X},\mathcal{Y},\mathcal{Z})$ as the $\mathbb{Z}_2\mathbb{Z}_4$-additive code generated by

$$\mathcal{G}_{BA} = \left(\begin{array}{cc|ccccc} \mathcal{G}_\mathcal{X}[b] & \mathcal{G}_\mathcal{X}[b] & 2\mathcal{G}_\mathcal{X}[b] & \mathcal{G}_\mathcal{X}[q] & \mathcal{G}_\mathcal{X}[q] & \mathcal{G}_\mathcal{X}[q] & \mathcal{G}_\mathcal{X}[q] \\ 0 & \mathcal{G}_\mathcal{Y}[b_2] & \mathcal{G}_\mathcal{Y}[b_2] & 0 & 2\mathcal{G}'_\mathcal{Y}[q_2] & \mathcal{G}'_\mathcal{Y}[q_2] & 3\mathcal{G}'_\mathcal{Y}[q_2] \\ 0 & \mathcal{G}_\mathcal{Y}[b_4] & \mathcal{G}_\mathcal{Y}[b_4] & 0 & \mathcal{G}_\mathcal{Y}[q_4] & 2\mathcal{G}_\mathcal{Y}[q_4] & 3\mathcal{G}_\mathcal{Y}[q_4] \\ \mathcal{G}_\mathcal{Y}[b_4] & \mathcal{G}_\mathcal{Y}[b_4] & 0 & 0 & 0 & \mathcal{G}_\mathcal{Y}[q_4] & \mathcal{G}_\mathcal{Y}[q_4] \\ 0 & \mathcal{G}_\mathcal{Z}[b] & 0 & 0 & 0 & 0 & \mathcal{G}_{\mathcal{Z}[q]} \end{array}\right),$$

where $\mathcal{G}'_\mathcal{Y}[q_2]$ is the matrix obtained from $\mathcal{G}_\mathcal{Y}[q_2]$ after switching twos and ones in their $\gamma_\mathcal{Y}$ rows of order two, and considering the ones from the third column of the construction as ones in $\mathbb{Z}_4$.

Theorem 6.14 *[51] For any r and $m \geq 2$, $0 < r < m$, the code $\mathcal{ARM}_s(r,m)$ obtained by applying the Plotkin construction on codes $\mathcal{ARM}_s(r,m-1)$ and $\mathcal{ARM}_s(r-1,m-1)$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(2\alpha, 2\beta; \gamma, \delta)$, where $\gamma = \gamma' + \gamma''$ and $\delta = \delta' + \delta''$; binary length 2^m; size 2^k, where $k = \sum_{i=0}^{r}\binom{m}{i}$; minimum distance 2^{m-r} and $\mathcal{ARM}_s(r-1,m) \subset \mathcal{ARM}_s(r,m)$. We consider $\mathcal{ARM}_s(0,m)$ be the repetition code with only one nonzero codeword (the vector with 2α ones and 2β twos) and $\mathcal{ARM}_s(m,m)$ be the whole space $\mathbb{Z}_2^{2\alpha} \times \mathbb{Z}_4^{2\beta}$.*

Theorem 6.15 *[51] For any r and $m \geq 3$, $0 < r < m$, $s > 0$, the code $\mathcal{ARM}_{s+1}(r,m+1)$ obtained by applying the BA-Plotkin construction on codes $\mathcal{ARM}_s(r,m-1)$, $\mathcal{ARM}_s(r-1,m-1)$ and $\mathcal{ARM}_s(r-2,m-1)$, respectively, to obtain a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(2\alpha, \alpha+4\beta; \gamma, \delta)$, where $\gamma = \gamma' + \gamma'''$, $\delta = \delta' + \gamma'' + 2\delta'' + \delta'''$; binary length 2^{m+1}; size 2^k, where $k = \sum_{i=0}^{r}\binom{m+1}{i}$; minimum distance 2^{m-r+1} and, moreover, $\mathcal{ARM}_{s+1}(r-1,m+1) \subset \mathcal{ARM}_{s+1}(r,m+1)$.*

6.4. Maximum distance separable $\mathbb{Z}_2\mathbb{Z}_4$-additive codes

We apply known upper bounds on the minimum distance of codes over rings to the case of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes. Two kinds of maximum distance separable codes are studied. We determine all possible parameters of these codes and characterize the codes in certain cases. The main results are also valid when $\alpha = 0$, namely for quaternary linear codes. The results in this subsection can be found in [6].

6.4.1. BOUNDS ON THE MINIMUM DISTANCE

The usual Singleton bound [62] for a code $\mathcal{C}$ of length n over an alphabet of size q is given by

$$d(\mathcal{C}) \leq n - \log_q |\mathcal{C}| + 1.$$

This is a combinatorial bound and does not rely on the algebraic structure of the code. It is well known [41] that for the binary case, $q = 2$, the only codes achieving this bound are the repetition codes (with $d(\mathcal{C}) = n$), codes with minimum distance 2 and size 2^{n-1}, or the trivial code containing all 2^n vectors. We remark that sometimes the singleton codes, i.e. codes with just one codeword, are also considered in this class, but it depends on the definition of minimum distance for such codes.

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ and let $C = \Phi(\mathcal{C})$. Since $d(\mathcal{C}) = d_H(C)$, we immediately obtain that

$$d(\mathcal{C}) \leq \alpha + 2\beta - \gamma - 2\delta + 1. \tag{20}$$

This version of the Singleton bound was previously stated for quaternary linear codes ($\alpha = 0$) in [27]. From [27], we also know that if $\mathcal{C}$ is a code of length n over a ring R with minimum distance $d(\mathcal{C})$, then

$$\left\lfloor \frac{d(\mathcal{C}) - 1}{2} \right\rfloor \leqslant n - rank(\mathcal{C}), \tag{21}$$

where $rank(\mathcal{C})$ is the minimal cardinality of a generating system for $\mathcal{C}$.

Theorem 6.16 *[6] Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$. Then,*

$$\frac{d(\mathcal{C}) - 1}{2} \leqslant \frac{\alpha}{2} + \beta - \frac{\gamma}{2} - \delta; \tag{22}$$

$$\left\lfloor \frac{d(\mathcal{C}) - 1}{2} \right\rfloor \leqslant \alpha + \beta - \gamma - \delta. \tag{23}$$

Let $\mathcal{C}$ be a $\mathbb{Z}_2\mathbb{Z}_4$-additive code. Recall that if $\mathcal{C} = \mathcal{C}_X \times \mathcal{C}_Y$, then $\mathcal{C}$ is called *separable*.

Theorem 6.17 *[6] If $\mathcal{C}$ is a $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ which is separable, then $d(\mathcal{C}) = \min\{d(\mathcal{C}_X), d(\mathcal{C}_Y)\}$ and $d(\mathcal{C}) \leq \min\{\alpha - \kappa + 1, \overline{d}\}$, where $\overline{d}$ is the maximum value satisfying both Bound (22) and Bound (23).*

6.4.2. MAXIMUM DISTANCE SEPARABILITY

We say that a $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ is maximum distance separable (MDS) if $d(\mathcal{C})$ meets the bound given in (22) or (23). In the first case, we say that $\mathcal{C}$ is MDS with respect to the Singleton bound, briefly MDSS. In the second case, $\mathcal{C}$ is MDS with respect to the rank bound, briefly MDSR.

Now, we give the characterization of all MDSS $\mathbb{Z}_2\mathbb{Z}_4$-additive codes. By the *even code* we mean the set of all even weight vectors. By the *repetition code* we mean the code such that its Gray map image is the binary repetition code with the all-zero and the all-one codewords.

Theorem 6.18 *[6] Let $\mathcal{C}$ be an MDSS $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ such that $1 < |\mathcal{C}| < 2^{\alpha+2\beta}$. Then, $\mathcal{C}$ is either*

(i) the repetition code of type $(\alpha, \beta; 1, 0; \kappa)$ and minimum distance $d(\mathcal{C}) = \alpha + 2\beta$, where $\kappa = 1$ if $\alpha > 0$ and $\kappa = 0$ otherwise; or

(ii) the even code with minimum distance $d(\mathcal{C}) = 2$ and type $(\alpha, \beta; \alpha - 1, \beta; \alpha - 1)$ if $\alpha > 0$, or type $(0, \beta; 1, \beta - 1; 0)$ otherwise.

Since the codes described in (i) and (ii) of Theorem 6.18 are additive dual codes, it is still true that the dual of an MDSS code is again MDSS, which is a well known property for linear codes over finite fields [41].

We can also give a strong condition for a $\mathbb{Z}_2\mathbb{Z}_4$-additive code to be MDSR.

Theorem 6.19 *[6] Let $\mathcal{C}$ be an MDSR $\mathbb{Z}_2\mathbb{Z}_4$-additive code of type $(\alpha, \beta; \gamma, \delta; \kappa)$ such that $1 < |\mathcal{C}| < 2^{\alpha+2\beta}$. Then, either*

(i) $\mathcal{C}$ is the repetition code as in (i) of Theorem 6.18 with $\alpha \leq 1$; or

(ii) $\mathcal{C}$ is of type $(\alpha, \beta; \gamma, \alpha + \beta - \gamma - 1; \alpha)$, where $\alpha \leq 1$ and $d(\mathcal{C}) = 4 - \alpha \in \{3, 4\}$; or

(iii) $\mathcal{C}$ is of type $(\alpha, \beta; \gamma, \alpha + \beta - \gamma; \alpha)$, where $\alpha \leq 1$ and $d(\mathcal{C}) \leq 2 - \alpha \in \{1, 2\}$.

Note that it is not true that the additive dual code of an MDSR code is again MDSR. See the examples below.

Recall that the *rank* of a binary code C is the dimension of the linear span of C. If C is linear, then the rank is simply the dimension of C. For MDS $\mathbb{Z}_2\mathbb{Z}_4$-additive codes, we can state which are the possible values for the rank of the Gray map images.

Corollary 6.20 *If $\mathcal{C}$ is an MDS $\mathbb{Z}_2\mathbb{Z}_4$-additive code, then $C = \Phi(\mathcal{C})$ is a linear code or it has rank equal to $\log_2 |C| + 1$. In this last case, $\mathcal{C}$ is an MDSR code with minimum distance 3 or 4.*

6.4.3. EXAMPLES

Examples 6.21 and 6.22 satisfy Bound (22). Example 6.21 is an MDS code with $\gamma = 0$, and Example 6.22 is an MDS code with $\alpha > 1$.

Example 6.21 *Consider a $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}_2$ of length 2 with generator matrix $\mathcal{G}_2 = (1|1)$. The code $\mathcal{C}_2$ is of type $(1, 1; 0, 1; 0)$ and $d(\mathcal{C}_2) = 2$. Applying Bound (22), we get that $\mathcal{C}_2$ is an MDSS code. In fact, it is the even code with $\alpha = \beta = 1$. Its additive dual code $\mathcal{C}_2^{\perp}$ is the repetition code $\{(0, 0), (1, 2)\}$, which is MDSS and MDSR. However, note that $\mathcal{C}_2$ is not an MDSR code.*

Example 6.22 *Consider a $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}_3$ with generator matrix*

$$\mathcal{G}_3 = \left(\begin{array}{cc|c} 1 & 1 & 0 \\ \hline 0 & 1 & 1 \end{array}\right).$$

The code $\mathcal{C}_3$ is of type $(2, 1; 1, 1; 1)$ and $d(\mathcal{C}_3) = 2$. This is again an MDSS code, which is the even code for $\alpha = 2$ and $\beta = 1$. The code $\mathcal{C}_3$ is not an MDSR code, but $\mathcal{C}_3^{\perp} = \{(0, 0, 0), (1, 1, 2)\}$ is again MDSS and MDSR.

The next example satisfies Bound (23).

Example 6.23 *Consider a $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}_4$ with generator matrix*

$$\mathcal{G}_4 = \left(\begin{array}{c|ccc} 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{array}\right).$$

The code $\mathcal{C}_4$ is of type $(1, 3; 3, 0; 1)$ and $d(\mathcal{C}_4) = 3$. Thus, $\mathcal{C}_4$ is an MDSR code (but not MDSS).

The next example gives a general construction for MDS codes meeting Bound (23) starting from binary MDS codes.

Example 6.24 *Let C be a binary $[n, k, d]$ MDS code. Applying χ to all but one coordinate gives a $\mathbb{Z}_2\mathbb{Z}_4$-additive code $\mathcal{C}$ with $\alpha = 1$, $\beta = n - 1$, $\gamma = k$, $\delta = 0$ and $d(\mathcal{C}) = 2d - 1$. Then, $d = n-k+1 = \alpha+\beta-\gamma+1$ so that $\lfloor \frac{d(\mathcal{C})-1}{2} \rfloor = \lfloor d - \frac{1}{2} \rfloor = d-1 = \alpha+\beta-(\gamma+\delta)$ and meets Bound (23). Of course, this construction works for the even binary code and the repetition binary code which are the possible binary linear MDS codes with more than one codeword.*

Finally, the next example shows an MDSR code which has a nonlinear Gray map image.

Example 6.25 *Let $\mathcal{C}_8$ be an MDSR $\mathbb{Z}_2\mathbb{Z}_4$-additive code given by the following generator matrix*

$$\mathcal{G}_8 = \left(\begin{array}{c|ccccccc} 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 \\ \hline 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right).$$

The code $\mathcal{C}_8$ is of type $(1, 7; 5, 2; 1)$ with $d(\mathcal{C}_8) = 3$, and it also meets Bound (23). Since $2(0|1110010) * (0|1001101) \notin \mathcal{C}_8$, where $*$ denotes the component-wise product, then from [29] the rank is 10 and, therefore, $\mathcal{C}_8$ has a nonlinear Gray map image.

As a conclusion, we have that all MDS $\mathbb{Z}_2\mathbb{Z}_4$-additive codes are zero or one error-correcting codes with the exception of the trivial repetition codes containing two codewords.

6.5. $\mathbb{Z}_4$-linear Preparata-like and Kerdock-like codes

In this subsection, we present some results about additive extended Preparata-like codes. The most important says that any additive extended Preparata-like code verifies that $\alpha = 0$, i.e., it is always a $\mathbb{Z}_4$-linear code. Moreover, we give the rank and dimension of the kernel of such Preparata-like codes and their $\mathbb{Z}_4$-dual codes, i.e., the $\mathbb{Z}_4$-linear Kerdock-like codes. The material in this subsection is a summary of the results presented in [14, 15, 38].

We denote by P an extended $\mathbb{Z}_4$-linear Preparata-like code with parameters $(n+1, d, M) = (2^{2m}, 6, 2^{n+1-4m})$, obtained as the Gray map image of an arbitrary quaternary linear Preparata-like code $\mathcal{P}$ [33]. The binary code with $d = 5$, obtained from P by deleting any position, is denoted by P^*. Denote by K an extended $\mathbb{Z}_4$-linear Kerdock-like code with parameters $(n+1 = 2^{2m}, (n+1)/2 - \sqrt{n+1}/2, 2^{4m})$, obtained as the Gray map image of $\mathcal{K}$, the $\mathbb{Z}_4$-dual of $\mathcal{P}$, that is, $\mathcal{K} = \{x \in \mathbb{Z}_4^{(n+1)/2} \mid \langle x, p\rangle = 0,\ \forall p \in \mathcal{P}\}$. We denote by C an extended perfect $\mathbb{Z}_4$-linear code with parameters $(n+1 = 2^{2m}, d = 4, M = 2^{n-2m-1})$ and by H its $\mathbb{Z}_4$-dual, a Hadamard $\mathbb{Z}_4$-linear code with parameters $(n+1 = 2^{2m}, d = (n+1)/2, M = 2(n+1))$.

Preparata-like codes are nonlinear. Concerning to their possible algebraic structure, we remark that the original Preparata codes [49] have a group propelinear structure [55] and the extended Preparata-like codes defined in [33] are $\mathbb{Z}_4$-linear and so, they are propelinear codes. In [15], it is proved the nonexistence of extended Preparata-like codes with other additive structures different of the $\mathbb{Z}_4$-linear ones.

Theorem 6.26 *[15] Let P be an additive extended Preparata-like code. Then, P is $\mathbb{Z}_4$-linear.*

Corollary 6.27 *[15] Let P be an additive extended Preparata-like code and let $K = \phi(\mathcal{P}^{\perp})$ be the corresponding additive extended Kerdock-like code. Then, K is $\mathbb{Z}_4$-linear.*

Given a Preparata-like code P^*, it is well known that the code C^* obtained as the union of P^* and the vectors at (maximum) distance 3 from P^* is a binary perfect code [61]. If P^* is a standard Preparata code, then C^* is linear, i.e., a Hamming code. If P is an extended Preparata-like code, then C is an extended perfect code. It is showed that if P is $\mathbb{Z}_4$-linear, then C is also $\mathbb{Z}_4$-linear. This allows to compute the ranks and kernels of P and its $\mathbb{Z}_4$-dual, the Kerdock-like code K. For the extended perfect $\mathbb{Z}_4$-linear codes the rank was computed in [38] and the kernel in [14].

Recall that Kerdock codes are defined [41] as the union of the first order Reed-Muller code $RM(1, 2m)$ and $2^{m-1} - 1$ cosets of $RM(1, 2m)$ in $RM(2, 2m)$ corresponding to quadratic bent functions such that the sum of any two is again a quadratic bent function. There are alternative (equivalent) descriptions of these cosets [37] but these cosets, or equivalently, their representatives, form what is called a Kerdock set [37]. There are many inequivalent choices of these Kerdock sets even for $\mathbb{Z}_4$-linear codes and thus many inequivalent $\mathbb{Z}_4$-linear Kerdock (and Preparata-like) codes [18]. It is proved in [15] that any $\mathbb{Z}_4$-linear Kerdock-like code has this same structure and that all such codes have the same rank as the classical Kerdock code [41].

Theorem 6.28 *[15] The rank of any $\mathbb{Z}_4$-linear Kerdock-like code K of length $n+1 = 2^{2m}$ is $rank(K) = 2m^2 + m + 1$.*

When P and C are additive (that is, $\mathbb{Z}_4$-linear), let $K = \phi(\mathcal{P}^{\perp})$ be the Kerdock code and $H = \phi(\mathcal{C}^{\perp})$ be the Hadamard code (the $\mathbb{Z}_4$-duals of P and C, respectively). From [33], the classical Kerdock code is additive, so the rank and dimension of the kernel are the same as the values computed for the general additive case.

	Additive Codes		Classical Codes	
	Rank	Kernel	Rank	Kernel
P	$2^{2m}-2m$	$2^{2m-1}-(2m-1)$	$2^{2m}-2m-1$	$2^{2m}-6m+1$
K	$2m^2+m+1$	$2m+1$	$2m^2+m+1$	$2m+1$
C	$2^{2m}-2m$	$2^{2m-1}+(2m-1)$	$2^{2m}-2m-1$	$2^{2m}-2m-1$
H	$2m+1$	$2m+1$		
R			$2m+1$	$2m+1$

Table 7: Rank and dimension of the kernel for codes P, K, C, H, R.

When P is the standard extended Preparata code (due to Preparata [49]), we know the kernel from its structure and it is easy to compute its dimension. The corresponding extended perfect code C is linear and it is possible to construct the binary Reed-Muller code $R = C^{\perp} = RM(1, 2m)$, the dual of the extended Hamming code C.

Note that codes H and R are constructed in different way, but R is unique in the class of linear codes, and H is also a linear code with the same parameters, so they are the same code.

As a summary, we present Table 7, where we compare the values of ranks and dimension of kernels for all these codes. Note that in the specific case $m = 2$, when Preparata codes and Kerdock codes coincide with the unique Nordstrom-Robinson code [41], the rank of the additive extended Preparata code is $rank(P) = 2^{2m} - 2m - 1$.

7. Application. $\mathbb{Z}_2\mathbb{Z}_4$-additive perfect codes in steganography

Steganography is an information hiding application which aims to hide secret data imperceptibly into a cover object. In [57] it is described a novel coding method based on $\mathbb{Z}_2\mathbb{Z}_4$-additive codes in which data is embedded by distorting each cover symbol by one unit at most (± 1-steganography). This method is optimal and solves the problem encountered by the most efficient methods known today, concerning the treatment of boundary values. The performance of this new technique is compared with that of the mentioned methods and with the well known rate-distortion upper bound to conclude that a higher payload can be obtained for a given distortion by using the proposed method.

Steganography is a scientific discipline within *data hiding*, which hides information imperceptibly into innocuous media. A comprehensive overview of the core principles and the mathematical methods that can be used for data hiding can be found in [42]. An interesting steganographic method is known as *matrix encoding*, introduced by Crandall [21] and analyzed by Bierbrauer et al. [5]. Matrix encoding requires the sender and the recipient to agree in advance on a parity-check matrix H, and the secret message is then extracted by the recipient as the syndrome (with respect to H) of the received cover object. This method was made popular by Westfeld [67], who incorporated a specific implementation using Hamming codes. The resulting method is known as the F5 algorithm and it can embed t bits of message in $2^t - 1$ cover symbols by changing, at most, one of them.

There are several parameters which are used to evaluate the performance of a steganographic

method over a cover message of N symbols: the *average distortion* $D = \frac{R_a}{N}$, where R_a is the expected number of changes over uniformly distributed messages; the *embedding rate* $E = \frac{t}{N}$, which is the amount of bits that can be hidden in a cover message; and some authors use instead the *embedding efficiency*, which is the average number of embedded bits per change. In our case we use the average distortion and the embedding rate. Given two methods with the same embedding rate, the one with smaller average distortion is said to perform better than the other. A scheme with block length N, embedding rate E, and average distortion D is called *optimal*, if all other schemes with the same block length N have embedding rate $E' \leq E$ or average distortion $D' \geq D$. Following the terminology used by Fridrich et al. [32], the tuple (D, E) is called CI*-rate*.

As Willems et al. in [66], we also assume that a discrete source produces a sequence $\mathbf{x} = (x_1, \ldots, x_N)$, where N is the block length, $x_i \in \aleph = \{0, 1, \ldots, 2^B - 1\}$, and $B \in \{8, 12, 16\}$ depends on the kind of source. The secret message $\mathbf{s} \in \{1, \ldots, M\}$ produces a composite sequence $\mathbf{y} = f(\mathbf{x}, \mathbf{s})$, where $\mathbf{y} = (y_1, \ldots y_N)$ and each $y_i \in \aleph$, by distorting $\mathbf{x}$. This distortion is assumed to be of squared-error type [66]. In these conditions, we may deal with "binary steganography", in which information is carried by the least significant bit (LSB) of each x_i and the appropriate solution comes from using binary Hamming codes [67], later improved using product Hamming codes [56]; or we may deal with "± 1-steganography", where $y_i = x_i + c$ for $c \in \{0, +1, -1\}$ and the information is carried by the two LSBs of x_i. Let the absolute value of c be the *amplitude* of an embedding change.

There are some steganographic techniques in which messages carrying hidden information are statistically indistinguishable from those not carrying hidden data. However, in general, the embedding becomes statistically detectable rather quickly with the increasing amplitude of embedding changes, and our interest goes to avoid changes of amplitude greater than one. With this assumption, the embedding rate of our ± 1-steganographic scheme is compared with the upper bound $H(D) + D$ [66], where $H(D)$ is the binary entropy function $H(D) = -D \log_2(D) - (1 - D) \log_2(1 - D)$ and $0 \leq D \leq 2/3$ is the average distortion. One of the purposes of steganographers is designing schemes in order to approach this upper bound.

In most papers, ± 1-steganography has been treated using ternary codes. Willems et al. [66] proposed a scheme based on ternary Hamming and Golay codes, which were proved to be optimal except for a remark which exposed a problem related to boundary values. Fridrich et al. [32] proposed a method based on rainbow colouring graphs using q-ary Hamming codes, where q is a prime power. This method performed better than the scheme from [66] when q is not a power of 3. However, the authors of both methods suggest making a change of magnitude greater than one in order to avoid having to apply the change $x_i - 1$ and $x_i + 1$ to a host sequence of value $x_i = 0$ and $x_i = 2^B - 1$, respectively. Note that this would introduce larger distortion and therefore make the embedding more detectable. The treatment of boundary grayscale values in steganography is important and, as far as we know, not many papers have paid attention to this issue.

In [57], we also consider ± 1-steganography. It is a new method based on $\mathbb{Z}_2\mathbb{Z}_4$-additive perfect codes which, although they are not linear, they have a representation using a parity-check matrix that makes them as computationally efficient as the Hamming codes. We show that this new method is optimal and performs better than the method obtained by direct sum of ternary Hamming codes from [66] and the method based on rainbow colouring of graphs using

q-Hamming codes [32] for the specific case $q = 3$. Furthermore, the proposed method also deals better with boundary grayscale values, because the magnitude of embedding changes is under no circumstances greater than one.

7.1. Steganography based on $\mathbb{Z}_2\mathbb{Z}_4$-additive perfect codes

Let us take a $\mathbb{Z}_2\mathbb{Z}_4$-additive perfect code and its additive dual, which is of type $(\alpha, \beta; \gamma, \delta)$. This gives a parity-check matrix H which has γ rows of order two and δ rows of order four.

For instance, by Corollary 6.5, for $m = 4$, there are three different $\mathbb{Z}_2\mathbb{Z}_4$-additive perfect codes of binary length $n = 2^4 - 1 = 15$ which correspond to the possible values of $\delta \in \{0, \ldots, \lfloor \frac{m}{2} \rfloor\} = \{0, 1, 2\}$. For $\delta = 0$, the corresponding $\mathbb{Z}_2\mathbb{Z}_4$-additive perfect code is the usual binary Hamming code, while for $\delta = 2$ the $\mathbb{Z}_2\mathbb{Z}_4$-additive perfect code has parameters $\alpha = 3$, $\beta = 6$, $\gamma = 0$, $\delta = 2$ and the following parity-check matrix:

$$H = \left(\begin{array}{ccc|cccccc} 2 & 0 & 2 & 0 & 1 & 1 & 1 & 1 & 2 \\ 2 & 2 & 0 & 1 & 0 & 1 & 2 & 3 & 1 \end{array} \right). \tag{24}$$

Let $\mathbf{h}_i$, for $i \in \{1, \ldots, \alpha + \beta\}$, denote the i-th column vector of H. Note that the all twos vector $\mathbf{2}$ is always one of the columns in H and, for the sake of simplicity, it is written as column $\mathbf{h}_1$. We group the remaining first α columns in H in such a way that, for any $2 \leq i \leq (\alpha+1)/2$, vector $\mathbf{h}_{2i}$ is paired up with its complementary vector $\bar{\mathbf{h}}_{2i} = \mathbf{h}_{2i+1}$, where $\bar{\mathbf{h}}_{2i} = \mathbf{h}_{2i} + \mathbf{2}$.

To use these $\mathbb{Z}_2\mathbb{Z}_4$-additive perfect codes in steganography, take $N = 2^{m-1} = \frac{\alpha+1}{2} + \beta$ and let $\mathbf{x} = (x_1, \ldots, x_N)$ be an N-length source of grayscale symbols such that $x_i \in \aleph = \{0, 1, \ldots, 2^B - 1\}$, where, for instance, $B = 8$ for grayscale images. We assume each grayscale symbol x_i is represented as a binary vector $(v_{(B-1)i}, \ldots, v_{1i}, v_{0i})$, obtained by first representing x_i in base 4 and then applying the Gray map to every quaternary symbol in that representation. For example, value 239 is represented as the quaternary vector (3233), which then gives rise to the binary vector (10111010) after applying the Gray map. We use the two least significant bits (LSBs), v_{1i}, v_{0i}, of every grayscale symbol x_i in the source, for $i > 1$, as well as the least significant bit v_{01} of symbol x_1 to embed the secret message. Each grayscale symbol x_i is associated with one or more columns $\mathbf{h}_i$ in H:

1. Symbol x_1 is associated with $\mathbf{h}_1$ by taking its least significant bit, v_{01}.

2. Symbol x_i, for $2 \leq i \leq (\alpha + 1)/2$, is associated with $\mathbf{h}_i$ and $\bar{\mathbf{h}}_i$, by taking, respectively, the two least significant bits, v_{1i}, v_{0i}, of x_i.

3. Symbol x_j, for $\alpha < j \leq N$, is associated with $\mathbf{h}_{j+(\alpha-1)/2}$ by taking its two least significant bits v_{1j}, v_{0j} and interpreting them as in $\mathbb{Z}_4$, after the Gray map.

In this way, the N-length packet $\mathbf{x}$ of symbols is translated into a vector $\mathbf{w} \in \mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$. The embedding process we propose is based on the matrix encoding method. The secret message can be any vector $\mathbf{s} \in \mathbb{Z}_2^\gamma \times \mathbb{Z}_4^\delta$. Vector $\epsilon \cdot \mathbf{h}_i$ indicates the changes needed to embed $\mathbf{s}$ within $\mathbf{x}$; that is $H\mathbf{w}^T + \epsilon \cdot \mathbf{h}_i = \mathbf{s}$, where ϵ is an integer whose value will be described bellow, $H\mathbf{w}^T$ is the syndrome vector of $\mathbf{w}$ and $\mathbf{h}_i$ is a column vector in H. We may have the following situations, depending on which column $\mathbf{h}_i$ needs to be modified:

- If $\mathbf{h}_i = \mathbf{h}_1$, then the embedder has to change the least significant bit of x_1 by adding or subtracting one unit to/from x_1, depending on which operation will flip its least significant bit, v_{01}.

- If $\mathbf{h}_i$ is among the first α column vectors in H and $2 \leq i \leq \alpha$, then ϵ can only be $\epsilon = 1$. In this case, since $\mathbf{h}_i$ was paired up with its complementary column vector $\bar{\mathbf{h}}_i$, then this situation is equivalent to make $(v_{1i}, 1 + v_{0i})$ or $(1 + v_{1i}, v_{0i})$, where v_{1i} and v_{0i} are the least significant bits of the symbol x_i which had been associated with those two column vectors. Hence, after the inverse of Gray map, by changing one or another we are actually adding or subtracting one unit to/from x_i. Note that a problem may crop up at this point if we need to add 1 to a symbol x_i of value $2^B - 1$ or subtract 1 from a symbol of value 0.

- If $\mathbf{h}_i$ is one of the last β columns in H, then this situation corresponds to add $\epsilon \in \{0, 1, 2, 3\}$. Note that because we are using a $\mathbb{Z}_2\mathbb{Z}_4$-additive perfect code, ϵ will never be 2. Hence, the embedder should add ($\epsilon = 1$) or subtract ($\epsilon = 3$) one unit to/from symbol $x_{i-(\alpha-1)/2}$. Once again, a problem may arise with boundary values.

Example 7.1 *Let* $\mathbf{x} = (239, 251, 90, 224, 226, 187, 229, 180)$ *be an* N*-length source of gray-scale symbols, where* $x_i \in \{0, \ldots, 255\}$ *and* $N = 8$*, and let* H *be the matrix in (24). The source* $\mathbf{x}$ *is then translated into the vector* $\mathbf{w} = (010|202310)$ *in the way specified above. Let* $\mathbf{s} = (02)^T$ *be the vector representing the secret message we want to embed in* $\mathbf{x}$*. We then compute* $H\mathbf{w}^T = (23)^T$ *and see, by the matrix encoding method, that* $\epsilon = 3$ *and* $\mathbf{h}_i = \mathbf{h}_9$*. According to the described method, we should subtract 1 from* x_8*. In this way,* x_8 *becomes* 179*, and then* $\mathbf{w}' = (010|202313)$*, which has the expected syndrome* $(02)^T$*.*

The problematic cases related to boundary values are also present in methods from [32] and [66], but their authors assume that the probability of gray value saturation is not too large. We argue that, though rare, this gray saturation can still occur.

Our method is able to hide any secret vector $\mathbf{s} \in \mathbb{Z}_2^\gamma \times \mathbb{Z}_4^\delta$ into the given N symbols. Hence, the embedding rate is $(\gamma + 2\delta)$ bits per N symbols, $E = \dfrac{\gamma + 2\delta}{N} = \dfrac{m}{2^{m-1}}$. Concerning the average distortion D, we are using a perfect code of binary length $2^m - 1$, which corresponds to $N = 2^{m-1}$ grayscale symbols. There are $N - 1$ symbols x_i, for $2 \leq i \leq N$, with a probability $2/2^m$ of being subjected to a change; a symbol x_1 with a probability $1/2^m$ of being the one changed; and, finally, there is a probability of $1/2^m$ that neither of the symbols will need to be changed to embed the secret message $\mathbf{s}$. Hence, $D = \dfrac{2N - 1}{N2^m} = \dfrac{2^m - 1}{2^{2m-1}}$. The described method has a CI-rate $(D_m, E_m) = \left(\dfrac{2N - 1}{2N^2}, \dfrac{1 + \log(N)}{N}\right)$, where $N = 2^{m-1}$ and m is any integer $m \geq 2$.

It is shown in [66] that the linear ternary perfect codes (Hamming or Golay) are optimal in the sense that they achieve the smallest possible distortion at a given embedding rate for a fixed block length. This property is not exclusive of these codes and, in fact, the method we have described using $\mathbb{Z}_2\mathbb{Z}_4$-additive perfect codes also satisfies it.

Proposition 7.2 *[57] The proposed embedding method based on* $\mathbb{Z}_2\mathbb{Z}_4$*-additive perfect codes is optimal.*

Note that we are only able to generate an embedding scheme for natural values of $m \geq 2$. However, we can use the direct sum of codes [41] to obtain codes whose CI-rates are convex combinations of CI-rates of boths codes. Thus given any nonallowable parameter D for the average distortion, we can take two codes with CI-rates (D_1, E_1) and (D_2, E_2), respectively, where $D_1 < D < D_2$, and their direct sum generates a code with a new CI-rate (D, E), with $D = \lambda D_1 + (1 - \lambda)D_2$ and $E = \lambda E_1 + (1 - \lambda)E_2$. From a graphic point of view, this is equivalent to draw a line between two contiguous points (D_1, E_1) and (D_2, E_2), as it is shown in Figure 1.

Proposition 7.3 *[57] For $m \geq 4$, the CI-rate given by the method based on $\mathbb{Z}_2\mathbb{Z}_4$-additive perfect codes improves the CI-rate obtained by direct sum of ternary Hamming codes with the same average distortion.*

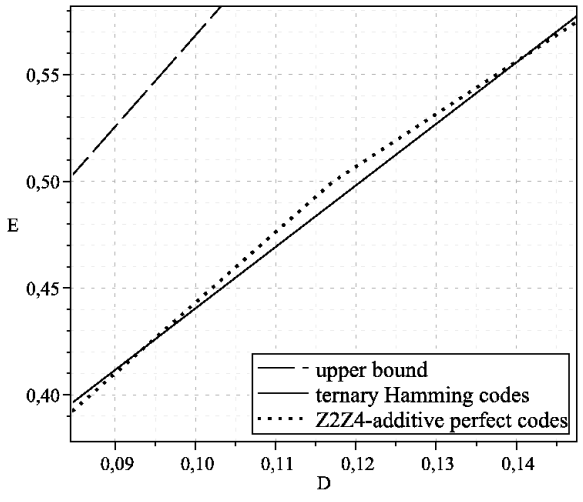

Figure 1: CI-rate (D, E), for $B = 8$, of steganographic methods based on ternary Hamming codes and on $\mathbb{Z}_2\mathbb{Z}_4$-additive perfect codes, compared with the upper bound $H(D) + D$, where E is the embedding rate, D is the average distortion and $H(D)$ is the binary entropy function.

7.2. Solving the extreme grayscale values problem

In Subsection 7.1, we described a problem which may arise when, according to our method, the embedder is required to add one unit to a source symbol x_i containing the maximum allowed value $(2^B - 1)$, or to subtract one unit from a symbol x_i containing the minimum allowed value, 0. To face this problem, we will use the complementary column vector $\bar{\mathbf{h}}_i$ of columns $\mathbf{h}_i$ in matrix H, where $\bar{\mathbf{h}}_i = 3\mathbf{h}_i + \mathbf{2}$ and $\mathbf{h}_i$ is among the last β columns in H. Note that $\mathbf{h}_i$ and $\bar{\mathbf{h}}_i$ can coincide.

The first α column vectors in H will be paired up as before, and the association between each x_i and each column vector $\mathbf{h}_i$ in H will be also the same as in Subsection 7.1. However, given an N-length source of grayscale symbols $\mathbf{x} = (x_1, \ldots, x_N)$, a secret message $\mathbf{s} \in \mathbb{Z}_2^\gamma \times \mathbb{Z}_4^\delta$

and the vector $\epsilon \cdot \mathbf{h}_i$, such that $H\mathbf{w}^T + \epsilon \cdot \mathbf{h}_i = \mathbf{s}$, indicating the changes needed to embed $\mathbf{s}$ within $\mathbf{x}$, we can now make some variations on the kinds of changes to be done for the specific problematic cases:

- If $\mathbf{h}_i$ is among the first α columns in H, for $2 \leq i \leq \alpha$, and the embedder is required to add 1 to a symbol $x_i = 2^B - 1$, then the embedder should instead subtract 1 from x_i as well as perform the appropriate operation ($+1$ or -1) over x_1 to have v_{01} flipped. Likewise, if the embedder is required to subtract 1 from a symbol $x_i = 0$, then (s)he should instead add 1 to x_i and also change x_1 to flip v_{01}.

- If $\mathbf{h}_i$ is one of the last β columns in H, and the embedder has to add 1 to a symbol $x_i = 2^B - 1$, (s)he should instead subtract 1 from the grayscale symbol associated to $\bar{\mathbf{h}}_i$ and also change x_1 to flip v_{01}. If the method requires subtracting 1 from $x_i = 0$, then we should instead add 1 to the symbol associated to $\bar{\mathbf{h}}_i$ and, again, change x_1 to flip v_{01}.

Example 7.4 *Let* $\mathbf{s}$ *and* $\mathbf{x}$ *be as in Example 1, except for the value of* x_8 *which is now* $x_8 = 0$. *The packet* $\mathbf{x}$ *is translated into vector* $\mathbf{w} = (010|202310)$. *However, now we are not able to make* $x_8 - 1$. *Instead of this, we will add one unit to* x_3, *which is the symbol associated with* $\bar{\mathbf{h}}_9 = \mathbf{h}_4$, *and subtract one unit from* x_1 *so as to have its LSB flipped. Therefore, we obtain* $\mathbf{x}' = (238, 251, 91, 224, 226, 187, 229, 0)$ *and then* $\mathbf{w}' = (110|302310)$

The method above described has the same embedding rate $E = \dfrac{m}{2^{m-1}}$ as the one from Subsection 7.1 but a slightly worse average distortion. We will take into account the squared-error distortion defined in [66] for our reasoning. As before, among the total number of grayscale symbols $N = 2^{m-1}$, there are $N - 1$ symbols x_i, for $2 \leq i \leq N$, with a probability $2/2^m$ of being changed; a symbol x_1 with a probability $1/2^m$ of being the one changed; and, finally, there is a probability of $1/2^m$ that neither of the symbols will need to be changed. As one may have noted in this scheme, performing a certain change to a symbol x_i, associated with a column $\mathbf{h}_i$ in H, has the same effect as performing the opposite change to the grayscale symbol associated with $\bar{\mathbf{h}}_i$ and also changing the least significant bit v_{01} of x_1. This means that with probability $\frac{2^B-2}{2^B}$ we will change a symbol x_i, for $2 \leq i \leq N$, a magnitude of 1; and with probability $\frac{2}{2^B}$ we will change two other symbols also a magnitude of 1. Therefore, $R_a = (N-1)\dfrac{2}{2^m}\left(\dfrac{2^B-2}{2^B} + 2\dfrac{2}{2^B}\right) + \dfrac{1}{2^m}$ and the average distortion is thus $D = \dfrac{2N - 1 + \frac{N-1}{2^{B-2}}}{N2^m}$. Hence, the described method has CI-rate:

$$(D_m, E_m) = \left(\frac{2N - 1 + \frac{N-1}{2^{B-2}}}{2N^2}, \frac{1 + \log(N)}{N}\right).$$

With the aim of providing a possible solution to the boundary grayscale values problem, the authors of [66] and [32] suggested to perform a change of magnitude greater than 1. However, the effects of doing this were out of the scope of ± 1-steganography.

8. Conclusions and further research

We have developed a general theory for $\mathbb{Z}_2\mathbb{Z}_4$-linear codes including generator matrices, parity-check matrices and duality. Such class of codes includes classical binary and quaternary linear codes generalizing them. There are some interesting classes of nonlinear binary codes that can be viewed as $\mathbb{Z}_2\mathbb{Z}_4$-linear codes but not as $\mathbb{Z}_4$-linear codes (e.g. some perfect single error-correcting codes). Moreover, $\mathbb{Z}_2\mathbb{Z}_4$-duality shows that $\mathbb{Z}_2\mathbb{Z}_4$-linear codes cannot be considered only as a variant of $\mathbb{Z}_4$-linear codes.

Further research on this topic could be the proper definition and study of $\mathbb{Z}_2\mathbb{Z}_4$-additive cyclic codes, such that the dual is also cyclic. In this case, a polynomial algebraic approach should be studied including generator and parity-check polynomials. Another research could be a deeply study on the $\mathbb{Z}_2\mathbb{Z}_4$-linear Reed-Muller codes structure with $\alpha \neq 0$, as well as the study of the automorphism groups of $\mathbb{Z}_2\mathbb{Z}_4$-additive codes for extended perfect, Hadamard and Reed-Muller codes, and their corresponding Gray map images.

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