VI – F – ENERGY-MOMENTUM TENSOR

Let us consider the Lagrangian density

$$L(x) = \frac{i}{2}\,\bar{\psi}(x)\,\gamma^\mu\,\partial_\mu\psi(x) - \frac{i}{2}\,\partial_\mu\bar{\psi}(x)\,\gamma^\mu\,\psi(x) - m\,\bar{\psi}(x)\,\psi(x) \tag{1}$$

Since

$$\frac{\partial L}{\partial\psi} = -m\,\bar{\psi}(x) - \frac{i}{2}\,\partial_\mu\bar{\psi}(x)\,\gamma^\mu \qquad \frac{\partial L}{\partial(\partial_\mu\psi)} = \frac{i}{2}\,\bar{\psi}(x)\,\gamma^\mu$$

$$\frac{\partial L}{\partial\bar{\psi}} = -m\,\psi(x) + \frac{i}{2}\,\gamma^\mu\,\partial_\mu\psi(x) \qquad \frac{\partial L}{\partial(\partial_\mu\bar{\psi})} = -\frac{i}{2}\,\gamma^\mu\,\psi(x)$$

and the equations of motion are

$$[\,i\gamma^\mu\partial_\mu - m\,]\,\psi(x) = 0 \qquad i\,\partial_\mu\bar{\psi}(x)\,\gamma^\mu + m\,\bar{\psi}(x) = 0 \tag{2}$$

Notice that if $\psi(x)$ and $\bar{\psi}(x)$ are solutions of (2) then $L(x) = 0$.

The cannonical energy-momentum density tensor is (II – 5.5)

$$\tilde{T}^{\mu\nu}(x) = -L(x)\,g^{\mu\nu} + \frac{\partial L}{\partial(\partial_\mu\psi)}\,\partial^\nu\psi + \partial^\nu\bar{\psi}\,\frac{\partial L}{\partial(\partial_\mu\bar{\psi})}$$

i.e.

$$\tilde{T}^{\mu\nu}(x) = -g^{\mu\nu}L(x) + \frac{i}{2}\,\bar{\psi}(x)\,\gamma^\mu\,\partial^\nu\psi(x) - \frac{i}{2}\,\partial^\nu\bar{\psi}(x)\,\gamma^\mu\,\psi(x) \tag{3}$$

and using the field equations (F.E)

$$\tilde{T}^{\mu\nu}(x) \overset{F.E}{=\!=} \frac{i}{2}\,\bar{\psi}(x)\,\gamma^\mu\,\partial^\nu\psi(x) - \frac{i}{2}\,\partial^\nu\bar{\psi}(x)\,\gamma^\mu\,\psi(x) \tag{4}$$

$$\tilde{T}^\mu{}_\mu(x) = -\frac{3i}{2}\,\bar{\psi}(x)\,\gamma^\mu\,\partial_\mu\psi(x) + \frac{3i}{2}\,\partial_\mu\bar{\psi}(x)\,\gamma^\mu\,\psi(x) + 4m\,\bar{\psi}(x)\,\psi(x) =$$

$$\overset{F.E}{=\!=} m\,\bar{\psi}(x)\,\psi(x) \tag{5}$$

$$\partial_\mu\tilde{T}^{\mu\nu}(x) = -\partial^\nu\bar{\psi}(x)\,[\,i\gamma^\mu\partial_\mu\psi(x) - m\,\psi(x)\,] + [\,i\partial_\mu\bar{\psi}(x)\,\gamma^\mu + m\,\bar{\psi}(x)\,]\,\partial^\nu\psi(x)$$

$$\overset{F.E}{=\!=} 0 \tag{6}$$

Notice that the cannonical tensor is not symmetric

$$\tilde{T}^{\mu\nu} - \tilde{T}^{\nu\mu} = \frac{i}{2} \left\{ \bar{\psi}(x) \gamma^\mu \partial^\nu \psi(x) - \bar{\psi}(x) \gamma^\nu \partial^\mu \psi(x) \right.$$

$$\left. - \partial^\nu \bar{\psi}(x) \gamma^\mu \psi(x) + \partial^\mu \bar{\psi}(x) \gamma^\nu \psi(x) \right\} \qquad (1)$$

Remember (VI-1.3) and (VI-2.7) that under infinitesimal Lorentz transformations

$$\psi(x) \longrightarrow \psi'(x') = \psi(x) + \frac{1}{2} S_{\mu\nu}\, \delta\omega^{\mu\nu} \psi(x) \qquad S_{\mu\nu} = -\frac{i}{2}\sigma_{\mu\nu}$$

$$\bar{\psi}(x) \longrightarrow \bar{\psi}'(x') = \bar{\psi}(x) + \frac{1}{2} \bar{\psi}(x) \bar{S}_{\mu\nu}\, \delta\omega^{\mu\nu} \qquad \bar{S}_{\mu\nu} = +\frac{i}{2}\sigma_{\mu\nu} \qquad (2)$$

The angular momentum density tensor is given by

$$\tilde{M}^\mu{}_{\nu\sigma}(x) = \tilde{T}^\mu{}_\nu(x)\, x_\sigma - \tilde{T}^\mu{}_\sigma(x)\, x_\nu - \frac{\partial L}{\partial(\partial_\mu \psi)} S_{\nu\sigma} \psi(x)$$

$$- \bar{\psi}(x) S_{\nu\sigma} \frac{\partial L}{\partial(\partial_\mu \bar{\psi})} \qquad (3)$$

i.e.

$$\tilde{M}^\mu{}_{\nu\sigma}(x) = \tilde{T}^\mu{}_\nu(x)\, x_\sigma - \tilde{T}^\mu{}_\sigma(x)\, x_\nu - \frac{1}{4} \bar{\psi}(x) \left[\gamma^\mu \sigma_{\nu\sigma} + \sigma_{\nu\sigma} \gamma^\mu \right] \psi(x) \qquad (4)$$

Since

$$\gamma^\mu \sigma_{\nu\sigma} + \sigma_{\nu\sigma}\gamma^\mu = 2i \left[\gamma^\mu \gamma_\nu \gamma_\sigma - g^\mu{}_\nu \gamma_\sigma + g^\mu{}_\sigma \gamma_\nu - g_{\sigma\nu} \gamma^\mu \right]$$

$$= 2 \epsilon^\mu{}_{\nu\sigma\rho}\, \gamma^\rho \gamma_5 \qquad (5)$$

The last term can be written is several alternative forms. Notice

$$\tilde{M}^\mu{}_{\nu\sigma}(x) = - \tilde{M}^\mu{}_{\sigma\nu}(x) \qquad (6)$$

Furthermore

$$\partial_\mu \tilde{M}^\mu{}_{\nu\sigma}(x) \overset{F.E}{=\!=} \tilde{T}_{\sigma\nu}(x) - \tilde{T}_{\nu\sigma}(x) - \frac{i}{2}\Big[\bar{\Psi}(x)\gamma_\sigma \partial_\nu \psi(x)$$

$$- \bar{\Psi}(x)\gamma_\nu \partial_\sigma \psi(x) + \partial_\sigma \bar{\Psi}(x)\gamma_\nu \psi(x) - \partial_\nu \bar{\Psi}(x)\gamma_\sigma \psi(x)\Big] = 0 \tag{1}$$

where the last equality is a consequence of (2.1)

In order to construct the Belinfante energy-momentum density tensor let us introduce

$$\tilde{M}^{(S)\mu}{}_{\nu\sigma}(x) \equiv -\frac{1}{4}\bar{\Psi}(x)\left[\gamma^\mu \sigma_{\nu\sigma} + \sigma_{\nu\sigma}\gamma^\mu\right]\psi(x) = -\tilde{M}^{(S)\mu}{}_{\sigma\nu}(x) \tag{2}$$

and

$$G^{\sigma\mu\nu}(x) = -\frac{1}{2}\left[\tilde{M}^{(S)\sigma\mu\nu} + \tilde{M}^{(S)\mu\nu\sigma} - \tilde{M}^{(S)\nu\sigma\mu}\right] \tag{3}$$

i.e.

$$G^{\sigma\mu\nu}(x) = \frac{1}{8}\bar{\Psi}(x)\left[\gamma^\sigma\sigma^{\mu\nu} + \sigma^{\mu\nu}\gamma^\sigma + \gamma^\mu\sigma^{\nu\sigma} + \sigma^{\nu\sigma}\gamma^\mu - \gamma^\nu\sigma^{\sigma\mu} - \sigma^{\sigma\mu}\gamma^\nu\right]\psi(x)$$

$$= \frac{i}{8}\bar{\Psi}(x)\left[\gamma^\mu\gamma^\nu\gamma^\sigma - \gamma^\sigma\gamma^\nu\gamma^\mu\right]\psi(x) = -G^{\mu\sigma\nu}(x) \tag{4}$$

And hence the Belinfante tensor is (II-8.2)

$$T^{\mu\nu}(x) = \tilde{T}^{\mu\nu}(x) + \partial_\sigma G^{\sigma\mu\nu}(x) \tag{5}$$

i.e.

$$T^{\mu\nu}(x) = -g^{\mu\nu}L(x) + \frac{i}{4}\Big[\bar{\Psi}(x)\gamma^\mu\partial^\nu\psi(x) + \bar{\Psi}(x)\gamma^\nu\partial^\mu\psi(x)$$

$$-\partial^\mu\bar{\Psi}(x)\gamma^\nu\psi(x) - \partial^\nu\bar{\Psi}(x)\gamma^\mu\psi(x)\Big] + \frac{1}{8}\left[i\partial_\sigma\bar{\Psi}(x)\gamma^\sigma\right](\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\psi(x)$$

$$+\frac{1}{8}\bar{\Psi}(x)(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\left[i\gamma^\sigma\partial_\sigma\psi(x)\right] \tag{6}$$

and using the equations of motion

$$T^{\mu\nu}(x) = \frac{i}{4}\Big[\bar\psi(x)\gamma^\mu\partial^\nu\psi(x) + \bar\psi(x)\gamma^\nu\partial^\mu\psi(x)$$

$$- \partial^\mu\bar\psi(x)\gamma^\nu\psi(x) - \partial^\nu\bar\psi(x)\gamma^\mu\psi(x)\Big] \qquad (1)$$

which clearly satisfies

$$T^{\mu\nu}(x) = T^{\nu\mu}(x) \quad , \quad \partial_\mu T^{\mu\nu}(x) = 0$$

$$T^\mu{}_\mu(x) = m\,\bar\psi(x)\psi(x) \qquad (2)$$

Furthermore the Belinfante angular-momentum tensor is

$$M^\mu{}_{\nu\sigma}(x) = T^\mu{}_\nu(x)\,x_\sigma - T^\mu{}_\sigma(x)\,x_\nu \qquad (3)$$

and clearly

$$M^\mu{}_{\nu\sigma}(x) = -M^\mu{}_{\sigma\nu}(x) \quad , \quad \partial_\mu M^\mu{}_{\nu\sigma}(x) = 0 \qquad (4)$$

The dimension of this field is $d = 3/2$ and therefore the field virial is

$$V^\mu = \frac{3}{2}\bar\psi\frac{\partial L}{\partial(\partial_\mu\bar\psi)} + \frac{3}{2}\frac{\partial L}{\partial(\partial_\mu\psi)}\psi + \frac{\partial L}{\partial(\partial_\nu\psi)}S^\mu{}_\nu\psi + \bar\psi\,\bar S^\mu{}_\nu\frac{\partial L}{\partial(\partial_\nu\bar\psi)}$$

$$= -\frac{3i}{4}\bar\psi\gamma^\mu\psi + \frac{3i}{4}\bar\psi\gamma^\mu\psi + \frac{1}{4}\bar\psi\left[\gamma^\nu\sigma^\mu{}_\nu + \sigma^\mu{}_\nu\gamma^\nu\right]\psi = 0 \quad \Rightarrow$$

$$V^\mu(x) = 0$$

and therefore

$$\Theta^{\mu\nu}(x) = T^{\mu\nu}(x)$$

VI - G - BAG MODEL

Consideremos una partícula de spin 1/2 en un pozo esférico $V(r)$ de tipo escalar. La ecuación de Dirac es

$$[\, i\gamma^\mu \partial_\mu - m - V(r)\,]\, \psi(x) = 0 \tag{1}$$

Procediendo exactamente como en el caso del Apéndice C se obtiene

$$\left.\begin{aligned} &\frac{d\,g(r)}{dr} + (1+\varkappa)\frac{1}{r}\, g(r) - [E + m + V(r)]\, f(r) = 0 \\ &\frac{d\,f(r)}{dr} + (1-\varkappa)\frac{1}{r}\, f(r) + [E - m - V(r)]\, g(r) = 0 \end{aligned}\right\} \tag{2}$$

Supongamos que en un intervalo de $[0, \infty)$, $V(r) = V = $ cte. Entonces de la primera ecuación

$$f(r) = \frac{1}{E+m+V}\left[\frac{d\,g(r)}{dr} + (1+\varkappa)\frac{1}{r}\, g(r)\right]$$

$$\frac{d\,f(r)}{dr} = \frac{1}{E+m+V}\left[\frac{d^2 g(r)}{dr^2} + (1+\varkappa)\frac{1}{r}\frac{d\,g(r)}{dr} - (1+\varkappa)\frac{1}{r^2}\, g(r)\right] \tag{3}$$

y sustituyendo en la segunda

$$\frac{1}{E+m+V}\left\{\frac{d^2 g(r)}{dr^2} + (1+\varkappa)\frac{1}{r}\frac{d\,g(r)}{dr} - (1+\varkappa)\frac{1}{r^2}\, g(r) + (1-\varkappa)\frac{1}{r}\frac{d g(r)}{dr} + (1-\varkappa^2)\frac{1}{r^2}\, g(r)\right\} + [E - m - V]\, g(r) = 0 \quad \Rightarrow$$

$$r^2\frac{d^2 g(r)}{dr^2} + 2r\frac{d\,g(r)}{dr} + \left\{[E^2 - (m+V)^2]\, r^2 - \varkappa(1+\varkappa)\right\} g(r) = 0 \tag{4}$$

Notemos que

$$\varkappa = \mp (J + 1/2) \quad \Rightarrow \quad \varkappa(1+\varkappa) = (J \mp 1/2)\,[\,(J \mp 1/2) + 1\,] \tag{5}$$

Consideremos ahora el pozo

$$V(r) = 0 \qquad r < R$$
$$V(r) = V_0 \qquad r > R \quad , \quad V_0 > 0 \qquad (1)$$

e introduzcamos

$$\alpha \equiv [E^2 - m^2]^{1/2}$$
$$\beta \equiv [(m + V_0)^2 - E^2]^{1/2} \qquad (2)$$

Entonces imponiendo las condiciones $g(0) = 0$ y $g(r) \to 0$ para $r \to \infty$ se obtiene inmediatamente que

$$\kappa = -(J + 1/2) \qquad g(r) = A\, j_{J-1/2}(\alpha r) \qquad r < R$$
$$g(r) = B\, h^{(1)}_{J-1/2}(i\beta r) \qquad r > R \qquad (3)$$

$$\kappa = +(J + 1/2) \qquad g(r) = A\, j_{J+1/2}(\alpha r) \qquad r < R$$
$$g(r) = B\, h^{(1)}_{J+1/2}(i\beta r) \qquad r > R \qquad (4)$$

Usando (1.3)

$$\kappa = -(J + 1/2) \qquad f(r) = -\frac{A\alpha}{E + m}\, j_{J+1/2}(\alpha r) \qquad r < R$$
$$f(r) = -\frac{iB\beta}{E + m + V_0}\, h^{(1)}_{J+1/2}(i\beta r) \qquad r > R \qquad (5)$$

$$\kappa = +(J + 1/2) \qquad f(r) = \frac{A\alpha}{E + m}\, j_{J-1/2}(\alpha r) \qquad r < R$$
$$f(r) = \frac{iB\beta}{E + m + V_0}\, h^{(1)}_{J-1/2}(i\beta r) \qquad r > R \qquad (6)$$

La condicion de continuidad de $g(r)$ y $f(r)$ en el punto $r = R$

$$\kappa = \mp(J + 1/2) \qquad \frac{\alpha}{E + m}\, \frac{j_{J\pm 1/2}(\alpha R)}{j_{J\mp 1/2}(\alpha R)} = \frac{i\beta}{E + m + V_0}\, \frac{h^{(1)}_{J\pm 1/2}(i\beta R)}{h^{(1)}_{J\mp 1/2}(i\beta R)} \qquad (7)$$

Consideremos ahora el caso de confinamiento total es decir $V_0 \to \infty$. Como

$$\lim_{V_0 \to \infty} \frac{i\beta}{E + m + V_0} \frac{h^{(1)}_{J \pm 1/2}(i\beta R)}{h^{(1)}_{J \mp 1/2}(i\beta R)} = \pm 1 \qquad (1)$$

La condición de confinamiento es pues

$$x = \mp (J + 1/2) \qquad \sqrt{E - m}\; j_{J \pm 1/2}(\alpha R) = \pm \sqrt{E + m}\; j_{J \mp 1/2}(\alpha R) \qquad (2)$$

o equivalentemente

$$j_L(\alpha R) = - \frac{x}{|x|} \sqrt{\frac{E - m}{E + m}}\; j_{L'}(\alpha r) \qquad \alpha = (E^2 - m^2)^{1/2}$$

$$x = \mp (J \pm 1/2) \qquad L = (J \mp 1/2) \qquad L' = (J \pm 1/2) \qquad (3)$$

Indicaremos por $E_{m,x}$ las soluciones de (3) para un x determinado. Si $m \equiv 0$ y $\omega_{m,x} \equiv E_{m,x} R$, se tiene para $\omega_{m,x} > 0$.

$$x = -1 \qquad j_1(\omega_{m,-1}) = j_0(\omega_{m,-1}) \Rightarrow \operatorname{tn} \omega_{m,-1} = \frac{\omega_{m,-1}}{1 - \omega_{m,-1}} \qquad (4)$$

$\omega_{1,-1} = 2.042\ 79$

$\omega_{2,-1} = 5.396\ 02$

$\omega_{3,-1} = 8.577\ 56$

$$\omega_{m,-1} \simeq \frac{\pi}{4}(4m - 1) - \frac{1}{\frac{\pi}{2}(4m-1) - 1} + \cdots$$

$$x = +1 \qquad j_1(\omega_{m,1}) = -j_0(\omega_{m,1}) \Rightarrow \operatorname{tn} \omega_{m,1} = \frac{\omega_{m,1}}{1 + \omega_{m,1}} \qquad (5)$$

$\omega_{1,1} = 3.811\ 54$

$\omega_{2,1} = 7.002\ 03$

$\omega_{3,1} = 10.163\ 32$

$$\omega_{m,1} = \frac{\pi}{4}(4m+1) - \frac{1}{\frac{\pi}{2}(4m+1)+1} + \cdots$$

$\kappa = -2$

$$j_2(\omega_{m,2}) = j_1(\omega_{m,-2}) \tag{1}$$

$$\omega_{1,-2} = 3.203\ 91$$

$$\omega_{2,-2} = 6.758\ 98$$

$$\omega_{3,-2} = 10.004\ 19$$

$$\omega_{m,-2} = \frac{\pi}{4}(4m+1) - \frac{\pi(4m+1)-3}{\frac{\pi^2}{8}(4m+1) - \frac{\pi}{2}(4m+1)+1} + \cdots$$

Notemos que $j_L(z) = (-1)^L j_L(-z)$ y por tanto estas ecuaciones tienen soluciones de energía negativa que indicaremos por

$$\omega_{-m,\kappa} = -\omega_{m,-\kappa} \tag{2}$$

y serán indicadas por $m<0$.

Recordar que los valores de (L, J) están en correspondencia biunívoca con κ y así por ejemplo

$S_{1/2}\quad \kappa=-1\ ;\quad P_{1/2}\quad \kappa=+1\ ;\quad P_{3/2}\quad \kappa=-2\ ,\quad D_{3/2}\quad \kappa=+2$ (3)

$D_{5/2}\quad \kappa=-3$

Las funciones de onda correspondientes a este modelo son nulas para $r > R$ y para $r < R$ vienen dadas por

$$\Psi^{\kappa}_{JM}(r) = A \begin{vmatrix} i\, j_L(\alpha r)\, \mathcal{Y}^{L}_{JM}(\hat{r}) \\ \frac{\kappa}{|\kappa|}\frac{\alpha}{E+m}\, j_{L'}(\alpha r)\, \mathcal{Y}^{L'}_{JM}(\hat{r}) \end{vmatrix} \tag{4}$$

$\kappa = \mp(J+1/2)\qquad L = J \mp 1/2\qquad L' = J \pm 1/2\qquad \alpha = \sqrt{E^2-m^2}$

Introduzcamos el cuadrivector $n^\mu \equiv (0, \hat{r})$, es decir normal a la superficie del saco; entonces en la superficie

$$-i\gamma_\mu n^\mu \Psi^{\kappa}_{JM}(\vec{r}) = -i \begin{vmatrix} 0 & -\vec{\sigma}.\hat{r} \\ \vec{\sigma}.\hat{r} & 0 \end{vmatrix} A \begin{vmatrix} i j_L(\alpha R)\, \mathcal{Y}^L_{JM}(\hat{r}) \\ \frac{\kappa}{|\kappa|}\frac{\alpha}{E+m} j_{L'}(\alpha R)\, \mathcal{Y}^{L'}_{JM}(\hat{r}) \end{vmatrix} =$$

$$= A \begin{vmatrix} i\frac{\kappa}{|\kappa|}\frac{\alpha}{E+m} j_{L'}(\alpha R)\, \vec{\sigma}.\hat{r}\, \mathcal{Y}^{L'}_{JM}(\hat{r}) \\ + j_L(\alpha R)\, \vec{\sigma}.\hat{r}\, \mathcal{Y}^L_{JM}(\hat{r}) \end{vmatrix} = \text{(usando (33))}$$

$$= A \begin{vmatrix} -i j_L(\alpha R)\, \vec{\sigma}.\hat{r}\, \mathcal{Y}^{L'}_{JM}(\hat{r}) \\ -\frac{\kappa}{|\kappa|}\frac{\alpha}{E+m} j_{L'}(\alpha R)\, \vec{\sigma}.\hat{r}\, \mathcal{Y}^L_{JM}(\hat{r}) \end{vmatrix} = \text{(usando (C.2.7))}$$

se obtiene

$$-i n_\mu \gamma^\mu \Psi^{\kappa}_{JM}(x) = -\Psi^{\kappa}_{JM}(x) \qquad r = R \tag{1}$$

$$\Rightarrow \; +i\, \bar{\Psi}^{\kappa}_{JM}(x)\, \gamma^\mu n_\mu = -\bar{\Psi}^{\kappa}_{JM}(x) \qquad r = R$$

Notemos que $\Psi^{\kappa}_{JM}(x) \neq 0$ para $r = R$, sin embargo esto no crea problemas. En efecto la corriente vectorial es

$$J^\mu(x) \equiv \bar{\psi}(x)\, \gamma^\mu\, \psi(x) \tag{2}$$

y se cumple

$$\partial_\mu J^\mu(x) = 0 \qquad r < R \tag{3}$$

pues $\psi(x)$ es solución de la ecuación de Dirac libre para $r<R$. Si queremos que la carga sea conservada debemos imponer

$$n_\mu J^\mu(x) = 0 \qquad r = R \tag{4}$$

Pero de (1) se deduce

$$i n_\mu J^\mu(x) = i n^\mu \bar{\psi}(x)\gamma_\mu \psi(x) = -\bar{\psi}(x)\psi(x) = +\bar{\psi}(x)\psi(x)$$

y por tanto

$$n_\mu J^\mu(x) = 0 \quad , \quad \bar{\psi}(x)\psi(x) = 0 \qquad r = R \tag{5}$$

Esto garantiza que ni la carga ni ningun otro número cuántico se pierde a traves de la superficie del bag.

Consideremos ahora el flujo de momento a través de la superficie. El tensor-energia-momento es (F-1.4)

$$T^{\mu\nu}(x) = \frac{i}{2}\bar{\psi}(x)\gamma^{\mu}\partial^{\nu}\psi(x) - \frac{i}{2}\partial^{\nu}\bar{\psi}(x)\gamma^{\mu}\psi(x) \qquad (1)$$

y dentro del saco $\partial_\mu T^{\mu\nu}(x) = 0$. El flujo del tensor momento a través de la superficie viene dado por

$$n_\mu T^{\mu\nu}(x)\Big|_{r=R} = \frac{1}{2}\bar{\psi}(x)\, i\gamma^{\mu} n_\mu \partial^{\nu}\psi(x) - \frac{1}{2}\partial^{\nu}\bar{\psi}(x)\, i n_\mu \gamma^{\mu}\psi(x)\Big|_{r=R} =$$

$$= \frac{1}{2}\bar{\psi}(x)\partial^{\nu}\psi(x) + \frac{1}{2}\partial^{\nu}\bar{\psi}(x)\psi(x)\Big|_{r=R} = \frac{1}{2}\partial^{\nu}[\bar{\psi}(x)\psi(x)]\Big|_{r=R}$$

y como $\bar{\psi}(x)\psi(x) = 0$ en la superficie la cantidad anterior es proporcional a n^ν y podemos escribir

$$n_\mu T^{\mu\nu}(x) = n^\nu P \neq 0 \qquad r=R \qquad (2)$$

$$P \equiv \frac{1}{2}(n_\mu \partial^\mu)\bar{\psi}(x)\psi(x)$$

donde P es la llamada presión de Dirac. En este modelo de un potencial escalar con $R = cte$ no se conserva el momento a traves de la frontera: no se hace ningún intento de calcular R dinamicamente.

Este modelo tan simple de bag es el de N.P. Bogoliubov [Ann. Inst. H. Poincaré 8, 163 (1967)]. En este modelo la masa del nucleón viene dada simplemente por

$$M_N = 3\,\omega_{1,-1}/R \qquad (3)$$

que reproduce el valor $M_N = 0.9389$ GeV para $R = 1.288$ fm. La primera excitación radial es la N(1470) con $M = 1.440$ GeV y en este modelo tiene una masa $M = (2\omega_{1,-1} + \omega_{2,-1})/R = 1.452$ GeV. "Dice Bogoliubov" une telle coincidence est même un peu surprenante".

En particular si $L=0 \Rightarrow \varkappa=-1$, $J=1/2$, $L'=1$

$$\Psi^{-1}_{1/2\,M}(r) = A \begin{vmatrix} i\, j_0(\alpha r)\, \mathcal{Y}^{0}_{1/2\,M}(\hat{r}) \\ -\dfrac{\alpha}{E+m}\, j_1(\alpha r)\, \mathcal{Y}^{1}_{1/2\,M}(\hat{r}) \end{vmatrix}$$

o equivalentemente como

$$\mathcal{Y}^{0}_{1/2\,M}(\hat{r}) = \sum_{\mu} C(1/2\ 0\ 1/2;\ \mu, M-\mu, M)\, \frac{1}{\sqrt{4\pi}}\, \delta_{M-\mu}\, \chi(\mu) = \frac{1}{\sqrt{4\pi}}\, \chi(M)$$

donde $\chi(\mu)$ son las funciones de onda de spin del electrón. Usando además (I.-2.7)

$$\Psi^{-1}_{1/2\,M}(r) = \frac{A}{\sqrt{4\pi}} \begin{vmatrix} i\, j_0(\alpha r)\, \chi(M) \\ -\dfrac{\alpha}{E+m}\, j_1(\alpha r)\, \vec{\sigma}.\hat{r}\, \chi(M) \end{vmatrix} \qquad (1)$$

y la constante de normalización A viene determinada por

$$I \equiv \int_{\text{esfera}} d^3r\ \Psi^{-1\,+}_{1/2\,M}\ \Psi^{-1}_{1/2\,M} = 1$$

$$I \equiv \frac{|A|^2}{4\pi} \int_{\text{esfera}} d^3r \left[-i\, j_0(\alpha r)\, \chi^+(M),\ -\frac{\alpha}{E+m}\, j_1(\alpha r)\, \chi^+(M)\, \vec{\sigma}.\hat{r} \right] \begin{vmatrix} i\, j_0(\alpha r)\, \chi(M) \\ -\dfrac{\alpha}{E+m}\, j_1(\alpha r)\, \vec{\sigma}.\hat{r}\, \chi(M) \end{vmatrix} =$$

$$= |A|^2 \int_0^R dr\, r^2 \left[j_0^2(\alpha r) + \frac{\alpha^2}{(E+m)^2}\, j_1^2(\alpha r) \right] =$$

$$= |A|^2 \frac{1}{\alpha^3} \left\{ \int_0^{\alpha R} dz\, z^2\, j_0^2(z) + \frac{\alpha^2}{(E+m)^2} \int_0^{\alpha R} dz\, z^2\, j_1^2(\alpha z) \right\}$$

$$I_0 \equiv \int_0^{\alpha R} dz\, z^2\, j_0^2(z) = \int_0^{\alpha R} dz\, \sin^2 z = -\frac{1}{2} \sin z \cos z + \frac{1}{2} z \Big|_0^{\alpha R}$$

$$= -\frac{1}{2} \sin \alpha R \cos \alpha R + \frac{1}{2} \alpha R$$

$$I_1 \equiv \int_0^{\alpha R} dz\, z^2\, j_1^2(z) = \int_0^{\alpha R} dz \left\{ \frac{1}{z^2} \sin^2 z + \cos^2 z - \frac{2}{z} \sin z \cos z \right\}$$

$$= \left\{ \frac{1}{2z}\cos^2 z - \frac{1}{2z}\sin^2 z - \frac{1}{2z} + \frac{1}{2}\sin z\cos z + \frac{z}{2} \right\}_0^{\alpha R} =$$

$$= \frac{1}{2\alpha R}\cos^2\alpha R - \frac{1}{2\alpha R}\sin^2\alpha R - \frac{1}{2\alpha R} + \frac{1}{2}\sin\alpha R\cos\alpha R + \frac{\alpha R}{2}$$

$$I = \frac{|A|^2}{\alpha^3}\left\{ -\frac{1}{2}\sin\alpha R\cos\alpha R + \frac{1}{2}\alpha R + \frac{\alpha^2}{(E+m)^2}\left[\frac{1}{2\alpha R}\cos^2\alpha R - \frac{1}{2\alpha R}\sin^2\alpha R - \right.\right.$$
$$\left.\left. - \frac{1}{2\alpha R} + \frac{1}{2}\sin\alpha R\cos\alpha R + \frac{1}{2}\alpha R \right]\right\}$$

$$|A|^{-2} = \frac{1}{\alpha^3}\left\{ -\frac{m}{E+m}\sin\alpha R\cos\alpha R + \frac{E}{E+m}\alpha R - \frac{E-m}{E+m}\frac{\sin^2\alpha R}{\alpha R} \right\}$$

Como

$$j_0(\alpha R) = \sqrt{\frac{E-m}{E+m}}\, j_1(\alpha R)$$

$$\cos\alpha R = \frac{\sin\alpha R}{\alpha R} - \sqrt{\frac{E+m}{E-m}}\sin\alpha R$$

de donde

$$|A|^{-2} = \frac{1}{\alpha^3}\left\{ -\frac{m}{E+m}\frac{\sin^2\alpha R}{\alpha R} + \frac{m}{E+m}\sqrt{\frac{E+m}{E-m}}\sin^2\alpha R + \frac{E}{E+m}\alpha R\sin^2\alpha R + \frac{E}{E+m}\frac{\sin^2\alpha R}{\alpha R} \right.$$
$$\left. + \frac{E}{E-m}\alpha R\sin^2\alpha R - 2\frac{E}{E+m}\sqrt{\frac{E+m}{E-m}}\sin^2\alpha R - \frac{E-m}{E+m}\frac{\sin^2\alpha R}{\alpha R} \right\}$$

and hence

$$|A|^{-2} = R^3 j_0^2(\alpha R)\,\frac{2E(E-1/R) + m/R}{(E^2-m^2)} \qquad (1)$$

que es la constante de normalización deseada, donde $\alpha = (E^2-m^2)^{1/2}$ y E es la solución de

$$\operatorname{tg}\alpha R = \frac{\alpha R}{1 - mR - \sqrt{(mR)^2 + (\alpha R)^2}} \qquad (2)$$

cuyas soluciones vienen dadas en la tabla

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mR	αR	mR	αR	mR	αR
0	2.042 787	2.6	2.676 344	5.2	2.866 065
0.2	2.132 432	2.8	2.698 650	5.4	2.874 778
0.4	2.211 002	3.0	2.719 134	5.6	2.882 987
0.6	2.280 274	3.2	2.737 991	5.8	2.890 731
0.8	2.341 656	3.4	2.755 392	6.0	2.898 048
1.0	2.396 287	3.6	2.771 485	6.2	2.904 970
1.2	2.445 103	3.8	2.786 403	6.4	2.911 527
1.4	2.488 883	4.0	2.800 259	6.6	2.917 747
1.6	2.528 281	4.2	2.813 156	6.8	2.923 653
1.8	2.563 852	4.4	2.825 184	7.0	2.929 267
2.0	2.596 067	4.6	2.836 422		
2.2	2.625 331	4.8	2.846 940		
2.4	2.651 990	5.0	2.856 803		

El operador del campo del quark lo indicaremos por $\psi(x)$ y usando la base de estados propios que acabamos de calcular se obtiene

$$\psi(x) = \sum_{m} \sum_{JLM} a_{JLM}(m)\, \Psi^{(m)}_{JLM}(\vec{r})\, e^{-i E^{(m)}_{JLM} t} \tag{1}$$

donde $m = \pm 1, \pm 2, \pm 3, \ldots$ indica las excitaciones radiales. Como es usual redefiniremos los operadores de aniquilación de quarks con $m<0$ como operadores creación de antiquarks con energía positiva

$$\psi(x) = \sum_{m=1}^{\infty} \sum_{JLM} \left[a_{JLM}(m)\, \Psi^{(m)}_{JLM}(\vec{r})\, e^{-i E^{(m)}_{JLM} t} + b^{+}_{JLM}(m)\, \Phi^{(m)}_{JLM}(\vec{r})\, e^{+i \tilde{E}^{(m)}_{JLM} t} \right] \tag{2}$$

$$\tilde{E}^{(m)}_{JLM} \equiv - E^{(-m)}_{JLM} \quad , \qquad \Phi^{(m)}_{JLM}(\vec{r}) \equiv \Psi^{(-m)}_{JLM}(\vec{r}) \qquad m = 1, 2, \ldots$$

$$[a_{JLM}(m),\ a^{+}_{J'L'M'}(m')]_{+} = [b_{JLM}(m),\ b^{+}_{J'L'M'}(m')]_{+} = \delta_{mm'}\delta_{JJ'}\delta_{LL'}\delta_{MM'}$$

y el vacío viene dado por

$$a_{JLM}(m)|0\rangle = b_{JLM}(m)|0\rangle = 0 \tag{3}$$

En este modelo el nucleón viene descrito por tres quarks de masa nula confinados en el estado $I=0$, $J=1/2$, $\kappa=-1$, $m=1$. y por tanto su masa es

$$M_m = M_p = \frac{3\,\omega_{1,-1}}{R} \tag{4}$$

Es bien sabido que el estado $p^{\uparrow}$ viene dado por

$$\begin{aligned} |p^{\uparrow}\rangle = \frac{1}{\sqrt{18}} \Big[& 2|u^{\uparrow} d^{\downarrow} u^{\uparrow}\rangle + 2|u^{\uparrow} u^{\uparrow} d^{\downarrow}\rangle + 2|d^{\downarrow} u^{\uparrow} u^{\uparrow}\rangle - \\ & -|u^{\uparrow} u^{\downarrow} d^{\uparrow}\rangle - |u^{\downarrow} u^{\uparrow} d^{\uparrow}\rangle - |d^{\uparrow} u^{\downarrow} u^{\uparrow}\rangle - |d^{\uparrow} u^{\uparrow} u^{\downarrow}\rangle \\ & -|u^{\uparrow} d^{\uparrow} u^{\downarrow}\rangle - |u^{\downarrow} d^{\uparrow} u^{\uparrow}\rangle \Big] \end{aligned} \tag{5}$$

y el neutrón es lo mismo con $u \leftrightarrow d$.

El operador momento magnético es

$$\vec{\mu} = \sum_i \int d^3r \ \frac{1}{2} \vec{r} \times [\bar{\Psi}_i(x) \ \vec{\gamma} \ Q_i \ \Psi_i(x)] \qquad (1)$$

donde Q_i es la carga del quark $i = u, d$. El momento magnético del protón es

$$\mu_p = \langle p \uparrow | \mu_3 | p \uparrow \rangle \qquad (2)$$

Entre dos estados quark $(m=0)$

$$\langle 1/2\, 0\, M'; m=1 | \vec{\mu} | 1/2\, 0\, M; m=1 \rangle = Q \int d^3r \ \frac{1}{2} \vec{r} \times \bar{\Psi}^{(1)}_{1/2\, 0\, M'}(\vec{r}) \ \vec{\gamma} \ \Psi^{(1)}_{1/2\, 0\, M}(\vec{r}) =$$

$$= Q \frac{|A|^2}{4\pi} \int d^3r \ \frac{1}{2} \vec{r} \times \left[-i j_0(\alpha r) \chi^+(M') \ , +j_1(\alpha r) \chi^+(M') \vec{\sigma}.\hat{r} \right] \begin{vmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{vmatrix} \begin{vmatrix} i j_0(\alpha r) \chi(M) \\ -j_1(\alpha r) \vec{\sigma}.\hat{r} \chi(M) \end{vmatrix} =$$

$$= Q \frac{|A|^2}{8\pi} \int d^3r \, r (+i) \ j_0(\alpha r) \ j_1(\alpha r) \ \chi^+(M') \left[(\hat{r} \times \vec{\sigma})(\vec{\sigma}.\hat{r}) - (\vec{\sigma}.\hat{r})(\hat{r} \times \vec{\sigma}) \right] \chi(M)$$

$$= -Q \frac{|A|^2}{4\pi} \int d^3r \ r \ j_0(\alpha r) \ j_1(\alpha r) \ \chi^+(M') \left[-\vec{\sigma} + \hat{r}.(\hat{r}.\vec{\sigma}) \right] \chi(M) =$$

$$= +Q \frac{|A|^2}{4\pi} \frac{2}{3} \int d^3r \ r \ j_0(\alpha r) \ j_1(\alpha r) \ \chi^+(M') \ \vec{\sigma} \ \chi(M)$$

$\Rightarrow$

$$\langle 1/2\, 0\, M'; m=1 | \mu_3 | 1/2\, 0\, M; m=1 \rangle = +Q \frac{|A|^2}{\cancel{4\pi}} \frac{2}{3} \cancel{4\pi} \ \delta_{MM'} \int_0^R dr \ r^3 \ j_0(\alpha r) \ j_1(\alpha r) \ (-1)^{M+1/2}$$

$$= +Q \frac{2}{3} \delta_{MM'} \frac{\alpha^2 R^2}{R^3 \sin^2 \alpha R} \frac{E}{2(E-1/R)} \frac{1}{\alpha^4} \int_0^{\alpha R} dz \ [\sin^2 z - z \sin z \cos z] \ (-1)^{M+1/2}$$

$$= + \delta_{MM'} Q \frac{(-1)^{M+1/2}}{3RE(E-1/R)} \frac{1}{\sin^2 ER} \left[-\frac{1}{2} \sin z \cos z + \frac{1}{2} z - \frac{1}{2} z \sin^2 z + \frac{1}{4} \sin z \cos z + \frac{1}{4} z \right]_0^{ER}$$

$$= + \delta_{MM'} Q \frac{(-1)^{M-1/2}}{3RE(E-1/R)} \frac{1}{\sin^2 ER} \frac{1}{4} \left[-3 \sin ER \cos ER - 2ER \sin^2 ER + 3ER \right]$$

$$= + \delta_{MM'} Q \frac{(-1)^{M-1/2}}{3RE(E-1/R)} \frac{1}{\sin^2 ER} \frac{1}{4} \Big\{ -3 \cancel{\frac{\sin^2 ER}{ER}} + 3 \sin^2 ER - 2ER \sin^2 ER$$

$$+ 3ER \sin^2 ER + 3 \cancel{\frac{\sin^2 ER}{ER}} - 6 \sin^2 ER + 3ER \sin^2 ER \Big\}$$

$$\langle 1/2\, 0\, M'; m=1 \,|\, \mu_3 \,|\, 1/2\, 0\, M; m=1 \rangle = +\delta_{MM'} \frac{Q}{12} \frac{4\omega_{1,-1} - 3}{\omega_{1,-1}(\omega_{1,-1} - 1)} R \,(-1)^{M-1/2} \qquad (1)$$

y por tanto usando (9.5) el momento magnético del protón es

$$\mu_p = \frac{R}{12} \frac{4\omega_{1,-1} - 3}{\omega_{1-1}(\omega_{1,-1} - 1)} \qquad (2)$$

y por tanto

$$g_p/2 = 2M_p \mu_p = \frac{4\cdot\omega_{1-1} - 3}{2(\omega_{1-1} - 1)} = 2.479 \qquad (3)$$

mientras que experimentalmente $g_p/2 = 2.7928456(11)$. Similarmente y teniendo en cuenta la función de onda del neutrón

$$(g_n/2) = -\frac{2}{3}(g_p/2) = -1.653 \qquad (4).$$

a comparar con $g_N/2 = -1.913\,0418\,4\,(88)$

Otra cantidad que puede calcularse con este modelo es el radio cuadrático del nucleón

$$\langle r^2 \rangle = \sum_i \int d^3r\, r^2\, \psi_i^+(\bar r)\, Q_i\, \psi_i(\bar r)$$

Notemos que

$$\langle 1/2\, 0\, M; m=-1 \,|\, Q r^2 \,|\, 1/2\, 0\, M; m=-1 \rangle = Q \int d^3r\, r^2\, \psi^{+(1)}_{1/2\,0\,M}(\bar r)\, \Psi^{(1)}_{1/2\,0\,M}(\bar r) =$$

$$= Q \frac{|A|^2}{4\pi} \int d^3r\, r^2 \left[j_0^2(\alpha r) + j_1^2(\alpha r) \right] = Q |A|^2 \frac{1}{\alpha^5} \int_0^{\alpha R} dz\, z^4 \left[j_0^2(z) + j_1^2(z) \right]$$

$$= Q \frac{|A|^2}{\alpha^5} \int_0^{\alpha R} dz\, z^4 \left\{ \frac{\sin^2 z}{z^2} + \frac{\sin^2 z}{z^4} - \frac{2 \sin z \cos z}{z^3} + \frac{\cos^2 z}{z^2} \right\}$$

$$= Q \frac{|A|^2}{\alpha^5} \int_0^{\alpha R} dz \left\{ z^2 + \sin^2 z - 2z \sin z \cos z \right\} =$$

$$= Q \frac{|A|^2}{\alpha^5} \left\{ \frac{1}{3} z^3 - \frac{1}{2} \sin z \cos z + \frac{1}{2} z - z \sin^2 z - \frac{1}{2} \sin z \cos z + \frac{1}{2} z \right\}_0^{R\alpha}$$

$$= Q \frac{|A|^2}{\alpha^5} \left\{ \frac{1}{3} z^3 + \quad z \quad - \quad \sin z \cos z - z \sin^2 z \right\}_0^{R\alpha}$$

$$= Q \frac{|A|^2}{\alpha^5} \left\{ \frac{1}{3} \alpha^3 R^3 + \quad \alpha R \quad - \quad \sin \alpha R \cos \alpha R - \alpha R \sin^2 \alpha R \right\}$$

De la condición (3.4) $(\alpha R \equiv w_{1,-1} \equiv w)$

$$\sin w = \frac{w}{\sqrt{D}} \qquad \cos w = \frac{1-w}{\sqrt{D}} \qquad D = 1 - 2w + 2w^2$$

se obtiene para la cantidad que calculamos

$$= Q R^2 \frac{D}{2 w^4 (w-1)} \left\{ \frac{1}{3} w^3 + \quad w \quad - \frac{1}{D} w (1-w) - w \frac{w^2}{D} \right\}$$

$$= Q R^2 \frac{1}{6 w^3 (w-1)} \left\{ w^2 D + 3D \quad - 3 \quad + w \quad - 3 w^2 \right\}$$

$$= Q R^2 \frac{1}{48 w^3 (w-1) D} \left\{ 8w^2 (4w^4 - 8w^3 + 8w^2 - 4w + 1) + 21 (4w^4 - 8w^3 + 8w^2 - 4w + 1) \right.$$
$$\left. - 6w^2 + 6w^3 - 21 (2w^2 - 2w + 1) + 21 w (2w^2 - 2w + 1) - 24 w^2 (2w^2 - 2w + 1) \right\} =$$

$$= Q R^2 \frac{1}{48 w^3 (w-1) D} \left\{ 32 w^6 - 64 w^5 + 64 w^4 - 32 w^3 + 8 w^2 + 84 w^4 - 168 w^3 + 168 w^2 \right.$$
$$\left. - 84 w + 21 - 6w^2 + 6 w^3 - 42 w^2 + 42 w - 21 + 42 w^3 - 42 w^2 + 21 w - 48 w^4 + 48 w^3 - 24 w^2 \right\}$$

$$= Q R^2 \frac{1}{48 w^3 (w-1) D} \left\{ 32 w^6 - 64 w^5 + 100 w^4 - 104 w^3 + 62 w^2 - 21 w \right\}$$

$$= Q R^2 \frac{1}{6 w^3 (w-1)} \left\{ 2w^4 - 2w^3 \overset{+4w^2}{-5w} - 0 \right\} = Q R^2 \frac{1}{6 w^2 (w-1)} (2w^3 - 2w^2 + 4w - 5)$$

y por tanto

$$\langle r^2 \rangle_p^{1/2} = \left\{ R^2 \frac{(2w^3 - 2w^2 + 4w - 5)}{6 w^2 (w-1)} \right\}^{1/2} = 0.674\ R = 0.868\ fm \qquad (1)$$

$$\langle r^2 \rangle_m = 0$$

a comparar con los valores experimentales

$$\langle r^2 \rangle_p^{1/2} = 0.88 \pm 0.03\ fm \qquad \langle r^2 \rangle_m = -0.12 \pm 0.01\ fm^2 \qquad (2)$$

Podemos tambien calcular la constante g_A del acoplamiento axial

$$g_A \equiv - \langle P\uparrow | \int d^3r \; \psi^+(\bar r) \, \tau_3 \, \Sigma_3 \, \psi(\bar r) \, | P\uparrow \rangle \qquad (1)$$

para ello necesitamos

$$\langle q; 1/2\, 0\, M; m=1 | \int d^3r \; \psi^+(\bar r) \, \tau_3 \, \Sigma_3 \, \psi(\bar r) \, | q; 1/2\, 0\, M; m=-1 \rangle = ($$

$$= (\delta_{qu} - \delta_{qd}) \int d^3r \; \bar\Psi^{(m=-1)}_{1/2\,0\,M}(\bar r) \, \Sigma_3 \, \bar\Psi^{(m=-1)}_{1/2\,0\,M}(\bar r) =$$

$$= (\delta_{qu} - \delta_{qd}) \frac{|A|^2}{4\pi} \int d^3r \left[j_0^2(\alpha r) \, \chi^+(M) \, \sigma_3 \, \chi(M) + j_1^2(\alpha r) \, \chi^+(M) (\bar\sigma \cdot \hat r) \, \sigma_3 \, (\bar\sigma \cdot \hat r) \chi(M) \right]$$

$$= (\delta_{qu} - \delta_{qd}) \frac{|A|^2}{4\pi} \int d^3r \left[j_0^2(\alpha r) \, \chi^+(M) \, \sigma_3 \, \chi(M) + j_1^2(\alpha r) \, \chi^+(M) \left[2\hat r (\hat r \cdot \bar\sigma) - \bar\sigma\right]_3 \chi(M) \right]$$

$$= (\delta_{qu} - \delta_{qd}) |A|^2 \int_0^R dr \, r^2 \left[j_0^2(\alpha r) \, (-1)^{M-1/2} - \frac{1}{3} (-1)^{M-1/2} j_1^2(\alpha r) \right]$$

$$= (\delta_{qu} - \delta_{qd}) |A|^2 (-1)^{M-1/2} \frac{1}{\alpha^3} \int_0^{\alpha R} dz \, z^2 \left[j_0^2(z) - \frac{1}{3} j_1^2(z) \right]$$

$$= (\delta_{qu} - \delta_{qd}) (-1)^{M-1/2} \frac{D}{2\omega^2(\omega-1)} \left\{ -\frac{1}{2} \sin\omega \cos\omega + \frac{1}{2}\omega - \frac{1}{6\omega}\cos^2\omega + \frac{1}{6\omega}\sin^2\omega + \frac{1}{6\omega} \right.$$

$$\left. - \frac{1}{6}\sin\omega\cos\omega - \frac{1}{6}\omega \right\} =$$

$$= (\delta_{qu} - \delta_{qd}) (-1)^{M-1/2} \frac{D}{2\omega^2(\omega-1)} \left\{ -\frac{\omega(1-\omega)}{2D} + \frac{1}{2}\omega - \frac{(1-\omega)^2}{6\omega D} + \frac{\omega}{6D} + \frac{1}{6\omega} - \frac{\omega(1-\omega)}{6D} - \frac{1}{6}\omega \right\}$$

$$= (\delta_{qu} - \delta_{qd}) (-1)^{M-1/2} \frac{D}{2\omega^2(\omega-1)} \frac{1}{6\omega D} \left\{ -3\omega^2(1-\omega) + 3\omega^2 D - (1-\omega)^2 + \omega^2 + D - \omega^2(1-\omega) - \omega^2 D \right\}$$

$$= (\delta_{qu} - \delta_{qd}) (-1)^{M-1/2} \frac{1}{12\omega^3(\omega-1)} \left\{ -4\omega^2 + 4\omega^3 + \omega^2 - 1 + 2\omega - \omega^2 + 2\omega^2 - 2\omega + 1 + 4\omega^4 - 4\omega^3 + 2\omega^2 \right\}$$

$$= (\delta_{qu} - \delta_{qd}) (-1)^{M-1/2} \frac{1}{12\omega^3(\omega-1)} \left[4\omega^4 \right] = (\delta_{qu} - \delta_{qd}) (-1)^{M-1/2} \frac{\omega}{3(\omega-1)}$$

y por tanto

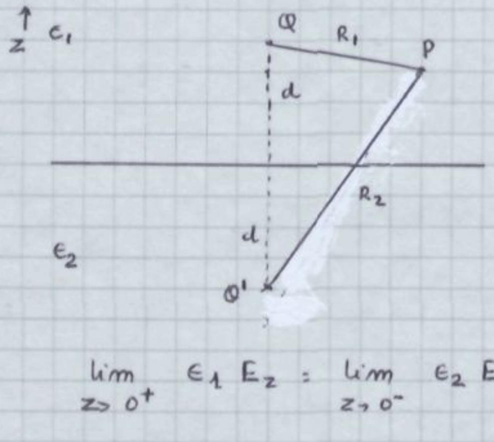
$$g_A = -\frac{5}{3} \left[1 - \frac{2\omega - 3}{3(\omega-1)} \right] = -1.088 \qquad (2)$$

Los quarks no son los únicos constituyentes de los hadrones. Se hallan acoplados a partículas vectoriales sin masa, o gluones, que son los mediadores de las interacciones entre quarks. En el límite $g \to 0$ los gluones satisfacen ecuaciones de campo idénticas a las de Maxwell

$$\vec{\nabla} \cdot \vec{E} = 0 \qquad \vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = 0$$
$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \tag{1}$$

donde $\vec{E}$ y $\vec{B}$ se denominan campo eléctrico y magnético gluónico, respectivamente. Supondremos que en el interior de la esfera de radio R hay una fase de vacío (fase hadrónica) con constante dieléctrica $\epsilon = 1$ y constante de permeabilidad magnética $\mu = 1$. Sin embargo fuera de la esfera el vacío físico actúa para los gluones como un extraño medio con $\epsilon = 0$ y $\mu = \infty$. Queremos ver que estas condiciones implican el confinamiento de los gluones.

La repulsión del campo gluónico desde la fase externa puede entenderse como un ejercicio de electróstatica. Consideremos una carga Q en un medio semiinfinito de constante dieléctrica ϵ_1 a una distancia d de un plano que separa esta región del espacio de otra con $\epsilon_2 = 0$. Tomaremos que este plano es el $z = 0$. El campo eléctrico viene determinado por



$$\epsilon_1 \, \vec{\nabla} \cdot \vec{E} = 4\pi \rho \,, \qquad z > 0$$
$$\epsilon_2 \, \vec{\nabla} \cdot \vec{E} = 0 \qquad z < 0 \tag{2}$$
$$\vec{\nabla} \times \vec{E} = 0 \qquad z \lessgtr 0$$

Además hay las condiciones de contorno

$$\lim_{z \to 0^+} \epsilon_1 E_z = \lim_{z \to 0^-} \epsilon_2 E_z \,, \qquad \lim_{z \to 0^+} E_{x,y} = \lim_{z \to 0^-} E_{x,y} \tag{3}$$

La solución al problema se puede lograr mediante el método de la carga imagen Q', colocada en una posición simétrica a Q con respecto a $z = 0$. El campo eléctrico es derivable de un potencial escalar ϕ que es

$$\phi = \frac{1}{\epsilon_1} \left(\frac{Q}{R_1} + \frac{Q'}{R_2} \right) \qquad z > 0 \tag{4}$$

Para $z<0$ el potencial es equivalente al de una carga Q'' en la posición de la carga real

$$\phi = \frac{1}{\epsilon_2}\;\frac{Q''}{R}$$

De aquí [$Q \equiv (0,0,d)$ $P \equiv (x,y,z)$]

$$\vec{E} = \frac{1}{\epsilon_1}\left(\frac{Qx}{R_1^3}+\frac{Q'x}{R_2^3}\,,\;\frac{Qy}{R_1^3}+\frac{Q'y}{R_2^3}\,,\;\frac{Q(z-d)}{R_1^3}+\frac{Q'(z+d)}{R_2^3}\right) \qquad z>0$$

$$\vec{E} = \frac{1}{\epsilon_2}\left(\frac{Q''x}{R^3}\,,\;\frac{Q''y}{R^3}\,,\;\frac{Q''(z-d)}{R^3}\right) \qquad z<0$$

De las condiciones de contorno $r \equiv (x^2+y^2+d^2)^{1/2}$

$$-\frac{Qd}{r^3}+\frac{Q'd}{r^3} = -\frac{Q''d}{r^3}\,, \qquad \frac{1}{\epsilon_1}\left(\frac{Qx}{R^3}+\frac{Q'x}{R^3}\right) = \frac{1}{\epsilon_2}\frac{Q''x}{R^3}$$

$$\Rightarrow\; -Qd+Q'd = -Q''d \qquad \epsilon_2 Q+\epsilon_2 Q' = \epsilon_1 Q''$$

$$\Rightarrow\; Q' = \frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}\,Q\,, \qquad Q'' = \frac{2\epsilon_2}{\epsilon_1+\epsilon_2}\,Q$$

y por tanto

$$\vec{E} = \left(\frac{1}{\epsilon_1}\frac{Qx}{R_1^3}+\frac{1}{\epsilon_1}\frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}\frac{Qx}{R_2^3}\,,\;\frac{1}{\epsilon_1}\frac{Qy}{R_1^3}+\frac{1}{\epsilon_1}\frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}\frac{Qy}{R_2^3}\,,\;\frac{1}{\epsilon_1}\frac{Q(z-d)}{R_1^3}+\frac{1}{\epsilon_1}\frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}\frac{Q(z+d)}{R_2^3}\right)$$

$$\vec{E} = \left(\frac{2}{\epsilon_1+\epsilon_2}\frac{Qx}{R^3}\,,\;\frac{2}{\epsilon_1+\epsilon_2}\frac{Qy}{R^3}\,,\;\frac{2}{\epsilon_1+\epsilon_2}\frac{Q(z-d)}{R^3}\right)$$

y en el límite $\epsilon_1 = 1$, $\epsilon_2 = 0$

$$\vec{E} = \left(\frac{Qx}{R_1^3}+\frac{Qx}{R_2^3}\,,\;\frac{Qy}{R_1^3}+\frac{Qy}{R_2^3}\,,\;\frac{Q(z-d)}{R_1^3}+\frac{Q(z+d)}{R_2^3}\right) \qquad z>0$$

$$\vec{E} = \left(2\frac{Qx}{R^3}\,,\;2\frac{Qy}{R^3}\,,\;2\frac{Q(z-d)}{R^3}\right) \qquad z<0$$

y para el vector desplazamiento

$$\vec{D} = \left(Qx\left(\frac{1}{R_1^3}+\frac{1}{R_2^3}\right),\;Qy\left(\frac{1}{R_1^3}+\frac{1}{R_2^3}\right),\;Q\left(\frac{z-d}{R_1^3}+\frac{z+d}{R_2^3}\right)\right), \quad z>0$$

$$\vec{D} = 0 \qquad z<0$$

es decir que $\vec{D}=0$ en el exterior y yace en la superficie $z=0$ para $z\neq 0$.

Lo mismo sucede con $\vec{B}$. y por tanto no puede producirse propagación de ondas en la fase exterior

Consideremos ahora la descripción de los gluones confinados en una esfera de radio R. En la region $r < R$ satisfacen las ecuaciones de Maxwell (14.1) con las condiciones frontera

$$\hat{r} \cdot \vec{E} = 0 \quad , \quad \hat{r} \times \vec{B} = 0 \qquad r = R \tag{1}$$

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$$\epsilon^{\dot{\alpha}\dot{\gamma}}\epsilon_{\dot{\alpha}\dot{\beta}} = -\delta^{\dot{\gamma}}_{\dot{\beta}}$$

$$\epsilon^{\alpha\gamma}\epsilon_{\alpha\beta} = -\delta^{\gamma}{}_{\beta}$$

II. - B - WEYL SPINORS

As it is well known we can establish different one to two homomorphisms between $L_+^\uparrow$ and $SL(2,C)$ (See Appendix A)

$$\begin{aligned} \Lambda &\longrightarrow M(\Lambda) \equiv [A^+(\Lambda)]^{-1} \\ \Lambda &\longrightarrow [M^T(\Lambda)]^{-1} \\ \Lambda &\longrightarrow M^*(\Lambda) \\ \Lambda &\longrightarrow [M^+(\Lambda)]^{-1} \end{aligned} \tag{1}$$

and in particular for $\Lambda^\mu{}_\nu = g^\mu{}_\nu + \delta\omega^\mu{}_\nu$ we have

$$M = I - \frac{i}{2} M^{\mu\nu}\delta\omega_{\mu\nu} \qquad M^{\mu\nu} \equiv \frac{i}{4}[\sigma^\mu\tilde{\sigma}^\nu - \sigma^\nu\tilde{\sigma}^\mu] \tag{2}$$

$$\tilde{M}^{\mu\nu} \equiv \frac{i}{4}[\tilde{\sigma}^\mu\sigma^\nu - \tilde{\sigma}^\nu\sigma^\mu] = M^{\mu\nu +}$$

and it is easy to check that

$$M^{-1}\sigma^\mu (M^{-1})^+ = \Lambda^\mu{}_\nu\sigma^\nu \quad , \quad M^+\tilde{\sigma}^\mu M = \Lambda^\mu{}_\nu\tilde{\sigma}^\nu \tag{3}$$

We define the spinors ψ_α, ψ^α, $\bar{\psi}_{\dot\alpha}$, $\bar{\psi}^{\dot\alpha}$ with the following transformation laws under $L_+^\uparrow$

$$\begin{aligned} \psi_\alpha &\longrightarrow \psi'_\alpha = M_\alpha{}^\beta\psi_\beta \quad , \quad \bar{\psi}_{\dot\alpha} \longrightarrow \bar{\psi}'_{\dot\alpha} = M^*{}_{\dot\alpha}{}^{\dot\beta}\bar{\psi}_{\dot\beta} \\ \psi^\alpha &\longrightarrow \psi'^\alpha = M^{-1}{}_\beta{}^\alpha\psi^\beta \quad , \quad \bar{\psi}^{\dot\alpha} \longrightarrow \bar{\psi}'^{\dot\alpha} = [M^*]^{-1}{}_{\dot\beta}{}^{\dot\alpha}\bar{\psi}^{\dot\beta} \end{aligned} \tag{4}$$

Similarly we can define the transformation laws for objects $\psi^{\alpha_1\dots\alpha_p\dot\alpha_1\dots\dot\alpha_q}_{\beta_1\dots\beta_r\dot\beta_1\dots\dot\beta_s}$ with $2^{p+q+r+s}$ components.

Let us introduce the antisymmetric tensor $\epsilon^{\alpha\beta}$, $\epsilon_{\alpha\beta}$, $\epsilon^{\dot\alpha\dot\beta}$, $\epsilon_{\dot\alpha\dot\beta}$

$$\begin{aligned} &\epsilon_{21} = \epsilon^{12} = +1 \quad , \quad \epsilon_{12} = \epsilon^{21} = -1 \quad , \quad \epsilon_{11} = \epsilon^{11} = \epsilon_{22} = \epsilon^{22} = 0 \\ &\epsilon_{\dot2\dot1} = \epsilon^{\dot1\dot2} = +1 \quad , \quad \epsilon_{\dot1\dot2} = \epsilon^{\dot2\dot1} = -1 \quad , \quad \epsilon_{\dot1\dot1} = \epsilon^{\dot1\dot1} = \epsilon_{\dot2\dot2} = \epsilon^{\dot2\dot2} = 0 \end{aligned} \tag{5}$$

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Ref E. M. CORSON "Introduction to Tensors, Spinors and Relativistic Wave equations" London. Blackie and Son Ltd. (1953)

W. THIRRING Suplemento del Nuovo Cimento 14, nº 2, 415 (1959).

Notice that

$$\epsilon_{\alpha\beta}\,\epsilon^{\beta\gamma} = \delta^{\gamma}{}_{\alpha} \quad , \quad \epsilon_{\dot\alpha\dot\beta}\,\epsilon^{\dot\beta\dot\gamma} = \delta^{\dot\gamma}_{\dot\alpha} \tag{1}$$

Furthermore it is immediate to check that these tensors are invariant under Lorentz transformations

$$\begin{aligned} &M_{\alpha}{}^{\gamma}\, M_{\beta}{}^{\delta}\, \epsilon_{\gamma\delta} = \epsilon_{\alpha\beta} && M^{-1}{}_{\gamma}{}^{\alpha}\, M^{-1}{}_{\delta}{}^{\beta}\, \epsilon^{\gamma\delta} = \epsilon^{\alpha\beta} \\ &M^{*}{}_{\dot\alpha}{}^{\dot\gamma}\, M^{*}{}_{\dot\beta}{}^{\dot\delta}\, \epsilon_{\dot\gamma\dot\delta} = \epsilon_{\dot\alpha\dot\beta} && [M^{*}]^{-1}{}_{\dot\gamma}{}^{\dot\alpha}\, [M^{*}]^{-1}{}_{\dot\delta}{}^{\dot\beta}\, \epsilon^{\dot\gamma\dot\delta} = \epsilon^{\dot\alpha\dot\beta} \end{aligned} \tag{2}$$

Furthermore

$$\begin{aligned} &M_{\alpha}{}^{\gamma}\, M_{\beta}{}^{\delta}\, \epsilon_{\gamma\delta} = \epsilon_{\alpha\beta} && M_{\gamma}{}^{\alpha}\, M_{\delta}{}^{\beta}\, \epsilon^{\gamma\delta} = \epsilon^{\alpha\beta} \\ &M^{*}{}_{\dot\alpha}{}^{\dot\gamma}\, M^{*}{}_{\dot\beta}{}^{\dot\delta}\, \epsilon_{\dot\gamma\dot\delta} = \epsilon_{\dot\alpha\dot\beta} && M^{*}{}_{\dot\gamma}{}^{\dot\alpha}\, M^{*}{}_{\dot\delta}{}^{\dot\beta}\, \epsilon^{\dot\gamma\dot\delta} = \epsilon^{\dot\alpha\dot\beta} \end{aligned} \tag{3}$$

Spinors with upper and lower indices are related through the ϵ - tensor

$$\begin{aligned} &\psi^{\alpha} = \epsilon^{\alpha\beta}\,\psi_{\beta} \quad , && \bar\psi^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\,\bar\psi_{\dot\beta} \\ &\psi_{\alpha} = \epsilon_{\alpha\beta}\,\psi^{\beta} \quad , && \bar\psi_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\,\bar\psi^{\dot\beta} \end{aligned} \tag{4}$$

These relations are Lorentz invariant as can be proved using (3) and this reflects the fact that M and $[M^{T}]^{-1}$, as well as M^{*} and $[M^{\dagger}]^{-1}$ are equivalent representations

It is possible to prove that the representations of $L_{+}^{\uparrow}$ generated by the spinor $\xi_{\alpha_1\dots\alpha_p\,\dot\alpha_1\dots\dot\alpha_q}$ which is fully symmetric for both kinds of indices is irreducible under $L_{+}^{\uparrow}$ and has dimension $(p+1)(q+1)$. These representations, denoted by $D^{(p/2,\, q/2)}$, are the unique irreducible representations of $L_{+}^{\uparrow}$ of finite dimension. A usual form of selecting the basis for the irreducible representation $D^{(j_1, j_2)}$ is

$$\psi^{j_1 j_2}_{m_1 m_2} = \frac{[\psi_1]^{j_1+m_1}\,[\psi_2]^{j_1-m_1}}{[(j_1+m_1)!\,(j_1-m_1)!\,]^{1/2}} \cdot \frac{[\bar\psi_{\dot 1}]^{j_2+m_2}\,[\bar\psi_{\dot 2}]^{j_2-m_2}}{[(j_2+m_2)!\,(j_2-m_2)!\,]^{1/2}} \tag{5}$$

where j_1 and j_2 are integer or half-integer and $m = -j, -j+1, \dots, j-1, j$. Under $SO(3) \subset L_{+}^{\uparrow}$, the representations M and M^{*} are equivalent and $D^{(j_1, j_2)}$ reduce

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$$(\sigma^b)_{\alpha\dot\beta}(\tilde\sigma_a)^{\dot\beta\beta} + (\sigma_a)_{\alpha\dot\beta}(\tilde\sigma^b)^{\dot\beta\beta} = 2 g^b_a \, \delta^\beta_\alpha$$

$$(\tilde\sigma_a)^{\dot\alpha\alpha}(\sigma^b)_{\alpha\dot\beta} + (\tilde\sigma^b)^{\dot\alpha\alpha}(\sigma_a)_{\alpha\dot\beta} = 2 g^b_a \, \delta^{\dot\alpha}_{\dot\beta}$$

$$(\tilde\sigma_c)^{\dot\lambda\lambda}(\sigma^c)_{\mu\dot\mu} = 2\delta^\lambda_\mu \, \delta^{\dot\lambda}_{\dot\mu}$$

$$(\sigma_a)_{\alpha\dot\alpha}(\sigma^a)_{\beta\dot\beta} = 2\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta} \qquad (\tilde\sigma_a)^{\dot\alpha\alpha}(\tilde\sigma^a)^{\dot\beta\beta} = 2\epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta}$$

$$\sigma^b\tilde\sigma^c\sigma^a = g^{bc}\sigma^a - g^{ba}\sigma^c + g^{ca}\sigma^b + i\epsilon^{bcae}\sigma_e$$

$$\tilde\sigma^b\sigma^c\tilde\sigma^a = g^{bc}\tilde\sigma^a - g^{ba}\tilde\sigma^c + g^{ca}\tilde\sigma^b - i\epsilon^{bcae}\tilde\sigma_e$$

as

$$D^{(j_1,j_2)} \longrightarrow D^{(j_1)} \otimes D^{(j_2)} = D^{(j_1+j_2)} \oplus \dots \oplus D^{(|j_1-j_2|)} \qquad (1)$$

The index structure of σ^μ and $\tilde{\sigma}^\mu$ is

$$[\sigma^0]_{\alpha\dot\alpha} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \quad [\sigma^1]_{\alpha\dot\alpha} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \quad [\sigma^2]_{\alpha\dot\alpha} = \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix}, \quad [\sigma^3]_{\alpha\dot\alpha} = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$$

$$[\tilde\sigma^0]^{\dot\alpha\alpha} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \quad [\tilde\sigma^1]^{\dot\alpha\alpha} = \begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix}, \quad [\tilde\sigma^2]^{\dot\alpha\alpha} = \begin{vmatrix} 0 & i \\ -i & 0 \end{vmatrix}, \quad [\tilde\sigma^3]^{\dot\alpha\alpha} = \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} \qquad (2)$$

with the relation

$$[\tilde\sigma^\mu]^{\dot\alpha\beta} = \epsilon^{\beta\alpha}\epsilon^{\dot\alpha\dot\beta}[\sigma^\mu]_{\alpha\dot\beta} \equiv [\sigma^\mu]^{\beta\dot\alpha} \qquad (3)$$

$$[\sigma^\mu]_{\beta\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\,\epsilon_{\beta\alpha}\,[\tilde\sigma^\mu]^{\dot\beta\alpha} = [\tilde\sigma^\mu]_{\dot\alpha\beta}$$

Obviously all relations given in the Appendix A are still valid. Furthermore we can easily check

$$\sigma^\mu{}_{\alpha\dot\alpha}\,\sigma^\nu{}_{\beta\dot\beta} - \sigma^\nu{}_{\alpha\dot\alpha}\,\sigma^\mu{}_{\beta\dot\beta} = -2i\left[(M^{\mu\nu}\epsilon)_{\alpha\beta}\,\epsilon_{\dot\alpha\dot\beta} + (\epsilon\tilde M^{\mu\nu})_{\dot\alpha\dot\beta}\,\epsilon_{\alpha\beta}\right]$$

$$\sigma^\mu{}_{\alpha\dot\alpha}\,\sigma^\nu{}_{\beta\dot\beta} + \sigma^\nu{}_{\alpha\dot\alpha}\,\sigma^\mu{}_{\beta\dot\beta} = +g^{\mu\nu}\,\epsilon_{\alpha\beta}\,\epsilon_{\dot\alpha\dot\beta} - 4\,[M^{\lambda\mu}\epsilon]_{\alpha\beta}\,[\epsilon\tilde M_\lambda{}^\nu]_{\dot\alpha\dot\beta} \qquad (4)$$

We can use all that to convert a vector in a bispinor and viceversa

$$v_{\alpha\dot\alpha} = v_\mu\,[\sigma^\mu]_{\alpha\dot\alpha} \qquad v^\mu = \frac{1}{2}\,[\tilde\sigma^\mu]^{\dot\alpha\alpha}\,v_{\alpha\dot\alpha}$$

$$v_{\dot\alpha\alpha} = v_\mu\,[\tilde\sigma^\mu]_{\dot\alpha\alpha} \qquad v^\mu = \frac{1}{2}\,[\sigma^\mu]^{\alpha\dot\alpha}\,v_{\dot\alpha\alpha} \qquad (5)$$

With these conventions of coefficients it is immediate to show that

$$\psi^\alpha\psi_\alpha\ , \quad \bar\psi_{\dot\alpha}\bar\psi^{\dot\alpha}\ , \quad \psi^\alpha\sigma^\mu{}_{\alpha\dot\alpha}\,\partial_\mu\bar\psi^{\dot\alpha} \qquad (6)$$

are Lorentz scalars.

We can define the Dirac matrices as $\gamma^\mu = \begin{vmatrix} 0 & \sigma^\mu \\ \tilde\sigma^\mu & 0 \end{vmatrix}$ (7)

which is the so called chiral or Weyl basis. In this basis, Dirac spinors contain two Weyl spinors*

$$\Psi_D = \begin{vmatrix} \chi_\alpha \\ \bar\psi^{\dot\alpha} \end{vmatrix} \qquad (8)$$

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* Dirac equation $(\not{p} - m)\psi = 0$

$$\begin{vmatrix} -m\,\delta_\alpha{}^\beta & (\sigma^\mu)_{\alpha\dot\beta}\,p_\mu \\ (\tilde\sigma^\mu)^{\dot\alpha\beta}\,p_\mu & -m\,\delta^{\dot\alpha}{}_{\dot\beta} \end{vmatrix} \begin{vmatrix} \chi_\beta \\ \bar\psi^{\dot\beta} \end{vmatrix} = 0$$

$$\Rightarrow \quad [\sigma^\mu]_{\alpha\dot\beta}\,p_\mu\,\bar\psi^{\dot\beta} = m\,\chi_\alpha$$

$$[\tilde\sigma^\mu]^{\dot\alpha\beta}\,p_\mu\,\chi_\beta = m\,\bar\psi^{\dot\alpha}$$

$$\gamma^\mu = \begin{vmatrix} 0 & [\sigma^\mu]_{\alpha\dot\beta} \\ [\tilde\sigma^\mu]^{\dot\alpha\beta} & 0 \end{vmatrix} \qquad \gamma_5 = \begin{vmatrix} \delta_\alpha{}^\beta & 0 \\ 0 & -\delta_{\dot\alpha}{}^{\dot\beta} \end{vmatrix} \qquad I = \begin{vmatrix} \delta^\beta_\alpha & 0 \\ 0 & \delta^{\dot\beta}_{\dot\alpha} \end{vmatrix}$$

$$\sigma^{\mu\nu} = \frac{i}{2}\begin{vmatrix} [\sigma^\mu]_{\alpha\dot\alpha}\,[\tilde\sigma^\nu]^{\dot\alpha\beta} & 0 \\ 0 & [\tilde\sigma^\mu]^{\dot\alpha\alpha}\,[\sigma^\nu]_{\alpha\dot\beta} \end{vmatrix} - (\mu\leftrightarrow\nu)$$

Hence

$$\sigma^{\mu\nu} = 2\begin{vmatrix} [M^{\mu\nu}]_\alpha{}^\beta & 0 \\ 0 & [\tilde M^{\mu\nu}]^{\dot\alpha}{}_{\dot\beta} \end{vmatrix}$$

Charge conjugation $\quad C \equiv \gamma^0\gamma^2 = \begin{vmatrix} [\sigma^0]_{\alpha\dot\alpha}\,[\tilde\sigma^2]^{\dot\alpha\beta} & 0 \\ 0 & [\tilde\sigma^0]^{\dot\alpha\alpha}\,[\sigma^2]_{\alpha\dot\beta} \end{vmatrix}$

$$\Rightarrow \quad C = -i\begin{vmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot\alpha\dot\beta} \end{vmatrix} \qquad C^+ = C \ , \ C^T = -C$$

$$C^{-1} = C$$

It is possible to define $\psi^C(x) \equiv i\,C\,\bar\psi^T(x) =$

$$\bar\psi^T(x) = \begin{vmatrix} \psi^\alpha \\ \bar\chi_{\dot\alpha} \end{vmatrix} \qquad \text{where} \qquad \psi_\alpha^* = \bar\psi_{\dot\alpha} \quad \text{for any spinor}$$

Then

$$\psi^C = \begin{vmatrix} \psi_\alpha \\ \bar\chi^{\dot\alpha} \end{vmatrix}$$

For a Majorana spin $\psi \equiv \chi \Rightarrow \psi^C \equiv \psi$

$$\psi_M = \begin{vmatrix} \chi_\alpha \\ \bar\chi^{\dot\alpha} \end{vmatrix}$$

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while Majorana spinors contain only one

$$\Psi_M = \begin{vmatrix} \chi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{vmatrix} \qquad \bar{\chi}_{\dot{\alpha}} \equiv (\chi_\alpha)^* \tag{1}$$

Notice that the last condition is consistent with Lorentz transformations.

Usually the following spinor summation conventions are used

$$\psi\chi \equiv \psi^\alpha \chi_\alpha = -\psi_\alpha \chi^\alpha = \chi^\alpha \psi_\alpha = \chi\psi$$
$$\bar{\psi}\bar{\chi} \equiv \bar{\psi}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}} = -\bar{\psi}^{\dot{\alpha}} \bar{\chi}_{\dot{\alpha}} = \bar{\chi}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}} = \bar{\chi}\bar{\psi} \tag{2}$$

where we have assumed, as always, that spinors anticommute. The definition of $\bar{\psi}\bar{\chi}$ is chosen in such a way that

$$(\chi\psi)^+ = (\chi^\alpha \psi_\alpha)^+ = \bar{\psi}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}} = \bar{\psi}\bar{\chi} \tag{3}$$

Let us consider some other interesting relations

i) $$\chi\sigma^\mu\bar{\psi} = -\bar{\psi}\tilde{\sigma}^\mu\chi \quad , \quad [\chi\sigma^\mu\bar{\psi}]^+ = \psi\sigma^\mu\bar{\chi} \tag{4}$$

Proof: $\chi\sigma^\mu\bar{\psi} = \chi^\alpha(\sigma^\mu\bar{\psi})_\alpha = \chi^\alpha(\sigma^\mu)_{\alpha\dot{\alpha}}\bar{\psi}^{\dot{\alpha}} = \epsilon^{\alpha\beta}\chi_\beta(\sigma^\mu)_{\alpha\dot{\alpha}}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{\psi}_{\dot{\beta}} =$

$= \chi_\beta \epsilon^{\beta\alpha}\epsilon^{\dot{\beta}\dot{\alpha}}(\sigma^\mu)_{\alpha\dot{\alpha}}\bar{\psi}_{\dot{\beta}} = \chi_\beta(\tilde{\sigma}^\mu)^{\dot{\beta}\beta}\bar{\psi}_{\dot{\beta}} = -\bar{\psi}_{\dot{\beta}}(\tilde{\sigma}^\mu)^{\dot{\beta}\beta}\chi_\beta = -\bar{\psi}\tilde{\sigma}^\mu\chi$

$(\chi\sigma^\mu\bar{\psi})^+ = [\chi^\alpha(\sigma^\mu)_{\alpha\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}]^+ = \psi^\alpha(\sigma^\mu)_{\alpha\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} = \psi_\alpha(\sigma^\mu)^{\alpha\dot{\alpha}}\bar{\chi}_{\dot{\alpha}} = \psi\sigma^\mu\bar{\chi}$

Also $$\chi M^{\mu\nu}\psi = -\psi M^{\mu\nu}\chi \quad , \quad \bar{\chi}\tilde{M}^{\mu\nu}\bar{\psi} = -\bar{\psi}\tilde{M}^{\mu\nu}\bar{\chi} \tag{4'}$$

ii) The Fierz transformation formula is

$$[\phi\psi]\bar{\chi}_{\dot{\beta}} = \frac{1}{2}[\phi\sigma^\mu\bar{\chi}][\psi\sigma_\mu]_{\dot{\beta}} \tag{5}$$

Proof: $(\phi\sigma^\mu\bar{\chi})(\psi\sigma_\mu)_{\dot{\beta}} = -(\bar{\chi}\tilde{\sigma}^\mu\phi)(\psi\sigma_\mu)_{\dot{\beta}} = -\bar{\chi}_{\dot{\alpha}}(\tilde{\sigma}^\mu)^{\dot{\alpha}\alpha}\phi_\alpha\psi^\beta(\sigma_\mu)_{\beta\dot{\beta}} =$

$= -2\delta^{\dot{\alpha}}{}_{\dot{\beta}}\delta^\alpha{}_\beta\bar{\chi}_{\dot{\alpha}}\phi_\alpha\psi^\beta = -2\bar{\chi}_{\dot{\beta}}\phi_\beta\psi^\beta = -2\phi_\beta\psi^\beta\bar{\chi}_{\dot{\beta}} = 2\phi^\beta\psi_\beta\bar{\chi}_{\dot{\beta}} = 2(\phi\psi)\bar{\chi}_{\dot{\beta}}$

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$$\psi_i \equiv \begin{vmatrix} \xi_{i\alpha} \\ \bar\eta_i^{\dot\alpha} \end{vmatrix} \qquad \bar\psi_i = | \eta_i^{\alpha}, \bar\xi_{i\dot\alpha} |$$

$$\bar\psi_1 \psi_2 = [\eta_1 \xi_2] + [\bar\eta_2 \bar\xi_1]$$

$$\bar\psi_1 \gamma_5 \psi_2 = [\eta_1 \xi_2] - [\bar\eta_2 \bar\xi_1]$$

$$\bar\psi_1 \gamma^\mu \psi_2 = [\bar\xi_1 \tilde\sigma^\mu \xi_2] - [\bar\eta_2 \tilde\sigma^\mu \eta_1]$$

$$\bar\psi_1 \gamma^\mu \gamma_5 \psi_2 = [\bar\xi_1 \tilde\sigma^\mu \xi_2] + [\bar\eta_2 \tilde\sigma^\mu \eta_1]$$

$$\bar\psi_1 \sigma^{\mu\nu} \psi_2 = 2 [\eta_1 M^{\mu\nu} \xi_2] - 2 [\bar\eta_2 \tilde M^{\mu\nu} \bar\xi_1]$$

We can introduce $P_L \equiv \frac{1}{2}(1+\gamma_5) \qquad P_R \equiv \frac{1}{2}(1-\gamma_5)$

$$P_L = \begin{vmatrix} \delta_\alpha{}^\beta & 0 \\ 0 & 0 \end{vmatrix} \qquad P_R = \begin{vmatrix} 0 & 0 \\ 0 & \delta^{\dot\alpha}{}_{\dot\beta} \end{vmatrix}$$

$$P_L \psi = \begin{vmatrix} \xi_\alpha \\ 0 \end{vmatrix} \qquad P_R \psi = \begin{vmatrix} 0 \\ \bar\eta^{\dot\alpha} \end{vmatrix} \qquad \psi = P_L \psi + P_R \psi \equiv \begin{vmatrix} \psi_L \\ \psi_R \end{vmatrix}$$

For a Majorana neutrino

$$\psi = \begin{vmatrix} \xi_\alpha \\ \bar\xi^{\dot\alpha} \end{vmatrix} = \begin{vmatrix} \xi_\alpha \\ \epsilon^{\dot\alpha\dot\beta} \bar\xi_{\dot\beta} \end{vmatrix} = \begin{vmatrix} \xi_\alpha \\ \epsilon^{\alpha\beta} \xi_\beta^* \end{vmatrix} = \begin{vmatrix} \psi_L \\ i\sigma_2 \psi_L^* \end{vmatrix}$$

Notice $\bar\psi_1 P_L \psi_2 = [\eta_1 \xi_2] \quad , \quad \bar\psi_1 P_R \psi_2 = [\bar\eta_2 \bar\xi_1]$

$$\bar\psi_1 \gamma^\mu P_L \psi_2 = [\bar\xi_1 \tilde\sigma^\mu \xi_2] \quad , \quad \bar\psi_1 \gamma^\mu P_R \psi_2 = -[\bar\eta_2 \tilde\sigma^\mu \eta_1]$$

Also for anticommuting Majorana spinors

$$\bar\psi_1 \psi_2 = \bar\psi_2 \psi_1 \quad , \quad \bar\psi_1 \gamma_5 \psi_2 = \bar\psi_2 \gamma_5 \psi_1$$

$$\bar\psi_1 \gamma^\mu \psi_2 = -\bar\psi_2 \gamma^\mu \psi_1 \quad , \quad \bar\psi_1 \gamma^\mu \gamma_5 \psi_2 = \bar\psi_2 \gamma^\mu \gamma_5 \psi_1$$

$$\bar\psi_1 \sigma^{\mu\nu} \psi_2 = -\bar\psi_2 \sigma^{\mu\nu} \psi_1$$

$$\bar\psi_1 \gamma^\mu P_L \psi_2 = -\bar\psi_2 \gamma^\mu P_R \psi_1$$

iii) Other useful relations

$$\theta^\alpha \theta^\beta = -\frac{1}{2}\epsilon^{\alpha\beta}(\theta\theta) \quad , \quad \theta_\alpha\theta_\beta = +\frac{1}{2}\epsilon_{\alpha\beta}(\theta\theta)$$

$$\bar\theta^{\dot\alpha}\bar\theta^{\dot\beta} = +\frac{1}{2}\epsilon^{\dot\alpha\dot\beta}(\bar\theta\bar\theta) \quad , \quad \bar\theta_{\dot\alpha}\bar\theta_{\dot\beta} = -\frac{1}{2}\epsilon_{\dot\alpha\dot\beta}(\bar\theta\bar\theta) \qquad (1)$$

$$(\theta\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\theta) = \frac{1}{2}g^{\mu\nu}(\theta\theta)(\bar\theta\bar\theta)$$

Proofs

$$(\theta\theta): \theta^\alpha\theta_\alpha = \theta^\alpha\epsilon_{\alpha\beta}\theta^\beta = -\theta^1\theta^2 + \theta^2\theta^1 = 2\theta^2\theta^1 = -2\theta^1\theta^2$$

$$(\theta\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\theta) = \frac{1}{2}\left[(\theta\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\theta) + (\theta\sigma^\nu\bar\theta)(\theta\sigma^\mu\bar\theta)\right] =$$

$$= -\frac{1}{2}\left[(\theta\sigma^\mu\bar\theta)(\bar\theta\tilde\sigma^\nu\theta) + (\theta\sigma^\nu\bar\theta)(\bar\theta\tilde\sigma^\mu\theta)\right] =$$

$$= -\frac{1}{2}\left[\theta^\alpha(\sigma^\mu)_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\bar\theta_{\dot\beta}(\tilde\sigma^\nu)^{\dot\beta\beta}\theta_\beta + \theta^\alpha(\sigma^\nu)_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\bar\theta_{\dot\beta}(\tilde\sigma^\mu)^{\dot\beta\beta}\theta_\beta\right] =$$

$$= \frac{1}{4}(\bar\theta\bar\theta)\theta^\alpha\left[(\sigma^\mu)_{\alpha\dot\alpha}\epsilon^{\dot\alpha}{}_{\dot\beta}(\tilde\sigma^\nu)^{\dot\beta\beta} + (\sigma^\nu)_{\alpha\dot\alpha}\epsilon^{\dot\alpha}{}_{\dot\beta}(\tilde\sigma^\mu)^{\dot\beta\beta}\right]\theta_\beta =$$

$$= \frac{1}{4}(\bar\theta\bar\theta)\theta^\alpha\, 2\,\delta_\alpha{}^\beta g^{\mu\nu}\theta_\beta = \frac{1}{2}g^{\mu\nu}(\bar\theta\theta)(\theta\theta)$$

iv) Similarly

$$(\theta\phi)(\theta\psi) = -\frac{1}{2}(\phi\psi)(\theta\theta)$$
$$(\bar\theta\bar\phi)(\bar\theta\bar\psi) = -\frac{1}{2}(\bar\phi\bar\psi)(\bar\theta\bar\theta) \qquad (2)$$

Proof

$$(\theta\phi)(\theta\psi) = \theta^\alpha\phi_\alpha\theta^\beta\psi_\beta = -\theta^\alpha\theta^\beta\phi_\alpha\psi_\beta = \frac{1}{2}(\theta\theta)\epsilon^{\alpha\beta}\phi_\alpha\psi_\beta =$$

$$= \frac{1}{2}(\theta\theta)\phi_\alpha\psi^\alpha = -\frac{1}{2}(\theta\theta)(\phi^\alpha\psi_\alpha) = -\frac{1}{2}(\theta\theta)(\phi\psi)$$

Let us now consider the group formed by Poincaré and supersymmetry transformations. In the same way that we use the parameters $\Lambda^\mu{}_\nu$ and a^μ to characterize the Poincaré transformations we will use the anticommuting parameters ξ and $\bar\xi$ to characterize the supersymmetric ones. Then we can define the action of the group as *

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$$R^\mu \equiv + i\,\theta\sigma^\mu\bar\xi - i\,\xi\sigma^\mu\bar\theta = - i\,\bar\xi\tilde\sigma^\mu\theta - i\,\xi\sigma^\mu\bar\theta$$

$$= - i\,\bar\xi_{\dot\alpha}[\tilde\sigma^\mu]^{\dot\alpha\beta}\theta_\beta - i\,\xi^\alpha[\sigma^\mu]_{\alpha\dot\beta}\bar\theta^{\dot\beta}$$

$$= - i\,[\xi^\alpha\ ,\ \bar\xi_{\dot\alpha}]\begin{vmatrix} 0 & [\sigma^\mu]_{\alpha\dot\beta} \\ [\tilde\sigma^\mu]^{\dot\alpha\beta} & \end{vmatrix}\begin{vmatrix}\theta_\beta \\ \bar\theta^{\dot\beta}\end{vmatrix} = - i\,\bar\xi\,\gamma^\mu\,\theta$$

Ref. J. WESS and B. ZUMINO. Nucl. Phys. B70, 39 (1974)

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Supersymmetry is a relativistic symmetry of lagrangian field theory which relates particle fields of different statistics. The final goal of supersymmetry practitioners is to have a unified theory encompassing all fundamental forces. Why supersymmetry in particle physics?

1) The only non-trivial extension of the space-time Poincaré symmetry which allows particle multiplets with different spins, statistics and internal quantum numbers

2) The symmetry between bosons and fermions implies a softening of quantum divergences because of mutual cancellations between boson and fermion loops. Therefore supersymmetry not only survives quantization but improves the ultraviolet behavior of relativistic quantum field theory

3) It provides a "raison d'être" for elementary scalar fields which are important in ordinary gauge theories in order to trigger the spontaneous symmetry breaking of the gauge symmetry.

4) It unifies "matter" with radiation (gauge vector particles) to the extend that gauge particles (including the graviton) necessarily have their associated fermionic partners (gluinos, photinos, gravitinos, etc).

5) It may ultimately leads to the unification of all fundamental interactions in the framework of the so called "extended supergravity theories" where all elementary fields, including the graviton, sit in a single irreducible multiplet

6) In the context of local supersymmetry (supergravity) where gravitational interactions are taken into account one can envisage a finite theory of gravity. In the most symmetric supergravity theories matter is unavoidably unified with

$$x^\mu \xrightarrow{(a,\Lambda,\xi,\bar\xi)} x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu - i\Lambda^\mu{}_\nu \theta\sigma^\nu\bar\xi + i\Lambda^\mu{}_\nu \xi\sigma^\nu\bar\theta$$

$$\theta^\alpha \longrightarrow \theta'^\alpha = [\theta^\beta + \xi^\beta][M^{-1}(\Lambda)]_\beta{}^\alpha \qquad (1)$$

$$\bar\theta^{\dot\alpha} \longrightarrow \bar\theta'^{\dot\alpha} = [M^+(\Lambda)]^{-1\,\dot\alpha}{}_{\dot\beta}\,[\bar\theta^{\dot\beta} + \bar\xi^{\dot\beta}]$$

Now let us consider the joint action of two transformations

$$x^\mu \xrightarrow{(a_1,\Lambda_1,\xi_1,\bar\xi_1)} x'^\mu = \Lambda_1{}^\mu{}_\nu x^\nu + a_1^\mu - i\Lambda_1{}^\mu{}_\nu \theta\sigma^\nu\bar\xi_1 + i\Lambda_1{}^\mu{}_\nu \xi_1\sigma^\nu\bar\theta$$

$$\theta^\alpha \longrightarrow \theta'^\alpha = [\theta^\beta + \xi_1{}^\beta][M^{-1}(\Lambda_1)]_\beta{}^\alpha$$

$$\bar\theta^{\dot\alpha} \longrightarrow \bar\theta'^{\dot\alpha} = [M^+(\Lambda_1)]^{-1\,\dot\alpha}{}_{\dot\beta}\,[\bar\theta^{\dot\beta} + \bar\xi_1{}^{\dot\beta}]$$

$$x'^\mu \xrightarrow{(a_2,\Lambda_2,\xi_2,\bar\xi_2)} x''^\mu = \Lambda_2{}^\mu{}_\nu x'^\nu + a_2^\mu - i\Lambda_2{}^\mu{}_\nu \theta'\sigma^\nu\bar\xi_2 + i\Lambda_2{}^\mu{}_\nu \xi_2\sigma^\nu\bar\theta'$$

$$\theta'^\alpha \longrightarrow \theta''^\alpha = [\theta'^\beta + \xi_2{}^\beta][M^{-1}(\Lambda_2)]_\beta{}^\alpha$$

$$\bar\theta'^{\dot\alpha} \longrightarrow \bar\theta''^{\dot\alpha} = [M^+(\Lambda_2)]^{-1\,\dot\alpha}{}_{\dot\beta}\,[\bar\theta'^{\dot\beta} + \bar\xi_2{}^{\dot\beta}]$$

Then

$$x''^\mu = \Lambda_2{}^\mu{}_\nu\,[\Lambda_1{}^\nu{}_\lambda x^\lambda + a_1^\nu - i\Lambda_1{}^\nu{}_\lambda \theta\sigma^\lambda\bar\xi_1 + i\Lambda_1{}^\nu{}_\lambda \xi_1\sigma^\lambda\bar\theta] + a_2^\mu$$

$$- i\Lambda_2{}^\mu{}_\nu\,[\theta+\xi_1]\,M^{-1}(\Lambda_1)\,\sigma^\nu\bar\xi_2 + i\Lambda_2{}^\mu{}_\nu\,\xi_2\sigma^\nu[M^+(\Lambda_1)]^{-1}(\bar\theta+\bar\xi_1) \quad \Rightarrow$$

$$x''^\mu = (\Lambda_2\Lambda_1)^\mu{}_\nu x^\nu + \Lambda_2{}^\mu{}_\nu a_1^\nu + a_2^\mu + i(\Lambda_2\Lambda_1)^\mu{}_\nu\,\xi_1\sigma^\nu M^+(\Lambda_1)\bar\xi_2$$

$$+ i(\Lambda_2\Lambda_1)^\mu{}_\nu\,\xi_2 M(\Lambda_1)\sigma^\nu\bar\xi_1 - i(\Lambda_2\Lambda_1)^\mu{}_\nu\,\theta\sigma^\nu[\bar\xi_1 + M^+(\Lambda_1)\bar\xi_2]$$

$$+ i(\Lambda_2\Lambda_1)^\mu{}_\nu\,[\xi_1 + \xi_2 M(\Lambda_1)]\sigma^\nu\bar\theta \qquad (2)$$

$$\theta'' = [\theta + \xi_1 + \xi_2 M(\Lambda_1)]\,M(\Lambda_2\Lambda_1)^{-1}$$

$$\bar\theta'' = [M^+(\Lambda_2\Lambda_1)]^{-1}\,[\bar\theta + \bar\xi_1 + M^+(\Lambda_1)\bar\xi_2]$$

If we denote by $U(a,\Lambda,\xi,\bar\xi)$ the unitary operator that implements the transformation (1), then we must have, at least locally, the following relation gravity. This means that all interactions are related by symmetry operations. In the maximally extended supergravity theory finiteness of the S-matrix elements has been proven up to seven loops.

7) The only present evidence of supersymmetry is a "theoretical one". It comes from the hierarchy problem of ordinary non-gravitational gauge interactions in which there is no natural explanation why the weak interactions scale $G_F^{-1/2}$ is so different from the ultraviolet natural cutoff of the theory i.e. $M_X \simeq 10^{15}-10^{16}$ GeV or $M_{Planck} \simeq 10^{19}$ GeV. Supersymmetry gives a possible solution to this problem because of the existence of non-renormalization theorems. If scalar fields are massless in the classical lagrangian, they will stay massless to any finite order of perturbation theory if supersymmetry is unbroken. If supersymmetry is broken they will acquire masses of the order of the supersymmetry breaking parameter. In fact in renormalizable theories scalar masses are not protected by any symmetry argument. So if the bare mass scale is zero and if the theory has a natural scale M, then the radiative corrections to the scalar masses will naturally be of order M.

Historical References

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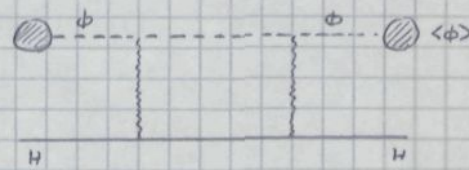
Gauge hierarchy problem

E. GILDENER Phys. Rev. D14, 1667 (1976)

In the context of grand unification there are two widely different mass scales: the $SU(2)\otimes U(1)$ breaking scale $M_W \simeq 10^2$ GeV and the GUT breaking scale $M_X \simeq 10^{15}$ GeV. In both cases the symmetry is broken by Higgs scalars with vacuum expectation values (v.e.v) $\langle H\rangle \sim 10^2$ GeV and $\langle\phi\rangle \sim 10^{15}$ GeV respectively. The scalar potential of the usual Higgs doublet is

$$V(H) = -\mu^2 |H|^2 + \lambda |H|^4 \quad\Rightarrow\quad \langle H\rangle = (\mu^2/2\lambda)^{1/2} \qquad (1)$$

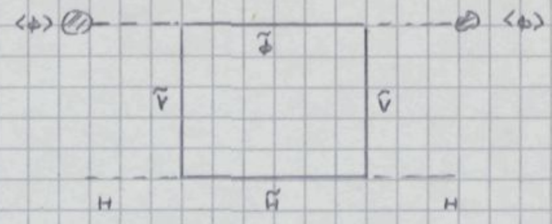
so that one obtains $M_W = (g/2)(\mu^2/\lambda)^{1/2}$. One has then to chose $|\mu| \sim 10^2$ GeV in order to get the experimental value for M_W. However, in the presence of a large mass scale like $\langle\phi\rangle \sim 10^{15}$ GeV, it is very difficult to keep a small value for $|\mu| \sim 10^2$ GeV. This is because the scalar potential in (1) has radiative corrections like the one shown in the figure which induce a large mass term for the Weinberg-Salam Higgs H: $m_H^2 \sim \langle\phi\rangle^2 \sim (10^{15}\,\mathrm{GeV})^2$. To be sure that $|\mu|$ is small enough we have to calculate the radiative corrections to the



scalar potential (1) up to several orders in perturbation theory and then fine-tune $|\mu| \sim 10^2$ GeV in order to get the desired value for M_W. This is very ugly unless there is a symmetry which protects the scalar H from adquiring a large mass. Thus the hierarchy problem is the problem of protecting a scalar field H from getting a huge mass $\sim M_X$ in a field theory where M_X is the natural scale for any mass parameter.

Note that a similar problem does not exist for fermions. In the standard model the fermion masses are forbidden by the $SU(2)\otimes U(1)$ gauge symmetry since the left-handed fermions transform as $SU(2)$ doublets and the right-handed ones like singlets. An explicit mass term would violate the $SU(2)\otimes U(1)$ symmetry. This is why quarks and leptons are light compared to M_X. They cannot acquire a mass until the $SU(2)\otimes U(1)$ symmetry is broken. One usually refers to this fact by saying that the quark-lepton content of the $SU(2)\otimes U(1)$ model is chiral. One cannot play this kind of game with scalars. Given a scalar φ which transforms as any representation of the gauge group one can always form a mass term which is gauge invariant $m^2\varphi^*\varphi$. Thus one cannot forbid a scalar mass on the basis of a gauge theory.

Supersymmetry is an exception to that rule. It introduces a concept of chirality for scalars by asigning a fermionic bispinor of definite handedness to each complex scalar. This is the simplest version of supersymmetry ($N=1$) in which any complex scalar has a fermionic partner. As we will see, this symmetry between bosons and fermions leads to cancellations which improve the ultraviolet behaviour of Green's functions. Thus for example, the contribution to the mass renormalization of the scalar H coming from the last figure (and similar other diagrams) gets exactly cancelled by the diagram in this figure in which supersymmetric partners of ϕ and of the gauge bosons V (denoted with a hat) circulate in the loop. The cancelation comes about because the coupling constants in both diagrams are equal and because of the relative minus sign of the fermionic loop.



We should nevertheless remark that SUSY by itself does not solve the gauge hierarchy problem. An explanation why the Weinberg-Salam Higgs doublet is practically massless compared to M_X is still lacking. SUSY just tells us that if we find a reason why the W-S doublet is light, its lightness will be protected from radiative corrections. This is quite a lot though. In order to have a consistent hierarchy in a non-SUSY GUT one has to calculate the effective scalar potential up to $(\alpha/\pi)^5 \sim 10^{-14}$ and the fine-tune the Higgs mass $|\mu| \sim 10^2$ GeV

$$U(a_2^\mu, \Lambda_2, \xi_2^\alpha, \bar\xi_2^{\dot\alpha})\ U(a_1^\mu, \Lambda_1, \xi_1^\alpha, \bar\xi_1^{\dot\alpha}) = \tag{1}$$

$$= U\big((\Lambda_2 a_1 + a_2)^\mu - i(\Lambda_2\Lambda_1)^\mu{}_\nu [\xi_1 \sigma^\nu M^+(\Lambda_1)\bar\xi_2 - \xi_2 M(\Lambda_1)\sigma^\nu \bar\xi_1],$$

$$\Lambda_2\Lambda_1,\ [\xi_1 + \xi_2 M(\Lambda_1)]^\alpha,\ [\bar\xi_1 + M^+(\Lambda_1)\bar\xi_2]^{\dot\alpha}\big)$$

For infinitesimal transformations we can write

$$U(a, \Lambda, \xi, \bar\xi) \equiv I + i a_\mu P^\mu - \frac{i}{2} M^{\mu\nu}\omega_{\mu\nu} + i(\xi Q) + i(\bar\xi \bar Q) \tag{2}$$

where $(\xi Q) \equiv \xi^\alpha Q_\alpha$, $(\bar\xi\bar Q) = \bar\xi_{\dot\alpha}\bar Q^{\dot\alpha}$. P^μ, $M^{\mu\nu}$, Q_α and $\bar Q^{\dot\alpha}$ are the generators.

Let us now try to derive the commutation rules for the generators. Let us start with

$$U(0, I, -\xi_2, -\bar\xi_2)\ U(0, I, -\xi_1, -\bar\xi_1)\ U(0, I, \xi_2, \bar\xi_2)\ U(0, I, \xi_1, \bar\xi_1) =$$

$$= U(-2i\,\xi_1 \sigma^\mu \bar\xi_2 + 2i\,\xi_2\sigma^\mu\bar\xi_1,\ I,\ 0,\ 0) \tag{3}$$

Hence

$$[I - i(\xi_2 Q) - i(\bar\xi_2\bar Q)]\,[I - i(\xi_1 Q) - i(\bar\xi_1 \bar Q)]\,[I + i(\xi_2 Q) + i(\bar\xi_2\bar Q)]$$

$$[I + i(\xi_1 Q) + i(\bar\xi_1 \bar Q)] = I - i\,[2i\,\xi_1\sigma^\mu\bar\xi_2 - 2i\,\xi_2\sigma^\mu\bar\xi_1]\,P_\mu \Rightarrow$$

$$\Rightarrow I - [(\xi_2 Q), (\bar\xi_1\bar Q)] + [(\bar\xi_1\bar Q), (\bar\xi_2\bar Q)] + [(\xi_1 Q), (\xi_2 Q)]$$

$$- [(\bar\xi_2\bar Q), (\xi_1 Q)] = I + 2(\xi_1\sigma^\mu\bar\xi_2)P_\mu - 2(\xi_2\sigma^\mu\bar\xi_1)P_\mu$$

i.e.

$$[(\xi_2 Q), (\bar\xi_1\bar Q)] = +2(\xi_2\sigma^\mu\bar\xi_1)P_\mu \tag{4}$$

$$[(\xi_2 Q), (\xi_1 Q)] = [(\bar\xi_2\bar Q), (\bar\xi_1\bar Q)] = 0$$

Similarly

$$U(-a, I, 0, 0)\ U(0, I, -\xi, -\bar\xi)\ U(a, I, 0, 0)\ U(0, I, \xi, \bar\xi) = U(0, I, 0, 0) \tag{5}$$

implies

$$[(\xi Q),\ a_\mu P^\mu] + [(\bar\xi\bar Q),\ a_\mu P^\mu] = 0$$

i.e.

$$[(\xi Q),\ P^\mu] = [(\bar\xi\bar Q),\ P^\mu] = 0 \qquad (1)$$

Also

$$U(0, I, -\xi, 0)\, U(0, \Lambda^{-1}, 0, 0)\, U(0, I, \xi, 0)\, U(0, \Lambda, 0, 0) = U(0, I, \xi M(\Lambda) - \xi, 0)$$

$$U(0, I, 0, -\bar\xi)\, U(0, \Lambda^{-1}, 0, 0)\, U(0, I, 0, \bar\xi)\, U(0, \Lambda, 0, 0) = U(0, I, 0, M^+(\Lambda)\bar\xi - \bar\xi)$$

$\Rightarrow$

$$I + \frac{1}{2}[(\xi Q),\ M^{\mu\nu}\omega_{\mu\nu}] = I + \frac{i}{4}\,\xi\,\sigma^\mu\tilde\sigma^\nu Q\,\omega_{\mu\nu}$$

$$I + \frac{1}{2}[(\bar\xi\bar Q),\ M^{\mu\nu}\omega_{\mu\nu}] = I + \frac{i}{4}\,\bar Q\,\tilde\sigma^\mu\sigma^\nu\bar\xi\,\omega_{\mu\nu}$$

Hence

$$[(\xi Q),\ M^{\mu\nu}] = \frac{i}{4}\,\xi\,(\sigma^\mu\tilde\sigma^\nu - \sigma^\nu\tilde\sigma^\mu)\,Q$$

$$[(\bar\xi\bar Q),\ M^{\mu\nu}] = -\frac{i}{4}\,\bar Q\,(\tilde\sigma^\mu\sigma^\nu - \tilde\sigma^\nu\sigma^\mu)\,\bar\xi \qquad (2)$$

and this with the usual commutation relations of P^μ and $M^{\mu\nu}$ among themselves constitute the desired commutation relations. They can also be written as *,

$$[P^\mu, P^\nu] = 0$$

$$[M^{\lambda\rho},\ P^\mu] = -i(g^{\lambda\mu}P^\rho - g^{\rho\mu}P^\lambda)$$

$$[M^{\lambda\rho},\ M^{\mu\nu}] = -i\,(g^{\lambda\mu}M^{\rho\nu} + g^{\rho\nu}M^{\lambda\mu} - g^{\rho\mu}M^{\lambda\nu} - g^{\lambda\nu}M^{\rho\mu})$$

$$\{Q_\alpha,\ \bar Q_{\dot\beta}\} = +2\,[\sigma^\mu]_{\alpha\dot\beta}P_\mu, \qquad \{Q_\alpha, Q_\beta\} = \{\bar Q_{\dot\alpha}, \bar Q_{\dot\beta}\} = 0 \qquad (3)$$

$$[Q_\alpha,\ P^\mu] = [\bar Q_{\dot\alpha},\ P^\mu] = 0$$

$$[Q_\alpha,\ M^{\mu\nu}] = \frac{i}{4}\,[\sigma^\mu\tilde\sigma^\nu - \sigma^\nu\tilde\sigma^\mu]_\alpha{}^\beta\, Q_\beta$$

$$[\bar Q_{\dot\alpha},\ M^{\mu\nu}] = -\frac{i}{4}\,[\tilde\sigma^\mu\sigma^\nu - \tilde\sigma^\nu\sigma^\mu]^{\dot\beta}{}_{\dot\alpha}\,\bar Q_{\dot\beta}$$

* Let us introduce $S \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} Q_1 \\ Q_2 \\ \bar{Q}^{\dot{1}} \\ \bar{Q}^{\dot{2}} \end{pmatrix}$ as well as $S^c \equiv \mathcal{C}\,\bar{S}^T = \frac{i}{\sqrt{2}} \begin{pmatrix} Q^2 \\ -Q^1 \\ -\bar{Q}_{\dot{2}} \\ \bar{Q}_{\dot{1}} \end{pmatrix}$

where we have used the chiral representation of the Dirac matrices with $\mathcal{C} \equiv \gamma^0 \gamma^2$. Since

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2 \begin{pmatrix} P_0 + P_3 & P_1 - i P_2 \\ P_1 + i P_2 & P_0 - P_3 \end{pmatrix} \qquad \{\bar{Q}^{\dot{\beta}}, Q^\alpha\} = 2 \begin{pmatrix} P_0 - P_3 & -P_1 + i P_2 \\ -P_1 - i P_2 & P_0 + P_3 \end{pmatrix}$$

we obtain immediately

$$\{S_\alpha, (S^c)_\beta\} = i \begin{pmatrix} 0 & 0 & -P_1 + i P_2 & P_0 + P_3 \\ 0 & 0 & -P_0 + P_3 & P_1 + i P_2 \\ -P_1 + i P_2 & -P_0 + P_3 & 0 & 0 \\ P_0 + P_3 & P_1 + i P_2 & 0 & 0 \end{pmatrix}$$

and this is equivalent to write

$$\{S_\alpha, (S^c)_\beta\} = -\gamma^\mu \mathcal{C}\, P_\mu$$

☒ R. Haag, J. Lopuszanski and M. Sohnius [Nucl. Phys. B88, 257 (1975)] have shown that the supersymmetry algebras, with the possible extensions to include central charges, are the only graded Lie algebras of symmetries of the S-matrix that are consistent with relativistic quantum field theory

A scalar superfield is a scalar function of x, θ and $\bar{\theta}$. We can always write it as

$$F(x,\theta,\bar{\theta}) = f(x) + [\theta\phi(x)] + [\bar{\theta}\bar{\chi}(x)] + [\theta\theta]\, m(x) + [\bar{\theta}\bar{\theta}]\, m(x) +$$

$$+ [\theta\sigma^\mu\bar{\theta}]\, v_\mu(x) + [\theta\theta][\bar{\theta}\bar{\lambda}(x)] + [\bar{\theta}\bar{\theta}][\theta\psi(x)] +$$

$$+ [\theta\theta][\bar{\theta}\bar{\theta}]\, d(x) \qquad (1)$$

The transformation law of the superfield under supersymmetry transformations is

$$\delta F(x,\theta,\bar{\theta}) \equiv F[x^\mu + i\theta\sigma^\mu\bar{\xi} + i\xi\sigma^\mu\bar{\theta},\ \theta+\xi,\ \bar{\theta}+\bar{\xi}] - F[x^\mu,\theta,\bar{\theta}] =$$

$$= \frac{\partial F}{\partial x^\mu}[-i\theta\sigma^\mu\bar{\xi} + i\xi\sigma^\mu\bar{\theta}] + \xi^\alpha\frac{\partial F}{\partial\theta^\alpha} + \bar{\xi}_{\dot{\alpha}}\frac{\partial F}{\partial\bar{\theta}_{\dot{\alpha}}} =$$

$$= \delta f(x) + [\theta\,\delta\phi(x)] + [\bar{\theta}\,\delta\bar{\chi}(x)] + [\theta\theta]\,\delta m(x) + [\bar{\theta}\bar{\theta}]\,\delta m(x) + [\theta\sigma^\mu\bar{\theta}]\,\delta v_\mu(x)$$

$$+ [\theta\theta][\bar{\theta}\,\delta\bar{\lambda}(x)] + [\bar{\theta}\bar{\theta}][\theta\,\delta\psi(x)] + [\theta\theta][\bar{\theta}\bar{\theta}]\,\delta d(x) \qquad (2)$$

Then

$$\delta F(x,\theta,\bar{\theta}) = -\partial_\mu f(x)[i\theta\sigma^\mu\bar{\xi} - i\xi\sigma^\mu\bar{\theta}] - [\theta\partial_\mu\phi(x)][i\theta\sigma^\mu\bar{\xi} - i\xi\sigma^\mu\bar{\theta}]$$

$$-[\bar{\theta}\partial_\mu\bar{\chi}(x)][i\theta\sigma^\mu\bar{\xi} - i\xi\sigma^\mu\bar{\theta}] - [\theta\theta]\partial_\mu m(x)\,[i\theta\sigma^\mu\bar{\xi} - i\xi\sigma^\mu\bar{\theta}] +$$

$$-[\bar{\theta}\bar{\theta}]\partial_\mu m(x)\,[i\theta\sigma^\mu\bar{\xi} - i\xi\sigma^\mu\bar{\theta}] - [\theta\sigma^\nu\bar{\theta}]\partial_\mu v_\nu(x)\,[i\theta\sigma^\mu\bar{\xi} - i\xi\sigma^\mu\bar{\theta}]$$

$$-[\theta\theta][\bar{\theta}\partial_\mu\bar{\lambda}(x)][i\theta\sigma^\mu\bar{\xi} - i\xi\sigma^\mu\bar{\theta}] - [\bar{\theta}\bar{\theta}][\theta\partial_\mu\psi(x)][i\theta\sigma^\mu\bar{\xi} - i\xi\sigma^\mu\bar{\theta}]$$

$$+[\theta\theta][\bar{\theta}\bar{\theta}]\partial_\mu d(x)\,[i\theta\sigma^\mu\bar{\xi} - i\xi\sigma^\mu\bar{\theta}] + [\xi\phi(x)] + [\bar{\xi}\bar{\chi}(x)] + 2[\theta\xi]\, m(x)$$

$$+ 2[\bar{\theta}\bar{\xi}]\, m(x) + [\xi\sigma^\mu\bar{\theta}]\, v_\mu(x) + [\theta\sigma^\mu\bar{\xi}]\, v_\mu(x) + 2[\theta\xi][\bar{\theta}\bar{\lambda}(x)]$$

$$+ [\theta\theta][\bar{\xi}\bar{\lambda}(x)] + 2[\bar{\theta}\bar{\xi}][\theta\psi(x)] + [\bar{\theta}\bar{\theta}][\xi\psi(x)] + 2[\theta\xi][\bar{\theta}\bar{\theta}]\, d(x)$$

$$+ 2[\theta\theta][\bar{\theta}\bar{\xi}]\, d(x)$$

and now using (2) we obtain immediately

$$\delta f(x) + [\theta\,\delta\phi(x)] + [\bar\theta\,\delta\bar\chi(x)] + [\theta\theta]\,\delta m(x) + [\bar\theta\bar\theta]\,\delta m(x) + [\theta\sigma^\mu\bar\theta]\,\delta v_\mu(x)$$

$$+ [\theta\theta][\bar\theta\,\delta\bar\lambda(x)] + [\bar\theta\bar\theta][\theta\,\delta\psi(x)] + [\theta\theta][\bar\theta\bar\theta]\,\delta d(x) =$$

$$= [\xi\phi(x)] + [\bar\xi\bar\chi(x)] + [i\theta\sigma^\mu\bar\xi]\,\partial_\mu f(x) + 2[\theta\xi]\,m(x) + [\theta\sigma^\mu\bar\xi]\,v_\mu(x)$$

$$+ i[\xi\sigma^\mu\bar\theta]\,\partial_\mu f(x) + 2[\bar\theta\bar\xi]\,m(x) + [\xi\sigma^\mu\bar\theta]\,v_\mu(x) + i[\theta\partial_\mu\phi(x)][\theta\sigma^\mu\bar\xi]$$

$$+ [\theta\theta][\bar\xi\bar\lambda(x)] + i[\bar\theta\partial_\mu\bar\chi(x)][\xi\sigma^\mu\bar\theta] + [\bar\theta\bar\theta][\xi\psi(x)] + i[\theta\partial_\mu\phi(x)][\xi\sigma^\mu\bar\theta]$$

$$+ i[\bar\theta\partial_\mu\bar\chi(x)][\theta\sigma^\mu\bar\xi] + 2[\theta\xi][\bar\theta\bar\lambda(x)] + 2[\bar\theta\bar\xi][\theta\psi(x)]$$

$$+ i[\theta\theta][\xi\sigma^\mu\bar\theta]\,\partial_\mu m(x) - i[\theta\sigma^\nu\bar\theta][\theta\sigma^\mu\bar\xi]\,\partial_\mu v_\nu(x) + 2[\theta\theta][\bar\theta\bar\xi]\,d(x)$$

$$- i[\bar\theta\bar\theta][\theta\sigma^\mu\bar\xi]\,\partial_\mu m(x) + i[\theta\sigma^\nu\bar\theta][\xi\sigma^\mu\bar\theta]\,\partial_\mu v_\nu(x) + 2[\theta\xi][\bar\theta\bar\theta]\,d(x)$$

$$+ i[\theta\theta][\bar\theta\partial_\mu\bar\lambda(x)][\xi\sigma^\mu\bar\theta] - i[\bar\theta\bar\theta][\theta\partial_\mu\psi(x)][\theta\sigma^\mu\bar\xi]$$

But

$$[\xi\sigma^\mu\bar\theta] = -[\bar\theta\tilde\sigma^\mu\xi]$$

$$[\theta\partial_\mu\phi(x)][\theta\sigma^\mu\bar\xi] = -\frac{1}{2}[\theta\theta][\partial_\mu\phi(x)\sigma^\mu\bar\xi]$$

$$[\bar\theta\partial_\mu\bar\chi(x)][\xi\sigma^\mu\bar\theta] = -\frac{1}{2}[\bar\theta\bar\theta][\xi\sigma^\mu\partial_\mu\bar\chi]$$

$$[\theta\partial_\mu\phi(x)][\xi\sigma^\mu\bar\theta] = -\frac{1}{2}[\theta\sigma^\mu\bar\theta][\partial_\lambda\phi(x)\sigma_\mu\tilde\sigma^\lambda\xi]$$

$$[\bar\theta\partial_\mu\bar\chi(x)][\theta\sigma^\mu\bar\xi] = -\frac{1}{2}[\theta\sigma^\mu\bar\theta][\bar\xi\tilde\sigma^\lambda\sigma_\mu\partial_\lambda\bar\chi(x)]$$

$$[\theta\xi][\bar\theta\bar\lambda(x)] = +\frac{1}{2}[\theta\sigma^\mu\bar\theta][\xi\sigma_\mu\bar\lambda(x)]$$

$$[\bar\theta\bar\xi][\theta\psi(x)] = +\frac{1}{2}[\theta\sigma^\mu\bar\theta][\psi(x)\sigma_\mu\bar\xi]$$

$$[\theta\theta][\xi\sigma^\mu\bar\theta] = -[\theta\theta][\bar\theta\tilde\sigma^\mu\xi]$$

$$[\theta\sigma^\nu\bar\theta][\theta\sigma^\mu\bar\xi] = +\frac{1}{2}[\theta\theta][\bar\theta\tilde\sigma^\nu\sigma^\mu\bar\xi]$$

$$[\theta\sigma^\nu\bar\theta][\xi\sigma^\mu\bar\theta] = +\frac{1}{2}[\bar\theta\bar\theta][\theta\sigma^\nu\tilde\sigma^\mu\xi]$$

$$[\theta\theta][\bar\theta\partial_\mu\bar\lambda(x)][\xi\sigma^\mu\bar\theta] = -\frac{1}{2}[\theta\theta][\bar\theta\bar\theta][\xi\sigma^\mu\partial_\mu\bar\lambda(x)]$$

$$[\bar\theta\bar\theta][\theta\partial_\mu\psi(x)][\theta\sigma^\mu\bar\xi] = -\frac{1}{2}[\theta\theta][\bar\theta\bar\theta][\partial_\mu\psi(x)\sigma^\mu\bar\xi]$$

An equalating equal powers in θ and $\bar\theta$ we get

$$\delta f(x) = [\xi \phi(x)] + [\bar{\xi} \bar{\chi}(x)]$$

$$\delta \phi(x) = -i\, \sigma^\mu \bar{\xi}\, \partial_\mu f(x) + 2 \xi\, m(x) + \sigma^\mu \bar{\xi}\, v_\mu(x)$$

$$\delta \bar{\chi}(x) = -i\, \tilde{\sigma}^\mu \xi\, \partial_\mu f(x) + 2 \bar{\xi}\, m(x) - \tilde{\sigma}^\mu \xi\, v_\mu(x)$$

$$\delta m(x) = + \frac{i}{2} [\partial_\mu \phi(x)\, \sigma^\mu \bar{\xi}] + [\bar{\xi} \bar{\lambda}(x)]$$

$$\delta m(x) = - \frac{i}{2} [\xi \sigma^\mu \partial_\mu \bar{\chi}(x)] + [\xi \psi(x)] \qquad (1)$$

$$\delta v_\mu(x) = - \frac{i}{2} [\partial_\lambda \phi(x)\, \sigma_\mu \tilde{\sigma}^\lambda \xi] + \frac{i}{2} [\bar{\xi} \tilde{\sigma}^\lambda \sigma_\mu\, \partial_\lambda \bar{\chi}(x)] +$$

$$+ [\bar{\xi} \sigma_\mu \bar{\lambda}(x)] + [\psi(x) \sigma_\mu \bar{\xi}]$$

$$\delta \bar{\lambda}(x) = -i\, \tilde{\sigma}^\mu \xi\, \partial_\mu m(x) + \frac{i}{2} \tilde{\sigma}^\nu \sigma^\mu \bar{\xi}\, \partial_\mu v_\nu(x) + 2 \bar{\xi}\, d(x)$$

$$\delta \psi(x) = -i\, \sigma^\mu \bar{\xi}\, \partial_\mu m(x) + \frac{i}{2} \sigma^\nu \tilde{\sigma}^\mu \xi\, \partial_\mu v_\nu(x) + 2 \xi\, d(x)$$

$$\delta d(x) = - \frac{i}{2} [\xi \sigma^\mu \partial_\mu \bar{\lambda}(x)] + \frac{i}{2} [\partial_\mu \psi(x)\, \sigma^\mu \bar{\xi}]$$

which are the transformation laws of the components of the scalar superfield under supersymmetry transformations.

It is easy to verify that linear combinations of superfields are again superfields. Similarly, products of superfields are again superfields. Thus we see that superfields form linear representations of the supersymmetry algebra. In general, however, the representations are highly reducible. We eliminate the extra component fields by imposing covariant constraints.

Notice that

$$\delta F(x, \bar{\theta}, \theta) = e^{-i(\xi Q + \bar{\xi} \bar{Q})} F(x, \theta, \bar{\theta}) - F(x, \theta, \bar{\theta}) =$$

$$= -i\, (\xi Q + \bar{\xi} \bar{Q})\, F(x, \theta, \bar{\theta}) \qquad (2)$$

Comparing (2) with (9.2)

$$-i\, \xi^\alpha Q_\alpha F(x, \theta, \bar{\theta}) - i\, \bar{\xi}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}} F(x, \theta, \bar{\theta}) = \xi^\alpha \left[\frac{\partial}{\partial \theta^\alpha} + i\, (\sigma^\mu)_{\alpha \dot{\beta}}\, \bar{\theta}^{\dot{\beta}} \partial_\mu \right] F(x, \theta, \bar{\theta})$$

$$+ \bar{\xi}_{\dot{\alpha}} \left[\frac{\partial}{\partial \bar{\theta}_{\dot{\alpha}}} + i (\tilde{\sigma}^\mu)^{\dot{\alpha} \beta}\, \theta_\beta\, \partial_\mu \right] F(x, \theta, \bar{\theta}) \qquad (3)$$

and therefore a representation of Q_α and $\bar{Q}^{\dot{\alpha}}$ is

$$Q_\alpha \equiv +i \frac{\partial}{\partial \theta^\alpha} - (\sigma^\mu)_{\alpha\dot{\beta}} \bar{\theta}^{\dot{\beta}} \partial_\mu \tag{1}$$

$$\bar{Q}^{\dot{\alpha}} \equiv +i \frac{\partial}{\partial \bar{\theta}_{\dot{\alpha}}} - (\tilde{\sigma}^\mu)^{\dot{\alpha}\beta} \theta_\beta \partial_\mu = +i \frac{\partial}{\partial \bar{\theta}_{\dot{\alpha}}} - \theta^\alpha (\sigma^\mu)_{\alpha\dot{\beta}} \epsilon^{\dot{\beta}\dot{\alpha}} \partial_\mu$$

and

$$\bar{Q}_{\dot{\alpha}} = -i \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} + \theta^\alpha (\sigma^\mu)_{\alpha\dot{\alpha}} \partial_\mu \tag{2}$$

where we have used the fact that

$$\epsilon^{\alpha\beta} \frac{\partial}{\partial \theta^\beta} = - \frac{\partial}{\partial \theta_\alpha} \quad , \quad \epsilon_{\dot{\alpha}\dot{\beta}} \frac{\partial}{\partial \bar{\theta}_{\dot{\beta}}} = - \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} \tag{3}$$

Let us check the commutation rules

$$Q_\alpha \bar{Q}_{\dot{\alpha}} + \bar{Q}_{\dot{\alpha}} Q_\alpha = \left\{ +i \frac{\partial}{\partial \theta^\alpha} - (\sigma^\mu)_{\alpha\dot{\beta}} \bar{\theta}^{\dot{\beta}} \partial_\mu \right\} \left\{ -i \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} + \theta^\beta (\sigma^\nu)_{\beta\dot{\alpha}} \partial_\nu \right\} +$$

$$+ \left\{ -i \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} + \theta^\beta (\sigma^\nu)_{\beta\dot{\alpha}} \partial_\nu \right\} \left\{ +i \frac{\partial}{\partial \theta^\alpha} - (\sigma^\mu)_{\alpha\dot{\beta}} \bar{\theta}^{\dot{\beta}} \partial_\mu \right\} =$$

$$= +i \frac{\partial \theta^\beta}{\partial \theta^\alpha} (\sigma^\mu)_{\beta\dot{\alpha}} \partial_\mu + i (\sigma^\mu)_{\alpha\dot{\beta}} \frac{\partial \bar{\theta}^{\dot{\beta}}}{\partial \bar{\theta}^{\dot{\alpha}}} \partial_\mu = +i (\sigma^\mu)_{\alpha\dot{\alpha}} \partial_\mu + i (\sigma^\mu)_{\alpha\dot{\alpha}} \partial_\mu =$$

$$= +2i (\sigma^\mu)_{\alpha\dot{\alpha}} \partial_\mu \equiv +2 (\sigma^\mu)_{\alpha\dot{\alpha}} P_\mu \tag{4}$$

That $P_\mu \equiv i \partial_\mu$ can be seen in the following way

$$e^{i a^\mu P_\mu} F(x) = F(x-a) \Rightarrow i a^\mu P_\mu F(x) \equiv - a^\mu \partial_\mu F(x) \Rightarrow P_\mu = i \partial_\mu \tag{5}$$

In the study of supersymmetries plays an important role the differential operators

$$D_\alpha \equiv \frac{\partial}{\partial \theta^\alpha} - i (\sigma^\mu)_{\alpha\dot{\beta}} \bar{\theta}^{\dot{\beta}} \partial_\mu$$

$$\bar{D}_{\dot{\alpha}} \equiv - \frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} + i \theta^\beta (\sigma^\mu)_{\beta\dot{\alpha}} \partial_\mu \tag{6}$$

which satisfy the commutation relations

$$\{D_\alpha, \bar{D}_{\dot\alpha}\} = +2i(\sigma^\mu)_{\alpha\dot\alpha}\partial_\mu$$

$$\{D_\alpha, D_\beta\} = \{\bar{D}_{\dot\alpha}, \bar{D}_{\dot\beta}\} = 0 \quad (1)$$

$$\{D_\alpha, Q_\beta\} = \{\bar{D}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = \{D_\alpha, \bar{Q}_{\dot\beta}\} = \{\bar{D}_{\dot\alpha}, Q_\beta\} = 0$$

Under supersymmetry transformations $D'_\alpha = D_\alpha$, $\bar{D}'_{\dot\alpha} = \bar{D}_{\dot\alpha}$ *

In order to reduce the scalar superfield we can impose the condition

$$\bar{D}_{\dot\alpha}\,\Phi(x,\theta,\bar\theta) = 0 \quad (2)$$

Let us see the general solution. Let us remember that the most general expression for the scalar superfield is given in (9.1). Hence

$$+i\theta^\beta(\sigma^\mu)_{\beta\dot\alpha}\partial_\mu f(x) + i\theta^\beta(\sigma^\mu)_{\beta\dot\alpha}\theta^\alpha\partial_\mu\phi_\alpha(x) + \bar\chi_{\dot\alpha}(x) +$$

$$-i\theta^\beta(\sigma^\mu)_{\beta\dot\alpha}\bar\theta^{\dot\beta}\partial_\mu\bar\chi_{\dot\beta} - i(\theta\theta)\,\theta^\beta(\sigma^\mu)_{\beta\dot\alpha}\partial_\mu m(x) + 2\bar\theta_{\dot\alpha}m(x)$$

$$+i(\bar\theta\bar\theta)\,\theta^\beta(\sigma^\mu)_{\beta\dot\alpha}\partial_\mu m(x) + \theta^\alpha(\sigma^\mu)_{\alpha\dot\alpha}v_\mu(x) + i\theta^\beta(\sigma^\nu)_{\beta\dot\alpha}\theta^\alpha(\sigma^\mu)_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_\nu v_\mu(x)$$

$$+(\theta\theta)\bar\lambda_{\dot\alpha}(x) + 2\bar\theta_{\dot\alpha}(\theta\psi(x)) + i(\bar\theta\bar\theta)\,\theta^\beta(\sigma^\mu)_{\beta\dot\alpha}(\theta\partial_\mu\psi(x))$$

$$+2(\theta\theta)\bar\theta_{\dot\alpha}\,d(x) = 0 \quad \Rightarrow$$

$$\bar\chi(x) \equiv 0$$

$$+i\theta^\beta(\sigma^\mu)_{\beta\dot\alpha}\partial_\mu f(x) + \theta^\alpha(\sigma^\mu)_{\alpha\dot\alpha}v_\mu(x) = 0 \quad \Rightarrow \quad v_\mu(x) \equiv -i\partial_\mu f(x)$$

$$m(x) \equiv 0$$

$$+i\theta^\beta(\sigma^\mu)_{\beta\dot\alpha}\theta^\alpha\partial_\mu\phi_\alpha(x) + (\theta\theta)\bar\lambda_{\dot\alpha}(x) = 0 \quad \Rightarrow \quad \bar\lambda_{\dot\alpha}(x) = -\frac{i}{2}(\sigma^\mu)^\alpha{}_{\dot\alpha}\partial_\mu\phi_\alpha(x)$$

$$\psi(x) \equiv 0$$

$$+i\theta^\beta(\sigma^\nu)_{\beta\dot\alpha}\theta^\alpha(\sigma^\mu)_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_\nu v_\mu(x) + 2(\theta\theta)\bar\theta_{\dot\alpha}d(x) = 0 \quad \Rightarrow \quad d(x) = -\frac{1}{4}\partial^\mu\partial_\mu f(x)$$

We can introduce

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$$D_\alpha \equiv \frac{\partial}{\partial\theta^\alpha} - i\,[\sigma^\mu]_{\alpha\dot\beta}\,\bar\theta^{\dot\beta}\,\frac{\partial}{\partial x^\mu} = \frac{\partial\theta'^\beta}{\partial\theta^\alpha}\frac{\partial}{\partial\theta'^\beta} + \frac{\partial x'^\mu}{\partial\theta^\alpha}\frac{\partial}{\partial x'^\mu} - i\,[\sigma^\mu]_{\alpha\dot\beta}\,\bar\theta^{\dot\beta}\,\frac{\partial x'^\nu}{\partial x^\mu}\frac{\partial}{\partial x'^\nu}$$

$$= \frac{\partial}{\partial\theta'^\alpha} - i\,[\sigma^\mu]_{\alpha\dot\beta}\,\bar\xi^{\dot\beta}\,\frac{\partial}{\partial x'^\mu} - i\,[\sigma^\mu]_{\alpha\dot\beta}\,\bar\theta^{\dot\beta}\,\frac{\partial}{\partial x'^\mu} = D'_\alpha$$

Ref J. WESS and B. ZUMINO Phys. Letters 59B, 52 (1975)

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$$f(x) \equiv A(x) \quad , \quad \phi(x) \equiv \sqrt{2}\,\psi(x) \quad , \quad \bar{\chi}(x) \equiv 0$$

$$m(x) = F(x) \quad , \quad n(x) = 0 \quad , \quad v_\mu(x) \equiv -i\,\partial_\mu A(x) \tag{1}$$

$$\bar{\lambda}(x) \equiv +i\frac{1}{\sqrt{2}}\,\partial_\mu\psi(x)\,\sigma^\mu \quad , \quad \psi(x) \equiv 0 \quad , \quad d(x) \equiv -\frac{1}{4}\,\partial^\mu\partial_\mu A(x)$$

and therefore the scalar superfield solution of (13.2) is

$$\Phi(x,\theta,\bar\theta) = A(x) + \sqrt{2}\,[\theta\,\psi(x)] + [\theta\theta]\,F(x) - i\,[\theta\sigma^\mu\bar\theta]\,\partial_\mu A(x)$$
$$+\frac{i}{\sqrt{2}}\,[\theta\theta]\,[\partial_\mu\psi(x)\,\sigma^\mu\bar\theta] - \frac{1}{4}\,(\theta\theta)(\bar\theta\bar\theta)\,\partial_\mu\partial^\mu A(x) \tag{2}$$

and from (11.1) we can read the transformation laws

$$\delta A(x) = +\sqrt{2}\,[\xi\,\psi(x)]$$
$$\delta\psi(x) = -\sqrt{2}\,i\,\sigma^\mu\bar\xi\,\partial_\mu A(x) + \sqrt{2}\,\xi\,F(x) \tag{3}$$
$$\delta F(x) = -i\sqrt{2}\,[\bar\xi\,\tilde\sigma^\mu\partial_\mu\psi(x)]$$

which are the desired commutation rules. Notice that the field (2) can also be written as

$$\Phi(x,\theta,\bar\theta) = A\,[x^\mu - i\theta\sigma^\mu\bar\theta] + \sqrt{2}\,\theta\,\psi\,[x^\mu - i\,\theta\sigma^\mu\bar\theta]$$
$$+ [\theta\theta]\,F\,[x^\mu - i\,\theta\sigma^\mu\bar\theta] \tag{4}$$

If we introduce $y^\mu \equiv x^\mu - i\,\theta\sigma^\mu\bar\theta$ then (12.5) can be written as

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} - 2i\,(\sigma^\mu)_{\alpha\dot\beta}\,\bar\theta^{\dot\beta}\,\frac{\partial}{\partial y^\mu}$$
$$\bar D_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} \tag{5}$$

Similarly we can define $\Phi^+(x,\theta,\bar\theta)$ which satisfies the constrain

$$D_\alpha\,\Phi^+(x,\theta,\bar\theta) = 0 \tag{6}$$

As before

$$-i(\sigma^\mu)_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_\mu f(x) + \phi_\alpha(x) - i(\sigma^\mu)_{\alpha\dot\beta}\bar\theta^{\dot\beta}(\theta\partial_\mu\phi) - i(\sigma^\mu)_{\alpha\dot\beta}\bar\theta^{\dot\beta}[\bar\theta\partial_\mu\bar\chi(x)]$$

$$+ 2\theta_\alpha m(x) - i(\sigma^\mu)_{\alpha\dot\beta}\bar\theta^{\dot\beta}(\theta\theta)\partial_\mu m(x) + \sigma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta}v_\mu(x) +$$

$$-i(\sigma^\mu)_{\alpha\dot\beta}\bar\theta^{\dot\beta}(\theta\sigma^\nu\bar\theta)\partial_\mu v_\nu(x) + 2\theta_\alpha(\bar\theta\bar\lambda) - i(\sigma^\mu)_{\alpha\dot\beta}\bar\theta^{\dot\beta}(\theta\theta)(\bar\theta\partial_\mu\bar\lambda)$$

$$+ (\bar\theta\bar\theta)\psi_\alpha + 2\theta_\alpha(\bar\theta\bar\theta)d(x) = 0 \quad \Rightarrow$$

$$\phi(x) \equiv 0$$

$$2\theta_\alpha m(x) = 0 \qquad m(x) \equiv 0$$

$$-i(\sigma^\mu)_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_\mu f(x) + \sigma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta}v_\mu(x) = 0 \quad \Rightarrow \quad v_\mu(x) \equiv +i\partial_\mu f(x)$$

$$-i(\sigma^\mu)_{\alpha\dot\beta}\bar\theta^{\dot\beta}\bar\theta_{\dot\alpha}\partial_\mu\bar\chi^{\dot\alpha}(x) + \bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha}\psi_\alpha(x) = 0 \quad \Rightarrow \quad \psi_\alpha(x) = -\frac{i}{2}(\sigma^\mu)_{\alpha\dot\alpha}\partial_\mu\bar\chi^{\dot\alpha}(x)$$

$$\bar\lambda(x) \equiv 0$$

$$-i(\sigma^\mu)_{\alpha\dot\beta}\bar\theta^{\dot\beta}\theta^\beta(\sigma^\nu)_{\beta\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu v_\nu(x) + 2\theta_\alpha(\bar\theta\bar\theta)d(x) = 0 \quad \Rightarrow \quad d(x) = -\frac{1}{4}\partial_\mu\partial^\mu f(x)$$

Now let us introduce

$$f(x) \equiv A^*(x), \qquad \phi(x) \equiv 0, \qquad \bar\chi(x) \equiv \sqrt{2}\,\bar\Psi(x), \qquad m(x) \equiv 0$$

$$m(x) \equiv F^*(x), \qquad v_\mu(x) = +i\partial_\mu A^*(x), \qquad \bar\lambda(x) \equiv 0 \tag{1}$$

$$\psi(x) = -\frac{i}{\sqrt{2}}\sigma^\mu\partial_\mu\bar\Psi(x), \qquad d(x) = -\frac{1}{4}\partial_\mu\partial^\mu A^*(x)$$

and hence

$$\Phi^+(x,\theta,\bar\theta) = A^*(x) + \sqrt{2}[\bar\theta\bar\Psi(x)] + [\bar\theta\bar\theta]F^*(x) + i[\theta\sigma^\mu\bar\theta]\partial_\mu A^*(x)$$

$$-\frac{i}{\sqrt{2}}[\bar\theta\bar\theta][\theta\sigma^\mu\partial_\mu\bar\Psi(x)] - \frac{1}{4}[\theta\theta][\bar\theta\bar\theta]\partial_\mu\partial^\mu A^*(x)$$

$$= A^*[x^\mu + i\theta\sigma^\mu\bar\theta] + \sqrt{2}[\bar\theta\bar\Psi(x^\mu + i\theta\sigma^\mu\bar\theta)] +$$

$$+ [\bar\theta\bar\theta]F^*(x^\mu + i\theta\sigma^\mu\bar\theta) \tag{2}$$

and using $y^{\mu+} \equiv x^\mu + i\theta\sigma^\mu\bar\theta$ we can write

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha}, \qquad \bar D_{\dot\alpha} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + 2i\theta^\beta(\sigma^\mu)_{\beta\dot\alpha}\frac{\partial}{\partial y^{+\mu}} \tag{3}$$

The transformation laws are

$$\delta A^*(x) = \sqrt{2}\,[\bar\xi\,\bar\psi(x)]$$

$$\delta\bar\psi(x) = -i\sqrt{2}\,\tilde\sigma^\mu\xi\,\partial_\mu A^*(x) + \sqrt{2}\,\bar\xi\,F^*(x) \qquad (1)$$

$$\delta F^*(x) = -i\sqrt{2}\,[\xi\sigma^\mu\partial_\mu\bar\psi(x)]$$

Notice that in both cases if $A(x)$ has dimension 1, then $\psi(x)$ has dimension 3/2 and $F(x)$ has dimension 2. Furthermore the F or F^+ component of a scalar superfield always transforms into a space time-derivative.

Let us now consider the product of two superfields

$$\Phi_i\Phi_j = A_iA_j + \sqrt{2}\,A_i(\theta\psi_j) + A_i(\theta\theta)F_j - iA_i(\theta\sigma^\mu\bar\theta)\partial_\mu A_j + \frac{i}{\sqrt{2}}(\theta\theta)A_i\,\partial_\mu\psi_j\sigma^\mu\bar\theta$$

$$-\frac{1}{4}(\theta\theta)(\bar\theta\bar\theta)A_i\partial_\mu\partial^\mu A_j + \sqrt{2}(\theta\psi_i)A_j + 2(\theta\psi_i)(\theta\psi_j) - i\sqrt{2}(\theta\psi_i)(\theta\sigma^\mu\bar\theta)\partial_\mu A_j$$

$$+(\theta\theta)F_iA_j - i(\theta\sigma^\mu\bar\theta)\partial_\mu A_i.A_j - i\sqrt{2}(\theta\sigma^\mu\bar\theta)\partial_\mu A_i(\theta\psi_j) - (\theta\sigma^\mu\bar\theta)\partial_\mu A_i(\theta\sigma^\nu\bar\theta)\partial_\nu A_j$$

$$+\frac{i}{\sqrt{2}}(\theta\theta)(\partial_\mu\psi_i\sigma^\mu\bar\theta)A_j - \frac{1}{4}(\theta\theta)(\bar\theta\bar\theta)\partial_\mu\partial^\mu A_i A_j \Rightarrow$$

$$\Phi_i\Phi_j = A_i(x)A_j(x) + \sqrt{2}\,\theta\,[A_i(x)\psi_j(x) + \psi_i(x)A_j(x)]$$

$$+[\theta\theta]\,[A_i(x)F_j(x) + F_i(x)A_j(x) - (\psi_i(x)\psi_j(x))]$$

$$-i[\theta\sigma^\mu\bar\theta]\,[A_i(x)\partial_\mu A_j(x) + \partial_\mu A_i(x)A_j(x)]$$

$$+\frac{i}{\sqrt{2}}[\theta\theta]\,\{A_i(x)\partial_\mu\psi_j(x)\sigma^\mu\bar\theta + \partial_\mu\psi_i(x)A_j(x)\sigma^\mu\bar\theta$$

$$+\psi_i(x)\partial_\mu A_j(x)\sigma^\mu\bar\theta + \partial_\mu A_i(x)\psi_j(x)\sigma^\mu\bar\theta\}$$

$$-\frac{1}{4}(\theta\theta)(\bar\theta\bar\theta)\,[A_i(x)\partial^\mu\partial_\mu A_j(x) + \partial_\mu\partial^\mu A_i(x)A_j(x) + 2\partial_\mu A_i(x)\partial^\mu A_j(x)]$$

$$\Phi_i(x,\theta,\bar\theta)\,\Phi_j(x,\theta,\bar\theta) = A_i(x)A_j(x) + \sqrt{2}\,\theta\,[A_i(x)\psi_j(x) + \psi_i(x)A_j(x)]$$

$$+[\theta\theta]\,[A_i(x)F_j(x) + F_i(x)A_j(x) - \psi_i(x)\psi_j(x)]$$

$$-i[\theta\sigma^\mu\bar\theta]\,\partial_\mu(A_i(x)A_j(x))$$

$$+\frac{i}{\sqrt{2}}[\theta\theta]\,\{\partial_\mu[A_i(x)\psi_j(x) + \psi_i(x)A_j(x)]\sigma^\mu\bar\theta\}$$

$$-\frac{1}{4}(\theta\theta)(\bar\theta\bar\theta)\,\partial_\mu\partial^\mu[A_i(x)A_j(x)] \qquad (2)$$

Therefore the product of two scalar superfields is again a scalar superfield. Notice that we can use the more compact notation (14.5) to obtain the same result

$$\Phi_i(x,\theta,\bar\theta)\,\Phi_j(x,\theta,\bar\theta) = A_i(y)A_j(y) + \sqrt{2}\,\theta\,[A_i(y)\psi_j(y) + \psi_i(y)A_j(y)]$$

$$+ [\theta\theta]\,[A_i(y)F_j(y) + F_i(y)A_j(y) - \psi_i(y)\psi_j(y)] \qquad (1)$$

In similar ways

$$\Phi_i^+(x,\theta,\bar\theta)\,\Phi_j^+(x,\theta,\bar\theta) = A_i^*(y^+)A_j^*(y^+) + \sqrt{2}\,[A_i^*(y^+)\bar\psi_j(y) + \bar\psi_i(y^+)A_j^*(y^+)]\,\bar\theta$$

$$+ [\bar\theta\bar\theta]\,[A_i^*(y^+)F_j^*(y^+) + F_i^*(y^+)A_j^*(y^+) - \bar\psi_i(y)\bar\psi_j(y)] \qquad (2)$$

Also

$$\Phi_i(x,\theta,\bar\theta)\,\Phi_j(x,\theta,\bar\theta)\,\Phi_k(x,\theta,\bar\theta) = A_i(y)A_j(y)A_k(y) +$$

$$+ \sqrt{2}\,\theta\,[A_i(y)\psi_j(y)A_k(y) + \psi_i(y)A_j(y)A_k(y) + A_i(x)A_j(x)\psi_k(x)]$$

$$+ [\theta\theta]\,[A_i(y)F_j(y)A_k(y) + F_i(y)A_j(y)A_k(y) + A_i(y)A_j(y)F_k(y)$$

$$- \psi_i(y)\psi_j(y)A_k(y) - A_i(x)\psi_j(y)\psi_k(y) - \psi_i(y)A_j(y)\psi_k(y)] \qquad (3)$$

Nevertheless the product $\Phi^+\Phi$ is not a scalar superfield. solution of (13.2) or (15.6)

$$\Phi_i^+\Phi_j = A_i^*A_j + \sqrt{2}\,A_i^*(\theta\psi_j) + (\theta\theta)A_i^*F_j - i(\theta\sigma^\mu\bar\theta)A_i^*\partial_\mu A_j + \frac{i}{\sqrt{2}}(\theta\theta)A_i^*(\partial_\mu\psi_j\sigma^\mu\bar\theta)$$

$$-\frac{1}{4}(\theta\theta)(\bar\theta\bar\theta)A_i^*(x)\partial_\mu\partial^\mu A_j(x) + \sqrt{2}(\bar\theta\bar\psi_i)A_j + 2(\bar\theta\bar\psi_i)(\theta\psi_j) + \sqrt{2}(\bar\theta\bar\psi_i)(\theta\theta)F_j$$

$$+ i\sqrt{2}(\bar\theta\bar\psi_i)(\theta\sigma^\mu\bar\theta)\partial_\mu A_j + i(\bar\theta\bar\psi_i)(\theta\theta)(\partial_\mu\psi_j\sigma^\mu\bar\theta) + (\bar\theta\bar\theta)F_i^*A_j + \sqrt{2}(\bar\theta\bar\theta)F_i^*(\theta\psi_i)$$

$$+ (\bar\theta\bar\theta)F_i^*(\theta\theta)F_j + i(\theta\sigma^\mu\bar\theta)\partial_\mu A_i^*A_j + i\sqrt{2}(\theta\sigma^\mu\bar\theta)\partial_\mu A_i^*(\theta\psi_j) +$$

$$+ (\theta\sigma^\mu\bar\theta)\partial_\mu A_i^*(\theta\sigma^\nu\bar\theta)\partial_\nu A_j - \frac{i}{\sqrt{2}}(\bar\theta\bar\theta)(\theta\sigma^\mu\partial_\mu\bar\psi_i)A_j(x) +$$

$$- i(\bar\theta\bar\theta)(\theta\sigma^\mu\partial_\mu\bar\psi_i)(\theta\psi_j) - \frac{1}{4}(\theta\theta)(\bar\theta\bar\theta)\partial_\mu\partial^\mu A_i^*A_j$$

Hence

$$\Phi_i^+ \Phi_j = A_i^*(x) A_j(x) + \sqrt{2} A_i^*(x) [\Theta \psi_j(x)] + \sqrt{2} [\bar\Theta \bar\psi_i(x)] A_j(x)$$

$$+ [\Theta\Theta][A_i^*(x) F_j(x)] + [\bar\Theta\bar\Theta] F_i^*(x) A_j(x) +$$

$$+ [\Theta\sigma^\mu\bar\Theta][-i A_i^*(x) \partial_\mu A_j(x) + i \partial_\mu A_i^*(x) A_j(x) + \psi_j(x) \sigma_\mu \bar\psi_i(x)]$$

$$+ [\Theta\Theta][\sqrt{2}\, \bar\Theta \bar\psi_i(x) F_j(x) - \frac{i}{\sqrt{2}} A_i^*(x) \bar\Theta \tilde\sigma^\mu \partial_\mu \psi_j(x) - \frac{i}{\sqrt{2}} \psi_j(x) \sigma^\mu \bar\Theta \partial_\mu A_i^*(x)]$$

$$+ [\bar\Theta\bar\Theta][\sqrt{2} F_i^*(x) \Theta \psi_j(x) - \frac{i}{\sqrt{2}} \Theta \sigma^\mu \partial_\mu \bar\psi_i(x) A_j(x) + \frac{i}{\sqrt{2}} \Theta \sigma^\mu \bar\psi_i(x) \partial_\mu A_j(x)]$$

$$+ [\Theta\Theta][\bar\Theta\bar\Theta][-\frac{1}{4} A_i^*(x) \partial_\mu \partial^\mu A_j(x) - \frac{1}{4} \partial_\mu \partial^\mu A_i^*(x) A_j(x) + F_i^*(x) F_j(x)$$

$$+ \frac{1}{2} \partial^\mu A_i^*(x) \partial_\mu A_j(x) + \frac{i}{2} \bar\psi_i(x) \tilde\sigma^\mu \partial_\mu \psi_j(x) + \frac{i}{2} \psi_j(x) \sigma^\mu \partial_\mu \bar\psi_i(x)] \qquad (1)$$

We are now ready to write the most general supersymmetric renormalizable Lagrangian density involving only chiral scalar superfields, i.e. solutions of (13.2) and (14.6)

$$L(x) = \Phi_i^+ \Phi_i \Big|_{(\Theta\Theta)(\bar\Theta\bar\Theta)\ \text{component}} + \Big\{ \Big[\frac{1}{2} m_{ij} \Phi_i \Phi_j + \frac{1}{3} g_{ijk} \Phi_i \Phi_j \Phi_k +$$

$$+ \lambda_i \Phi_i \Big] \Big|_{(\Theta\Theta)\ \text{component}} + \text{h.c.} \Big\} \qquad (1)$$

where m_{ij} and g_{ijk} are symmetric in their indices. Notice that this garantees that under supersymmetric transformations $L(x)$ changes only by a total derivative. Then

$$L(x) = -i \partial_\mu \bar\psi_i(x) \tilde\sigma^\mu \psi_i(x) - A_i^*(x) \partial_\mu \partial^\mu A_i(x) + F_i^*(x) F_i(x)$$

$$+ \Big\{ m_{ij} A_i(x) F_j(x) - \frac{1}{2} m_{ij} \psi_i(x) \psi_j(x) + g_{ijk} A_i(x) A_j(x) F_k(x)$$

$$- g_{ijk} \psi_i(x) \psi_j(x) A_k(x) + \lambda_i F_i(x) + \text{h.c.} \Big\} \qquad (2)$$

where we have dropped all total derivatives. The equation of motion can easily be derived. In particular

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$$L(x) = \partial^\mu A^+(x)\partial_\mu A(x) + \frac{i}{2}\bar{\Psi}(x)\tilde{\sigma}^\mu\partial_\mu\psi(x) + \frac{i}{2}\psi(x)\sigma^\mu\partial_\mu\bar{\Psi}(x) + F^+(x)F(x) + m A(x)F(x) + m A^+(x)F^+(x)$$

$$-\frac{1}{2}m\psi(x)\psi(x) - \frac{1}{2}m\bar{\Psi}(x)\bar{\Psi}(x) + g A^2(x)F(x) + g A^{+2}(x)F^+(x) - g\psi(x)\psi(x)A(x) - g\bar{\Psi}(x)\bar{\Psi}(x)A^+(x)$$

$$+\lambda F(x) + \lambda F^+(x)$$

$$F(x) + m A^+(x) + g A^{+2}(x) + \lambda = 0 \qquad F^+(x) + m A(x) + g A^2(x) + \lambda = 0 \qquad \psi(x) = \begin{vmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{vmatrix} \qquad \bar{\Psi}(x) = |\psi^\alpha, \bar{\psi}_{\dot{\alpha}}|$$

$$L(x) = \partial^\mu A^+(x)\partial_\mu A(x) + \frac{i}{2}\bar{\Psi}(x)\gamma^\mu\partial_\mu\psi(x) + F^+(x)F(x) + m A(x)F(x) + m A^+(x)F^+(x) - \frac{1}{2}m\bar{\Psi}(x)\psi(x)$$

$$+ g A^2(x)F(x) + g A^{+2}(x)F^+(x) - \frac{1}{2}g\bar{\Psi}(x)\psi(x)[A(x)+A^+(x)] - \frac{1}{2}g\bar{\Psi}(x)\gamma_5\psi(x)[A(x)-A^+(x)] + \lambda F(x) + \lambda F^+(x)$$

$$L(x) = \partial_\mu A^+(x)\partial^\mu A(x) - m^2 A^+(x)A(x) + \frac{i}{2}\bar{\Psi}(x)\gamma^\mu\partial_\mu\psi(x) - \frac{1}{2}m\bar{\Psi}(x)\psi(x)$$

$$-\lambda g A^2(x) - \lambda g A^{+2}(x) - mg A^2(x)A^+(x) - mg A^{+2}(x)A(x) - g^2 A^{+2}(x)A^2(x)$$

$$-\lambda m A(x) - \lambda m A^+(x) - \lambda^2 - \frac{1}{2}g\bar{\Psi}(x)\psi(x)[A(x)+A^+(x)] - \frac{1}{2}g\bar{\Psi}(x)\gamma_5\bar{\Psi}(x)[A(x)-A^+(x)]$$

Sf $\quad A(x) \rightarrow \frac{1}{\sqrt{2}}[A(x) + iB(x)] \qquad A^+(x) \rightarrow \frac{1}{\sqrt{2}}[A - iB(x)]$

ip. $\quad A(x) \leftarrow \frac{1}{\sqrt{2}}[A(x) + A^+(x)] \qquad B(x) \leftarrow -\frac{i}{\sqrt{2}}[A(x) - A^+(x)] \qquad g \rightarrow \sqrt{2}g$

$$L(x) = \frac{1}{2}\partial_\mu A(x)\partial^\mu A(x) - \frac{1}{2}m^2 A(x)A(x) + \frac{1}{2}\partial_\mu B(x)\partial^\mu B(x) - \frac{1}{2}m^2 B(x)B(x)$$

$$+\frac{i}{2}\bar{\Psi}(x)\gamma^\mu\partial_\mu\psi(x) - \frac{1}{2}m\bar{\Psi}(x)\psi(x) - mg A^3(x) - mg B^2(x)A(x)$$

$$-\frac{1}{2}g^2 A^4(x) - \frac{1}{2}g^2 B^4(x) - g^2 A^2(x)B^2(x) - g\bar{\Psi}(x)\psi(x)A(x) - ig\bar{\Psi}(x)\gamma_5\psi(x)B(x)$$

$$-\sqrt{2}\lambda g A^2(x) + \sqrt{2}\lambda g B^2(x) - \sqrt{2}\lambda m A(x) - \lambda^2.$$

where α is a real constant and n is called the R-character of the superfield. For the components

$$R\ :\ A(x) \longrightarrow e^{2in\alpha} A(x)$$
$$\psi(x) \longrightarrow e^{2i(n-1/2)\alpha}\psi(x) \qquad (1)$$
$$F(x) \longrightarrow e^{2i(n-1)\alpha} F(x)$$

Mass terms or potentials are R-invariant only if the R-characters of their respective superfields add up to one*

Let us consider an arbitrary superfield

$$U(x,\theta,\bar\theta) \equiv f(x) + [\theta\phi(x)] + [\bar\theta\bar\chi(x)] + [\theta\theta]\, m(x) + [\bar\theta\bar\theta]\, n(x)$$
$$+ [\theta\sigma^\mu\bar\theta]\, v_\mu(x) + [\theta\theta][\bar\theta\bar\lambda(x)] + [\bar\theta\bar\theta][\theta\psi(x)] + [\theta\theta][\bar\theta\bar\theta]\, d(x) \qquad (2)$$

Let us see that $\bar D\bar D\, U(x,\theta,\bar\theta) \equiv \bar D_{\dot\alpha}\bar D^{\dot\alpha}\, U(x,\theta,\bar\theta) = \Phi(x,\theta,\bar\theta)$ is a chiral superfield

$$\Phi(x,\theta,\bar\theta) = \epsilon^{\dot\alpha\dot\beta}\,\bar D_{\dot\alpha}\bar D_{\dot\beta}\, U(x,\theta,\bar\theta) \qquad (3)$$

$$\bar D_{\dot\beta} U(x,\theta,\bar\theta) = + i\,\theta^\beta(\sigma^\mu)_{\beta\dot\beta}\,\partial_\mu f(x) + i\,\theta^\beta(\sigma^\mu)_{\beta\dot\beta}\,(\theta\partial_\mu\phi(x))$$
$$+ \bar\chi_{\dot\beta} + i\,\theta^\beta(\sigma^\mu)_{\beta\dot\beta}\,[\bar\theta\partial_\mu\bar\chi(x)]$$
$$+ 2\bar\theta_{\dot\beta}\, n(x) + i\,\theta^\beta(\sigma^\mu)_{\beta\dot\beta}\,[\bar\theta\bar\theta]\,\partial_\mu m(x) + \theta^\alpha(\sigma^\mu)_{\alpha\dot\beta}\, v_\mu(x)$$
$$+ i\,\theta^\beta(\sigma^\mu)_{\beta\dot\beta}\,[\theta\sigma^\nu\bar\theta]\,\partial_\mu v_\nu(x) + (\theta\theta)\,\bar\lambda_{\dot\beta}(x)$$
$$+ 2\bar\theta_{\dot\beta}\,(\theta\psi(x)) + i\,\theta^\beta(\sigma^\mu)_{\beta\dot\beta}\,(\bar\theta\bar\theta)(\theta\partial_\mu\psi(x)) + 2(\theta\theta)\,\bar\theta_{\dot\beta}\, d(x)$$

$$\Phi(x,\theta,\bar\theta) = + [\theta\theta]\,\partial^\mu\partial_\mu f(x) + i[\theta\sigma^\mu\partial_\mu\bar\chi(x)] + i[\theta\sigma^\mu\partial_\mu\bar\chi(x)]$$
$$- [\theta\theta][\bar\theta\partial^\mu\partial_\mu\bar\chi(x)] - \quad - 4n(x) + 2i[\theta\sigma^\mu\bar\theta]\,\partial_\mu m(x)$$
$$+ 2i[\theta\sigma^\mu\bar\theta]\,\partial_\mu m(x) + [\theta\theta][\bar\theta\bar\theta]\,\partial^\mu\partial_\mu m(x) - i[\theta\theta]\,\partial^\mu v_\mu(x)$$
$$- i[\theta\theta]\,\partial^\mu v_\mu(x) \quad - 4[\theta\psi] + 2i[\theta\sigma^\mu\bar\theta][\theta\partial_\mu\psi(x)]$$

$$R : L \to L(x) = -i\, \partial_\mu \bar{\psi}_i(x)\, \tilde{\sigma}^\mu \psi_i(x) - A_i^*(x)\, \partial_\mu \partial^\mu A_i(x) + F_i^*(x) F_i(x)$$

$$+ e^{2i\alpha(m_i + m_j - 1)} \left[m_{ij} A_i F_j - \frac{1}{2} m_{ij} \psi_i \psi_j \right] + h.c.$$

$$+ e^{2i\alpha(m_i + m_j + m_k - 1)} \left[g_{ijk} A_i A_j F_k - g_{ijk} \psi_i \psi_j A_k \right] + \lambda_i e^{2i(m_i - 1)} F_i$$

$$+ 2i\,[\theta\sigma^\mu\bar\theta]\,[\theta\,\partial_\mu\psi(x)] - 4\,[\theta\theta]\,d(x)$$

Hence

$$\Phi(x,\theta,\bar\theta) = -4m(x) + \theta\left(-4\psi(x) + 2i\,\sigma^\mu\partial_\mu\bar\chi(x)\right)$$

$$+[\theta\theta]\left(+\partial^\mu\partial_\mu f(x) - 2i\,\partial^\mu v_\mu(x) - 4\,d(x)\right)$$

$$+i\,[\theta\sigma^\mu\bar\theta]\left(+4\,\partial_\mu m(x)\right) + i\,[\theta\theta]\,[-2\,\partial_\nu\psi + i\,\sigma^\mu\partial_\mu\partial_\nu\bar\chi(x)]\,\sigma^\nu\bar\theta$$

$$+(\theta\theta)(\bar\theta\bar\theta)\,[\partial_\mu\partial^\mu m(x)] \qquad (1)$$

and this is a chiral field.

Let us now introduce the so called vector superfields. They are scalar superfields such that

$$V(x,\theta,\bar\theta) = V^+(x,\theta,\bar\theta) \qquad (2)$$

and therefore their general expression is

$$V(x,\theta,\bar\theta) = C(x) + i\,[\theta\chi(x)] - i\,[\bar\theta\bar\chi(x)] + \frac{i}{2}\,[\theta\theta]\,[M(x) + i\,N(x)]$$

$$-\frac{i}{2}\,[\bar\theta\bar\theta]\,[M(x) - i\,N(x)] + [\theta\sigma^\mu\bar\theta]\,v_\mu(x) +$$

$$+ i\,[\theta\theta]\,[\bar\theta\,(\bar\lambda(x) + \frac{i}{2}\,\tilde\sigma^\mu\partial_\mu\chi(x))]$$

$$- i\,[\bar\theta\bar\theta]\,[\theta\,(\lambda(x) + \frac{i}{2}\,\sigma^\mu\partial_\mu\bar\chi(x))]$$

$$+\frac{1}{2}\,[\theta\theta]\,[\bar\theta\bar\theta]\,[D(x) - \frac{1}{2}\,\partial^\mu\partial_\mu C(x)] \qquad (3)$$

where all fields must be real in order to satisfy (2). The vector field $v_\mu(x)$ lends its name to the entire multiplet. We have choosen very particular combinations of fields as coefficients of the last three terms. Our choice was dictated by the hermitian field $\Phi + \Phi^+$, where Φ and Φ^+ are scalar fields (chiral)

$$\Phi(x,\theta,\bar\theta) + \Phi^+(x,\theta,\bar\theta) = A(x) + A^*(x) + \sqrt{2}\,\{[\theta\psi(x)] + [\bar\theta\bar\psi(x)]\}$$

$$+[\theta\theta]\,F(x) + [\bar\theta\bar\theta]\,F^*(x) - i\,[\theta\sigma^\mu\bar\theta]\,[\partial_\mu A(x) - \partial_\mu A^*(x)]$$

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$$- \frac{i}{\sqrt{2}} \; [\theta\theta][\bar{\theta}\,\tilde{\sigma}^\mu \partial_\mu \psi(x)] \; - \; \frac{i}{\sqrt{2}} \; [\bar{\theta}\bar{\theta}]\,[\theta\,\sigma^\mu \partial_\mu \bar{\psi}(x)]$$

$$- \frac{1}{4} \; [\theta\theta][\bar{\theta}\bar{\theta}]\, \partial_\mu \partial^\mu \, (A(x) + A^*(x)) \qquad (1)$$

This combination has the gradient $i\,\partial_\mu [A - A^*]$ as coefficient of $[\theta\sigma^\mu\bar{\theta}]$, motivating us to define the following supersymmetric generalization of a gauge transformation

$$V(x,\theta,\bar{\theta}) \longrightarrow V(x,\theta,\bar{\theta}) + \Phi(x,\theta,\bar{\theta}) + \Phi^+(x,\theta,\bar{\theta}) \qquad (2)$$

and under this transformation

$$C(x) \longrightarrow C(x) + A(x) + A^*(x)$$

$$\chi(x) \longrightarrow \chi(x) - i\sqrt{2}\,\psi(x)$$

$$M(x) + i\,N(x) \longrightarrow M(x) + i\,N(x) - 2i\,F(x) \qquad (3)$$

$$v_\mu(x) \longrightarrow v_\mu(x) - i\,\partial_\mu [A(x) - A^*(x)]$$

$$\lambda(x) \longrightarrow \lambda(x)$$

$$D(x) \longrightarrow D(x)$$

The choice of components in (21.3) renders $\lambda(x)$ and $D(x)$ gauge invariant.

From (3) we see that there is a special gauge (Wess-Zumino or WZ-gauge) in which $C(x)$, $\chi(x)$, $M(x)$ and $N(x)$ are all zero. Fixing this gauge breaks supersymmetry but still allows the usual gauge transformations $v_\mu(x) \rightarrow v_\mu(x) + \partial_\mu a(x)$ where $a(x) \equiv \frac{1}{2}\,Im\,[A(x)]$
It is very easy to compute powers of $V(x)$ in this gauge

$$V(x) = + [\theta\sigma^\mu\bar{\theta}]\,v_\mu(x) + i\,[\theta\theta][\bar{\theta}\bar{\lambda}(x)] - i\,[\bar{\theta}\bar{\theta}][\theta\lambda(x)] +$$

$$+ \frac{1}{2}\,[\theta\theta][\bar{\theta}\bar{\theta}]\,D(x)$$

$$V^2(x) = \frac{1}{2}\,[\theta\theta][\bar{\theta}\bar{\theta}]\,v^\mu(x)\,v_\mu(x) \qquad (3)$$

$$V^3(x) = 0$$

Thus we may view the vector field $V(x)$ as a supersymmetric generalization of the Yang-Mills potential. To construct the corresponding supersymmetric field strength, we observe that $\lambda(x)$ and $\bar{\lambda}(x)$ are the lowest-dimensional gauge invariant component fields in $V(x)$. Let us introduce the superfields

$$W_\alpha \equiv -\frac{1}{4}\bar{D}\bar{D}D_\alpha V \quad , \quad \bar{W}_{\dot{\alpha}} = -\frac{1}{4}DD\bar{D}_{\dot{\alpha}}V \qquad (1)$$

Notice that these fields are chiral

$$\bar{D}_{\dot{\beta}}W_\alpha = 0 \quad , \quad D_\beta \bar{W}_{\dot{\alpha}} = 0 \qquad (1)$$

which is an immediate consequence of (23.3) Furthermore they are gauge invariant:

$$W_\alpha \longrightarrow -\frac{1}{4}\bar{D}\bar{D}D_\alpha(V+\Phi+\Phi^+) = W_\alpha - \frac{1}{4}\bar{D}\bar{D}D_\alpha\Phi - \frac{1}{4}\bar{D}\bar{D}D_\alpha\Phi^+ =$$

$$= W_\alpha - \frac{1}{4}\bar{D}\bar{D}D_\alpha\Phi = W_\alpha - \frac{1}{4}\bar{D}(\bar{D}D_\alpha + D_\alpha\bar{D})\Phi = W_\alpha$$

Let us compute the components of these fields in the WZ-gauge where

$$V(x) = +[\theta\sigma^\mu\bar{\theta}]v_\mu(x) + i[\theta\theta][\bar{\theta}\bar{\lambda}(x)] - i[\bar{\theta}\bar{\theta}][\theta\lambda(x)] + \frac{1}{2}[\theta\theta][\bar{\theta}\bar{\theta}]D(x)$$

and using $y^\mu \equiv x^\mu - i\theta\sigma^\mu\bar{\theta}$

$$V = +[\theta\sigma^\mu\bar{\theta}]v_\mu(y) + i[\theta\theta][\bar{\theta}\bar{\lambda}(y)] - i[\bar{\theta}\bar{\theta}][\theta\lambda(x)] + \frac{1}{2}[\theta\theta][\bar{\theta}\bar{\theta}][D(x) + i\partial^\mu v_\mu(x)]$$

$$D_\alpha V = +(\sigma^\mu)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}v_\mu(y) - 2i(\sigma^\mu)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}[\theta\sigma^\nu\bar{\theta}]\partial_\mu v_\nu(y) + 2i\theta_\alpha[\bar{\theta}\bar{\lambda}(y)]$$

$$+2(\sigma^\mu)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}(\theta\theta)(\bar{\theta}\partial_\mu\bar{\lambda}(y)) - i[\bar{\theta}\bar{\theta}]\lambda_\alpha(y) + \theta_\alpha[\bar{\theta}\bar{\theta}][D(y) + i\partial^\mu v_\mu(y)]$$

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}D_\alpha V = -\frac{1}{4}\bar{D}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}}D_\alpha V = -\frac{1}{4}\epsilon^{\dot{\alpha}\dot{\beta}}\bar{D}_{\dot{\alpha}}\bar{D}_{\dot{\beta}}D_\alpha V =$$

$$= -\frac{1}{4}\epsilon^{\dot{\alpha}\dot{\beta}}\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}\frac{\partial}{\partial\bar{\theta}^{\dot{\beta}}}D_\alpha V = -i(\sigma^\mu)_{\alpha\dot{\alpha}}(\tilde{\sigma}^\nu)^{\dot{\alpha}\rho}\theta_\rho\partial_\mu v_\nu(y) - [\theta\theta](\sigma^\mu)_{\alpha\dot{\alpha}}\partial_\mu\bar{\lambda}^{\dot{\alpha}}(y)$$

$$+ i\lambda_\alpha(y) + \theta_\alpha[D(y) + i\partial^\mu v_\mu(y)]$$

$$W_\alpha = -i\lambda_\alpha(y) + \theta_\alpha D(y) - \frac{i}{2}[\sigma^\mu]_{\alpha\dot\alpha}[\tilde\sigma^\nu]^{\dot\alpha\beta}\theta_\beta\,[\partial_\mu v_\nu(y) - \partial_\nu v_\mu(y)]$$

$$- [\theta\theta]\,[\sigma^\mu]_{\alpha\dot\alpha}\,\partial_\mu\bar\lambda^{\dot\alpha}(y) \qquad (1)$$

Similarly using $y^{+\mu} = x^\mu + i\,\theta\sigma^\mu\bar\theta$ we can write

$$V = +[\theta\sigma^\mu\bar\theta]\,v_\mu(y^+) + i\,[\theta\theta][\bar\theta\bar\lambda(y^+)] - i\,[\bar\theta\bar\theta][\theta\lambda(y^+)] +$$

$$+\frac{1}{2}[\theta\theta][\bar\theta\bar\theta][D(x) - i\partial^\mu v_\mu(y)]$$

$$\bar D_{\dot\alpha} V = +\theta^\alpha[\sigma^\mu]_{\alpha\dot\alpha}\,v_\mu(y^+) + 2i\,\theta^\beta[\sigma^\nu]_{\beta\dot\alpha}\,[\theta\sigma^\mu\bar\theta]\,\partial_\nu v_\mu(y^+) + i\,[\theta\theta]\,\bar\lambda_{\dot\alpha}(y^+)$$

$$-2i\,\bar\theta_{\dot\alpha}\,[\theta\lambda(y^+)] + 2\,\theta^\beta[\sigma^\mu]_{\beta\dot\alpha}\,[\bar\theta\bar\theta][\theta\partial_\mu\lambda(y^+)] + [\theta\theta]\,\bar\theta_{\dot\alpha}\,[D(y^+) - i\partial^\mu v_\mu(y^+)]$$

$$\bar W_{\dot\alpha} = -\frac{1}{4}DD\bar D_{\dot\alpha}V = +\frac{1}{4}\epsilon^{\alpha\beta}D_\alpha D_\beta\bar D_{\dot\alpha}V = \frac{1}{4}\epsilon^{\alpha\beta}\frac{\partial}{\partial\theta^\alpha}\frac{\partial}{\partial\theta^\beta}\bar D_{\dot\alpha}V =$$

$$= +i\,\bar\theta^{\dot\beta}[\tilde\sigma^\mu]_{\dot\beta\alpha}[\sigma^\nu]^\alpha{}_{\dot\alpha}\,\partial_\nu v_\mu(y^+) + i\,\bar\lambda_{\dot\alpha}(y^+) + [\sigma^\mu]^\rho{}_{\dot\alpha}\,[\bar\theta\bar\theta]\,\partial_\mu\lambda_\rho(y^+)$$

$$+\bar\theta_{\dot\alpha}\,[D(y^+) - i\partial^\mu v_\mu(y^+)]$$

$$\bar W_{\dot\alpha} = i\,\bar\lambda_{\dot\alpha}(y^+) + \bar\theta_{\dot\alpha}D(y^+) - \frac{i}{2}\bar\theta_{\dot\beta}[\tilde\sigma^\mu]^{\dot\beta\alpha}[\sigma^\nu]_{\alpha\dot\alpha}[\partial_\mu v_\nu(y^+) - \partial_\nu v_\mu(y^+)]$$

$$-[\bar\theta\bar\theta]\,\partial_\mu\lambda^\alpha(y^+)\,[\sigma^\mu]_{\alpha\dot\alpha} \qquad (2)$$

It is clear that the components of W_α and $W_{\dot\alpha}$ are all gauge invariant fields $D(x)$, $\lambda(x)$, $v_{\mu\nu} \equiv \partial_\mu v_\nu(x) - \partial_\nu v_\mu(x)$.

Notice furthermore that

$$D^\alpha W_\alpha = \bar D_{\dot\alpha}\bar W^{\dot\alpha} \qquad (3)$$

Proof $D^\alpha W_\alpha = -\frac{1}{4}D^\alpha\bar D_{\dot\beta}\bar D^{\dot\beta}D_\alpha V = -\frac{1}{4}\epsilon^{\alpha\beta}\epsilon^{\dot\beta\dot\alpha}D_\beta\bar D_{\dot\beta}\bar D_{\dot\alpha}D_\alpha V$

(13.1)
$$= -\frac{i}{2}[\sigma^\mu]^{\alpha\dot\alpha}\bar D_{\dot\alpha}D_\alpha\partial_\mu V + \frac{1}{4}\epsilon^{\alpha\beta}\epsilon^{\dot\beta\dot\alpha}\bar D_{\dot\beta}D_\beta\bar D_{\dot\alpha}D_\alpha V$$

$$= -\frac{i}{2}[\sigma^\mu]^{\alpha\dot\alpha}\bar D_{\dot\alpha}D_\alpha\partial_\mu V + \frac{i}{2}[\sigma^\mu]^{\alpha\dot\alpha}\bar D_{\dot\alpha}D_\alpha\partial_\mu V - \frac{1}{4}\epsilon^{\alpha\beta}\epsilon^{\dot\beta\dot\alpha}\bar D_{\dot\beta}D_\beta D_\alpha\bar D_{\dot\alpha}V$$

$$= \bar D_{\dot\alpha}\bar W^{\dot\alpha}$$

For $\theta = \bar{\theta} = 0$, this relation simply expresses the fact that the component field D is real. It may be shown that (24.1) and (24.2) represents the most general solutions of the chirality conditions (23.1) and the constraint (24.3). Furthermore

$$W^\alpha W_\alpha = -\frac{1}{4} \bar{D}\bar{D} W^\alpha D_\alpha V \tag{1}$$

Since W_α is chiral, the $[\theta\theta]$ component of $W^\alpha W_\alpha$ transforms as a total derivative

$$W^\alpha W_\alpha \Big|_{[\theta\theta]} = \epsilon^{\alpha\beta} W_\beta W_\alpha \Big|_{[\theta\theta]} = + i2[\lambda(x)\sigma^\mu \partial_\mu \bar{\lambda}(x)] + D^2(x)$$

$$- \frac{1}{2} v_{\mu\nu}(x) v^{\mu\nu}(x) - \frac{i}{4} \epsilon^{\mu\nu\rho\sigma} v_{\mu\nu}(x) v_{\rho\sigma}(x) \tag{2}$$

$$\bar{W}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} \Big|_{[\bar{\theta}\bar{\theta}]} = \epsilon^{\dot{\alpha}\dot{\beta}} \bar{W}_{\dot{\alpha}} \bar{W}_{\dot{\beta}} \Big|_{[\bar{\theta}\bar{\theta}]} = - i2[\partial_\mu \lambda(x) \sigma^\mu \bar{\lambda}(x)] + D^2(x)$$

$$- \frac{1}{2} v_{\mu\nu}(x) v^{\mu\nu}(x) + \frac{i}{4} \epsilon^{\mu\nu\rho\sigma} v_{\mu\nu}(x) v_{\rho\sigma}(x)$$

From (2) we see that

$$L(x) = \frac{1}{4} \left\{ W^\alpha W_\alpha \Big|_{[\theta\theta]} + \bar{W}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} \Big|_{[\bar{\theta}\bar{\theta}]} \right\} \tag{3}$$

is the supersymmetric gauge-invariant generalization of the Lagrangian for a free vector field.

$$L(x) = \frac{1}{2} D^2(x) - \frac{1}{4} v^{\mu\nu}(x) v_{\mu\nu}(x) + i [\lambda(x) \sigma^\mu \partial_\mu \bar{\lambda}(x)] \tag{4}$$

where total derivatives have been supressed.

This Lagrangian can also be obtained as (valid in any gauge)

$$L(x) = \frac{1}{4} \left\{ W^\alpha D_\alpha V + \bar{W}_{\dot{\alpha}} \bar{D}^{\dot{\alpha}} V \right\}_{[\theta\theta][\bar{\theta}\bar{\theta}]} \tag{5}$$

due to (1) and the fact that $\bar{D}$ is equivalent to $\frac{\partial}{\partial\bar{\theta}}$ but for an x-space derivative.

We can always add the mass term $m^2 V^2$ to the Lagrangian (25.4). This term is not gauge invariant and cannot be computed in the W-Z gauge. From (21.3)

$$V^2\Big|_{[\theta\theta][\bar\theta\bar\theta]} = \Big\{ \frac{1}{2} C [\theta\theta][\bar\theta\bar\theta] [D - \frac{1}{2}\partial_\mu\partial^\mu C] +$$

$$+ [\theta\chi][\bar\theta\bar\theta][\theta(\lambda + \frac{i}{2}\sigma^\mu\partial_\mu\bar\chi)] + [\bar\theta\bar\chi][\theta\theta][\bar\theta(\bar\lambda + \frac{i}{2}\tilde\sigma^\mu\partial_\mu\chi)]$$

$$+ \frac{1}{4}[\theta\theta][M + iN][\bar\theta\bar\theta][M - iN] + \frac{1}{4}[\bar\theta\bar\theta][M - iN][\theta\theta][M + iN]$$

$$+ [\theta\sigma^\mu\bar\theta] v_\mu [\theta\sigma^\nu\bar\theta] v_\nu + [\theta\theta][\bar\theta(\bar\lambda + \frac{i}{2}\tilde\sigma^\mu\partial_\mu\chi)][\bar\theta\bar\chi]$$

$$+ [\bar\theta\bar\theta][\theta(\lambda + \frac{i}{2}\sigma^\mu\partial_\mu\bar\chi)][\theta\chi] + \frac{1}{2}[\theta\theta][\bar\theta\bar\theta][D - \frac{1}{2}\partial^\mu\partial_\mu C] C \Big\} / [\theta\theta][\bar\theta\bar\theta]$$

$$V^2 = \frac{1}{2} v^\mu(x) v_\mu(x) - [\chi(x)\lambda(x)] - [\bar\chi(x)\bar\lambda(x)] + \frac{1}{2}[M^2(x) + N^2(x)]$$

$$- \frac{i}{2}[\chi(x)\sigma^\mu\partial_\mu\bar\chi(x)] - \frac{i}{2}[\bar\chi(x)\tilde\sigma^\mu\partial_\mu\chi(x)] -$$

$$- \frac{1}{2} C(x)\partial^\mu\partial_\mu C(x) + C(x) D(x) \qquad (1)$$

It is interesting to note that this term not only gives mass to the vector field $v_\mu(x)$ but also introduce the additional degrees of freedom $C(x)$ and $\chi(x)$ required for a massive multiplet. Then (25.4) together with (1) describes a vector field, two spin-1/2 field and one scalar field, all of equal mass

Let us finally compute the transformation laws of the vector superfield under supersymmetry transformations. V given in (21.3) is a field of the type (1) with

$$f(x) \equiv C(x), \quad \phi(x) = i\chi(x), \quad \bar\chi(x) = -i\bar\chi(x), \quad m(x) = \frac{i}{2}[M(x) + iN(x)]$$

$$n(x) = -\frac{i}{2}[M(x) - iN(x)], \quad v_\mu(x) = + v_\mu(x), \quad \bar\lambda(x) = i[\bar\lambda(x) + \frac{i}{2}\tilde\sigma^\mu\partial_\mu\chi(x)]$$

$$\psi(x) = -i[\lambda(x) + \frac{i}{2}\sigma^\mu\partial_\mu\bar\chi(x)], \quad d(x) = \frac{1}{2}[D(x) - \frac{1}{2}\partial^\mu\partial_\mu C(x)]$$

and using (11.1) we obtain

$$\delta C(x) = i\,[\xi\,\chi(x)] - i\,[\bar\xi\,\bar\chi(x)]$$

$$\delta \chi(x) = \sigma^\mu \bar\xi\, \partial_\mu C(x) + \xi\,[M(x) + iN(x)] - i\sigma^\mu \bar\xi\, v_\mu(x)$$

$$\delta \bar\chi(x) = -\tilde\sigma^\mu \xi\, \partial_\mu C(x) + \bar\xi\,[M(x) - iN(x)] - i\tilde\sigma^\mu \xi\, v_\mu(x)$$

$$\delta M(x) = [\xi\,\lambda(x)] + [\bar\xi\,\bar\lambda(x)] + i\,[\xi\,\sigma^\mu \partial_\mu \bar\chi] + i\,[\bar\xi\,\tilde\sigma^\mu \partial_\mu \chi] \qquad (1)$$

$$\delta N(x) = i\,[\xi\,\lambda(x)] - i\,[\bar\xi\,\bar\lambda(x)] - [\xi\,\sigma^\mu \partial_\mu \bar\chi] + [\bar\xi\,\tilde\sigma^\mu \partial_\mu \chi(x)]$$

$$\delta v_\mu(x) = +i\,[\xi\,\sigma_\mu\,\bar\lambda(x)] + i\,[\bar\xi\,\tilde\sigma_\mu\,\lambda(x)] - [\xi\,\partial_\mu \chi] - [\bar\xi\,\partial_\mu \bar\chi(x)]$$

$$\delta \bar\lambda(x) = -i\,\bar\xi\, D(x) + \frac{1}{2}\tilde\sigma^\mu \sigma^\nu\, \bar\xi\, v_{\mu\nu}(x)$$

$$\delta \lambda(x) = i\,\xi\, D(x) - \frac{1}{2}\sigma^\mu \tilde\sigma^\nu\, \xi\, v_{\mu\nu}(x)$$

$$\delta D(x) = -\xi\,\sigma^\mu \partial_\mu \bar\lambda(x) + \bar\xi\,\tilde\sigma^\mu \partial_\mu \lambda(x)$$

and hence

$$\delta v_{\mu\nu}(x) = -i\,(\xi\,[\sigma_\mu \partial_\nu - \sigma_\nu \partial_\mu]\,\bar\lambda(x)) - i\,(\bar\xi\,(\tilde\sigma_\mu \partial_\nu - \tilde\sigma_\nu \partial_\mu)\,\lambda(x)) \qquad (2)$$

Now we would like to discuss the gauge invariant interactions of scalar and vector multiplets. Let us start with the $U(1)$ case. Scalar superfields Φ_e transform by a phase under global $U(1)$ transformations

$$\Phi_e \longrightarrow \Phi_e' = e^{-it_e\lambda}\,\Phi_e \qquad (3)$$

The t_e are the $U(1)$ charges appropriate to the Φ_e, while λ is the group parameter. Both are real quantities. Since $D_\alpha \lambda = \bar D_{\dot\alpha} \lambda = 0$ it is clear that Φ_e' is also a chiral scalar superfield. It is easy to construct a lagrangian density invariant under (3) for constant parameter λ:

$$L(x) = L_{K.E.}(x) + L_{P.E}(x)$$

$$L_{K.E}(x) = \Phi_e^+ \Phi_e \Big|_{[\theta\theta][\bar\theta\bar\theta]} \qquad (4)$$

$$L_{P.E}(x) = \left[\frac{1}{2} m_{ij}\, \Phi_i \Phi_j + \frac{1}{3} g_{ijk}\, \Phi_i \Phi_j \Phi_k\right]\Bigg|_{[\theta\theta]} + h.c$$

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U(1) invariance requires m_{ij} or $g_{ijk} = 0$ whenever $t_i + t_j$ or $t_i + t_j + t_k \neq 0$.

Equation (27.1) takes a scalar quiral superfield into another when λ is a constant. When λ depends on x, the situation becomes more complicated. In this case λ must be promoted to a full scalar multiplet

$$\Phi_e' = e^{-i t_e \Lambda} \Phi_e \qquad \bar{D}_{\dot\alpha} \Lambda = 0$$
$$\Phi_e'^{+} = e^{+i t_e \Lambda^+} \Phi_e^+ \qquad D_\alpha \Lambda^+ = 0 \tag{1}$$

Only then will the Φ_e' and $\Phi_e'^{+}$ remain chiral scalar superfields. The Lagrangian (27.5) is not invariant under such local transformations. In particular

$$L_{K.E}(x) \longrightarrow L'_{K.E}(x) = \Phi_e^+ \Phi_e \, e^{i t_e (\Lambda^+ - \Lambda)} \tag{2}$$

It is easy to see that $L_{K.E.}(x)$ may be rendered invariant by introducing a vector superfield V with its transformation law

$$V \longrightarrow V' = V + i(\Lambda - \Lambda^+) \tag{3}$$

With this addition, the full Lagrangian becomes

$$L(x) = \frac{1}{4}\left\{ W^\alpha W_\alpha \Big|_{\theta\theta} + \bar{W}_{\dot\alpha} \bar{W}^{\dot\alpha} \Big|_{\bar\theta\bar\theta} \right\} + \Phi_e^+ e^{t_e V} \Phi_e \Big|_{[\theta\theta][\bar\theta\bar\theta]} +$$
$$+ \left[\left(\frac{1}{2} m_{ij} \Phi_i \Phi_j + \frac{1}{3} g_{ijk} \Phi_i \Phi_j \Phi_k \right)_{(\theta\theta)} + h.c \right] \tag{5}$$

It is invariant under U(1) transformations if m_{ij} or $g_{ijk} = 0$ whenever $t_i + t_j$ or $t_i + t_j + t_k = 0$. At first, (5) looks non renormalizable. It may however be evaluated in the WZ gauge

$$\Phi^+ e^{tV} \Phi \Big|_{[\theta\theta][\bar\theta\bar\theta]} = \Phi^+ \Phi \left(1 + tV + \tfrac{1}{2} t^2 V^2\right) \Big|_{[\theta\theta][\bar\theta\bar\theta]} =$$

$$= F(x) F^*(x) - A(x) \partial_\mu \partial^\mu A^*(x) - i \partial_\mu \psi(x) \sigma^\mu \bar\psi(x) +$$
$$+ t \left\{ \frac{1}{2} A^*(x) A(x) D(x) + \frac{i}{\sqrt{2}} A(x) [\bar\lambda(x) \bar\psi(x)] + \frac{i}{\sqrt{2}} A^*(x) [\lambda(x) \psi(x)] \right.$$
$$\left. + \frac{1}{2} v_\mu(x) \psi(x) \sigma^\mu \bar\psi(x) - \frac{i}{2} v^\mu(x) A^*(x) \partial_\mu A(x) + \frac{i}{2} v^\mu(x) \partial^\mu A^*(x) A(x) \right\}$$
$$+ \frac{1}{4} t^2 v^\mu(x) v_\mu(x) A^*(x) A(x) \tag{5}$$

In this gauge, the Lagrangian contains no terms of dimension higher than four*

The supersymmetric extension of Q.E.D is constructed in terms of two scalar superfields $\Phi_\pm$ with gauge transformation laws

$$\Phi_\pm \longrightarrow \Phi'_\pm = e^{\mp i e \Lambda} \Phi_\pm \tag{1}$$

with Lagrangian

$$L_{QED}(x) = \frac{1}{4}\left\{ W^\alpha W_\alpha \Big|_{[\theta\theta]} + \bar{W}_{\dot\alpha} \bar{W}^{\dot\alpha}\Big|_{[\bar\theta\bar\theta]} \right\} + \Phi_+^\dagger e^{2eV} \Phi_+ \Big|_{[\theta\theta][\bar\theta\bar\theta]} +$$

$$+ \Phi_-^\dagger e^{-2eV} \Phi_- \Big|_{[\theta\theta][\bar\theta\bar\theta]} + m\left\{ \Phi_+ \Phi_- \Big|_{[\theta\theta]} + \Phi_+^\dagger \Phi_-^\dagger \Big|_{[\bar\theta\bar\theta]} \right\} \tag{2}$$

In terms of component fields

$$L_{QED} = \frac{1}{2} D^2(x) - \frac{1}{4} v^{\mu\nu}(x) v_{\mu\nu}(x) + i\lambda(x)\sigma^\mu \partial_\mu \bar\lambda(x) +$$

$$+ m\left\{ A_+(x) F_-(x) + A_-(x) F_+(x) + A_+^*(x) F_-^*(x) + A_-^*(x) F_+^*(x) - \psi_+(x)\psi_-(x) - \bar\psi_+(x)\bar\psi_-(x)\right\}$$

$$- A_+^*(x)\partial^\mu\partial_\mu A_+(x) - A_-^*(x)\partial^\mu\partial_\mu A_-(x) + F_+^*(x) F_+(x) + F_-^*(x) F_-(x)$$

$$- i\partial_\mu \bar\psi_+(x)\tilde\sigma^\mu \psi_+(x) - i\partial_\mu\bar\psi_-(x)\tilde\sigma^\mu\psi_-(x) +$$

$$+ 2e v^\mu(x)\left[-\frac{1}{2}\bar\psi_+\tilde\sigma_\mu\psi_+ + \frac{1}{2}\bar\psi_-\tilde\sigma_\mu\psi_- - \frac{i}{2}A_+^*\partial_\mu A_+ + \frac{i}{2}\partial_\mu A_+^* A_+ - \right.$$

$$\left. + \frac{i}{2} A_-^*\partial_\mu A_- - \frac{i}{2}\partial_\mu A_-^* A_-\right]$$

$$- ie 2\frac{1}{\sqrt{2}}\left[A_+\bar\psi_+\bar\lambda - A_+^*\psi_+\lambda - A_-\bar\psi_-\bar\lambda + A_-^*\psi_-\lambda\right]$$

$$+ \frac{1}{2} 2e\, D\left[A_+^* A_+ - A_-^* A_-\right] + \frac{1}{4} 4e^2 v^\mu(x) v_\mu(x)\left[A_+^* A_+ + A_-^* A_-\right] \tag{3}$$

This is written is the WZ gauge which breaks supersymmetries but allows the usual gauge transformation $v_\mu \to v_\mu + \partial_\mu a(x)$. It contains a vector field $v^\mu(x)$, two Weyl spinor fields $\psi_+(x)$, $\psi_-(x)$, the spinor field $\lambda(x)$, the scalar fields $A_-(x)$ $A_+(x)$ and the auxiliar fields $D(x)$, $F_-(x)$ and $F_+(x)$.

Let us now deduce the field equations for the auxiliar fields

$$D = -2\frac{e}{2}\left[A_+^* A_+ - A_-^* A_-\right] \qquad F_\pm = -m A_\mp^* \qquad F_\pm^* = -m A_\mp \tag{4}$$

* It is easy to check that the terms containing $D(x)$ are

$$L_D(x) = \frac{1}{2} D^2(x) + \frac{1}{2} A_i^*(x)\, t_i\, A_i(x)\, D(x)$$

and therefore

$$D(x) = -\frac{1}{2} \left[A_i^*(x)\, t_i\, A_i(x) \right]$$

and this adds a term to the scalar potential given in p. 19

$$V(A, A^*) = \sum_k \left| \frac{\partial W}{\partial A_k} \right|^2 + \frac{1}{8} \sum_k \left[t_k \left(A_k^* A_k \right) \right]^2$$

In general if T_i are the generators of the gauge group acting on the (possible reducible) scalar representation, then the last term is

$$\frac{1}{8} \sum_i \left[t_i \left(A^* T_i A \right) \right]^2 \qquad A \equiv \begin{vmatrix} A_1(x) \\ A_2(x) \\ \vdots \end{vmatrix}$$

where the sum runs over all generators, and t_i is the coupling constant associated with the generator T_i

Now the condition for supersymmetry to be unbroken at the tree level is that for each field and for each generator we require

$$\frac{\partial W}{\partial A_i} = 0 \qquad \sum_{km} A_k^* (T_i)_{km} A_m = 0$$

If these equations have simultaneous solutions, supersymmetry is unbroken at tree level. Otherwise, supersymmetry is spontaneously broken.

Then up to total derivative

$$L_{QED}(x) = -\frac{1}{4} v^{\mu\nu}(x)\, v_{\mu\nu}(x) + \frac{i}{2}\left[\lambda \sigma^\mu \partial_\mu \bar\lambda + \bar\lambda \tilde\sigma^\mu \partial_\mu \lambda\right]$$

$$+ i \psi_+ \sigma^\mu \partial_\mu \bar\psi_+ + i \bar\psi_- \tilde\sigma^\mu \partial_\mu \psi_- - m \psi_+ \psi_- - m \bar\psi_+ \bar\psi_-$$

$$+ \partial^\mu A_+^* \partial_\mu A_+ - m^2 A_+^* A_+ + \partial^\mu A_-^* \partial_\mu A_- - m^2 A_-^* A_-$$

$$+ e v^\mu \left[+ \psi_+ \sigma_\mu \bar\psi_+ + \bar\psi_- \tilde\sigma_\mu \psi_- - i A_+^* \partial_\mu A_+ + i \partial_\mu A_+^* A_+ + i A_-^* \partial_\mu A_- - i \partial_\mu A_-^* A_- \right]$$

$$- i\sqrt{2}\, e \left[A_+ \bar\psi_+ \bar\lambda - A_+^* \psi_+ \lambda - A_- \bar\psi_- \bar\lambda + A_-^* \psi_- \lambda \right]$$

$$+ e^2 v_\mu v^\mu \left[A_+^* A_+ + A_-^* A_- \right] - \frac{e^2}{2}\left[A_+^* A_+ - A_-^* A_- \right]^2 \qquad (1)$$

Let us now introduce the Dirac field

$$\psi(x) \equiv \begin{vmatrix} \psi_{-\alpha}(x) \\ \bar\psi_+^{\dot\alpha}(x) \end{vmatrix} \qquad \bar\psi(x) = \left| \psi_+^\alpha(x) ,\ \bar\psi_{-\dot\alpha}(x) \right| \qquad \gamma^\mu = \begin{vmatrix} 0 & +[\sigma^\mu]_{\alpha\dot\beta} \\ +[\tilde\sigma^\mu]^{\dot\alpha\beta} & 0 \end{vmatrix}$$

Then

$$-m \bar\psi \psi = -m \left[\psi_+^\alpha \psi_{-\alpha} + \bar\psi_{-\dot\alpha} \bar\psi_+^{\dot\alpha} \right] = -m \psi_+ \psi_- - m \bar\psi_- \bar\psi_+$$

$$i \bar\psi \gamma^\mu \partial_\mu \psi = + i \left[\psi_+^\alpha ,\ \bar\psi_{-\dot\alpha} \right] \begin{vmatrix} 0 & [\sigma^\mu]_{\alpha\dot\beta} \\ [\tilde\sigma^\mu]^{\dot\alpha\beta} & 0 \end{vmatrix} \begin{vmatrix} \partial_\mu \psi_{-\beta} \\ \partial_\mu \bar\psi_+^{\dot\beta} \end{vmatrix} =$$

$$= + i \psi_+^\alpha [\sigma^\mu]_{\alpha\dot\beta} \partial_\mu \bar\psi_+^{\dot\beta} - i \bar\psi_{-\dot\alpha} [\tilde\sigma^\mu]^{\dot\alpha\beta} \partial_\mu \psi_{-\beta} =$$

$$= + i \psi_+ \sigma^\mu \partial_\mu \bar\psi_+ + i \bar\psi_- \tilde\sigma^\mu \partial_\mu \psi_-$$

Similarly we can introduce the Majorana spinor

$$\Lambda(x) = \begin{vmatrix} \lambda_\alpha(x) \\ \bar\lambda^{\dot\alpha}(x) \end{vmatrix} \qquad \bar\Lambda(x) = \left| \lambda^\alpha(x) ,\ \bar\lambda_{\dot\alpha}(x) \right|$$

$$i \bar\Lambda \gamma^\mu \partial_\mu \Lambda = + i \lambda \sigma^\mu \partial_\mu \bar\lambda + i \bar\lambda \tilde\sigma^\mu \partial_\mu \lambda$$

$$\bar\Lambda \psi = \lambda \psi_- + \bar\lambda \bar\psi_+ , \qquad \bar\psi \Lambda = \psi_+ \lambda + \bar\psi_- \bar\lambda$$

$$\gamma_5 = \begin{vmatrix} I_\alpha{}^\beta & 0 \\ 0 & -I^{\dot\alpha}{}_{\dot\beta} \end{vmatrix} \qquad \bar\Lambda \gamma_5 \psi = \lambda \psi_- - \bar\lambda \bar\psi_+$$

$$\bar\psi \gamma_5 \Lambda = \lambda \psi_+ - \bar\lambda \bar\psi_-$$

$$\bar\psi_+ \bar\lambda = \frac{1}{2} \bar\Lambda (1 - \gamma_5) \psi , \qquad \lambda \psi_- = \frac{1}{2} \bar\Lambda (1 + \gamma_5) \psi$$

$$\lambda \psi_+ = \frac{1}{2} \bar\psi (1 + \gamma_5) \Lambda , \qquad \bar\lambda \bar\psi_- = \frac{1}{2} \bar\psi (1 - \gamma_5) \Lambda$$

Then

$$L_{QED}(x) = -\frac{1}{4} v^{\mu\nu}(x) v_{\mu\nu}(x) + \frac{i}{2} \bar{\Lambda}(x) \gamma^\mu \partial_\mu \Lambda(x) + i \bar{\psi}(x) \gamma^\mu \partial_\mu \psi(x)$$

$$- m \bar{\psi}(x) \psi(x) + \partial^\mu A_+^*(x) \partial_\mu A_+(x) - m^2 A_+^*(x) A_+(x) + \partial^\mu A_-^*(x) \partial_\mu A_-(x)$$

$$- m^2 A_-^*(x) A_-(x) + e \bar{\psi}(x) \gamma_\mu \psi(x) v_\mu(x) - ie [A_+^*(x) \partial_\mu A_+(x) - \partial_\mu A_+^*(x) A_+(x)] v^\mu(x)$$

$$+ ie [A_-^*(x) \partial_\mu A_-(x) - \partial_\mu A_-^*(x) A_-(x)] v^\mu(x) - \frac{ie}{\sqrt{2}} [A_+(x) \bar{\Lambda}(x) (1-\gamma_5) \psi(x)$$

$$- A_+^*(x) \bar{\psi}(x) (1+\gamma_5) \Lambda(x)] + \frac{ie}{\sqrt{2}} [A_-(x) \bar{\psi}(x) (1-\gamma_5) \Lambda(x) - A_-^*(x) \bar{\Lambda} (1+\gamma_5) \psi(x)]$$

$$+ e^2 v_\mu(x) v^\mu(x) [A_+^*(x) A_+(x) + A_-^*(x) A_-(x)] -$$

$$- \frac{e^2}{2} [A_+^*(x) A_+(x) - A_-^*(x) A_-(x)]^2 \tag{1}$$

which is the desired result. There is the usual photon field $v_\mu(x)$ and the electron field $\psi(x)$; furthermore there is a Majorana partner, $\Lambda(x)$, of the photon field (the photino) without mass and two scalar partners of the electrons $A_+(x)$, $A_-(x)$ with opposite charges. *

It is straightforward to generalize all that to non-Abelian compact groups.

$$\Phi \longrightarrow \Phi' = e^{-i\Lambda} \Phi \quad , \quad \Phi^+ \longrightarrow \Phi'^+ = e^{+i\Lambda} \Phi^+ \tag{2}$$

where Λ is a matrix $\Lambda_{ij} = (T^a)_{ij} \Lambda_a$, where T^a are the generators of the gauge group in the representation defined by the scalar chiral field Φ. In the adjoint representation, we normalize our generators as follows

$$\mathrm{Tr} [T^a T^b] = k \delta^{ab} \qquad k > 0 \tag{3}$$

With this convention, the structure constants f^{abc}

$$[T^a , T^b] = i f^{abc} T^c \tag{4}$$

are fully antisymmetric.

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* Let us now introduce the scalar fields

$$A(x) \equiv \frac{1}{\sqrt{2}} \left[A_+(x) + A_-^*(x) \right] \qquad B(x) \equiv \frac{1}{\sqrt{2}} \left[A_+(x) - A_-^*(x) \right]$$

Then

$$L_{QED}(x) = -\frac{1}{4} v^{\mu\nu}(x)\, v_{\mu\nu}(x) + \frac{i}{2} \bar{\Lambda}(x) \gamma^\mu \partial_\mu \Lambda(x) + i \bar{\psi}(x) \gamma^\mu \partial_\mu \psi(x)$$

$$- m \bar{\psi}(x) \psi(x) + \partial^\mu A^*(x) \partial_\mu A(x) - m^2 A^*(x) A(x) + \partial^\mu B^*(x) \partial_\mu B(x)$$

$$- m^2 B^*(x) B(x) + e \bar{\psi}(x) \gamma^\mu \psi(x) v_\mu(x)$$

$$- ie \left[A^*(x) \partial_\mu A(x) - \partial_\mu A^*(x) A(x) \right] v^\mu(x)$$

$$- ie \left[B^*(x) \partial_\mu B(x) - \partial_\mu B^*(x) B(x) \right] v^\mu(x)$$

$$+ ie \bar{\Lambda}(x) \psi(x) A(x) - ie \bar{\psi}(x) \Lambda(x) A^*(x)$$

$$- ie \bar{\Lambda}(x) \gamma_5 \psi(x) B(x) - ie \bar{\psi}(x) \gamma_5 \Lambda(x) B^*(x)$$

$$+ e^2 v^\mu(x) v_\mu(x) \left[A^*(x) A(x) + B^*(x) B(x) \right]$$

$$- \frac{e^2}{2} \left[A^*(x) B(x) + B^*(x) A(x) \right]^2$$

Let us define R-parity as $R \equiv (-1)^{2S + 3B + L}$ — S = spin, B = barionic number, L = leptonic number

$R_e = +1$, $R_\gamma = +1$

$R_{\tilde{e}} = -1$, $R_{\tilde{\gamma}} = -1$

In general R(particles) = +1, R(superparticles) = -1

It is a multiplicatively conserved quantum number

The Lagrangian (28.4) is invariant under non-Abelian gauge transformations, provided we extend the transformation law (28.3):

$$e^{V'} = e^{-i\Lambda^+} e^{V} e^{i\Lambda} \tag{1}$$

$$\Lambda_{ij} \equiv (T^a)_{ij}\, \Lambda_a \quad , \quad V_{ij} \equiv (T^a)_{ij}\, V_a$$

When computing the r.h.s of (1) we encounter only commutators of group generators and using (31.4) we can express $V' = T^a V_a'$. This shows that the transformation law (1) is independent of any specific representation for the generators T^a. Furthermore we can write

$$V' = V + i\,(\Lambda - \Lambda^+) + \dots \tag{2}$$

so non-Abelian theories also allow a WZ gauge where $V^3 = 0$.

Eq. (1) can be evaluated for infinitesimal gauge transformations with the following form of Haussdorf's formula:

$$e^{A} e^{B} = \exp\left\{ A + \mathcal{L}_{\frac{A}{2}} \cdot [B + \coth(\mathcal{L}_{\frac{A}{2}}) \cdot B] + \dots \right\} \tag{3}$$

This expression contains all terms linear in B. $\mathcal{L}_{A/2} \cdot B \equiv [\frac{A}{2}, B]$. The hyperbolic tangent must be understood in terms of its power series expansion, where

$$c_n\, [\mathcal{L}_{\frac{A}{2}}]^n \cdot B \equiv c_n \left[\frac{A}{2}, \left[\frac{A}{2}, \left[\dots, \left[\frac{A}{2}, B\right] \dots \right]\right]\right] \tag{4}$$

with n factors $A/2$. Then

$$\delta V = V' - V = i\, \mathcal{L}_{\frac{V}{2}} \cdot [(\Lambda + \Lambda^+) + \cot g\, [\mathcal{L}_{V/2}]\, (\Lambda - \Lambda^+)] \tag{5}$$

The supersymmetric field strength W^α (23.1) must be generalized

$$W_\alpha \equiv -\frac{1}{4} \bar{D}\bar{D}\, e^{-V} D_\alpha\, e^{+V} \tag{6}$$

where V are matrices with the generators in the adjoint representation of the gauge group. It is easy to verify that

$$W_\alpha \longrightarrow W_\alpha' = e^{-i\Lambda}\, W_\alpha\, e^{+i\Lambda} \tag{7}$$

Proof $W'_\alpha = -\frac{1}{4}\bar{D}\bar{D}\, e^{-V'} D_\alpha e^{V'} = -\frac{1}{4}\bar{D}\bar{D}\, e^{-i\Lambda} e^{-V} e^{+i\Lambda^+} D_\alpha e^{-i\Lambda^+} e^{V} e^{i\Lambda} =$

$= e^{-i\Lambda} W_\alpha e^{i\Lambda} - \frac{1}{4} e^{-i\Lambda}\bar{D}\{\bar{D}, D_\alpha\} e^{i\Lambda} = e^{-i\Lambda} W_\alpha e^{+i\Lambda}$ Q.E.D.

We are now ready to write down the most general Lagrangian for the supersymmetric renormalizable interaction of scalar, spinor and vector fields.

$$L(x) = \frac{1}{4k}\,\mathrm{Tr}\left\{ W^\alpha W_\alpha\Big|_{[\theta\theta]} + \bar{W}_{\dot\alpha}\bar{W}^{\dot\alpha}\Big|_{[\bar\theta\bar\theta]}\right\} + \Phi^+ e^V \Phi\Big|_{[\theta\theta][\bar\theta\bar\theta]} +$$

$$+ \left[\left(\frac{1}{2} m_{ij}\Phi_i\Phi_j + \frac{1}{3} g_{ijk}\Phi_i\Phi_j\Phi_k\right)\Big|_{[\theta\theta]} + \text{h.c.}\right] \qquad (1)$$

Gauge invariance requires the mass matrix m_{ij} and the coupling constants g_{ijk} to be fully symmetric invariant tensors with respect to the internal symmetry group.

If supersymmetric gauge theories are to find realistic application in high energy physics, both supersymmetry and gauge symmetry must be broken spontaneously. The spontaneous breaking of ordinary gauge symmetry is well understood, but supersymmetry imposes additional conditions which need further discussion. These restrictions rest on the property

$$H \equiv P^0 = \frac{1}{4}\left[\bar{Q}_{\dot 1} Q_1 + Q_1\bar{Q}_{\dot 1} + \bar{Q}_{\dot 2} Q_2 + Q_2\bar{Q}_{\dot 2}\right] \qquad (2)$$

as can be obtained directly from the commutation relations. Eq. (2) tells us that $\langle\Psi|H|\Psi\rangle \geq 0$ for every state $|\Psi\rangle$. Furthermore, it tells us that states with vanishing energy density are supersymmetric ground states of the theory. Such states are ground states because the expectation value of H may never be negative; they are supersymmetric because $\langle 0|H|0\rangle = 0 \Rightarrow Q_\alpha|0\rangle = \bar{Q}^{\dot\alpha}|0\rangle = 0$. Ground states of zero energy preserves supersymmetry, while those of positive energy break it spontaneously. Now we will discuss models which exhibit the general properties of spontaneous symmetry breaking in supersymmetric theories. The supersymmetry is unbroken if and only if $E_{vac} = 0$.

We first consider a supersymmetric model, constructed from scalar superfields, in which the ground state breaks supersymmetry. We know from (19.2) that

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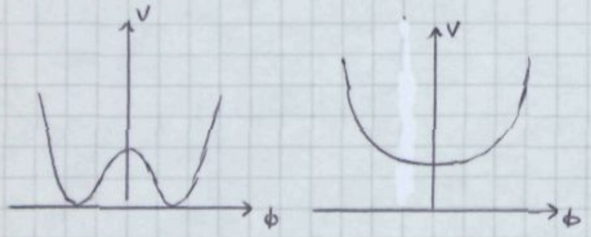
Ref. P. FAYET, and J. ILIOPOULOS Phys. Letters <u>51B</u>, 461 (1974)

P. FAYET. Nucl. Phys. <u>B90</u>, 105 (1975)

* In this respect supersymmetry is quite different from ordinary symmetries. With an ordinary internal symmetry, a symmetric state may exist without being the ground state. In Fig a) is shown a potential scalar which describes a theory with two ground states. In each ground state the scalar field has a vacuum expectation value, possibly breaking some internal symmetry. However, supersymmetry is not spontaneously broken, because the ground state energy, the minimum value of the potential, is zero for each of the two possible states. In Fig b), on the other hand, there is a unique ground state, the scalar field does not have an expectation value, and no internal symmetry is spontaneously broken. However supersymmetry is spontaneously broken because the minimum value of the potential is positive. A state with $V=0$ does not exists.



a) b)

** The spontaneously broken case deserves further discussion. Then $Q_\alpha|0\rangle \equiv |\psi_\alpha\rangle$ which is a fermionic state. We can write $\langle\bar\psi_{\dot\beta}|J^\mu_\alpha|0\rangle = f(\tilde\sigma^\mu)_{\dot\beta\alpha}$ where J^μ_α is the conserved supercurrent with $Q_\alpha \equiv \int d^3x\, J^\mu_\alpha(0,\vec x)$. The supercurrent thus create a fermion out of the vacuum with coupling f, which is a measure for the breakdown of supersymmetry. This situation is analogues to the case of ordinary spontaneously broken symmetries where Goldstone bosons can be created out of the vacuum. In fact an analogue of the Goldstone theorem can be proved for supersymmetry. It proves the existence of a massless fermion in theories with spontaneously broken supersymmetry. This particle is called Goldstone fermion or goldstino. f is the Goldstino decay constant. Notice

$$\{Q_\alpha, \bar Q_{\dot\beta}\} = 2[\sigma^\mu]_{\alpha\dot\beta} P_\mu \Rightarrow \{J^0_\alpha, \bar Q_{\dot\beta}\}_F\ 2[\sigma^\mu]_{\alpha\dot\beta} T_{\mu 0} \Rightarrow T^{\mu\nu}(x) = \frac{1}{2}(\tilde\sigma^\mu)^{\dot\beta\alpha}\{J^\nu_\alpha(x), \bar Q_{\dot\beta}\}$$

Then the vacuum energy density is $E_{vac} = \langle 0|T^{00}(x)|0\rangle = 2\,\mathrm{Re}\, f.$, and E_{vac} measures the strenght of supersymmetry breakdown.

The potential energy in such models takes the form

$$V \equiv F_k^* F_k \qquad F_k^* \equiv - \left[\lambda_k + m_{ik} A_i(x) + g_{ijk} A_i(x) A_j(x) \right] \tag{1}$$

Vacuum expectation values a_i of A_i for which $\langle F_k \rangle = 0$ signal supersymmetric minima of the potential. To break supersymmetry we must choose special values for the parameters λ_k, m_{ik}, g_{ijk} such that the equations

$$0 = \lambda_k + m_{ik} a_i + g_{ijk} a_j a_k \tag{2}$$

has no solution in a_i. Such models have been constructed by O'Raifeartaigh [Nucl. Phys. B96, 331 (1975)]. He found that 3 scalar superfields are required to break supersymmetry, the simplest model being given by $\lambda_0 = \lambda$, $m_{12} = m_{21} = m$, $g_{011} = g_{101} = g_{110} = g$, with all other constant zero* $\lambda + g a_1^2 = 0$, $m a_2 + 2 g a_0 a_1 = 0$, $m a_1 = 0$

P. Fayet and J. Iliopoulos [Phys. Lett. 51B, 461 (1974)] have shown how spontaneously break supersymmetry in gauge theories with Abelian gauge groups. They observed that the $[\theta\theta][\bar\theta\bar\theta]$-component of the vector superfield is both supersymmetric and gauge invariant. They add this term to the Lagrangian (29.2) and find that it spontaneously breaks supersymmetry.

$$L = \frac{1}{4} \left\{ W^\alpha W_\alpha \Big|_{[\theta\theta]} + \bar W_{\dot\alpha} \bar W^{\dot\alpha} \Big|_{[\bar\theta\bar\theta]} \right\} + \Phi_+^+ e^{2eV} \Phi_+ \Big|_{[\theta\theta][\bar\theta\bar\theta]} + \Phi_-^+ e^{-2eV} \Phi_-^+ \Big|_{[\theta\theta][\bar\theta\bar\theta]}$$

$$+ m \left\{ \Phi_+ \Phi_- \Big|_{[\theta\theta]} + \Phi_+^+ \Phi_-^+ \Big|_{[\bar\theta\bar\theta]} \right\} + 2\varkappa V \Big|_{[\theta\theta][\bar\theta\bar\theta]} \tag{3}$$

$$2\varkappa V \Big|_{[\theta\theta][\bar\theta\bar\theta]} = \varkappa D$$

In this model, the potential is given by

$$\mathcal{V} = \frac{1}{2} D^2(x) + F_+^*(x) F_+(x) + F_-^*(x) F_-(x) \tag{4}$$

where $D(x)$, $F_+(x)$, $F_-(x)$ are solutions of the Euler equations

$$D(x) + \varkappa + e \left[A_+^+(x) A_+(x) - A_-^+(x) A_-(x) \right] = 0$$

$$F_+(x) + m A_-^*(x) = 0 \qquad F_-(x) + m A_+^*(x) = 0 \tag{5}$$

There is no solution of (5) which leaves $\mathcal{V} = 0$, so supersymmetry is broken spontaneously.

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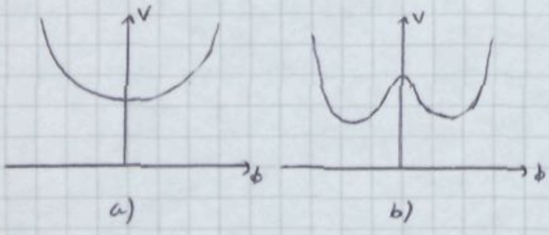
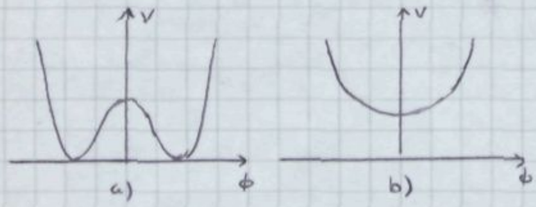
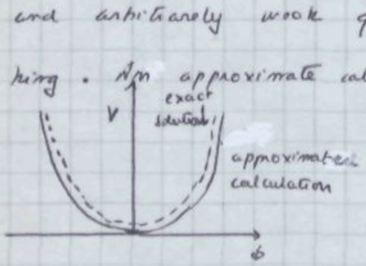
Let us ask under what conditions quantum corrections can change the pattern of symmetry breaking that one finds at the tree level. A simple argument shows that this ordinarely cannot happen in the case of internal symmetries. If an internal symmetry is unbroken at the tree level, this means (Fig a) that the classical potential has its minimum at a symmetrical



point. Broken symmetry means that the classical potential has its minimum away from the origin. It is not possible by means of arbitrary small corrections to turn a potential of type a) into one of type b), or vice-versa. Therefore, sufficiently weak quantum corrections will not break a symmetry that is unbroken at the tree level, nor will they restore broken symmetry. [Some unnatural possible exceptions should be noted E. WITTEN Nucl. Phys. B188, 513 (1981)].



In supersymmetry, these issues must be reconsidered, because the criteria for supersymmetry breaking are rather different. Fig a) corresponds to breaking some internal symmetry but supersymmetry is not broken. Fig b) internal symmetries are not spontaneously broken but supersymmetry is spontaneously broken. It follows from this that if supersymmetry is broken in the tree approximation, then it really is broken in the exact theory, at least if the coupling is weak enough. Arbitrarely weak corrections cannot shift the minimum of the potential from the non-zero value of b) to zero. This could occur, if at all, only if the coupling constant exceeds some critical value. In this respect supersymmetry resembles internal symmetries. If we assume instead that supersymmetry is not broken at the tree level, the situation is very different, and arbitrarely weak quantum corrections could conceivable induce supersymmetry breaking. An approximate calculation including many effects and showing that the vacuum energy is



zero in a certain approximation always leaves open the possibility that even smaller effects that have been neglected break supersymmetry.

Nevertheless if supersymmetry is spontaneously broken, there must exists goldstone fermions. Weak quantum corrections will not bring into being a massless fermion if one does not already exists. If all fermions have non-zero mass at the tree level, weak quantum corrections will not shift any of the fermion masses to zero. Consequently, in any theory in which supersymmetry is not broken at the tree level and in which all fermions have non-zero masses at the tree level, the supersymmetry must be truly unbroken, at least for weak enough coupling. [E. WITTEN]. The small effects that can break supersymmetry must be non-perturbative since it has been shown that if the potential vanishes at some point in field space, then it vanishes at that point to all finite orders of perturbation theory [S. WEINBERG P.L. 62B, 111 (1976)]

Let us examine the potential in more detail. We can write

$$\mathcal{V} = \frac{1}{2}\kappa^2 + [m^2 + \kappa e] A_+^*(x) A_+(x) + [m^2 - \kappa e] A_-^*(x) A_-(x)$$

$$+ \frac{1}{2} e^2 [A_+^*(x) A_+(x) - A_-^*(x) A_-(x)]^2 \qquad (1)$$

We must distinguish between two cases $m^2 > \kappa e$ and $m^2 < \kappa e$.

When $m^2 > \kappa e$, both A_+ and A_- have real masses. The model describes two complex scalar fields one of mass $m_+^2 = m^2 + \kappa e$, the other of mass $m_-^2 = m^2 - \kappa e$, as well as tree spinor fields ψ_+, ψ_-, λ and one vector field. The masses of the spinor and vector fields are unchanged by the symmetry breaking. In particular, the fields $\psi_\pm$ retains its mass m, while λ and v^μ remain massless. Note that $m_+^2 + m_-^2 = 2m^2$. The vector field $v^\mu(x)$ plays the role of gauge field for the unbroken $U(1)$ symmetry group, while λ becomes the Goldstone fermion arising from spontaneously broken supersymmetry. From the transformation law for $\lambda(x)$ (27.1)

$$\delta\lambda(x) = i\xi D(x) - \frac{1}{2}\sigma^\mu\tilde{\sigma}^\nu \xi v_{\mu\nu}(x) \qquad (2)$$

we see that $\lambda(x)$ transforms inhomogeneously as soon as D acquires a vacuum expectation value

$$\delta\lambda(x) = -i\xi\kappa + \ldots \qquad (3)$$

This identifies λ as the Goldstone fermion. Non-zero vacuum expectation values of the auxiliary fields induce the spontaneous breakdown of supersymmetry.

When $m^2 < e\kappa$, $A_+ = A_- = 0$ no longer minimizes the potential (1). To find the minimum we must solve the equations

$$\frac{\partial\mathcal{V}}{\partial A_+^*} = [m^2 + \kappa e] A_+(x) + e^2 [A_+^*(x) A_+(x) - A_-^*(x) A_-(x)] A_+(x) = 0$$

$$\frac{\partial\mathcal{V}}{\partial A_-^*} = [m^2 - \kappa e] A_-(x) - e^2 [A_+^*(x) A_+(x) - A_-^*(x) A_-(x)] A_-(x) = 0 \qquad (1)$$

That gives a minimum at $A_+(x) = 0$ $A_-(x) = v$ where v is the solution of

$$|v|^2 e^2 + (m^2 - \kappa e) = 0 \qquad |v|^2 = -\frac{1}{e^2}(m^2 - \kappa e) \qquad (2)$$

We can choose v real by an adequate gauge transformation. Expanding the potential around its minimum spontaneously breaks the U(1) symmmetry. In terms of $A \equiv A_+(x)$ and $B = A_-(x) - v$ we have

$$L(x) = -\frac{1}{4} v^{\mu\nu}(x)\, v_{\mu\nu}(x) + i\lambda(x)\sigma^\mu\partial_\mu\bar\lambda(x) + \partial^\mu A_+^*(x)\,\partial_\mu A_+(x) +$$

$$+ \partial^\mu A_-^*(x)\,\partial_\mu A_-(x) - i\partial_\mu\bar\psi_+(x)\,\tilde\sigma^\mu\psi_+(x) - i\partial_\mu\bar\psi_-(x)\,\tilde\sigma^\mu\psi_-(x) - m\,\psi_+(x)\,\psi_-(x)$$

$$- m\,\bar\psi_+(x)\,\bar\psi_-(x) + ie\sqrt2\,[A_+\bar\psi_+\bar\lambda - A_+^*\psi_+\lambda - A_-\bar\psi_-\bar\lambda + A_-^*\psi_-\lambda]$$

$$+ e^2 v^\mu v_\mu\,[A_+^*A_+ + A_-^*A_-] + 2ev^\mu\,[-\tfrac12\bar\psi_+\tilde\sigma_\mu\psi_+ + \tfrac12\bar\psi_-\tilde\sigma_\mu\psi_-$$

$$-\tfrac{i}{2}A_+^*\partial_\mu A_+ + \tfrac{i}{2}\partial_\mu A_+^* A_+ + \tfrac{i}{2}A_-^*\partial_\mu A_- - \tfrac{i}{2}\partial_\mu A_-^* A_-] - \mathcal{V}$$

$$\Rightarrow L(x) = -\frac14 v^{\mu\nu}v_{\mu\nu} + i\lambda\sigma^\mu\partial_\mu\bar\lambda + \partial^\mu A^*\partial_\mu A + \partial^\mu B^*\partial_\mu B$$

$$-i\partial_\mu\bar\psi_+\tilde\sigma^\mu\psi_+ - i\partial_\mu\bar\psi_-\tilde\sigma^\mu\psi_- - m\psi_+\psi_- - m\bar\psi_+\bar\psi_- + ie\sqrt2\,[A\bar\psi_+\bar\lambda - A^*\psi_+\lambda]$$

$$+ ie\sqrt2\,[-B\bar\psi_-\bar\lambda + B^*\psi_-\lambda] + ie\sqrt2\,[-v\bar\psi_-\bar\lambda + v\psi_-\lambda] + e^2 v^\mu v_\mu A^*A$$

$$+ e^2 v^\mu v_\mu\,[v^2 + vB + vB^* + BB^*] + 2ev^\mu\,[-\tfrac12\bar\psi_+\tilde\sigma_\mu\psi_+ + \tfrac12\bar\psi_-\tilde\sigma_\mu\psi_-$$

$$-\tfrac{i}{2}A^*\partial_\mu A + \tfrac{i}{2}\partial_\mu A^* A + \tfrac{i}{2}v\partial_\mu B + \tfrac{i}{2}B^*\partial_\mu B - \tfrac{i}{2}\partial_\mu B^* v - \tfrac{i}{2}\partial_\mu B^* B]$$

$$-\frac{m^2}{2e^2}(2e\kappa - m^2) - 2m^2A^*A - \frac12 2e^2v^2\left[\frac{1}{\sqrt2}(B+B^*)\right]^2 - \frac12 e^2A^{*2}A^2$$

$$-\frac12 e^2 B^2 B^{*2} - e^2 v B^2 B^* - e^2 v B^{*2} B + e^2 v A^*AB + e^2 v B^*A^*B + e^2 A^*AB^*B \qquad (1)$$

Now we can see several things

i) The constant $(2e\kappa - m^2)m^2/2e^2 > 0$; both supersymmetry and gauge symmetry are broken spontaneously

ii) The field $A(x)$ has a mass $m_A = (2m^2)^{1/2}$

iii) The real field $[B(x) + B^*(x)]/\sqrt2$ has a mass $m_B = \sqrt{2e^2v^2}$

iv) The vector field adquieres a mass $m_v = \sqrt{2e^2v^2}$ by eating the Goldstone boson field $[B - B^*]/i\sqrt2$ leaving the total number of degrees of freedom unchanged

v) The spinor mass term is now

$$-m\psi_+\psi_- - m\bar\psi_+\bar\psi_- + iev\sqrt2\,[\psi_-\lambda - \bar\psi_-\bar\lambda]$$

Introducing the fields

$$\psi \equiv \psi_-$$

$$\tilde{\psi} = [m^2 + 2e^2v^2]^{-1/2}[m\psi_+ - i\sqrt{2}ev\lambda] \qquad (1)$$

$$\tilde{\lambda} = [m^2 + 2e^2v^2]^{-1/2}[m\lambda - i\sqrt{2}ev\psi_+]$$

The mass term becomes

$$-[m^2 + 2e^2v^2]^{1/2}[\psi\tilde{\psi} + \bar{\psi}\bar{\tilde{\psi}}] \qquad (2)$$

The Goldstone spinor $\tilde{\lambda}$ remains massless. Note that $\tilde{\lambda}$ transforms inhomogeneously

$$\delta\tilde{\lambda} = -i\frac{m}{e}\xi[ex - m^2]^{1/2} + \cdots \qquad (3)$$

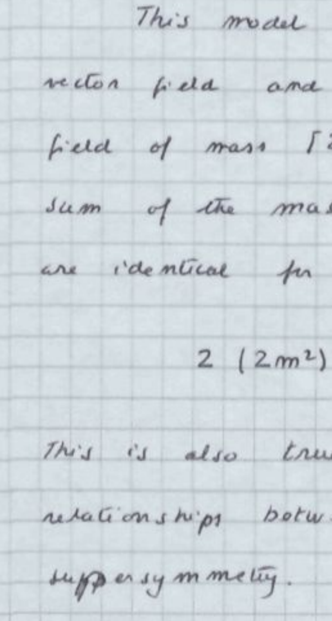
as expected for a Goldstone field. (This is obtained from (1) and (35.2) (27.1)).

This model describes two spinor fields of mass $[m^2 + 2e^2v^2]^{1/2}$, one vector field and one scalar field, each of mass $[2e^2v^2]^{1/2}$, one complex scalar field of mass $[2m^2]^{1/2}$, and one massless Goldstone spinor. Note that the sum of the masses squared weighted by the number of degrees of freedom are identical for bosonic and fermionic modes:

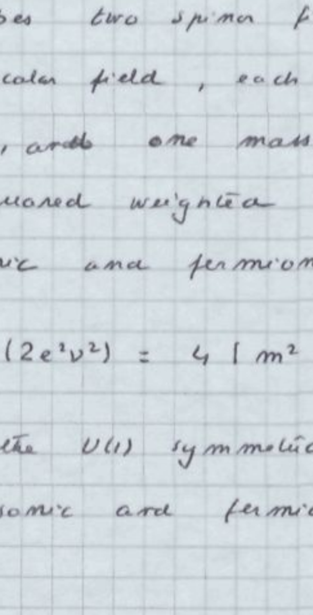
$$2(2m^2) + 4(2e^2v^2) = 4[m^2 + 2e^2v^2]$$

This is also true for the U(1) symmetric case described earlier. In fact, such relationships between bosonic and fermionic masses are to be expected from supersymmetry.

The situation for the Fayet-Iliopoulos model is sketched in the figure.



$m^2 > ex$. Supersymmetry alone is broken



$m^2 < ex$, both gauge symmetry and supersymmetry are broken

Non-vanishing vacuum expectation values of auxiliary fields induce supersymmetry breaking, while non-zero vacuum expectation values of dynamical scalar fields lead to the breaking of gauge symmetry.

After having seen a model in which supersymmetry and gauge symmetry are broken spontaneously, one might wish to construct a model in which only the gauge symmetry is broken. We first discuss such models with scalar superfields. In this case we must find a solution a_i to (34.2) which is not left invariant under the internal symmetry group. As a simple example, we consider the group U(1) with three scalar superfields one neutral, one positive, and one negative. The Lagrangian

$$L_{P.E} = \frac{1}{2} m^2 \Phi^2 + \mu^2 \Phi_+ \Phi_- + \lambda \Phi + g \Phi \Phi_+ \Phi_- + \text{h.c.} \qquad (1)$$

is U(1) invariant. The equations (34.2) becomes. [See (19.2)]

$$\begin{aligned} &\lambda + m^2 a + g a_+ a_- = 0 \\ &a_- (\mu^2 + g a) = 0 \\ &a_+ (\mu^2 + g a) = 0 \end{aligned} \qquad (2)$$

This set of equations has two solutions

$$\text{a)} \quad a_+ = a_- = 0 \qquad a = -\frac{\lambda}{m^2} \qquad (3)$$

$$\text{b)} \quad a_+ a_- = -\frac{1}{g} \left[\lambda - \frac{m^2 \mu^2}{g} \right] \qquad a = -\frac{\mu^2}{g}$$

The first does not break the U(1) symmetry, but the second does. In the second solution, only the product $a_+ a_-$ is determined. This stems from the fact that $L_{P.E.}$ is invariant not only under the U(1) group, but also its complex extension. For any solution a_+, a_- of (3), there exists an entire class of solutions $e^{\lambda} a_+$, $e^{-\lambda} a_+$ for arbitrary complex λ. The ground state has a larger degeneracy that required by the initial symmetry group.

If we gauge the Lagrangian (1) we must introduce a vector superfield V, coupling to Φ_+ and Φ_- as in (34.3). This results in the following trilinear coupling between the scalar fields $A_\pm$ and the vector multiplet

$$2e\, V\, [\, A_+^* A_+ - A_-^* A_- \,] \qquad (4)$$

For the symmetry breaking solution, this contributes a piece to the D-term

$$2eV\, [\, a_+^* a_+ - a_-^* a_- + \frac{\varkappa}{e} \,] \qquad (5)$$

Such term would ordinarily break supersymmetry. Because of the degeneracy $a_\pm \to e^{\pm\lambda} a_\pm$, however, it is possible to transform away this term for any choice of κ. In this model, D-terms do not induce the spontaneous breakdown of supersymmetry.

The mass term associated to (38.3) is given by

$$2e^2 \left[a_+^* a_+ + a_-^* a_- \right] V^2 \tag{1}$$

It cannot be transformed away. It gives a mass to the vector field v_μ. Comparing (1) with (25.4), we see that spontaneous gauge symmetry breaking in supersymmetric theories gives rise to an entire massive vector multiplet. This is the supersymmetric extension of the Higgs-Kibble mechanism.

These models are easily extended to non-Abelian symmetry groups. Supersymmetric solutions require $F_k = 0$ where F_k is given by (34.1). The parameters m, g and λ are restricted by the internal symmetry group. In gauge theories, the additional equation

$$V^c a_i^+ T_{ik}^c a_k = 0 \tag{2}$$

must also be satisfied for supersymmetric minima of the potential. (See comments on pag. 34) *

We have found superfields very useful for the construction of supersymmetry representations and invariant Lagrangians. Now we shall see that they also simplify the calculation of radiative corrections in quantized supersymmetric theories. The Feynman rules for supersymmetric theories may be stated in terms of superfield vertices and propagators. Many component-field Feynman diagrams are contained in one superfield diagram, so many miraculous cancellations between component diagrams are manifest in one superfield diagram. For this reason alone one would like to find a superfield formulation of supersymmetric theories.

To derive superfield propagators we must first introduce the concept of integration in superspace. An indefinite integral over a Grassman variable η is defined as follows

$$\int d\eta = 0 \qquad\qquad \int d\eta\ \eta = 1 \tag{3}$$

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Ref.. S. FERRARA and P. PIGUET Nucl. Phys. B93, 261 (1975)

M.T. GRISARU, W. SIEGEL and M. ROČEK. Nucl. Phys. B159, 429 (1979)

* These models are easily extended to non-Abelian symmetry groups. Supersymmetric solutions require

$$F_k^* = -[\lambda_k + m_{ik} a_i + g_{ijk} a_i a_j] = 0 \qquad (1)$$

The parameters λ, m, and g are restricted by the internal symmetry group. In gauge theories, supersymmetric minima must also satisfy

$$D^e \equiv a_i^+ T^e_{ik} a_k = 0 \qquad (2)$$

The Fayet - Iliopoulos D-term is not gauge invariant and cannot appear in the non-Abelian sector of supersymmetric models.

In the remainder of this note we shall show that (1) determines the supersymmetric breaking of non-Abelian theories. That is, if (1) has a solution a_i, then it is always possible to find a solution $\hat{a}_i$ which satisfies (2) as well. We shall demonstrate this for the case of a semisimple gauge group G. To begin, let us suppose we have found a solution a_i such that $F_k^*(a_i) = 0$. We may then compute

$$D^e = a_i^+ T^e_{ik} a_k \qquad (3)$$

The vector D^e specifies a certain direction in the regular representation. There is always a group element which transforms this vector into a linear combination of vectors in the Cartan subalgebra. Because (1) is invariant under G, this transformation rotates the a_i into another solution $\tilde{a}_i$. The vector D^e transforms into a vector $\tilde{D}^e$ whose nonvanishing components lie in the Cartan subalgebra. We may now perform a linear transformation within the Cartan subalgebra such that the direction $\tilde{D}^e$ defines a single generator with eigenvalues μ_i. In this basis, the only non-vanishing component of $\tilde{D}^e$ is $\tilde{D}$:

$$\tilde{D} = \tilde{a}_i^+ \mu_i a_i \qquad (4)$$

The equations (1) are also invariant under gauge transformations with complex group parameters. This is because the complex conjugate representations of the scalar fields never enter F^*. We are free to perform such a transformation in the direction $\tilde{D}$:

$$\hat{a}_i = \exp\{\mu_i \eta\} \tilde{a}_i \qquad (5)$$

The parameters $\hat{a}_i$ solve (1) for all values of η. Taking η real, we find

$$\hat{D} = \tilde{a}_i^+ \mu_i e^{2\mu_i \eta} \tilde{a}_i \qquad (6)$$

We now distinguish two cases. In the first case, all the μ_i (for which $\tilde{a}_i \neq 0$) are of the same sign, say positive. We then let $\eta \to -\infty$ to find $\hat{D} = 0$. In the second case, the μ_i take both signs. We shall show that there is still a value of η where $\hat{D} = 0$. In particular we note that

$$\hat{D} = \frac{1}{2}\frac{\partial}{\partial\eta} \tilde{a}_i^+ e^{2\mu_i\eta} \tilde{a}_i \qquad (7)$$

Considering $\tilde{a}_i^+ e^{2\mu_i\eta}\tilde{a}_i$ as a function of η, we see that it tends to $+\infty$ as $\eta \to \pm\infty$. Therefore →

Any function of η is a polynomial $f(\eta) = c + \Delta\eta$, so definition (39.3) extends immediately to arbitrary functions of grassman variables

$$f(\eta) = c + \Delta\eta \qquad \int d\eta\, f(\eta) = \Delta \quad , \quad \int d\eta\, \eta\, f(\eta) = c \tag{4}$$

Notice that $\eta' = a\eta$ implies $d\eta' = d\eta/a$. Since $\int d\eta \frac{\partial f(\eta)}{\partial\eta} = 0$, partial integration is always possible. Note that integration and differentiation give the same result on functions of grassman variables. Delta functions are defined by the integral

$$\int d\eta\, \delta(\eta)\, f(\eta) = f(0) \tag{5}$$

From (4) it follows $\delta(\eta) \equiv \eta$ so consequently $\delta(\eta)\,\delta(\eta) = 0$. Defining the volume elements in superspace as

$$d^2\theta \equiv -\frac{1}{4}\epsilon_{\alpha\beta}\, d\theta^\alpha d\theta^\beta \quad , \quad d^2\bar\theta \equiv -\frac{1}{4}\epsilon^{\dot\alpha\dot\beta}\, d\bar\theta_{\dot\alpha}\, d\bar\theta_{\dot\beta} \quad , \quad d^2\theta\, d^2\bar\theta \equiv d^4\theta \tag{6}$$

we find

$$\int d^2\theta\; \theta\theta = -\frac{1}{4}\epsilon_{\alpha\beta}\,\epsilon_{\gamma\delta}\int d\theta^\alpha d\theta^\beta\, \theta^\gamma\theta^\delta =$$

$$= \frac{1}{4}\left\{ -\int d\theta^1 d\theta^2\, \theta^1\theta^2 + \int d\theta^1 d\theta^2\, \theta^2\theta^1 + \int d\theta^2 d\theta^1\, \theta^1\theta^2 + \int d\theta^2 d\theta^1\, \theta^2\theta^1 \right\} = 1$$

i.e.

$$\int d^2\theta\; \theta\theta = 1 \qquad \int d^2\bar\theta\; \bar\theta\bar\theta = 1 \tag{7}$$

This allows us to write the Lagrangian (27.4) as an integral over superspace

$$L(x) = \int d^2\theta\, d^2\bar\theta \left\{ \Phi_i^+ \Phi_i + \frac{1}{2} m_{ij}\, \Phi_i \Phi_j\, \delta(\bar\theta) + \frac{1}{2} m_{ij}^*\, \Phi_i^+ \Phi_j^+\, \delta(\theta) \right.$$

$$\left. + \frac{1}{3}\lambda_{ijk}\, \Phi_i \Phi_j \Phi_k\, \delta(\bar\theta) + \frac{1}{3}\lambda^*_{ijk}\, \Phi_i^+ \Phi_j^+ \Phi_k^+\, \delta(\bar\theta) \right\} \tag{8}$$

Perturbation theory in superspace may be developed as a direct extension of ordinary perturbation theory. In particular, one would like to calculate superfield Green's functions

$$\langle 0 | T\{ \Phi(x^1, \theta^1, \bar\theta^1) \;\ldots\ldots\; \Phi(x^\ell \theta^\ell \bar\theta^\ell)\, \Phi^+(x^{\ell+1} \theta^{\ell+1} \bar\theta^{\ell+1}) \ldots \Phi^+(x^r, \theta^r, \bar\theta^r)\} | 0 \rangle \tag{9}$$

It has minimum for some value of g. At this point the derivative vanishes and $\hat{D}=0$

This completes the proof. We have shown that spontaneous supersymmetry breaking in non-Abelian models is controlled by F-terms. Supersymmetry is spontaneously broken if and only if the equations $F_k^* = 0$ have no solution. This is the O'Raifeartaigh mechanism for supersymmetry breaking.

From these one recovers the component-field Green's function by power series expansion in $\theta^1, \bar{\theta}^1, \ldots \theta^r, \bar{\theta}^r$.

As in any field theory, we begin our analysis by evaluating the free field two-point functions, the propagators. For chiral fields these are derived from the free-field part of the Lagrangian (40.8).

$$L_0(x) = \int d^2\theta \, d^2\bar{\theta} \left\{ \Phi^+ \Phi + \frac{1}{2} m \, \Phi \Phi \, \delta(\bar{\theta}) + \frac{1}{2} m \, \Phi^+ \Phi^+ \delta(\theta) \right\}$$

$$= \partial^\mu A^*(x) \, \partial_\mu A(x) - i \partial_\mu \bar{\psi}(x) \, \tilde{\sigma}^\mu \psi(x) + F^*(x) F(x)$$

$$+ m [A(x) F(x) + A^*(x) F^*(x)] - \frac{1}{2} m [\psi(x) \psi(x) + \bar{\psi}(x) \bar{\psi}(x)] \qquad (1)$$

Let us begin considering the scalar sector. Let us introduce real fields

$$A(x) = \frac{1}{\sqrt{2}} [A_1(x) + i A_2(x)] \qquad F(x) = \frac{1}{\sqrt{2}} [F_1(x) + i F_2(x)]$$

$$L_{OB}(x) = \frac{1}{2} \partial^\mu A_1(x) \partial_\mu A_1(x) + \frac{1}{2} \partial^\mu A_2(x) \partial_\mu A_2(x) + \frac{1}{2} F_1^2(x) + \frac{1}{2} F_2^2(x)$$

$$+ m A_1(x) F_1(x) - m A_2(x) F_2(x) \equiv \frac{1}{2} \phi_i(x) W_{ij} \phi_j(x)$$

$$\phi_i(x) \equiv [A_1(x), A_2(x), F_1(x), F_2(x)] \qquad W_{ij} = \begin{vmatrix} \overleftarrow{\partial}^\mu \overrightarrow{\partial}_\mu & 0 & m & 0 \\ 0 & \overleftarrow{\partial}^\mu \overrightarrow{\partial}_\mu & 0 & -m \\ m & 0 & 1 & 0 \\ 0 & -m & 0 & 1 \end{vmatrix}$$

$$\hat{W}_{ij} = \begin{vmatrix} p^2 & 0 & m & 0 \\ 0 & p^2 & 0 & -m \\ m & 0 & 1 & 0 \\ 0 & -m & 0 & 1 \end{vmatrix}$$

$$\hat{W}_{ij} = \frac{1}{p^2 - m^2 + i\epsilon} \begin{vmatrix} 1 & 0 & -m & 0 \\ 0 & 1 & 0 & +m \\ -m & 0 & p^2 & 0 \\ 0 & +m & 0 & p^2 \end{vmatrix}$$

$$\langle 0|T(A(x)A^*(y))|0\rangle = \frac{i}{(2\pi)^4}\int d^4p\, e^{-ip(x-y)} \frac{1}{p^2-m^2+i\epsilon}$$

$$\langle 0|T(A(x)F(y))|0\rangle = \langle 0|T(A^*(x)F^*(y))|0\rangle = \frac{i}{(2\pi)^4}\int d^4p\, e^{-ip(x-y)} \frac{-m}{p^2-m^2+i\epsilon}$$

$$\langle 0|T(F(x)F^*(y))|0\rangle = \frac{i}{(2\pi)^4}\int d^4p\, e^{-ip.(x-y)} \frac{p^2}{p^2-m^2+i\epsilon}$$

Using

$$\Delta_F(x;m^2) \equiv \frac{1}{(2\pi)^4}\int d^4p\, e^{-ip.x} \frac{1}{p^2-m^2+i\epsilon} \quad (1)$$

we get as only non-zero propagators

$$\langle 0|T(A(x)A^*(y))|0\rangle = i\,\Delta_F(x-y;m^2)$$

$$\langle 0|T(A(x)F(y))|0\rangle = \langle 0|T(A^*(x)F^*(y))|0\rangle = -im\,\Delta_F(x-y;m^2) \quad (2)$$

$$\langle 0|T(F(x)F^*(y))|0\rangle = -i\,\partial^\mu\partial_\mu\,\Delta_F(x-y;m^2)$$

Now let us consider the fermionic sector

$$L_{0F}(x) = -i\partial_\mu\bar\psi(x)\,\tilde\sigma^\mu\,\psi(x) - \frac{1}{2}m\,\psi(x)\psi(x) - \frac{1}{2}m\,\bar\psi(x)\bar\psi(x) =$$

$$= -\frac{i}{2}\partial_\mu\bar\psi(x)\,\tilde\sigma^\mu\psi(x) + \frac{i}{2}\psi(x)\sigma^\mu\partial_\mu\bar\psi(x) - \frac{1}{2}m\,\psi(x)\psi(x) - \frac{1}{2}m\,\bar\psi(x)\bar\psi(x)$$

Using the Fourier transform $\vec\partial_\mu \to -ip_\mu$ $\overleftarrow\partial_\mu \to +ip_\mu$ we can write

$$L_{0F}(x) = \frac{1}{2}\bar\psi_{\dot\alpha}(p_\mu\tilde\sigma^\mu)^{\dot\alpha\beta}\psi_\beta + \frac{1}{2}\psi^\alpha(p_\nu\sigma^\nu)_{\alpha\dot\beta}\bar\psi^{\dot\beta}$$

$$- \frac{1}{2}m\,\psi^\alpha\psi_\alpha - \frac{1}{2}m\,\bar\psi_{\dot\alpha}\bar\psi^{\dot\alpha} =$$

$$= \frac{1}{2}[\psi^\alpha, \bar\psi_{\dot\alpha}] \begin{vmatrix} -m\,\delta_\alpha{}^\beta & (p_\nu\sigma^\nu)_{\alpha\dot\beta} \\ (p_\mu\tilde\sigma^\mu)^{\dot\alpha\beta} & -m\,\delta^{\dot\alpha}{}_{\dot\beta}\end{vmatrix} \begin{vmatrix}\psi_\beta \\ \bar\psi^{\dot\beta}\end{vmatrix}$$

$$(p_\nu\sigma^\nu) \equiv \begin{vmatrix} p_0+p_3 & p_1-ip_2 \\ p_1+ip_2 & p_0-p_3\end{vmatrix} \qquad (p_\mu\tilde\sigma^\mu) \equiv \begin{vmatrix} p_0-p_3 & -p_1+ip_2 \\ -p_1-ip_2 & p_0+p_3\end{vmatrix}$$

Hence

$$\hat{W}_{ij} = \begin{vmatrix} -m & 0 & p_0+p_3 & p_1-ip_2 \\ 0 & -m & p_1+ip_2 & p_0-p_3 \\ p_0-p_3 & -p_1+ip_2 & -m & 0 \\ -p_1-ip_2 & p_0+p_3 & 0 & -m \end{vmatrix}$$

$$\hat{W}^{-1}_{ij} = \frac{1}{p^2-m^2+i\epsilon} \begin{vmatrix} m & 0 & p_0+p_3 & p_1-ip_2 \\ 0 & m & p_1+ip_2 & p_0-p_3 \\ p_0-p_3 & -p_1+ip_2 & m & 0 \\ -p_1-ip_2 & p_0+p_3 & 0 & m \end{vmatrix}$$

$$\hat{W}^{-1} = \frac{1}{p^2-m^2+i\epsilon} \begin{vmatrix} m\,\delta_\alpha{}^\beta & (p_\nu\sigma^\nu)_{\alpha\dot{\beta}} \\ (p_\mu\tilde{\sigma}^\mu)^{\dot{\alpha}\beta} & m\,\delta^{\dot{\alpha}}{}_{\dot{\beta}} \end{vmatrix}$$

$$\langle 0|T(\psi_\alpha(x)\,\psi^\beta(y))|0\rangle = \frac{i}{(2\pi)^4}\int d^4p\; e^{-ip(x-y)}\;\frac{m\,\delta_\alpha{}^\beta}{p^2-m^2+i\epsilon}$$

$$\langle 0|T(\bar{\psi}^{\dot{\alpha}}(x)\,\bar{\psi}_{\dot{\beta}}(y))|0\rangle = \frac{i}{(2\pi)^4}\int d^4p\; e^{-ip(x-y)}\;\frac{m\,\delta^{\dot{\alpha}}{}_{\dot{\beta}}}{p^2-m^2+i\epsilon}$$

$$\langle 0|T(\psi_\alpha(x)\,\bar{\psi}_{\dot{\beta}}(y))|0\rangle = \frac{i}{(2\pi)^4}\int d^4p\; e^{-ip(x-y)}\;\frac{(p_\mu\sigma^\mu)_{\alpha\dot{\beta}}}{p^2-m^2+i\epsilon}$$

$$\langle 0|T(\bar{\psi}^{\dot{\alpha}}(x)\,\psi^\beta(y))|0\rangle = \frac{i}{(2\pi)^4}\int d^4p\; e^{-ip\cdot(x-y)}\;\frac{(p_\mu\tilde{\sigma}^\mu)^{\dot{\alpha}\beta}}{p^2-m^2+i\epsilon}$$

Hence

$$\langle 0|T(\psi_\alpha(x)\,\psi^\beta(y))|0\rangle = i m\,\delta_\alpha{}^\beta\,\Delta_F(x-y;m^2)$$

$$\langle 0|T(\bar{\psi}^{\dot{\alpha}}(x)\,\bar{\psi}_{\dot{\beta}}(y))|0\rangle = i m\,\delta^{\dot{\alpha}}{}_{\dot{\beta}}\,\Delta_F(x-y;m^2) \qquad (1)$$

$$\langle 0|T(\psi_\alpha(x)\,\bar{\psi}_{\dot{\beta}}(y))|0\rangle = i\,[+i\,(\sigma^\mu)_{\alpha\dot{\beta}}\,\partial_\mu]\,\Delta_F(x-y;m^2)$$

$$\langle 0|T(\bar{\psi}^{\dot{\alpha}}(x)\,\psi^\beta(y))|0\rangle = i\,[+i\,(\tilde{\sigma}^\mu)^{\dot{\alpha}\beta}\,\partial_\mu]\,\Delta_F(x-y;m^2)$$

Hence (42.2) and (43.1) are all the non-zero propagators for the free fields.

We may use these component propagators to construct the superfields propagators. For example

$\langle 0| T(\Phi(y,\theta)\ \Phi(y',\theta'))|0\rangle =$

$\langle 0| T\{ [A(y) + \sqrt{2}\,\theta\psi(y) + (\theta\theta)F(y)][A(y') + \sqrt{2}\,\theta'\psi(y') + (\theta'\theta')F(y')]\}|0\rangle$

$= (\theta'\theta')\ \langle 0|T(A(y)F(y'))|0\rangle + 2\,\theta^\alpha\theta'^\beta\ \langle 0|T(\psi_\alpha(y)\psi_\beta(y'))|0\rangle$

$+ (\theta\theta)\ \langle 0|T(F(y)A(y'))|0\rangle =$

$= (\theta'\theta')(-i)m\,\Delta_F(y-y';m^2) + 2\theta^\alpha\theta'_\beta\, i m\,\delta_\alpha{}^\beta\,\Delta_F(y-y';m^2)$

$+ (\theta\theta)(-i)m\,\Delta_F(y-y';m^2) =$

$= i m\,\Delta_F(y-y';m^2)\,[-(\theta'\theta') + 2(\theta\theta') - (\theta\theta)]$

Hence

$$\langle 0|T(\Phi(y,\theta),\Phi(y',\theta'))|0\rangle = -im\,(\theta-\theta')^2\,\Delta_F(y-y';m^2)$$

$$\langle 0|T(\Phi^+(y^+,\bar\theta),\Phi^+(y'^+,\bar\theta'))|0\rangle = -im\,(\bar\theta-\bar\theta')^2\,\Delta_F(y^+-y'^+;m^2) \qquad (1)$$

$$\langle 0|T(\Phi(y,\theta),\Phi^+(y'^+\bar\theta'))|0\rangle = \langle 0|T(\Phi^+(y'^+\bar\theta')\ \Phi(y\,\theta))|0\rangle =$$

$$= i\,\{ I + 2i\,\theta\sigma^\mu\bar\theta'\,\partial_\mu - (\theta\theta)(\bar\theta'\bar\theta')\,\partial^\mu\partial_\mu\}\,\Delta_F(y-y'^+;m^2)$$

From $y^\mu = x^\mu + i\theta\sigma^\mu\bar\theta$ $\quad y^{\mu+} = x^\mu + i\theta\sigma^\mu\bar\theta$ we get

$\langle 0|T(\Phi(x,\theta,\bar\theta),\ \Phi(x',\theta',\bar\theta'))|0\rangle =$

$= -im\,\delta(\theta-\theta')\exp\{-i[\theta\sigma^\mu\bar\theta - \theta'\sigma^\mu\bar\theta']\partial_\mu\}\,\Delta_F(x-x';m^2)$

$\langle 0|T(\Phi^+(x,\theta,\bar\theta)\ ,\ \Phi^+(x',\theta',\bar\theta'))|0\rangle = \qquad (2)$

$= -im\,\delta(\bar\theta-\bar\theta')\exp\{+i[\theta\sigma^\mu\bar\theta - \theta'\sigma^\mu\bar\theta']\partial_\mu\}\,\Delta_F(x-x';m^2)$

$\langle 0|T(\Phi(x,\theta,\bar\theta)\ \Phi^+(x',\theta',\bar\theta'))|0\rangle = \langle 0|T(\Phi^+(x'\theta'\bar\theta')\ \Phi(x\,\theta\,\bar\theta))|0\rangle =$

$= i\exp\{-i[\theta\sigma^\mu\bar\theta + \theta'\sigma^\mu\bar\theta' - 2\theta\sigma^\mu\bar\theta']\partial_\mu\}\,\Delta_F(x-x';m^2)$

Now we can define the superpropagator in momentum space in the usual way

$$S_{\Phi\Phi}(p;\theta\bar\theta,\theta'\bar\theta') \equiv -i\int d^4x\, e^{ip(x-x')} \langle 0|T(\Phi(x,\theta,\bar\theta),\Phi(x',\theta',\bar\theta'))|0\rangle \tag{1}$$

Hence

$$S_{\Phi\Phi}(p;\theta\bar\theta,\theta'\bar\theta') = -m\,\delta(\theta-\theta')\frac{1}{(2\pi)^4}\int d^4x\, d^4q\,\frac{1}{q^2-m^2+i\epsilon}\,e^{ip(x-x')}$$

$$\exp\left[-i(\theta\sigma^\mu\bar\theta-\theta'\sigma^\mu\bar\theta')\partial^x_\mu\right] e^{-iq\cdot(x-x')} =$$

$$= -m\,\delta(\theta-\theta')\frac{1}{(2\pi)^4}\int d^4x\, d^4q\,\frac{1}{q^2-m^2+i\epsilon}\,e^{i(p-q)(x-x')}\exp\left[-(\theta\sigma^\mu\bar\theta-\theta'\sigma^\mu\bar\theta')q_\mu\right]$$

Therefore

$$S_{\Phi\Phi}(p;\theta\bar\theta,\theta'\bar\theta') = -m\,\delta(\theta-\theta')\frac{1}{p^2-m^2+i\epsilon}\,e^{-(\theta\sigma^\mu\bar\theta-\theta'\sigma^\mu\bar\theta')p_\mu}$$

$$S_{\Phi^+\Phi^+}(p;\theta\bar\theta,\theta'\bar\theta') = -m\,\delta(\bar\theta-\bar\theta')\frac{1}{p^2-m^2+i\epsilon}\,e^{+(\theta\sigma^\mu\bar\theta-\theta'\sigma^\mu\bar\theta')p_\mu} \tag{2}$$

$$S_{\Phi\Phi^+}(p,\theta\bar\theta,\theta'\bar\theta') = \frac{1}{p^2-m^2+i\epsilon}\,e^{-[\theta\sigma^\mu\bar\theta+\theta'\sigma^\mu\bar\theta'-2\theta\sigma^\mu\bar\theta']p_\mu}$$

With these propagators we can evaluate the superfield green's functions to any order in perturbation theory writting all in terms of free fields and using Wick's theorem. Let us take as interaction

$$L_I(x) = \frac{g}{3!}\int d^2\theta\, d^2\bar\theta' \left\{ \delta(\bar\theta)\,\Phi^3(x,\theta,\bar\theta) + \delta(\theta)\,\Phi^{+3}(x,\theta,\bar\theta)\right\} \tag{3}$$

As an example we would like to consider the one loop correction to the propagator. We have

$$\langle 0|T(\Phi(x\theta\bar\theta)\,\Phi(x'\theta'\bar\theta'))|0\rangle_c^{(2)} \equiv$$

$$= -\frac{g^2}{72}\int d^4x_1\, d^4x_2\, d^2\theta_1\, d^2\bar\theta_1\, d^2\theta_2\, d^2\bar\theta_2 \Big\{ \langle 0|T\big(\Phi(x)\Phi(x')\Phi^3(x_1)\Phi^3(x_2)\,\delta(\bar\theta_1)\delta(\bar\theta_2)$$

$$+\Phi(x)\Phi(x')\Phi^3(x_1)\Phi^{+3}(x_2)\,\delta(\bar\theta_1)\delta(\theta_2) + \Phi(x)\Phi(x')\Phi^{+3}(x_1)\Phi^3(x_2)\,\delta(\theta_1)\delta(\bar\theta_2)$$

$$+\Phi(x)\Phi(x')\Phi^{+3}(x_1)\Phi^{+3}(x_2)\,\delta(\theta_1)\delta(\theta_2)\big)|0\rangle \Big\} =$$

$$= -\frac{g^2}{2}\int d^4x_1\, d^4x_2\, d^2\theta_1\, d^2\bar\theta_1\, d^2\theta_2\, d^2\bar\theta_2 \,.$$

$$\Big\{ \delta(\bar\theta_1)\delta(\bar\theta_2)\big[\Phi(x)\Phi(x_1)\,\Phi(x_1)\Phi(x_2)\,\Phi(x_2)\Phi(x_2)\,\Phi(x_1)\Phi(x') + \Phi(x)\Phi(x_1)\,\Phi(x_1)\Phi(x_2)\,\Phi(x_2)\Phi(x_1)\,\Phi(x_2)\Phi(x')\big]$$

$$+\delta(\theta_1)\delta(\theta_2)\big[\Phi(x)\Phi^+(x_1)\,\Phi^+(x_1)\Phi^+(x_2)\,\Phi^+(x_2)\Phi^+(x_2)\,\Phi^+(x_1)\Phi(x') + \Phi(x)\Phi^+(x_1)\,\Phi^+(x_1)\Phi^+(x_2)\,\Phi^+(x_2)\Phi^+(x_1)\,\Phi^+(x_2)\Phi(x')\big]$$

$$+\delta(\bar\theta_1)\delta(\theta_2)\big[\Phi(x)\Phi(x_1)\,\Phi(x')\Phi(x_1)\,\Phi(x_1)\Phi^+(x_2)\,\Phi^+(x_2)\Phi^+(x_2) + \Phi(x)\Phi(x_1)\,\Phi(x')\Phi^+(x_2)\,\Phi(x_1)\Phi^+(x_2)\,\Phi(x_1)\Phi^+(x_2)\big]$$

$$+\delta(\bar\theta_1)\delta(\theta_2)\big[\Phi(x)\Phi^+(x_2)\,\Phi(x')\Phi^+(x_2)\,\Phi(x_1)\Phi(x_1)\,\Phi(x_1)\Phi^+(x_2) + \Phi(x)\Phi^+(x_2)\,\Phi(x')\Phi(x_1)\,\Phi(x_1)\Phi^+(x_2)\,\Phi(x_1)\Phi^+(x_2)\big]\Big\}$$

Notice that the 1st, 3rd, 5th and 7th terms correspond to tadpole graphs and they give contributions proportional to $\delta(\theta_1-\theta_1)=\delta(0)=0$. Therefore there are no non-vanishing tadpole graphs in this theory. The second term contains $\delta(\theta_1-\theta_2)\,\delta(\theta_1-\theta_2)=0$. It is easy to see that all closed-loop diagrams vanish when they contain only $\Phi\Phi$ (or $\Phi^+\Phi^+$) propagators. This follows immediately from the fact that they are proportional to $\delta(0)$ in θ ($\bar\theta$) space. The two remaining terms correspond to the Feynman superdiagrams

$$S^{(2)}_{\Phi\Phi}(p;\theta\bar\theta,\theta'\bar\theta') = +\frac{ig^2}{2}\int d^4x\, d^4x_1\, d^4x_2\, d^2\theta_1\, d^2\bar\theta_1\, d^2\theta_2\, d^2\bar\theta_2\, \delta(\bar\theta_1)\delta(\theta_2)\, e^{ip.(x-x')}$$

$$\Big\{\Phi(x\theta\bar\theta)\Phi(x_1\theta_1\bar\theta_1)\,\Phi(x'\theta'\bar\theta')\Phi^+(x_2\theta_2\bar\theta_2)\,\Phi(x_1\theta_1\bar\theta_1)\Phi^+(x_2\theta_2\bar\theta_2)\,\Phi(x_1\theta_1\bar\theta_1)\Phi^+(x_2\theta_2\bar\theta_2) +$$

$$+\Phi(x\theta\bar\theta)\Phi^+(x_2\theta_2\bar\theta_2)\,\Phi(x'\theta'\bar\theta')\Phi(x_1\theta_1\bar\theta_1)\,\Phi(x_1\theta_1\bar\theta_1)\Phi^+(x_2\theta_2\bar\theta_2)\,\Phi(x_1\theta_1\bar\theta_1)\Phi^+(x_2\theta_2\bar\theta_2)\Big\}$$

and equivalent way of writting this is

$$S^{(2)}_{\Phi\Phi}(p,\theta\bar\theta,\theta'\bar\theta') = \frac{1}{2} i g^2 \int d^4x\, d^4x_1\, d^4x_2\, d^2\theta_1\, d^2\bar\theta_1\, d^2\theta_2\, d^2\bar\theta_2\, \delta(\bar\theta_1)\,\delta(\theta_2)\, e^{-ip.(x-x')}$$

$$\Big\{ \underline{\Phi(x\theta\bar\theta)\,\Phi(x_1\theta_1\bar\theta_1)}\ \underline{\Phi(x_1\theta_1\bar\theta_1)\,\Phi^+(x_2\theta_2\bar\theta_2)}\ \underline{\Phi^+(x_2\theta_2\bar\theta_2)\,\Phi(x_1\theta_1\bar\theta_1)}\ \underline{\Phi^+(x_2\theta_2\bar\theta_2)\,\Phi(x'\theta'\bar\theta')} +$$

$$+ \underline{\Phi(x\theta\bar\theta)\,\Phi^+(x_2\theta_2\bar\theta_2)}\ \underline{\Phi^+(x_2\theta_2\bar\theta_2)\,\Phi(x_1\theta_1\bar\theta_1)}\ \underline{\Phi(x_1\theta_1\bar\theta_1)\,\Phi^+(x_2\theta_2\bar\theta_2)}\ \underline{\Phi(x_1\theta_1\bar\theta_1)\,\Phi(x'\theta'\bar\theta')} \Big\}$$

$$= \frac{ig^2}{2} \int d^4x\, d^4x_1\, d^4x_2\, d^2\theta_1\, d^2\bar\theta_1\, d^2\theta_2\, d^2\bar\theta_2\, \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4p_3}{(2\pi)^4}\frac{d^4p_4}{(2\pi)^4}\, \delta(\bar\theta_1)\,\delta(\theta_2)\,.$$

$$\Big\{ S_{\Phi\Phi}(p_1;\theta\bar\theta,\theta_1\bar\theta_1)\, e^{-ip_1(x-x_1)}\, S_{\Phi\Phi^+}(p_2;\theta_1\bar\theta_1,\theta_2\bar\theta_2)\, e^{-ip_2(x_1-x_2)}\, S_{\Phi^+\Phi}(p_3;\theta_2\bar\theta_2,\theta_1\bar\theta_1)$$

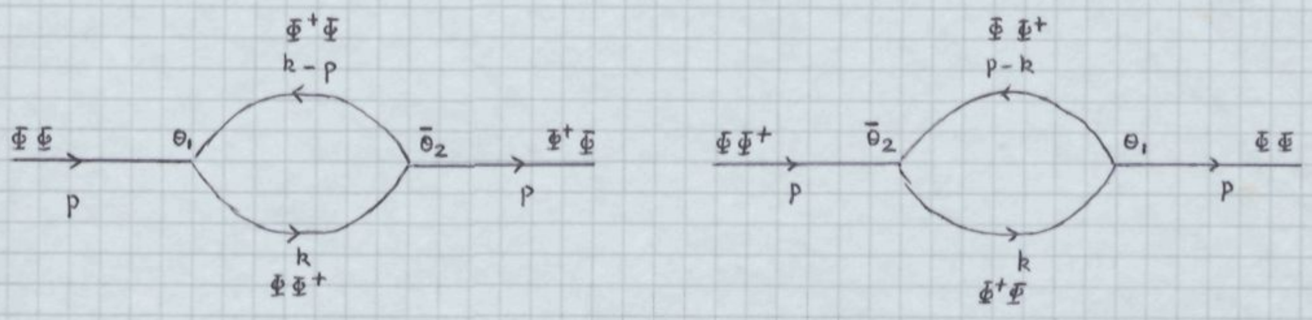
$$e^{-ip_3(x_2-x_1)}\, S_{\Phi^+\Phi}(p_4;\theta_2\bar\theta_2,\theta'\bar\theta')\, e^{-ip_4(x_2-x')} + S_{\Phi\Phi^+}(p_1;\theta\bar\theta,\theta_2\bar\theta_2)\, e^{-ip_1(x-x_2)}$$

$$S_{\Phi^+\Phi}(p_2;\theta_2\bar\theta_2;\theta_1\bar\theta_1)\, e^{-ip_2(x_2-x_1)}\, S_{\Phi\Phi^+}(p_3;\theta_1\bar\theta_1,\theta_2\bar\theta_2)\, e^{-ip_3(x_1-x_2)}\, S_{\Phi\Phi}(p_4;\theta_1\bar\theta_1,\theta'\bar\theta')\, e^{-ip_4(x_1-x')} \Big\}$$

$$S^{(2)}_{\Phi\Phi}(p;\theta\bar\theta,\theta'\bar\theta') = \frac{ig^2}{2}\int d^2\theta_1\, d^2\bar\theta_1\, d^2\theta_2\, d^2\bar\theta_2\, \frac{d^4k}{(2\pi)^4}\, \delta(\bar\theta_1)\,\delta(\theta_2)\,.$$

$$\Big\{ S_{\Phi\Phi}(p;\theta\bar\theta,\theta_1\bar\theta_1)\, S_{\Phi\Phi^+}(k;\theta_1\bar\theta_1,\theta_2\bar\theta_2)\, S_{\Phi^+\Phi}(k-p;\theta_2\bar\theta_2,\theta_1\bar\theta_1)\, S_{\Phi^+\Phi}(p;\theta_2\bar\theta_2,\theta'\bar\theta') +$$

$$+ S_{\Phi\Phi^+}(p;\theta\bar\theta,\theta_2\bar\theta_2)\, S_{\Phi^+\Phi}(k;\theta_2\bar\theta_2,\theta_1\bar\theta_1)\, S_{\Phi\Phi^+}(k-p;\theta_1\bar\theta_1,\theta_2\bar\theta_2)\, S_{\Phi\Phi}(p;\theta_1\bar\theta_1,\theta'\bar\theta') \Big\} \quad (1)$$



Notice that (1) correspond to the diagrams given above. If a vertex has θ_i it means that there is a factor $\delta(\bar\theta_1)$

Let us take into account that

$$S_{\Phi^+\Phi}(p;\theta\bar\theta;\theta'\bar\theta') = S_{\Phi\Phi^+}(-p;\theta'\bar\theta',\theta\bar\theta) \quad (2)$$

Let us for the moment consider only the θ dependence

$$A = \int d^2\theta_1\, d^2\bar\theta_1\, d^2\theta_2\, d^2\bar\theta_2\; \delta(\bar\theta_1)\,\delta(\theta_2)\Big\{\delta(\theta-\theta_1)\, e^{-(\theta\sigma^\mu\bar\theta)p_\mu}\, e^{+2(\theta_1\sigma^\mu\bar\theta_2)k_\mu}$$

$$e^{-2(\theta_1\sigma^\mu\bar\theta_2)(k_\mu-p_\mu)}\, e^{+(\theta'\sigma^\mu\bar\theta' - 2\theta'\sigma^\mu\bar\theta_2')p_\mu} + e^{-(\theta\sigma^\mu\bar\theta - 2\theta\sigma^\mu\bar\theta_2)p_\mu}$$

$$e^{-2(\theta_1\sigma^\mu\bar\theta_2)k_\mu}\, e^{+2(\theta_1\sigma^\mu\bar\theta_2)(k_\mu-p_\mu)}\; \delta(\theta_1-\theta')\, e^{(\theta'\sigma^\mu\bar\theta')p_\mu}\Big\}$$

Since

$$e^{(\theta\sigma^\mu\bar\theta)p_\mu}\, e^{(\theta'\sigma^\mu\bar\theta')k_\mu} = e^{(\theta\sigma^\mu\bar\theta)p_\mu + (\theta'\sigma^\mu\bar\theta')k_\mu} \qquad (1)$$

we can write

$$A = \int d^2\theta_1\, d^2\bar\theta_1\, d^2\theta_2\, d^2\bar\theta_2\; \delta(\bar\theta_1)\,\delta(\theta_2)\Big\{\delta(\theta-\theta_1)\, e^{-(\theta\sigma^\mu\bar\theta)p_\mu}$$

$$e^{+2(\theta_1\sigma^\mu\bar\theta_2)p_\mu}\, e^{(\theta'\sigma^\mu\bar\theta' - 2\theta'\sigma^\mu\bar\theta_2)p_\mu} + e^{-(\theta\sigma^\mu\bar\theta - 2\theta\sigma^\mu\bar\theta_2)p_\mu}$$

$$e^{-2(\theta_1\sigma^\mu\bar\theta_2)p_\mu}\; \delta(\theta_1-\theta')\, e^{(\theta'\sigma^\mu\bar\theta')p_\mu}\Big\} = \text{Integrating } \bar\theta_1 \text{ and } \theta_2 \text{ and changing } \bar\theta_2\to\bar\theta_1$$

$$= \int d^2\theta_1\, d^2\bar\theta_1\Big\{\delta(\theta-\theta_1)\, e^{-(\theta\sigma^\mu\bar\theta)p_\mu}\, e^{2(\theta_1\sigma^\mu\bar\theta_1)p_\mu}\, e^{(\theta'\sigma^\mu\bar\theta' - 2\theta'\sigma^\mu\bar\theta_1)p_\mu}$$

$$+ e^{-(\theta\sigma^\mu\bar\theta - 2\theta\sigma^\mu\bar\theta_1)p_\mu}\, e^{-2(\theta_1\sigma^\mu\bar\theta_1)p_\mu}\; \delta(\theta_1-\theta')\, e^{(\theta'\sigma^\mu\bar\theta')p_\mu}\Big\}$$

$$= \int d^2\theta_1\, d^2\bar\theta_1\Big\{\delta(\theta-\theta_1)\, e^{-[(\theta\sigma^\mu\bar\theta) - (\theta_1\sigma^\mu\bar\theta_1)]p_\mu}\, e^{[\theta'\sigma^\mu\bar\theta' + \theta_1\sigma^\mu\bar\theta_1 - 2\theta'\sigma^\mu\bar\theta_1]p_\mu}$$

$$+ e^{-(\theta\sigma^\mu\bar\theta + \theta_1\sigma^\mu\bar\theta_1 - 2\theta\sigma^\mu\bar\theta_1)p_\mu}\; \delta(\theta_1-\theta')\, e^{[(\theta'\sigma^\mu\bar\theta' - \theta_1\sigma^\mu\bar\theta_1]p_\mu}\Big\}$$

We get

$$S^{(2)}_{\Phi\Phi}(p;\theta\bar\theta;\theta'\bar\theta') = \int d^2\theta_1\, d^2\bar\theta_1\Big\{S_{\Phi\Phi}(p;\theta\bar\theta,\theta_1\bar\theta_1)\, S_{\Phi^+\Phi}(p;\theta_1\bar\theta_1,\theta'\bar\theta')$$

$$+ S_{\Phi\Phi^+}(p;\theta\bar\theta,\theta_1\bar\theta_1)\, S_{\Phi\Phi}(p;\theta_1\bar\theta_1,\theta'\bar\theta')\Big\}\Big\{\frac{ig^2}{2}\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{[k^2-m^2+i\epsilon][(k-p)^2-m^2+i\epsilon]}\Big\}$$

Notice that all the divergent parts are only logarithmically divergent. The fact that the self-energy corrections to the boson propagator are only logarithmically divergent is a fundamental result in supersymmetric theories.

The superfield propogators (44.1) may be obtained directly as superspace Green's functions for the free field equation. Before trying to do this let us derive some needed relations. From (20.2) and (21.1) it is easy to check that

$$\int d^4x\, d^2\theta\, d^2\bar{\theta}\; U(x,\theta,\bar{\theta}) = \int d^4x\, d^2\theta \left[-\frac{1}{4}\bar{D}\bar{D}\right] U(x,\theta,\bar{\theta})$$

$$= \int d^4x\, d^2\bar{\theta}\left[-\frac{1}{4}DD\right] U(x,\theta,\bar{\theta}) \qquad (1)$$

i.e. $d^2\bar{\theta}$ is equivalent to $-\frac{1}{4}\bar{D}\bar{D}$ under x-integration and $d^2\theta$ to $-\frac{1}{4}DD$

Furthermore let us consider a chiral field. Φ

$$\Phi(x,\theta,\bar{\theta}) = A(x) + \sqrt{2}\,[\theta\psi(x)] + [\theta\theta]F(x) - i\,[\theta\sigma^\mu\bar{\theta}]\,\partial_\mu A(x)$$

$$+ \frac{i}{\sqrt{2}}[\theta\theta]\,[\partial_\mu\psi(x)\sigma^\mu\bar{\theta}] - \frac{1}{4}[\theta\theta][\bar{\theta}\bar{\theta}]\,\partial_\mu\partial^\mu A(x) \qquad (2)$$

Then

$$DD\Phi = D^\alpha D_\alpha\Phi = \epsilon^{\alpha\beta}D_\beta D_\alpha\Phi = \epsilon^{\alpha\beta}D_\beta\left\{\frac{\partial}{\partial\theta^\alpha} - i(\sigma^\mu)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu\right\}\Phi =$$

$$= \epsilon^{\alpha\beta}D_\beta\Big\{-i(\sigma^\mu)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu A(x) + \sqrt{2}\,\psi_\alpha(x) - i\sqrt{2}\,(\sigma^\mu)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\theta^{\beta}\partial_\mu\psi_\beta(x) +$$

$$+ 2\theta_\alpha F(x) - i[\theta\theta]\,(\sigma^\mu)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu F(x) - i\,(\sigma^\mu)_{\alpha\dot{\beta}}\theta^{\dot{\beta}}\partial_\mu A(x) -$$

$$- [\theta\sigma^\nu\bar{\theta}]\,(\sigma^\mu)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu\partial_\nu A(x) + i\sqrt{2}\,\theta_\alpha\,[\partial_\mu\psi(x)\sigma^\mu\bar{\theta}] +$$

$$+ \frac{1}{\sqrt{2}}[\theta\theta]\,(\sigma^\mu)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu\partial_\nu\psi^\beta(x)\,(\sigma^\mu)_{\beta\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} - \frac{1}{2}\theta_\alpha[\bar{\theta}\bar{\theta}]\,\partial_\mu\partial^\mu A(x)\Big\} =$$

$$= [\bar{\theta}\bar{\theta}]\,\partial^\mu\partial_\mu A(x) - i\sqrt{2}\,[\partial_\mu\psi(x)\sigma^\mu\bar{\theta}] - i\sqrt{2}\,[\partial_\mu\psi(x)\sigma^\mu\bar{\theta}] + \sqrt{2}\,[\bar{\theta}\bar{\theta}][\theta\partial_\mu\partial^\mu\psi(x)]$$

$$- 4F(x) - 2i\,[\theta\sigma^\mu\bar{\theta}]\,\partial_\mu F(x) - 2i\,[\theta\sigma^\mu\bar{\theta}]\,\partial_\mu F(x) + [\theta\theta][\bar{\theta}\bar{\theta}]\,\partial^\mu\partial_\mu F(x)$$

$$+ [\bar{\theta}\bar{\theta}]\,\partial^\mu\partial_\mu A(x) + [\bar{\theta}\bar{\theta}]\,\partial^\mu\partial_\mu A(x) - i2\sqrt{2}\,[\partial_\mu\psi(x)\sigma^\mu\bar{\theta}]$$

$$+ \frac{\sqrt{2}}{2}[\bar{\theta}\bar{\theta}]\,[\theta\partial^\mu\partial_\mu\psi(x)] + \frac{\sqrt{2}}{2}[\bar{\theta}\bar{\theta}]\,[\theta\partial^\mu\partial_\mu\psi(x)] + [\bar{\theta}\bar{\theta}]\,\partial_\mu\partial^\mu A(x) =$$

$$DD\Phi(x,\theta,\bar{\theta}) = -4F(x) - i4\sqrt{2}\,[\partial_\mu\psi(x)\sigma^\mu\bar{\theta}] + 4[\bar{\theta}\bar{\theta}]\,\partial^\mu\partial_\mu A(x)$$

$$- 4i\,[\theta\sigma^\mu\bar{\theta}]\,\partial_\mu F(x) + 2\sqrt{2}\,[\bar{\theta}\bar{\theta}][\theta\partial_\mu\partial^\mu\psi(x)] + [\theta\theta][\bar{\theta}\bar{\theta}]\,\partial^\mu\partial_\mu F(x)$$

Taking into account the result (21.1) we get

$$(\bar{D}\bar{D})(DD)\,\Phi(x,\theta,\bar{\theta}) = -16\,\partial^\mu\partial_\mu A(x) - 16\sqrt{2}\,[\theta\,\partial^\mu\partial_\mu \psi(x)] - 16\,[\theta\theta]\,\partial^\mu\partial_\mu F(x)$$

$$+ 16i\,(\theta\sigma^\mu\bar{\theta})\,\partial_\nu[\partial^\mu\partial_\mu A(x)] - i\frac{16}{\sqrt{2}}\,[\theta\theta][\partial_\nu(\partial^\mu\partial_\mu\psi(x))\sigma^\nu\bar{\theta}]$$

$$+ 4\,[\theta\theta][\bar{\theta}\bar{\theta}]\,\partial^\nu\partial_\nu\partial^\mu\partial_\mu A(x) = -16\,\partial^\mu\partial_\mu\,\Phi(x,\theta,\bar{\theta}) \qquad (1)$$

Then we can introduce the chiral projection operators $\Box \equiv \partial^\mu\partial_\mu$

$$P_1 = -\frac{1}{16\Box}(DD)(\bar{D}\bar{D}) \qquad\qquad P_2 = -\frac{1}{16\Box}(\bar{D}\bar{D})(DD)$$

$$P_1\,\Phi^+(x,\theta,\bar{\theta}) = \Phi^+(x,\theta,\bar{\theta}) \qquad\qquad P_2\,\Phi(x,\theta,\bar{\theta}) = \Phi(x,\theta,\bar{\theta}) \qquad (2)$$

$$P_1\,\Phi(x,\theta,\bar{\theta}) = 0 \qquad\qquad P_2\,\Phi^+(x,\theta,\bar{\theta}) = 0$$

$$+ \qquad D\,P_1 = 0 \qquad\qquad \bar{D}\,P_2 = 0$$

Notice furthermore that from (17.2) it is evident that

$$A \equiv \int d^4x\, d^2\theta\, d^2\bar{\theta}\;\Phi^+\Phi^+\,\delta(\theta) = \int d^4x\;[2A^*(x)F^+(x) - \bar{\psi}(x)\bar{\psi}(x)]$$

while

$$B \equiv \int d^4x\, d^2\theta\, d^2\bar{\theta}\;\Phi^+\bar{D}\bar{D}\,\Phi^+ = \int d^4x\, d^2\theta\, d^2\bar{\theta}\;\Phi^+\Big\{-4F^+(x) + 4i\sqrt{2}\,[\theta\sigma^\mu\partial_\mu\bar{\psi}(x)]$$

$$+ 4[\theta\theta]\,\partial^\mu\partial_\mu A^*(x) + 4i\,[\theta\sigma^\mu\bar{\theta}]\,\partial_\mu F^+(x) + 2\sqrt{2}\,[\theta\theta][\bar{\theta}\,\partial^\mu\partial_\mu\bar{\psi}(x)] +$$

$$+ [\theta\theta][\bar{\theta}\bar{\theta}]\,\partial^\mu\partial_\mu F^+(x)\Big\} = \int d^4x\, d^2\theta\, d^2\bar{\theta}\;\Big\{[\theta\theta][\bar{\theta}\bar{\theta}]\,A^*(x)\,\partial^\mu\partial_\mu F^+(x) +$$

$$+ 4[\theta\theta](\bar{\theta}\,\partial^\mu\partial_\mu\bar{\psi}(x)][\bar{\theta}\bar{\psi}(x)] + 4[\theta\theta][\bar{\theta}\bar{\theta}]\,F^*(x)\,\partial^\mu\partial_\mu A^*(x) -$$

$$- 4[\theta\sigma^\mu\bar{\theta}][\theta\sigma^\nu\bar{\theta}]\,\partial_\mu A^+(x)\,\partial_\nu F^*(x) + 4[\bar{\theta}\bar{\theta}][\theta\sigma^\mu\partial_\mu\psi(x)][\theta\sigma^\nu\partial_\nu\bar{\psi}(x)]$$

$$+ [\theta\theta][\bar{\theta}\bar{\theta}]\,F^*(x)\,\partial_\mu\partial^\mu A^+(x)\Big\} = \int d^4x\, d^2\theta\, d^2\bar{\theta}\;\Big\{[\theta\theta][\bar{\theta}\bar{\theta}]\,A^+(x)\,\partial^\mu\partial_\mu F^+(x) -$$

$$- 2[\theta\theta][\bar{\theta}\bar{\theta}][\bar{\psi}(x)\,\partial^\mu\partial_\mu\bar{\psi}(x)] + 4[\theta\theta][\bar{\theta}\bar{\theta}]\,F^*(x)\,\partial^\mu\partial_\mu A^+(x) - 2[\theta\theta][\bar{\theta}\bar{\theta}]\,\partial_\mu A^+(x)\,\partial^\mu F^*(x)$$

$$- 2[\theta\theta][\bar{\theta}\bar{\theta}][\bar{\psi}(x)\,\partial^\mu\partial_\mu\bar{\psi}(x)] + [\theta\theta][\bar{\theta}\bar{\theta}]\,F^*(x)\,\partial_\mu\partial^\mu A^*(x)\Big\} =$$

$$= \int d^4x\;\Big\{8A^*(x)\,\partial^\mu\partial_\mu F^+(x) - 4(\bar{\psi}(x)\,\partial^\mu\partial_\mu\bar{\psi}(x))\Big\}$$

where parts integration has been used when needed. Hence we can write

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$D\mathcal{L}_1 = -\frac{1}{16} D(DD)(\bar{D}\bar{D})$

Acting on an arbitrary superfield $\bar{D}\bar{D}U \sim \Phi$, $(\bar{D}D)\Phi \sim \Phi^+$, $D\Phi^+ = 0$

$$\frac{1}{4}\int d^4x\, d^2\theta\, d^2\bar\theta\ \Phi^+ \frac{\bar D\bar D}{\Box}\Phi^+ = -\int d^4x\, d^2\theta\, d^2\bar\theta\ \Phi^+\Phi^+\,\delta(\theta) \tag{1}$$

$$\frac{1}{4}\int d^4x\, d^2\theta\, d^2\bar\theta\ \Phi \frac{DD}{\Box}\Phi = -\int d^4x\, d^2\theta\, d^2\bar\theta\ \Phi\Phi\,\delta(\bar\theta)$$

The free field Lagrangian is

$$L_0 \equiv \int d^4x\, d^2\theta\, d^2\bar\theta \left\{ \Phi^+\Phi + \frac{m}{2}\Phi\Phi\,\delta(\bar\theta) - \frac{m}{2}\Phi^+\Phi^+\,\delta(\theta)\right\} \tag{2}$$

and using (1)

$$L_0 = \int d^4x\, d^2\theta\, d^2\bar\theta \left\{ \Phi^+\Phi + \frac{m}{8}\left[\Phi\frac{DD}{\Box}\Phi + \Phi^+\frac{\bar D\bar D}{\Box}\Phi^+\right]\right\} =$$

$$= \int d^4x\, d^2\theta\, d^2\theta\ \frac{1}{2}[\Phi, \Phi^+]\ \mathcal{M} \begin{vmatrix}\Phi\\ \Phi^+\end{vmatrix} \qquad \mathcal{M} \equiv \begin{vmatrix} +\frac{m}{4\Box}DD & 1 \\ 1 & +\frac{m}{4\Box}\bar D\bar D\end{vmatrix} \tag{3}$$

*

Using the projection operators P_1 and P_2

$$L_0 = \int d^4x\, d^2\theta\, d^2\bar\theta\ \frac{1}{2}[\Phi, \Phi^+] \begin{vmatrix} P_1 & 0 \\ 0 & P_2\end{vmatrix} \mathcal{M} \begin{vmatrix} P_2 & 0 \\ 0 & P_1\end{vmatrix} \begin{vmatrix}\Phi\\ \Phi^+\end{vmatrix} \tag{4}$$

With the projection operators in place, Φ and Φ^+ may be varied at will, leading directly to the free-field Euler-Lagrange equations **

$$\begin{vmatrix} P_1 & 0 \\ 0 & P_2\end{vmatrix} \begin{vmatrix} +\frac{m}{4\Box}DD & 1 \\ 1 & +\frac{m}{4\Box}\bar D\bar D\end{vmatrix} \begin{vmatrix} P_2 & 0 \\ 0 & P_1\end{vmatrix} \begin{vmatrix}\Phi\\ \Phi^+\end{vmatrix} = 0 \tag{5}$$

For interacting fields, we replace the right-hand side of (5) by the appropriate current J:

$$\begin{vmatrix} P_1 & 0 \\ 0 & P_2\end{vmatrix} \mathcal{M} \begin{vmatrix} P_2 & 0 \\ 0 & P_1\end{vmatrix} \begin{vmatrix}\Phi\\ \Phi^+\end{vmatrix} = \begin{vmatrix} J \\ J^+\end{vmatrix} \tag{6}$$

We define the Green's function for this system as follows

$$\begin{vmatrix} P_1 & 0 \\ 0 & P_2\end{vmatrix} \mathcal{M} \begin{vmatrix} P_2 & 0 \\ 0 & P_1\end{vmatrix} \Delta = \begin{vmatrix} P_1 & 0 \\ 0 & P_2\end{vmatrix} \delta(x-x')\,\delta(\theta-\theta')\,\delta(\bar\theta-\bar\theta') \tag{7}$$

* If we wish to derive field equations from (3) by a variational principle, we must take into account the fact that the chiral fields Φ and Φ^+ are subjects to constraints. We do this by varying Φ and Φ^+ in the y and y^+ bases:

$$\frac{\delta}{\delta\Phi(y,\theta)}\,\Phi(y',\theta') = \delta(y-y')\,\delta(\theta-\theta') \qquad (1)$$

In these bases the field variations automatically remain chiral. We may use this result to find the variations of Φ and Φ^+ under full superspace integrations.

$$\frac{\delta}{\delta\Phi(x,\theta,\bar\theta)}\int d^4x'\,d^2\theta'\,d^2\bar\theta'\;\Phi(x',\theta',\bar\theta')\,F(x',\theta',\bar\theta') = \frac{\delta}{\delta\Phi(y,\theta)}\int d^4y'\,d^2\theta'\,d^2\bar\theta'\;\Phi(y',\theta')\,F(y'+i\theta'\sigma\bar\theta',\theta',\bar\theta')$$

$$= \int d^4y'\,d^2\theta'\,d^2\bar\theta'\;\delta(y-y')\,\delta(\theta-\theta')\,F(y'+i\theta'\sigma\bar\theta',\theta',\bar\theta') = \int d^2\bar\theta\;F(y+i\theta\sigma\bar\theta,\theta,\bar\theta) = \quad (49.1)$$

$$= -\frac{1}{4}\bar D\bar D\,F(x,\theta,\bar\theta) \qquad (2)$$

Eq. (2) may be summarized by a formal rule

$$\frac{\delta}{\delta\Phi(x,\theta,\bar\theta)}\,\Phi(x',\theta',\bar\theta') = -\frac{1}{4}\bar D\bar D\;\delta(\theta-\theta')\,\delta(\bar\theta-\bar\theta')\,\delta(x-x') \qquad (3)$$

This rule reproduces (2):

$$\frac{\delta}{\delta\Phi(x,\theta,\bar\theta)}\int d^4x'\,d^2\theta'\,d^2\bar\theta'\;\Phi(x',\theta',\bar\theta')\,F(x',\theta',\bar\theta') = -\frac{1}{4}\int d^4x'\,d^2\theta'\,d^2\bar\theta'\;\bar D\bar D\,\delta(\theta-\theta')\,\delta(\bar\theta-\bar\theta')\,\delta(x-x')\,F(x',\theta',\bar\theta')$$

$$= -\frac{1}{4}\bar D\bar D\,F(x,\theta,\bar\theta)$$

where we have integrated by parts to obtain the final result.

The free-field Euler-Lagrange equations are found by varying (51.3) according to (3) ***

** It is easy to check that from here we get

$$\left.\begin{aligned} &\ell_1 \quad \frac{m}{4\Box}DD\,\Phi + \Phi^+ = 0\\ &\ell_2 \quad \frac{m}{4\Box}\bar D\bar D\,\Phi^+ + \Phi = 0\end{aligned}\right\}$$

Multiply for $(\bar D\bar D)$

$$\Rightarrow\ \left.\begin{aligned}&\frac{m}{4\Box}(DD)\Phi + \Phi^+ = 0\\ &\frac{m}{4\Box}(\bar D\bar D)\Phi^+ + \Phi = 0\end{aligned}\right\} \quad \left.\begin{aligned}&\frac{m}{4}(DD)\Phi + \Box\Phi^+ = 0\\ &\frac{m}{4}(\bar D\bar D)\Phi^+ + \Box\Phi = 0\end{aligned}\right\} \quad \begin{aligned}&m\Phi^+ - \frac{1}{4}(DD)\Phi = 0\\ &m\Phi - \frac{1}{4}(\bar D\bar D)\Phi^+ = 0.\end{aligned}$$

***

$$-\frac{1}{4}\begin{vmatrix}(\bar D\bar D) & 0\\ 0 & (DD)\end{vmatrix}\,\mathcal{M}\begin{vmatrix}\Phi\\ \Phi^+\end{vmatrix} = 0 \;\Rightarrow\; \left.\begin{aligned}&\frac{m}{4\Box}(\bar D\bar D)(DD)\Phi + (\bar D\bar D)\Phi^+ = 0\\ &(DD)\Phi + \frac{m}{4\Box}(DD)(\bar D\bar D)\Phi^+ = 0\end{aligned}\right\} \quad \left.\begin{aligned}&m\Phi - \frac{1}{4}(\bar D\bar D)\Phi^+ = 0\\ &-\frac{1}{4}(DD)\Phi + m\Phi^+ = 0\end{aligned}\right\}$$

The δ-functions are multiplied by the projection operators because (51.6) has a solution only if

$$\begin{vmatrix} P_2 & 0 \\ 0 & P_1 \end{vmatrix} \begin{vmatrix} J \\ J^+ \end{vmatrix} = \begin{vmatrix} J \\ J^+ \end{vmatrix} \qquad \text{(Chiral sources)} \tag{1}$$

In order to solve for the Green's functions, we must exploit further properties of the projection operators

$$P_1 = -\frac{1}{16\Box}(DD)(\bar{D}\bar{D}) \qquad P_2 = -\frac{1}{16\Box}(\bar{D}\bar{D})(DD) \tag{2}$$

From Ekein properties

$$P_1 P_1 = P_1 \quad , \quad P_2 P_2 = P_2 \quad , \quad P_1 P_2 = P_2 P_1 = 0 \tag{3}$$

Let us furthermore introduce the additional operators

$$P_+ \equiv \frac{i}{4\Box^{1/2}}(DD) \qquad P_- \equiv \frac{i}{4\Box^{1/2}}(\bar{D}\bar{D}) \qquad P_T \equiv \frac{1}{8\Box} D^\alpha (\bar{D}\bar{D}) D_\alpha \tag{4}$$

Now using (13.1)

$$(DD)(\bar{D}\bar{D}) = D^\alpha D_\alpha \bar{D}_{\dot\alpha} \bar{D}^{\dot\alpha} = 2i\, D^\alpha (\sigma^\mu)_{\alpha\dot\alpha} \bar{D}^{\dot\alpha} \partial_\mu - D^\alpha \bar{D}_{\dot\alpha} D_\alpha \bar{D}^{\dot\alpha} =$$

$$= 4i\, D^\alpha (\sigma^\mu)_{\alpha\dot\alpha} \bar{D}^{\dot\alpha} \partial_\mu + D^\alpha (\bar{D}\bar{D}) D_\alpha$$

$$(\bar{D}\bar{D})(DD) = \bar{D}^{\dot\alpha} \bar{D}_{\dot\alpha} D_\alpha D^\alpha = 2i\, \bar{D}^{\dot\alpha} (\tilde\sigma^\mu)_{\dot\alpha\alpha} D^\alpha \partial_\mu - \bar{D}^{\dot\alpha} D_\alpha D_{\dot\alpha} D^\alpha =$$

$$= 4i\, \bar{D}^{\dot\alpha} (\tilde\sigma^\mu)_{\dot\alpha\alpha} D^\alpha \partial_\mu + D^\alpha (\bar{D}\bar{D}) D_\alpha$$

$$\Rightarrow (DD)(\bar{D}\bar{D}) + (\bar{D}\bar{D})(DD) - 2 D^\alpha (\bar{D}\bar{D}) D_\alpha = 4i\, D^\alpha (\sigma^\mu)_{\alpha\dot\alpha} \bar{D}^{\dot\alpha} \partial_\mu + 4i\, \bar{D}^{\dot\alpha} (\tilde\sigma^\mu)_{\dot\alpha\alpha} D^\alpha \partial_\mu$$

$$= -8 (\sigma^\mu)^{\alpha\dot\alpha} (\tilde\sigma^\nu)_{\dot\alpha\alpha} \partial_\mu \partial_\nu = -16\, \partial^\mu \partial_\mu$$

Hence

$$P_1 + P_2 + P_T = 1 \tag{5}$$

$$\Rightarrow \quad P_1 P_T = P_2 P_T = P_T P_1 = P_T P_2 = 0 \quad , \quad P_T^2 = P_T$$

Now it is immediate to check the following multiplication table. With the help of this multiplication table, we may readily express the differential operator of eq (51.7) in the following way

	P_1	P_2	P_+	P_-	P_T
P_1	P_1	0	P_+	0	0
P_2	0	P_2	0	P_-	0
P_+	0	P_+	0	P_1	0
P_-	P_-	0	P_2	0	0
P_T	0	0	0	0	P_T

$$\begin{vmatrix} P_1 & 0 \\ 0 & P_2 \end{vmatrix} \mathcal{M} \begin{vmatrix} P_2 & 0 \\ 0 & P_1 \end{vmatrix} = \begin{vmatrix} P_1 & 0 \\ 0 & P_2 \end{vmatrix} \begin{vmatrix} -\frac{im}{\Box^{1/2}} P_+ & \delta \\ \delta & -\frac{im}{\Box^{1/2}} P_- \end{vmatrix} \begin{vmatrix} P_2 & 0 \\ 0 & P_1 \end{vmatrix} = \begin{vmatrix} P_1 & 0 \\ 0 & P_2 \end{vmatrix} \mathcal{M} \tag{1}$$

Now let us see that the solution of (51.7) is

$$\Delta = \begin{vmatrix} P_2 & 0 \\ 0 & P_1 \end{vmatrix} \mathcal{M}^{-1} \delta(x-x') \delta(\theta-\theta') \delta(\bar\theta - \bar\theta') \tag{2}$$

We have

$$\begin{vmatrix} P_1 & 0 \\ 0 & P_2 \end{vmatrix} \mathcal{M} \begin{vmatrix} P_2 & 0 \\ 0 & P_1 \end{vmatrix} \Delta = \begin{vmatrix} P_1 & 0 \\ 0 & P_2 \end{vmatrix} \mathcal{M} \begin{vmatrix} P_2 & 0 \\ 0 & P_1 \end{vmatrix} \mathcal{M}^{-1} \delta(x-x') \delta(\theta-\theta') \delta(\bar\theta-\bar\theta')$$

$$= \begin{vmatrix} P_1 & 0 \\ 0 & P_2 \end{vmatrix} \delta(x-x') \delta(\theta-\theta') \delta(\bar\theta - \bar\theta')$$

Notice that $\mathcal{M}$ must be inverted. The trick is to expand it in terms of P-operators

$$\mathcal{M} = \begin{vmatrix} -\frac{im}{\Box^{1/2}} & 0 \\ 0 & 0 \end{vmatrix} P_+ + \begin{vmatrix} 0 & 0 \\ 0 & -\frac{im}{\Box^{1/2}} \end{vmatrix} P_- + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} (P_1 + P_2 + P_T) \tag{3}$$

The inverse of any operator of the type

$$\Sigma = A P_1 + D P_2 + B P_+ + C P_- + E P_T \tag{4}$$

is given by

$$\Sigma^{-1} = [A - BD^{-1}C]^{-1} P_1 + [D - CA^{-1}B]^{-1} P_2 - [A^{-1}B][D - CA^{-1}B]^{-1} P_+$$

$$- D^{-1}C [A - BD^{-1}C]^{-1} P_- + E^{-1} P_T$$

provided A, D, E are all invertible. We have

$$\bar{X}\bar{X}^{-1} = A(A-BD^{-1}C)^{-1}P_1 - B(D-CA^{-1}B)^{-1}P_+ + D(D-CA^{-1}B)^{-1}P_2$$

$$- C(A-BD^{-1}C)^{-1}P_- + B(D-CA^{-1}B)^{-1}P_+ - BD^{-1}C(A-BD^{-1}C)^{-1}P_1$$

$$+ C(A-BD^{-1}C)^{-1}P_- - CA^{-1}B(D-CA^{-1}B)^{-1}P_2 + P_T = P_1 + P_2 + P_T = 1$$

Then we find in order to invert $\mathcal{M}$

$$A = D = E = A^{-1} = D^{-1} = E^{-1} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \qquad B = \begin{vmatrix} -\frac{im}{\Box^{1/2}} & 0 \\ 0 & 0 \end{vmatrix} \qquad C = \begin{vmatrix} 0 & 0 \\ 0 & -\frac{im}{\Box^{1/2}} \end{vmatrix}$$

$$A - BD^{-1}C = \begin{vmatrix} 0 & \frac{1}{\Box}(\Box + m^2) \\ 1 & 0 \end{vmatrix} \qquad D - CA^{-1}B = \begin{vmatrix} 0 & 1 \\ \frac{1}{\Box}(\Box + m^2) & 0 \end{vmatrix}$$

$$(A-BD^{-1}C)^{-1} = \begin{vmatrix} 0 & 1 \\ \frac{\Box}{\Box + m^2} & 0 \end{vmatrix} \qquad (D-CA^{-1}B)^{-1} = \begin{vmatrix} 0 & \frac{\Box}{\Box + m^2} \\ 1 & 0 \end{vmatrix}$$

$$\mathcal{M}^{-1} = \begin{vmatrix} 0 & 1 \\ \frac{\Box}{\Box + m^2} & 0 \end{vmatrix} P_1 + \begin{vmatrix} 0 & \frac{\Box}{\Box + m^2} \\ 1 & 0 \end{vmatrix} P_2 + \begin{vmatrix} 0 & 0 \\ 0 & +\frac{im}{\Box^{1/2}}\frac{\Box}{\Box + m^2} \end{vmatrix} P_+ +$$

$$+ \begin{vmatrix} +\frac{im}{\Box^{1/2}}\frac{\Box}{\Box + m^2} & 0 \\ 0 & 0 \end{vmatrix} P_- + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} P_T$$

$$\mathcal{M}^{-1} = \begin{vmatrix} +\frac{im}{\Box^{1/2}}\frac{\Box}{\Box + m^2} P_- & P_1 + \frac{\Box}{\Box + m^2} P_2 + P_T \\ \frac{\Box}{\Box + m^2} P_1 + P_2 + P_T & +\frac{im}{\Box^{1/2}}\frac{\Box}{\Box + m^2} P_+ \end{vmatrix}$$

Hence

$$\Delta = \begin{vmatrix} +\frac{im}{\Box^{1/2}}\frac{\Box}{\Box + m^2} P_- & \frac{\Box}{\Box + m^2} P_2 \\ \frac{\Box}{\Box + m^2} P_1 & +\frac{im}{\Box^{1/2}}\frac{\Box}{\Box + m^2} P_+ \end{vmatrix} \delta(x-x')\,\delta(\theta-\theta')\,\delta(\bar{\theta}-\bar{\theta}')$$

Hence

$$\Delta = -\frac{1}{\Box + m^2}\begin{vmatrix} +\frac{m}{4}(\bar D\bar D) & \frac{1}{16}(\bar D\bar D)(DD) \\ \frac{1}{16}(DD)(\bar D\bar D) & +\frac{m}{4}(DD)\end{vmatrix}\delta(x-x')\delta(\theta-\theta')\delta(\bar\theta-\bar\theta') \quad (1)$$

To compare this result to the previous propagators we must compute the covariant derivatives of the δ-functions. Let us consider

$$(D_1 D_1)(\theta_1-\theta_2)^2 = \epsilon^{\alpha\beta} D_{1\beta} D_{1\alpha}(\theta_1-\theta_2)^2 = \epsilon^{\alpha\beta} D_{1\beta}\left\{\frac{\partial}{\partial\theta_1^\alpha} - i(\sigma^\mu)_{\alpha\dot\alpha}\bar\theta_1^{\dot\alpha}\partial_\mu\right\}(\theta_1-\theta_2)^2$$

$$= \epsilon^{\alpha\beta}\left\{\frac{\partial}{\partial\theta_1^\beta} - i(\sigma^\nu)_{\beta\dot\beta}\bar\theta_1^{\dot\beta}\partial_\nu\right\}\left\{2(\theta_1-\theta_2)_\alpha - i(\sigma^\mu)_{\alpha\dot\alpha}\bar\theta_1^{\dot\alpha}(\theta_1-\theta_2)^2\partial_\mu\right\} =$$

$$= -4 + 2i(\theta_1-\theta_2)\sigma^\mu\bar\theta_1\partial_\mu + 2i(\theta_1-\theta_2)\sigma^\mu\bar\theta_1\partial_\mu + (\theta_1-\theta_2)^2(\bar\theta_1\bar\theta_1)\partial_\mu\partial^\mu =$$

$$= -4\left\{1 + i(\theta_1-\theta_2)\sigma^\mu\bar\theta_1\partial_\mu + \frac{1}{4}(\theta_1-\theta_2)^2(\bar\theta_1\bar\theta_1)\partial_\mu\partial^\mu\right\}$$

$$= -4\left\{1 + i(\theta_1-\theta_2)\sigma^\mu\bar\theta_1\partial_\mu + \frac{1}{2}[i(\theta_1-\theta_2)\sigma^\mu\bar\theta_1\partial_\mu]^2\right\}$$

$$= -4\exp\left\{i(\theta_1-\theta_2)\sigma^\mu\bar\theta_1\partial_\mu\right\}$$

Hence

$$(D_1 D_1)(\theta_1-\theta_2)^2 = -4\, e^{i(\theta_1-\theta_2)\sigma^\mu\bar\theta_1\partial_\mu}$$
$$(\bar D_1\bar D_1)(\bar\theta_1-\bar\theta_2)^2 = -4\, e^{-i\theta_1\sigma^\mu(\bar\theta_1-\bar\theta_2)\partial_\mu} \quad (2)$$

Similarly

$$(\bar D_1\bar D_1)(D_1 D_1)(\theta_1-\theta_2)^2(\bar\theta_1-\bar\theta_2)^2 = (\bar D_1\bar D_1)(-4)(\bar\theta_1-\bar\theta_2)^2\left[1 + i\theta_1\sigma^\mu\bar\theta_1\partial_\mu - i\theta_2\sigma^\mu\bar\theta_1\partial_\mu\right.$$
$$\left. - \frac{1}{4}(\theta_1\theta_1)(\bar\theta_1\bar\theta_1)\partial_\mu\partial^\mu + \frac{1}{2}(\theta_1\theta_2)(\bar\theta_1\bar\theta_1)\partial_\mu\partial^\mu - \frac{1}{4}(\theta_2\theta_2)(\bar\theta_1\bar\theta_1)\partial_\mu\partial^\mu\right] =$$

$$= -4(\bar D_1\bar D_1)\left\{(\bar\theta_1\bar\theta_1) - 2(\bar\theta_1\bar\theta_2) + (\bar\theta_2\bar\theta_2) + i(\bar\theta_1\bar\theta_1)(\theta_1\sigma^\mu\bar\theta_1)\partial_\mu - 2i(\bar\theta_1\bar\theta_2)(\theta_1\sigma^\mu\bar\theta_1)\partial_\mu\right.$$
$$+ i(\bar\theta_2\bar\theta_2)(\theta_1\sigma^\mu\bar\theta_1)\partial_\mu + 2i(\bar\theta_1\bar\theta_2)(\theta_2\sigma^\mu\bar\theta_1)\partial_\mu - i(\bar\theta_2\bar\theta_2)(\theta_2\sigma^\mu\bar\theta_1)\partial_\mu -$$
$$\left. - \frac{1}{4}(\theta_1\theta_1)(\bar\theta_2\bar\theta_2)(\bar\theta_1\bar\theta_1)\partial_\mu\partial^\mu + \frac{1}{2}(\theta_1\theta_2)(\bar\theta_1\bar\theta_1)(\bar\theta_2\bar\theta_2)\partial_\mu\partial^\mu - \frac{1}{4}(\theta_2\theta_2)(\bar\theta_1\bar\theta_1)(\bar\theta_2\bar\theta_2)\partial_\mu\partial^\mu\right\}$$

$$= -4\epsilon^{\dot\alpha\dot\beta}\bar D_{\dot\alpha}\left(-\frac{\partial}{\partial\bar\theta_1^{\dot\beta}} + i\theta_1^\beta(\sigma^\nu)_{\beta\dot\beta}\partial_\nu\right)\left\{(\bar\theta_1\bar\theta_1) - 2(\bar\theta_1\bar\theta_2) + (\bar\theta_2\bar\theta_2) - 2i(\bar\theta_1\bar\theta_2)(\theta_1\sigma^\mu\bar\theta_1)\partial_\mu\right.$$
$$+ i(\bar\theta_2\bar\theta_2)(\theta_1\sigma^\mu\bar\theta_1)\partial_\mu + 2i(\bar\theta_1\bar\theta_2)(\theta_2\sigma^\mu\bar\theta_1)\partial_\mu - i(\bar\theta_2\bar\theta_2)(\theta_2\sigma^\mu\bar\theta_1)\partial_\mu -$$
$$\left. - \frac{1}{4}(\theta_1\theta_1)(\bar\theta_2\bar\theta_2)(\bar\theta_1\bar\theta_1)\partial_\mu\partial^\mu + \frac{1}{2}(\theta_1\theta_2)(\bar\theta_1\bar\theta_1)(\bar\theta_2\bar\theta_2)\partial_\mu\partial^\mu - \frac{1}{4}(\theta_2\theta_2)(\bar\theta_1\bar\theta_1)(\bar\theta_2\bar\theta_2)\partial_\mu\partial^\mu\right\}$$

and keeping only the terms in first space derivatives (just to check signs)

$$= -4\epsilon^{\dot\alpha\dot\beta}\left(-\frac{\partial}{\partial\bar\theta_1^{\dot\alpha}} + i\,\theta_1^{\alpha}(\sigma^\rho)_{\alpha\dot\alpha}\,\partial_\rho\right)\Big| + 2\bar\theta_{1\dot\beta} + i\,\theta_1^{\beta}(\sigma^\mu)_{\beta\dot\beta}(\bar\theta_1\bar\theta_1)\,\partial_\mu$$

$$-2\bar\theta_{2\dot\beta} - 2i\,\theta_1^{\beta}(\sigma^\mu)_{\beta\dot\beta}(\bar\theta_1\bar\theta_2)\,\partial_\mu + i\,\theta_1^{\beta}(\sigma^\mu)_{\beta\dot\beta}(\bar\theta_2\bar\theta_2)\,\partial_\mu +$$

$$-2i\,\bar\theta_{2\dot\beta}(\theta_1\sigma^\mu\bar\theta_1)\partial_\mu - 2i(\bar\theta_1\bar\theta_2)\,\theta_1^{\beta}(\sigma^\mu)_{\beta\dot\beta}\,\partial_\mu + i(\bar\theta_2\bar\theta_2)\,\theta_1^{\beta}(\sigma^\mu)_{\beta\dot\beta}\,\partial_\mu$$

$$+ 2i\,\bar\theta_{2\dot\beta}(\theta_2\sigma^\mu\bar\theta_1)\,\partial_\mu + 2i(\bar\theta_1\bar\theta_2)\,\theta_2^{\beta}(\sigma^\mu)_{\beta\dot\beta}\,\partial_\mu - i(\bar\theta_2\bar\theta_2)\,\theta_2^{\beta}(\sigma^\mu)_{\beta\dot\beta}\,\partial_\mu + \cdots\Big\}$$

$$= 16 - 8i(\theta_1\sigma^\mu\bar\theta_1)\partial_\mu - 8i(\theta_1\sigma^\mu\bar\theta_1)\partial_\mu + 8i(\theta_1\sigma^\mu\bar\theta_2)\partial_\mu + 8i(\theta_1\sigma^\mu\bar\theta_2)\partial_\mu$$

$$+ 8i(\theta_1\sigma^\mu\bar\theta_2)\partial_\mu + 8i(\theta_1\sigma^\mu\bar\theta_2)\partial_\mu - 8i(\theta_2\sigma^\mu\bar\theta_2)\partial_\mu - 8i(\theta_2\sigma^\mu\bar\theta_2)\partial_\mu + \cdots$$

Hence

$$(\bar D_1\bar D_1)(D_1D_1)(\theta_1-\theta_2)^2(\bar\theta_1-\bar\theta_2)^2 = 16\,e^{-i[\theta_1\sigma^\mu\bar\theta_1 + \theta_2\sigma^\mu\bar\theta_2 - 2\theta_1\sigma^\mu\bar\theta_2]\partial_\mu} \tag{1}$$

$$(D_1D_1)(\bar D_1\bar D_1)(\theta_1-\theta_2)^2(\bar\theta_1-\bar\theta_2)^2 = 16\,e^{+i[\theta_1\sigma^\mu\bar\theta_1 + \theta_2\sigma^\mu\bar\theta_2 - 2\theta_2\sigma^\mu\bar\theta_1]\partial_\mu}$$

and therefore the propagator is

$$\Delta = \frac{1}{-\Box - m^2}\begin{vmatrix}\Delta_{11} & \Delta_{12}\\ \Delta_{21} & \Delta_{22}\end{vmatrix}\delta(x_1-x_2)$$

$$\Delta_{11} = -m\,\delta(\theta_1-\theta_2)\,e^{-i[\theta_1\sigma^\mu\bar\theta_1 - \theta_2\sigma^\mu\bar\theta_2]\partial_\mu} \tag{1}$$

$$\Delta_{12} = e^{-i[\theta_1\sigma^\mu\bar\theta_1 + \theta_2\sigma^\mu\bar\theta_2 - 2\theta_1\sigma^\mu\bar\theta_2]\partial_\mu}$$

$$\Delta_{21} = e^{+i[\theta_1\sigma^\mu\bar\theta_1 + \theta_2\sigma^\mu\bar\theta_2 - 2\theta_2\sigma^\mu\bar\theta_1]\partial_\mu}$$

$$\Delta_{22} = -m\,\delta(\bar\theta_1-\bar\theta_2)\,e^{+i[\theta_1\sigma^\mu\bar\theta_1 - \theta_2\sigma^\mu\bar\theta_2]\partial_\mu}$$

which coincides with the result (44.2) and (45.3)

Having gained some experience with superfields methods, we shall now compute the propagator for vector superfields. We start with the usual Lagrangian

$$L = \int d^4x \left\{ \frac{1}{4} W^\alpha W_\alpha \Big|_{\theta\theta} + \frac{1}{4} \bar{W}_{\dot\alpha} \bar{W}^{\dot\alpha} \Big|_{\bar\theta\bar\theta} + m^2 V^2 \Big|_{\theta\theta\bar\theta\bar\theta} \right\} \quad (1)$$

To this we add the gauge fixing term

$$- \frac{\xi}{8} \left[(\bar D \bar D) V \right] \left[(DD) V \right] \Big|_{\theta\theta\bar\theta\bar\theta} \quad (2)$$

Notice

$$(DD)V = D^\alpha D_\alpha V = \epsilon^{\alpha\beta} D_\beta D_\alpha V = \epsilon^{\alpha\beta} \left\{ \frac{\partial}{\partial\theta^\beta} - 2i (\sigma^\rho)_{\beta\dot\beta} \bar\theta^{\dot\beta} \frac{\partial}{\partial y^\rho} \right\}$$

$$\Big\{ (\sigma^\mu)_{\alpha\dot\alpha} \bar\theta^{\dot\alpha} v_\mu(y) - 2i (\sigma^\mu)_{\alpha\dot\alpha} \bar\theta^{\dot\alpha} (\theta\sigma^\nu\bar\theta) \partial_\mu v_\nu(y) + 2i\theta_\alpha [\bar\theta\bar\lambda(y)]$$

$$+ 2(\sigma^\mu)_{\alpha\dot\alpha} \bar\theta^{\dot\alpha} (\theta\theta)(\bar\theta\partial_\mu\bar\lambda(y)) - i(\bar\theta\bar\theta)\lambda_\alpha(y) + \theta_\alpha(\bar\theta\bar\theta)[D(y) + i\partial^\mu v_\mu(y)] \Big\} =$$

$$= 2i(\bar\theta\tilde\sigma^\nu\sigma^\mu\bar\theta)\partial_\nu v_\mu(y) + 2i(\bar\theta\tilde\sigma^\mu\sigma^\nu\bar\theta)\partial_\mu v_\nu(y) - 4i(\bar\theta\bar\lambda(y))$$

$$+ 4(\theta\sigma^\mu\bar\theta)(\bar\theta\partial_\mu\bar\lambda(y)) + 4(\theta\sigma^\mu\bar\theta)(\bar\theta\partial_\mu\lambda(y)) - 2(\bar\theta\bar\theta)[D(y) + i\partial^\mu v_\mu(y)]$$

$$(DD)V = 2i(\bar\theta\bar\theta)\partial^\mu v_\mu(y) - 2(\bar\theta\bar\theta)D(y) - 4i(\bar\theta\bar\lambda(y)) + 8(\theta\sigma^\mu\bar\theta)(\bar\theta\partial_\mu\bar\lambda(y))$$

$$(\bar D\bar D)V = -2i(\theta\theta)\partial^\mu v_\mu(y^+) - 2(\theta\theta)D(y^+) + 4i(\theta\lambda(y^+)) + 8(\theta\sigma^\mu\bar\theta)(\theta\partial_\mu\lambda(y^+)) \quad (3)$$

Hence (2) picks a piece proportional to $-\frac{\xi}{2}[\partial_\mu v^\mu(x)]^2$ as desired. Then

$$L = \int d^4x\, d^2\theta\, d^2\bar\theta \left\{ \frac{1}{4} W^\alpha W_\alpha \delta(\bar\theta) + \frac{1}{4} W_{\dot\alpha} W^{\dot\alpha} \delta(\theta) + m^2 V^2 - \frac{\xi}{8} [(\bar D\bar D)V][(DD)V] \right\}$$

Notice (25.4)

$$A = \int d^4x\, d^2\theta\, d^2\bar\theta \left\{ \frac{1}{4} W^\alpha W_\alpha \delta(\bar\theta) + \frac{1}{4} W_{\dot\alpha} W^{\dot\alpha} \delta(\theta) \right\} =$$

$$= \int d^4x \left\{ \frac{1}{2} D^2(x) - \frac{1}{4} v^{\mu\nu}(x) v_{\mu\nu}(x) + i [\lambda(x)\sigma^\mu\partial_\mu\bar\lambda(x)] \right\}$$

and

$$\int d^4x\, d^2\theta\, d^2\bar\theta \;\; V\, D^\alpha (\bar D \bar D) D_\alpha V = -4 \int d^4x\, d^2\theta\, d^2\bar\theta \;\; V\, D^\alpha W_\alpha =$$

$$= +4 \int d^4x\, d^2\theta\, d^2\bar\theta \;\; V \left\{ \frac{\partial}{\partial\theta^\alpha} - 2i\,(\sigma^\rho)_{\alpha\dot\beta}\, \bar\theta^{\dot\beta} \frac{\partial}{\partial y^\rho} \right\} \left\{ -i\lambda^\alpha(y) + \theta^\alpha D(y) - \right.$$

$$\left. - \frac{i}{2} [\sigma^\mu]^\alpha{}_{\dot\alpha} [\tilde\sigma^\nu]^{\dot\alpha\beta} \theta_\beta\, v_{\mu\nu}(y) - (\theta\theta)(\sigma^\mu)^\alpha{}_{\dot\alpha}\, \partial_\mu \bar\lambda^{\dot\alpha}(y) \right\} =$$

$$= 4 \int d^4x\, d^2\theta\, d^2\bar\theta \;\; V \left\{ -2(\bar\theta\tilde\sigma^\mu \partial_\mu \lambda(y)) + 2D(y) + 2i(\theta\sigma^\mu\bar\theta)\partial_\mu D(y) \right.$$

$$\left. - (\bar\theta\tilde\sigma^\rho\sigma^\mu\tilde\sigma^\nu\theta)\,\partial_\rho v_{\mu\nu}(y) + 2(\theta\sigma^\mu\partial_\mu\bar\lambda(y)) + 2i(\theta\theta)(\bar\theta\tilde\sigma^\nu\sigma^\mu\partial_\rho\partial_\nu\bar\lambda^{\dot\alpha}) \right\}$$

$$= 4 \int d^4x\, d^2\theta\, d^2\bar\theta \left\{ (\theta\sigma^\rho\bar\theta)\, v_\rho(x) + i(\theta\theta)(\bar\theta\bar\lambda(x)) - i(\bar\theta\bar\theta)(\theta\lambda(x)) + \frac{1}{2}(\theta\theta)(\bar\theta\bar\theta)D(x) \right\}$$

$$\cdot \left\{ -2(\bar\theta\tilde\sigma^\mu\partial_\mu\lambda(x)) + 2(\theta\sigma^\mu\partial_\mu\bar\lambda(x)) + 2D(x) - (\bar\theta\tilde\sigma^\tau\sigma^\mu\tilde\sigma^\nu\theta)\,\partial_\tau v_{\mu\nu}(x) \right\}$$

$$= 4 \int d^4x\, d^2\theta\, d^2\bar\theta \left\{ -(\theta\sigma^\rho\bar\theta)(\bar\theta\tilde\sigma^\tau\sigma^\mu\tilde\sigma^\nu\theta)\, v_\rho(x)\,\partial_\tau v_{\mu\nu}(x) - 2i(\theta\theta)(\bar\theta\bar\lambda(x))(\bar\theta\tilde\sigma^\mu\partial_\mu\lambda(x)) \right.$$

$$\left. - 2i(\bar\theta\bar\theta)(\theta\lambda(x))(\theta\sigma^\mu\partial_\mu\bar\lambda(x)) + (\theta\theta)(\bar\theta\bar\theta)D^2(x) \right\} =$$

$$= 4 \int d^4x \left\{ -\frac{1}{2} v^{\mu\nu}(x)\, v_{\mu\nu}(x) - 2i\,\bar\lambda(x)\,\tilde\sigma^\mu\,\partial_\mu\lambda(x) + D^2(x) \right\}$$

Hence

$$\int d^4x\, d^2\theta\, d^2\bar\theta \left\{ \frac{1}{4} W^\alpha W_\alpha\, \delta(\bar\theta) + \frac{1}{4} W_{\dot\alpha} W^{\dot\alpha}\, \delta(\theta) \right\} =$$

$$= \frac{1}{8} \int d^4x\, d^2\theta\, d^2\bar\theta \;\; V\, D^\alpha (\bar D\bar D) D_\alpha V \qquad (1)$$

Then we can write the Lagrangian as

$$L = \int d^4x\, d^2\theta\, d^2\bar\theta \left\{ \frac{1}{8} V D^\alpha (\bar D\bar D) D_\alpha V + m^2 V^2 - \right.$$

$$\left. - \frac{\xi}{16} V \left[(DD)(\bar D\bar D) + (\bar D\bar D)(DD) \right] V \right\} \qquad (2)$$

and using the operators defined on pag. 42 we can write

$$L = \int d^4x \, d^2\theta \, d^2\bar{\theta} \; V \left\{ \Box P_T + m^2 (P_1 + P_2 + P_T) + \xi \Box (P_1 + P_2) \right\} V$$

$$\equiv \int d^4x \, d^2\theta \, d^2\bar{\theta} \; V \, \mathcal{W} \, V \qquad (1)$$

As before the Euler - Lagrange equations are

$$\mathcal{W} V = 0 \qquad (2)$$

while the superfield Green's functions are defined in the usual way

$$\mathcal{W} \Delta = \delta(x - x') \, \delta(\theta - \theta') \, \delta(\bar{\theta} - \bar{\theta}') \qquad (3)$$

Note that we choose to invert $\mathcal{W}$ and not $2\mathcal{W}$ as might be expected from (1), since is precisely this choice that leads to the usual normalization of the Green's functions of the component fields. Now

$$\mathcal{W}^{-1} = \frac{1}{\Box + m^2} P_T + \frac{1}{m^2 + \xi \Box} (P_1 + P_2) \qquad (4)$$

Now we obtain using (56.1)

$$(P_1 + P_2) \, \delta(x-x') \, \delta(\theta_1 - \theta_2) \, \delta(\bar{\theta}_1 - \bar{\theta}_2) =$$

$$= -\frac{1}{16 \Box} \left[(\bar{D}_1 \bar{D}_1)(D_1 D_1) + (\bar{D}_1 \bar{D}_1)(\bar{D}_1 \bar{D}_1) \right] \delta(x-x') \, \delta(\theta_1 - \theta_2) \, \delta(\bar{\theta}_1 - \bar{\theta}_2) =$$

$$= -\frac{1}{\Box} \left\{ \exp\left\{ -i \left[\theta_1 \sigma^\mu \bar{\theta}_1 + \theta_2 \sigma^\mu \bar{\theta}_2 - 2 \theta_1 \sigma^\mu \bar{\theta}_2 \right] \partial_\mu \right\} + \right.$$

$$\left. + \exp\left\{ +i \left[\theta_1 \sigma^\mu \bar{\theta}_1 + \theta_2 \sigma^\mu \bar{\theta}_2 - 2 \theta_2 \sigma^\mu \bar{\theta}_1 \right] \partial_\mu \right\} \right\} \delta(x - x') =$$

$$= -\frac{1}{\Box} \left\{ \exp\left\{ -i \left[(\theta_1 - \theta_2) \sigma^\mu (\bar{\theta}_1 - \bar{\theta}_2) - \theta_1 \sigma^\mu \bar{\theta}_2 + \theta_2 \sigma^\mu \bar{\theta}_1 \right] \partial_\mu \right\} + \right.$$

$$\left. + \exp\left\{ +i \left[(\theta_1 - \theta_2) \sigma^\mu (\bar{\theta}_1 - \bar{\theta}_2) + \theta_1 \sigma^\mu \bar{\theta}_2 - \theta_2 \sigma^\mu \bar{\theta}_1 \right] \partial_\mu \right\} \right\} \delta(x-x') \equiv S$$

Let us introduce

$$A \equiv (\theta_1 - \theta_2) \sigma^\mu (\bar{\theta}_1 - \bar{\theta}_2) \partial_\mu \qquad B \equiv \left[\theta_1 \sigma^\mu \bar{\theta}_2 - \theta_2 \sigma^\mu \bar{\theta}_1 \right] \partial_\mu$$

Then

$$S \equiv -\frac{1}{\Box}\left\{ e^{+i(B-A)} + e^{i(B+A)} \right\} =$$

$$= -\frac{2}{\Box}\left\{ \left[1 + iB + \frac{i^2}{2!}B^2 + \frac{i^3}{3!}B^3 + \frac{i^4}{4!}B^4 \right] + \frac{i^2}{2!}A^2\left[1 + iB + \frac{i^2}{2!}B^2 \right] \right.$$

$$\left. + \frac{i^4}{4!}A^4 \right\} =$$

$$= -\frac{2}{\Box} \exp[iB] \left\{ 1 - \frac{1}{2}A^2 + \frac{1}{4!}A^4 \right\}$$

But

$$A^2 = \frac{1}{2}\left[(\theta_1-\theta_2)(\theta_1-\theta_2)(\bar\theta_1-\bar\theta_2)(\bar\theta_1-\bar\theta_2) \right]\Box \qquad A^4 = 0$$

Then

$$(P_1 + P_2)\,\delta(x-x')\,\delta(\theta_1-\theta_2)\,\delta(\bar\theta_1-\bar\theta_2) = \tag{1}$$

$$= -\frac{1}{2\Box}\exp\left\{ i\left[\theta_1\sigma^\mu\bar\theta_2 - \theta_2\sigma^\mu\bar\theta_1 \right]\partial_\mu \right\}\left\{ 4 - (\theta_1-\theta_2)^2(\bar\theta_1-\bar\theta_2)^2\Box \right\}\delta(x_1-x')$$

Furthermore

$$P_T\,\delta(x-x')\,\delta(\theta_1-\theta_2)\,\delta(\bar\theta_1-\bar\theta_2) = (1 - P_1 - P_2)\,\delta(x-x')\,\delta(\theta_1-\bar\theta_1)\,\delta(\theta_2-\bar\theta_2) =$$

$$= +\frac{1}{2\Box}\exp\left\{ i\left[\theta_1\sigma^\mu\bar\theta_2 - \theta_2\sigma^\mu\bar\theta_1 \right]\partial_\mu \right\}\left\{ 4 + (\theta_1-\theta_2)^2(\bar\theta_1-\bar\theta_2)^2\Box \right\}\delta(x-x') \tag{2}$$

and therefore the propagator of the vector superfield is

$$\Delta = \frac{1}{2\Box}\, e^{i\left[\theta_1\sigma^\mu\bar\theta_2 - \theta_2\sigma^\mu\bar\theta_1 \right]\partial_\mu}\,.$$

$$\left\{ \frac{1}{\Box + m^2}\left[4 + (\theta_1-\theta_2)^2(\bar\theta_1-\bar\theta_2)^2\Box \right] - \right.$$

$$\left. - \frac{1}{\xi\Box + m^2}\left[4 - (\theta_1-\theta_2)^2(\bar\theta_1-\bar\theta_2)^2\Box \right] \right\}\delta(x_1-x_2) \tag{3}$$

which is the desired result.

Feynman rules for superfields

M. T. GRISARU, M. ROČEK and W. SIEGEL Nucl. Phys. B159, 429 (1979)

B. A. OVRUT and J. WESS Phys. Rev. D25, 409 (1982)

Now we would like to derive the Feynman rules for the supersymmetric Φ^3 model

$$L(x) = \int d^2\theta\, d^2\bar\theta\ \Phi^+(x,\theta,\bar\theta)\ \Phi(x,\theta,\bar\theta) + \\ + \left\{ \int d^2\theta \left[\frac{1}{2} m^2\ \Phi^2(x,\theta,\bar\theta) + \frac{1}{3} g\ \Phi^3(x,\theta,\bar\theta) \right] + h.c \right\} \qquad (1)$$

These rules may be applied to all chiral models and extended to supersymmetric gauge theories as well. We have obtained before the free-field two-point functions $[\, z \equiv (x,\theta,\bar\theta)\,]$

$$\langle 0 |\, T\, (\Phi_0(z)\ \Phi_0(z'))\, | 0\rangle = \frac{i}{-\Box - m^2}\ \Delta_{11}(z,z')\ \delta(x-x')$$

$$\langle 0 |\, T\, (\Phi_0^+(z)\ \Phi_0^+(z'))\, | 0\rangle = \frac{i}{-\Box - m^2}\ \Delta_{22}(z,z')\ \delta(x-x') \qquad (2)$$

$$\langle 0 |\, T\, (\Phi_0(z)\ \Phi_0^+(z'))\, | 0\rangle = \frac{i}{-\Box - m^2}\ \Delta_{12}(z,z')\ \delta(x-x')$$

$$\langle 0 |\, T\, (\Phi_0^+(z)\ \Phi_0(z'))\, | 0\rangle = \frac{i}{-\Box - m^2}\ \Delta_{21}(z,z')\ \delta(x-x')$$

(see eq (44.2) and (56.1)). These two-point functions may all be obtained from the free generating functional

$$Z_0[J,J^+] \equiv \langle 0 |\, T \left\{ \exp\left[i \int d^4x\, d^2\theta\, d^2\bar\theta \left[J(z) \left(\frac{1}{4\Box} (DD) \right) \Phi_0(z) + \right.\right.\right. \\ \left.\left.\left. + J^+(z) \left(\frac{1}{4\Box} (\bar D \bar D) \right) \Phi_0^+(z) \right] \right] \right\} | 0\rangle \qquad (3)$$

and using (2) it is easy to check that

$$Z_0[J,J^+] = \exp\left\{ -\frac{1}{2} \int d^4x\, d^2\theta\, d^2\bar\theta\ d^4x'\, d^2\theta'\, d^2\bar\theta'\ \big(J(z),\ J^+(z)\big) \right. \\ \left. \begin{bmatrix} \frac{1}{4\Box}(DD) & 0 \\ 0 & \frac{1}{4\Box}(\bar D\bar D) \end{bmatrix} \frac{i}{-\Box - m^2} \begin{bmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{bmatrix} \delta(x-x') \begin{bmatrix} \frac{1}{4\Box'}(D'D') & 0 \\ 0 & \frac{1}{4\Box'}(\bar D'\bar D') \end{bmatrix} \begin{bmatrix} J(z') \\ J^+(z') \end{bmatrix} \right\} \qquad (4)$$

This functional generates all free-field Green's functions as sums of products of two-point functions. In general we have

$$G_0^{(N)}(z^1, \dots z^M ; z^{M+1}, \dots z^N) =$$

$$= (-i)^N \frac{\delta}{\delta J(z^1)} \cdots \frac{\delta}{\delta J(z^M)} \frac{\delta}{\delta J^+(z^{M+1})} \cdots \frac{\delta}{\delta J^+(z^N)} Z_0[J, J^+] \Big|_{J = J^+ = 0} \qquad (1)$$

Another equivalent way of writting (63.4) can be obtained using (55.1)

$$Z_0[J, J^+] = \exp\left\{ -\frac{1}{2} \int dz\, dz' \,(J(z), J^+(z)) \begin{vmatrix} \frac{1}{4\Box}(DD) & 0 \\ 0 & \frac{1}{4\Box}(\bar D \bar D) \end{vmatrix} \right.$$

$$\left. \frac{i}{-\Box - m^2} \begin{vmatrix} \frac{m}{4}(\bar D \bar D) & \frac{1}{16}(\bar D \bar D)(DD) \\ \frac{1}{16}(DD)(\bar D \bar D) & \frac{m}{4}(DD) \end{vmatrix} \delta(z-z') \begin{vmatrix} \frac{1}{4\Box'}(D'D') & 0 \\ 0 & \frac{1}{4\Box'}(\bar D' \bar D') \end{vmatrix} \begin{vmatrix} J(z') \\ J^+(z') \end{vmatrix} \right\}$$

$-m P_1 \qquad -\frac{1}{4} P_1 (DD)$

$$= \exp\left\{ -\frac{1}{2} \int dz\, dz' \,(J(z), J^+(z)) \frac{i}{-\Box - m^2} \begin{vmatrix} \frac{m}{16\Box}(DD)(\bar D \bar D) & \frac{1}{64\Box}(DD)(\bar D \bar D)(DD) \\ \frac{1}{64\Box}(\bar D \bar D)(DD)(\bar D \bar D) & \frac{m}{16\Box}(\bar D \bar D)(DD) \end{vmatrix} \right.$$

$$\left. \delta(z-z') \begin{vmatrix} \frac{1}{4\Box'}(D'D') & 0 \\ 0 & \frac{1}{4\Box'}(\bar D' \bar D') \end{vmatrix} \begin{vmatrix} J(z') \\ J^+(z') \end{vmatrix} \right\}$$

Using (52.1)

$$Z_0[J, J^+] = \exp\left\{ -\frac{1}{2} \int dz\, dz' \,(J(z), J^+(z)) \frac{i}{-\Box - m^2} \begin{vmatrix} -m & -\frac{1}{4}(DD) \\ -\frac{1}{4}(\bar D \bar D) & -m \end{vmatrix} \delta(z-z') \right.$$

$$\left. \begin{vmatrix} \frac{1}{4\Box'}(D'D') & 0 \\ 0 & \frac{1}{4\Box'}(\bar D' \bar D') \end{vmatrix} \begin{vmatrix} J(z') \\ J^+(z') \end{vmatrix} \right\} =$$

integrating by parts and using (52.1)

$$Z_0[J, J^+] = \exp\left\{ -\frac{i}{2} \int dz\, dz' \,(J(z), J^+(z)) \frac{1}{-\Box - m^2} \begin{vmatrix} -\frac{m}{4\Box}(DD) & -\frac{1}{16\Box}(DD)(\bar D \bar D) \\ -\frac{1}{16\Box}(DD)(\bar D \bar D) & -\frac{m}{4\Box}(\bar D \bar D) \end{vmatrix} \right.$$

$$\left. \delta(z-z') \begin{vmatrix} J(z') \\ J^+(z') \end{vmatrix} \right\}$$

and hence

$$Z_0[J, J^+] = \exp\left\{ -\frac{i}{2} \int d^4x\, d^2\theta\, d^2\bar{\theta}\, d^4x'\, d^2\theta'\, d^2\bar{\theta}' \, (J(z), J^+(z))\, \Delta_{GRS}(z,z') \begin{pmatrix} J(z') \\ J^+(z') \end{pmatrix} \right\} \qquad (1)$$

where Δ_{GRS} is the propagator introduced by Grisaru, Roček and Siegel

$$\Delta_{GRS}(z,z') = \frac{1}{-\Box - m^2} \left| \begin{matrix} -\frac{m}{4\Box}(DD) & 1 \\ 1 & -\frac{m}{4\Box}(\bar{D}\bar{D}) \end{matrix} \right| \qquad (2)$$

Now we can differentiate Z_0 with respect to J and J^+ using the rule (51.book, 3)

$$\frac{1}{i} \left| \begin{matrix} \frac{\delta}{\delta J(z)} \\ \frac{\delta}{\delta J^+(z)} \end{matrix} \right| Z_0[J, J^+] = -\int d^4x'\, d^2\theta'\, d^2\bar{\theta}' \, \frac{1}{-\Box - m^2} \left| \begin{matrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{matrix} \right| \delta(x-x')$$

$$\frac{1}{4\Box} \left| \begin{matrix} (DD)\, J(z) \\ (\bar{D}\bar{D})\, J^+(z) \end{matrix} \right| Z_0[J, J^+] \qquad (3)$$

Hence

$$\frac{1}{4} \left| \begin{matrix} (\bar{D}\bar{D}) & 0 \\ 0 & (DD) \end{matrix} \right| \mathcal{M} \, \frac{1}{i} \left| \begin{matrix} \frac{\delta}{\delta J(z)} \\ \frac{\delta}{\delta J^+(z)} \end{matrix} \right| Z_0[J, J^+] =$$

$$= -\int d^4x\, d^2\theta'\, d^2\bar{\theta}' \, \frac{1}{4} \left| \begin{matrix} (\bar{D}\bar{D}) & 0 \\ 0 & (DD) \end{matrix} \right| \mathcal{M} \underbrace{\frac{1}{-\Box - m^2} \left| \begin{matrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{matrix} \right| \delta(x-x')}_{\Delta} \, \frac{1}{4\Box} \left| \begin{matrix} (DD)\, J(z) \\ (\bar{D}\bar{D})\, J^+(z) \end{matrix} \right| Z_0[J, J^+]$$

It is easy to check that

$$\frac{1}{4} \left| \begin{matrix} (\bar{D}\bar{D}) & 0 \\ 0 & (\bar{D}D) \end{matrix} \right| \mathcal{M}\, \Delta = \frac{1}{4} \left| \begin{matrix} (\bar{D}\bar{D}) & 0 \\ 0 & (DD) \end{matrix} \right| \delta(x-x')\, \delta(\theta-\theta')\, \delta(\bar{\theta}-\bar{\theta}') \qquad (4)$$

and we find a functional equation for Z_0

$$\frac{1}{4} \left| \begin{matrix} (\bar{D}\bar{D}) & 0 \\ 0 & (DD) \end{matrix} \right| \mathcal{M} \, \frac{1}{i} \left| \begin{matrix} \frac{\delta}{\delta J(z)} \\ \frac{\delta}{\delta J^+(z)} \end{matrix} \right| Z_0[J, J^+] = \left| \begin{matrix} J(z) \\ J^+(z) \end{matrix} \right| Z_0[J, J^+] \qquad (5)$$

We may generalize this equation to the case of interacting fields. For the Φ^3 model the Lagrangian can be written as

$$L = \int d^4x\, d^2\theta\, d^2\bar\theta \left\{ \frac{1}{2} [\Phi, \Phi^+]\, \mathcal{M} \begin{vmatrix} \Phi \\ \Phi^+ \end{vmatrix} + \right. \tag{1}$$

$$\left. + \frac{1}{3} [\Phi, \Phi^+] \begin{vmatrix} \frac{1}{4\Box}(DD) & 0 \\ 0 & \frac{1}{4\Box}(\bar D\bar D) \end{vmatrix} \begin{vmatrix} \Phi^2 \\ \Phi^{+2} \end{vmatrix} + [\Phi, \Phi^+] \begin{vmatrix} \frac{1}{4\Box}(DD) & 0 \\ 0 & \frac{1}{4\Box}(\bar D\bar D) \end{vmatrix} \begin{vmatrix} J(z) \\ J^+(z) \end{vmatrix} \right\}$$

and hence the field equations are

$$\frac{1}{4} \begin{vmatrix} (\bar D\bar D) & 0 \\ 0 & (DD) \end{vmatrix} \mathcal{M} \begin{vmatrix} \Phi \\ \Phi^+ \end{vmatrix} - g \begin{vmatrix} \Phi^2 \\ \Phi^{+2} \end{vmatrix} = \begin{vmatrix} J(z) \\ J^+(z) \end{vmatrix} \tag{2}$$

and they lead to the following equation for Z

$$\frac{1}{4} \begin{vmatrix} (\bar D\bar D) & 0 \\ 0 & (DD) \end{vmatrix} \mathcal{M} \frac{1}{i} \begin{vmatrix} \frac{\delta}{\delta J(z)} \\ \frac{\delta}{\delta J^+(z)} \end{vmatrix} Z[J, J^+] =$$

$$= \left\{ \begin{vmatrix} J(z) \\ J^+(z) \end{vmatrix} + g \begin{vmatrix} \left[P_2 \frac{1}{i} \frac{\delta}{\delta J(z)} \right]^2 \\ \left[P_1 \frac{1}{i} \frac{\delta}{\delta J^+(z)} \right]^2 \end{vmatrix} \right\} Z[J, J^+] \tag{3}$$

Notice that we have introduced the projection operators in (3). We could have done this in (2), but there it is obvious that Φ is chiral. The chirality of the functional derivative is less explicit, so we choose to keep P_1 and P_2 in (3).

To solve this equation for Z, we first compute the commutator

$$\left[\left(P_2 \frac{\delta}{\delta J(z)} \right)^3, J(z') \right] = 3 \left(-\frac{1}{4} \right) \left[P_2 \frac{\delta}{\delta J(z)} \right]^2 (\bar D\bar D)\, \delta(z-z') =$$

$$= 3 \left(-\frac{1}{4} \bar D\bar D \right) \left[P_2 \frac{\delta}{\delta J(z)} \right]^2 \delta(z-z') \tag{4}$$

where the last step is possible because $\bar D P_2 = 0$. Integrating over $d^4x\, d^2\theta$,

$$\left[\int d^4x\, d^2\theta \left(P_2 \frac{\delta}{\delta J(z)} \right)^3, J(z') \right] = 3 \int d^4x\, d^2\theta\, d^2\bar\theta \left[P_2 \frac{\delta}{\delta J(z)} \right]^2 \delta(z-z') = 3 \left[P_2' \frac{\delta}{\delta J(z')} \right]^2$$

Furthermore

$$\exp\left\{ i\,\frac{g}{3}\int d^4x'\,d^2\theta'\left(P_2\frac{1}{i}\frac{\delta}{\delta J(z')}\right)^3\right\} J(z)\ \exp\left\{-i\,\frac{g}{3}\int d^4x'\,d^2\theta'\left(P_2\,\frac{1}{i}\,\frac{\delta}{\delta J(z')}\right)^3\right\} =$$

$$= J(z) + g\left(P_2\,\frac{1}{i}\,\frac{\delta}{\delta J(z)}\right)^2 \qquad (1)$$

Now we cast (64.3) in the following form

$$\frac{1}{4}\begin{vmatrix} (\bar D\bar D) & 0 \\ 0 & (DD)\end{vmatrix}\ \mathcal{M}\ \frac{1}{i}\begin{vmatrix}\dfrac{\delta}{\delta J(z)} \\ \dfrac{\delta}{\delta J^+(z)}\end{vmatrix}\ Z[J,J^+] = \exp\left\{ i\int d^4x'\,L_{int}\left(\frac{\delta}{\delta J(z')},\frac{\delta}{\delta J^+(z')}\right)\right\}.$$

$$* \begin{vmatrix} J(z) \\ J^+(z)\end{vmatrix}\ \exp\left\{-i\int d^4x'\,L_{int}\left(\frac{\delta}{\delta J(z')},\ \frac{\delta}{\delta J^+(z')}\right)\right\}\ Z[J,J^+] \qquad (2)$$

This shows that

$$Z_0[J,J^+] = \exp\left\{-i\int d^4x'\,L_{int}\left(\frac{\delta}{\delta J},\ \frac{\delta}{\delta J^+}\right)\right\}\ Z[J,J^+] \qquad (3)$$

since the righ-hand side satisfies the free equations (65.5). No normalization factor is needed because of the fact that all vacuum-to-vacuum diagrams vanish. With (3) we have proved

$$Z[J,J^+] = \exp\left\{ i\int d^4x\,L_{int}\left(\frac{\delta}{\delta J},\frac{\delta}{\delta J^+}\right)\right\}\ Z_0[J,J^+] \qquad (4)$$

and solved for the generating functional of an interacting chiral supersymmetric theory.

Having found the generating functional, we shall now derive the Feynman rules. We begin by recalling the relation between Green's functions and the generating functional.

$$G^{(N)}(z^1,\dots,z^M;\,z^{M+1},\dots,z^N) = (-i)^N\,\frac{\delta}{\delta J_1}\,\frac{\delta}{\delta J_2}\cdots\frac{\delta}{\delta J_M}\,\frac{\delta}{\delta J^+_{M+1}}\cdots\frac{\delta}{\delta J^+_N}$$

$$\sum_{k=0}^{\infty}\frac{i^k}{k!}\left\{\int d^4x^k\,L_{int}\left(\frac{\delta}{\delta J_k},\frac{\delta}{J^+_k}\right)\right\}^k Z_0[J,J^+]\Big|_{J=J^+=0} \qquad (5)$$

The factors of L_{int} generate vertices at z^k. The derivatives in L_{int} act on previous derivatives and on Z_0 itself. Each derivative acting on Z_0 creates a new propagator at z^k. Each derivative acting on a previous derivative connects an existing propagator to z^k. In this way every new vertex is completely saturated with propagators.

The effective action is computed from the one particle irreducible (1PI) Green's functions. In general, 1PI diagrams have at least two internal lines leaving each vertex. The external legs of the 1PI diagrams are amputated with inverse propagators. They are then multiplied by the superfield amplitudes $\Phi(z)$ and $\Phi^+(z)$. This leads to the following Feynman rules:

1) For each external line, write a chiral superfield $\Phi(z)$, $\Phi^+(z)$
2) At each Φ^3 vertex with two [three] internal lines, include factors of $-(\bar{D}\bar{D})/4$ acting on one [two] internal propagators. At each Φ^{+3} vertex, include similar factors of $-(DD)/4$.
3) Write a factor $g/3$ for each vertex, and integrate $d^4x\, d^2\theta\, d^2\bar{\theta}$ over each vertex
4) Use Grisaru - Roček - Siegel propagators for $\Phi\Phi$, $\Phi^+\Phi^+$ and $\Phi^+\Phi$ internal lines.
5) Compute the usual combinatoric factors of a A^3 theory.

Let us now use these rules to follow the θ-integrations around an arbitrary closed loop. The Feynman rules and the GRS propagators combine to give an expression of the following form

$$(D_1 D_1)^{l_1} (\bar{D}_1 \bar{D}_1)^{k_1} \delta(12)\, (D_2 D_2)^{l_2} (\bar{D}_2 \bar{D}_2)^{k_2} \delta(23) \ldots\ldots (D_m D_m)^{l_m} (\bar{D}_m \bar{D}_m)^{k_m} \delta(m1) \qquad (1)$$

The exponents l_i, k_i are either zero or one, and $\delta(12) = \delta(\theta_1 - \theta_2)\, \delta(\bar{\theta}_1 - \bar{\theta}_2)$. For a general loop, the D and $\bar{D}$ factors might appear in the opposite order. However, any higher powers of D^2 and $\bar{D}^2$ may be reduced to the above form up to powers of $\Box$

$$\bar{D}^2 D^2 \bar{D}^2 = -16 \Box \bar{D}^2 \quad , \quad D^2 \bar{D}^2 D^2 = -16 \Box D^2 \qquad (2)$$

Of course, for the effective action, the above expression is multiplied by superfields for external legs and GRS propagators for adjoining closed loops. It is also integrated over $d^4x_1\, d^2\theta_1\, d^2\bar{\theta}_1 \ldots d^4x_m\, d^2\theta_m\, d^2\bar{\theta}_m$. The final expression is evaluated by removing the D and $\bar{D}$ derivatives from one δ-function after another by partial integration. This introduces new derivatives on the lines that leave the loop. It also introduces a certain number of derivatives on the last δ-function, say $\delta(m1)$. All but one of the θ-integrations may be performed with the aid of the δ-functions $\delta(12) \ldots \delta(m-1\, m)$. This leaves a factor of

$$\int d^2\theta_m\, d^2\bar\theta_m \;(DD)^{\ell}\,(\bar D\bar D)^{k}\;\delta(\theta_m-\theta_1)\,\delta(\bar\theta_m-\bar\theta_1)\Big|_{\substack{\theta_1=\theta_m\\ \bar\theta_1=\bar\theta_m}} \qquad (1)$$

or

$$\int d^2\theta_m\, d^2\bar\theta_m \;(\bar D\bar D)^{k}\,(DD)^{\ell}\;\delta(\theta_m-\theta_1)\,\delta(\bar\theta_m-\bar\theta_1)\Big|_{\substack{\theta_1=\theta_m\\ \bar\theta_1=\bar\theta_m}} \qquad (2)$$

These expressions vanish unless $k=\ell=1$. In this case, we find

$$\int d^2\theta_m\, d^2\bar\theta_m \;(DD)(\bar D\bar D)\;\delta(\theta_m-\theta_1)\,\delta(\bar\theta_m-\bar\theta_1)\Big|_{\substack{\theta_1=\theta_m\\ \bar\theta_1=\bar\theta_m}} =$$

$$= \int d^2\theta_m\, d^2\bar\theta_m \;(\bar D\bar D)(DD)\;\delta(\theta_m-\theta_1)\,\delta(\bar\theta_m-\bar\theta_1)\Big|_{\substack{\theta_1=\theta_m\\ \bar\theta_1=\bar\theta_m}} = 16\int d^2\theta_m\, d^2\bar\theta_m \qquad (3)$$

as follows from (56.1). The whole loop in θ-space has shrunk to one $d^2\theta d^2\bar\theta$ integration. This process can now be carried over to the next loop and we finally arrive at the result

$$\int d^2\theta\, d^2\bar\theta \int d^4x_1 \ldots d^4x_m\;F_1(x_1,\theta,\bar\theta)\ldots\; F_m(x_m,\theta,\bar\theta)\;G(x_1\ldots x_m) \qquad (4)$$

The function $G(x_1 \ldots x_m)$ is translationally invariant and the F's are products of superfields and their derivatives. No factors of $1/\Box$ appear in the F's, so for chiral operators the $d^2\theta\, d^2\bar\theta$ integration cannot be converted into a $d^2\theta$ integration without introducing space time derivatives. This leads to the surprising result that mass and coupling terms of the $d^2\theta$ form are not renormalized in supersymmetric theories. Furthermore, no higher dimensional momentum-independent chiral operators are induced in the effective superpotential to any order in perturbation theory. Eq (4) also implies that all vacuum-to-vacuum diagrams vanish. This is because expressions of the type (4) without any superfields are immediately annihilated by the $d^2\theta\, d^2\bar\theta$ integration.*

We have seen already that there are not counterterms for the superpotential. For possible other divergences we have the following power counting rules. The degree of divergence of any graph is given by [S. FERRARA and G. PIGUET Nucl. Phys B93, 261 (1975)][K. SIBOLD. 'Renormalization Rigid N=1 Supersymmetry'. Cours de 3ème Cycle a la Suisse Romande (1983)] (except for contributions to the superpotential)

$$d = 2 - E_c - I_c$$

where E_c is the number of external chiral lines and I_c is the number of massive internal

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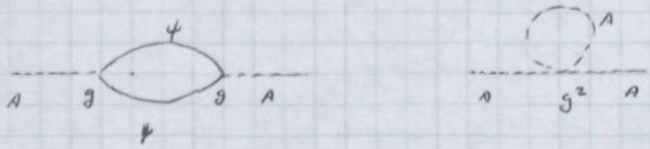
* Thus e.g. in the Wess-Zumino model the mass of the scalars gets no renormalization because of the cancelation of the quadratic divergences in the graphs



The fact that the parameters g and m do not get loop corrections does not mean that they do not "run". They depend on the energy scale at which they are measured because there is still wave function renormalization of the fields since the kinetic term in the lagrangian does receive logarithmically divergent loop corrections. Only in higher supersymmetries (N=4 or certain N=2 theories) do these loops corrections also vanish giving rise to completely finite theories [S. MANDELSTAM Nucl. Phys. B213, 149 (1983)] [A. PARKES and P. WEST Phys. Lett. B122, 365 (1983) and Nucl. Phys. B222, 269 (1983)].

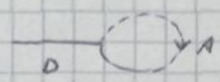
chiral propagators. From this one then deduces the possible divergent contributions to the effective action. They are

$$\int d^2\theta\, d^2\bar\theta\ \Phi^+ \Phi\ , \qquad \int d^2\theta\, d^2\bar\theta\ \Phi^+ V \Phi\ , \qquad \int d^2\theta\, d^2\bar\theta\ V \tag{1}$$

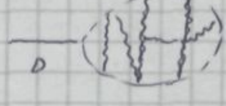
$$\int d^2\theta\, d^2\bar\theta\ VV\ , \qquad \int d^2\theta\, d^2\bar\theta\ VVV$$

where in the last two terms two D and two $\bar D$ factors should be included at arbitrary positions. We have assumed that V appears as a gauge vector multiplet. The factors Φ have dimension 1, D_α dimension 1/2, V is dimensionless and $d^2\theta\, d^2\bar\theta$ has dimension two. As a result all contributions in (1) except for $\int d^2\theta\, d^2\bar\theta\ V$ are only logarithmically divergent (remember that the last two term contain $D^2 \bar D^2$). They correspond to the usual wave function and gauge coupling constant renormalizations. The contribution $\int d^2\theta\, d^2\bar\theta\ V$ to the Fayet-Iliopoulos term is quadratically divergent. It is gauge invariant only if the gauge group under consideration is abelian (or contains a U(1) factor). We will come back to it later and assume for the moment that it is absent. We thus have only logarithmic divergences. In particular the usual quadratic divergence of the masses of scalar particles is absent in a supersymmetric theory.

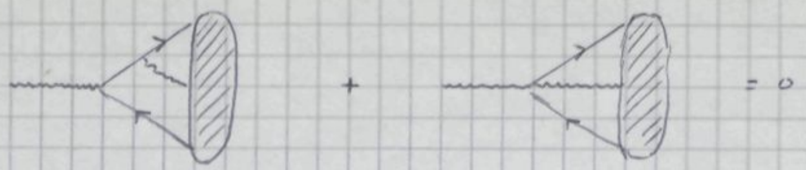
Let us come back to the Fayet-Iliopoulos term which can only appear in the presence of an abelian gauge symmetry. It receives a quadratic divergent contribution in perturbation theory by the graph



and there is no other contribution to this order in the gauge coupling constant that could cancel this contribution. The graph, however, implies a sum of the contributions of all scalar particles in the theory weighted by the charges which define the coupling to the D-field. If one would now, for example, impose a parity symmetry these contributions would add up to zero (D is a pseudoscalar). More general the contribution at the one loop level is absent if the trace of the charges $\mathrm{Tr}\, Q$ of the scalar particles vanishes. But this takes only care of the first order of perturbation theory. Contributions in higher order would still lead to quadratic divergences. It actually turns out that this does not happen. The imposition of the constraint $\mathrm{Tr}\, Q = 0$



is sufficient to forbid the appearance of $\int d^2\theta\, d^2\bar\theta\ V$ in perturbation theory. [E. WITTEN Nucl. Phys. B188, 513 (1981)] [W. FISCHLER, H. P. NILLES, J. POLCHINSKI, S. RABY and L. SUSSKIND Phys. Rev. Lett. 47, 757 (1981)]. This result can be best seen by the use of supergraphs



The two contributions cancel. The blob denotes an arbitrary supergraph with external Φ, Φ^+ and V external lines and it is of course understood that the blobs in the two contributions are identical for the cancellation to hold.

We can thus summarize the results for the supersymmetric theories under consideration. Provided that $Tr\,Q = 0$, there are no quadratic divergences. The existing logarithmic divergences correspond to wave function renormalizations of the chiral and vector field as well as the gauge coupling renormalization. The parameters in the superpotential (terms like $\int d^2\theta$) are not renormalized at all. The same is true for the vacuum energy reflecting the fact that supersymmetry remains well defined in perturbation theory.

Supersymmetry is free of quadratic divergences. One might now ask the question whether all theories that do not have quadratic divergences are automatically supersymmetric. The answer to this question is no. Supersymmetry can be broken explicitly and there are still no quadratic divergences in field quantities [L. GIRARDELLO and M. T. GRISARU Nucl. Phys. B194, 65 (1982)] [S. DIMOPOULOS and H. GEORGI. Nucl. Phys. B193, 150 (1981)] [K. HARADA and N. SAKAI Prog. Theor. Phys. 67, 1877 (1982)] [N. SAKAI Zeit f. Physik C11, 153 (1982)]. The nature of these allowed breakings is however restricted. The breakings that lead to theories where quadratic divergences are still absent are called soft breakings. They have been completely listed by Girardello and Grisaru and are of the form.

i) Any dimension two operator. This means scalar mass terms of two different types

$$m^2 A^+(x)\, A(x) \quad , \quad m^2 \,[\, A(x)\, A(x) + h.c.\,] \qquad (1)$$

ii) Gaugino mass terms

$$m\, \lambda(x)\, \lambda(x) \qquad (2)$$

iii) Trilinear scalar couplings $\quad \mu\, [A^3(x) + h.c.\,] \qquad (3)$

Any of these term may be added to the SUSY Lagrangian without generating quadratic divergences in loop calculations. However, other dimension three terms like explicit chiral fermion masses may induce those divergences, i.e. they are not soft.

It is striking that the term $m\lambda\lambda$ is soft whereas terms like $m\psi\psi$, whereas ψ is the fermion of a chiral multiplet, are not soft. This can be explained by gauge symmetry and the fact that $\lambda\lambda$ is the "lowest" member of the $W^\alpha W_\alpha$ multiplet.

With this explicit breaking parameters now logarithmic divergences do usually appear (e.g. logarithmically divergent mass counterterm) and there is also a quadratic divergence of the vacuum energy.

Let us consider a non-abelian gauge group G with Lie algebra

$$[T_a, T_b] = i f_{ijk} T_c \qquad , \quad a, b, c = 1, 2, \dots, N \tag{1}$$

where $T_a = T_a^+$ are the group generators and f_{abc} is real and fully antisymmetric. We will assume

$$\mathrm{Tr}\,[T_a T_b] = k\,\delta_{ab} \qquad , \quad k > 0 \tag{2}$$

We would like to construct the most general Lagrangian density corresponding to a theory with the following characteristics

i) Must be renormalizable

ii) Must be invariant under supersymmetry transformations

iii) Must be invariant under the local gauge transformations associated with the group G

iv) Must involve only spin 0, 1/2 and 1 fields.

Up to this end we will introduce a set of chiral superfields $\Phi_i(x,\theta,\bar\theta)$ and $\Phi_i^+(x,\theta,\bar\theta)$ which, under local gauge transformations, transform as

$$\Phi(x,\theta,\bar\theta) \longrightarrow \Phi'(x,\theta,\bar\theta) = e^{-i\Lambda(x,\theta,\bar\theta)}\,\Phi(x,\theta,\bar\theta)$$

$$\Phi^+(x,\theta,\bar\theta) \longrightarrow \Phi'^+(x,\theta,\bar\theta) = \Phi^+(x,\theta,\bar\theta)\,e^{+i\Lambda^+(x,\theta,\bar\theta)} \tag{3}$$

where

$$\Lambda(x,\theta,\bar\theta) \equiv 2g\,T_a\,\Lambda_a(x,\theta,\bar\theta)$$

$$\Lambda^+(x,\theta,\bar\theta) \equiv 2g\,T_a\,\Lambda_a^+(x,\theta,\bar\theta) \tag{4}$$

being $\Lambda_a(x,\theta,\bar\theta)$ and $\Lambda_a^+(x,\theta,\bar\theta)$ chiral superfields and T_a corresponds to the representation R of G.

We will introduce furthermore vector superfields $V_a(x,\theta,\bar\theta)$ and for them we will assume the following transformation law

$$e^{V(x,\theta,\bar\theta)} \longrightarrow e^{V'(x,\theta,\bar\theta)} = e^{-i\Lambda^+(x,\theta,\bar\theta)}\, e^{V(x,\theta,\bar\theta)}\, e^{+i\Lambda(x,\theta,\bar\theta)}$$

$$V(x,\theta,\bar\theta) \equiv -2g\,T_a\,V_a(x,\theta,\bar\theta) \tag{5}$$

Notice that when the r.h.s of the last equation is evaluated using the Baker - Haussdorf formula we see clearly, using the group algebra that the transformation law for the vector fields is independent of the representation R used. Furthermore

$$V(x,\theta,\bar{\theta}) \longrightarrow V'(x,\theta,\bar{\theta}) = V(x,\theta,\bar{\theta}) + i\,[\Lambda(x\theta\bar{\theta}) - \Lambda^{+}(x\theta\bar{\theta})] + \dots \qquad (1)$$

and therefore also here we can use the Wess - Zumino (WZ) gauge.

We will need the superfield strengths defined as

$$W_\alpha \equiv -\frac{1}{4}\bar{D}\bar{D}\,e^{-V} D_\alpha e^{+V} \quad , \quad \bar{W}_{\dot\alpha} \equiv -\frac{1}{4} DD\, e^{+V}\bar{D}_{\dot\alpha} e^{-V} \qquad (2)$$

and we obtain easely

$$W_\alpha \longrightarrow W'_\alpha = e^{-i\Lambda} W_\alpha e^{+i\Lambda}$$

$$\bar{W}_{\dot\alpha} \longrightarrow \bar{W}'_{\dot\alpha} = e^{-i\Lambda^{+}} \bar{W}_{\dot\alpha} e^{+i\Lambda^{+}} \qquad (3)$$

With all that the most general Lagrangian density of the required type is

$$L(x) = \frac{1}{16\,k\,g^2}\,\mathrm{Tr}\left\{ W^\alpha W_\alpha\Big|_{[\theta\theta]} + \bar{W}_{\dot\alpha}\bar{W}^{\dot\alpha}\Big|_{[\bar\theta\bar\theta]} \right\} + \Phi^{+} e^{V}\Phi\Big|_{[\theta\theta][\bar\theta\bar\theta]}$$

$$+\left[\left(\frac{1}{2} m_{ij}\,\Phi_i\Phi_j + \frac{1}{3} g_{ijk}\,\Phi_i\Phi_j\Phi_k\right)\Big|_{[\theta\theta]} + h.c.\right] \qquad (4)$$

where m_{ij} and g_{ijk} are fully symmetric and they must be invariant tensors under gauge transformations.

Let us now try to write (4) in terms of the component fields. We will carry our analysis in the WZ gauge. Then

$$\Phi^{+} e^{V}\Phi = \Phi_i^{+}\,\{\delta_{ij} - 2g\,(T_a)_{ij} V_a + 2g^2\,(T_aT_b)_{ij} V_a V_b\}\,\Phi_j \qquad (5)$$

From (18.1)

$$\Phi_i^+ \Phi_i \Big|_{[\theta\theta][\bar\theta\bar\theta]} = \partial_\mu A_i^*(x)\, \partial^\mu A_i(x) + F_i^*(x)\, F_i(x) + i\, \bar\Psi_i(x)\, \tilde\sigma^\mu \partial_\mu \psi_i(x) \tag{1}$$

Furthermore

$$-2g\,(T_a)_{ij}\, \Phi_i^+ \Phi_j V_a \Big|_{[\theta\theta][\bar\theta\bar\theta]} = -2g\,(T_a)_{ij}\, \Phi_i^+ \Phi_j \Big\{ [\theta\sigma_\mu\bar\theta]\, v_a^\mu(x) + i[\theta\theta][\bar\theta\bar\lambda_a(x)]$$

$$- i[\bar\theta\bar\theta][\theta\lambda_a(x)] + \frac{1}{2}[\theta\theta][\bar\theta\bar\theta]\, D_a(x) \Big\}\Big|_{[\theta\theta][\bar\theta\bar\theta]} =$$

$$= -2g\,(T_a)_{ij} \Big\{ \frac{1}{2} A_i^*(x)\, A_j(x)\, [\theta\theta][\bar\theta\bar\theta]\, D_a(x) - i\sqrt{2}\, A_i^*(x)\, [\theta\psi_j(x)][\bar\theta\bar\theta][\theta\lambda_a(x)]$$

$$+ i\sqrt{2}\, [\bar\theta\bar\Psi_i(x)]\, A_j(x)\, [\theta\theta][\bar\theta\bar\lambda_a(x)] +$$

$$+ [\theta\sigma_\nu\bar\theta][-i A_i^*(x)\, \partial^\nu A_j(x) + i\, \partial^\nu A_i^*(x)\, A_j(x) + \psi_j(x)\, \sigma^\nu \bar\Psi_i(x)][\theta\sigma_\mu\bar\theta]\, v_a^\mu(x) \Big\}\Big|_{[\theta\theta][\bar\theta\bar\theta]} \Rightarrow$$

$$-2g\,(T_a)_{ij}\, \Phi_i^+ \Phi_j V_a \Big|_{[\theta\theta][\bar\theta\bar\theta]} = -g\,(T_a)_{ij}\, A_i^*(x)\, A_j(x)\, D_a(x)$$

$$- i\sqrt{2}\, g\,(T_a)_{ij}\, A_i^*(x)\, [\psi_j(x)\lambda_a(x)] + i\sqrt{2}\, g\,(T_a)_{ij}\, A_j(x)\, [\bar\Psi_i(x)\bar\lambda_a(x)]$$

$$+ i g\,(T_a)_{ij}\, [A_i^*(x)\, \partial_\mu A_j(x) - \partial_\mu A_i^*(x)\, A_j(x)]\, v^\mu_a(x)$$

$$+ g\,(T_a)_{ij}\, [\bar\Psi_i(x)\, \tilde\sigma_\mu\, \psi_j(x)]\, v^\mu_a(x) \tag{2}$$

Finally

$$2g^2\,(T_a T_b)_{ij}\, \Phi_i^+ \Phi_j V_a V_b \Big|_{[\theta\theta][\bar\theta\bar\theta]} = 2g^2\,(T_a T_b)_{ij}\, \Phi_i^+ \Phi_j \Big\{ [\theta\sigma_\mu\bar\theta]\, v_a^\mu \cdot$$

$$\cdot [\theta\sigma_\nu\bar\theta]\, v_b^\nu(x) \Big\}\Big|_{[\theta\theta][\bar\theta\bar\theta]} = 2g^2\,(T_a T_b)_{ij}\, A_i^*(x)\, A_j(x)\, \frac{1}{2}[\theta\theta][\bar\theta\bar\theta]\, v_a^\mu(x)\, v^b_\mu(x) \Big|_{[\theta\theta][\bar\theta\bar\theta]} \Rightarrow$$

$$2g^2\,(T_a T_b)_{ij}\, \Phi_i^+ \Phi_j V_a V_b \Big|_{[\theta\theta][\bar\theta\bar\theta]} = g^2\,(T_a T_b)_{ij}\, A_i^*(x)\, A_j(x)\, v_a^\mu(x)\, v_\mu^b(x) \tag{3}$$

Let us now consider the field strengths. We have

$$W_\alpha = -\frac{1}{4}\bar D\bar D\, e^{2g T_a V_a} D_\alpha\, e^{-2g T_a V_a} =$$

$$= -\frac{1}{4}\bar{D}\bar{D}\left\{1 + 2g\,T_a V_a + 2g^2\, T_a T_b V_a V_b\right\} D_\alpha \left\{-2g\,T_c V_c + 2g^2\, T_c T_d V_c V_d\right\} =$$

$$= +\frac{1}{2}g\,T_a\,\bar{D}\bar{D}\,D_\alpha V_a + g^2\,T_a T_b\,\bar{D}\bar{D}\,V_a D_\alpha V_b - \frac{1}{2}g^2\,T_a T_b\,\bar{D}\bar{D}\,D_\alpha V_a V_b =$$

$$= \frac{1}{2}g\,T_a\,\bar{D}\bar{D}\,D_\alpha V_a + \frac{1}{2}g^2\,T_a T_b\,\bar{D}\bar{D}\,V_a D_\alpha V_b - \frac{1}{2}g^2\,T_a T_b\,\bar{D}\bar{D}\,V_b D_\alpha V_a \Rightarrow$$

$$W_\alpha = +\frac{1}{2}g\,T_a\left\{\bar{D}\bar{D}\,D_\alpha V_a + i g\,f_{abc}\,\bar{D}\bar{D}\,V_b D_\alpha V_c\right\}$$

(1)

$$\bar{W}_{\dot\alpha} = -\frac{1}{2}g\,T_a\left\{DD\,\bar{D}_{\dot\alpha} V_a - i g\,f_{abc}\,DD\,V_b \bar{D}_{\dot\alpha} V_c\right\}$$

From (24.1) and (24.2)

$$\bar{D}\bar{D}\,D_\alpha V_a = 4i\,\lambda^a_\alpha(y) - 4\theta_\alpha D_a(y) + 2i\,[\sigma^\mu]_{\alpha\dot\alpha}[\tilde\sigma^\nu]^{\dot\alpha\beta}\theta_\beta\left[\partial_\mu v^a_\nu(y) - \partial_\nu v^a_\mu(y)\right]$$

$$+ 4[\theta\theta][\sigma^\mu]_{\alpha\dot\alpha}\partial_\mu\bar\lambda^{\dot\alpha}_a(y) \qquad y^\mu \equiv x^\mu - i[\theta\sigma^\mu\bar\theta]$$

(2)

$$DD\,\bar{D}_{\dot\alpha} V_a = -4i\,\bar\lambda^a_{\dot\alpha}(y^+) - 4\bar\theta_{\dot\alpha} D_a(y^+) + 2i\,\bar\theta_{\dot\beta}[\tilde\sigma^\mu]^{\dot\beta\alpha}[\sigma^\nu]_{\alpha\dot\alpha}\left[\partial_\mu v^a_\nu(y^+) - \partial_\nu v^a_\mu(y^+)\right]$$

$$+ 4[\bar\theta\bar\theta]\,\partial_\mu\lambda^\alpha_a(y^+)[\sigma^\mu]_{\alpha\dot\alpha} \qquad y^{+\mu} \equiv x^\mu + i[\theta\sigma^\mu\bar\theta]$$

Furthermore

$$\bar{D}\bar{D}\,V_b D_\alpha V_c = \epsilon^{\dot\gamma\dot\delta}\frac{\partial}{\partial\bar\theta^{\dot\gamma}}\frac{\partial}{\partial\bar\theta^{\dot\delta}}\Big\{[\theta\sigma^\mu\bar\theta]v^b_\mu(y) + i[\theta\theta][\bar\theta\bar\lambda_b(y)] - i[\bar\theta\bar\theta][\theta\lambda_b(y)]$$

$$+\frac{1}{2}[\theta\theta][\bar\theta\bar\theta]\left[D_b(y) + i\,\partial_\mu v^\mu_b(y)\right]\Big\}\Big\{(\sigma^\nu)_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}v^c_\nu(y) - 2i(\sigma^\nu)_{\alpha\dot\beta}\bar\theta^{\dot\beta}(\theta\sigma^\rho\bar\theta)\partial_\mu v^c_\rho(y)$$

$$+2i\,\theta_\alpha[\bar\theta\bar\lambda_c(y)] + 2(\sigma^\nu)_{\alpha\dot\beta}\bar\theta^{\dot\beta}(\theta\theta)(\bar\theta\,\partial_\nu\bar\lambda_c(y)) - i[\theta\theta]\lambda^c_\alpha(y) + \theta_\alpha[\bar\theta\bar\theta][D_c(y) + i\,\partial^\nu v^c_\nu(y)]\Big\} =$$

$$= \epsilon^{\dot\gamma\dot\delta}\frac{\partial}{\partial\bar\theta^{\dot\gamma}}\frac{\partial}{\partial\bar\theta^{\dot\delta}}\Big\{\theta^\beta(\sigma^\mu)_{\beta\dot\beta}\bar\theta^{\dot\beta}v^b_\mu(y)(\sigma^\nu)_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}v^c_\nu(y) + 2i\,\theta^\beta(\sigma^\mu)_{\beta\dot\beta}\bar\theta^{\dot\beta}v^b_\mu(y)\,\theta_\alpha\bar\theta_{\dot\epsilon}\bar\lambda^{\dot\epsilon}_c(y)$$

$$+ i[\theta\theta]\bar\theta_{\dot\beta}\bar\lambda^{\dot\beta}_b(y)[\sigma^\mu]_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}v^c_\mu(y)\Big\} =$$

$$= \epsilon^{\dot\gamma\dot\delta}\frac{\partial}{\partial\bar\theta^{\dot\gamma}}\Big\{-\theta^\beta(\sigma^\mu)_{\beta\dot\delta}v^b_\mu(y)(\sigma^\nu)_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}v^c_\nu(y) + \theta^\beta(\sigma^\mu)_{\beta\dot\beta}\bar\theta^{\dot\beta}v^b_\mu(y)(\sigma^\nu)_{\alpha\dot\delta}v^c_\nu(y)$$

$$-2i\,\theta^\beta(\sigma^\mu)_{\beta\dot\delta}v^b_\mu(y)\,\theta_\alpha\bar\theta_{\dot\epsilon}\bar\lambda^{\dot\epsilon}_c(y) + 2i\,\theta^\beta(\sigma^\nu)_{\beta\dot\beta}\bar\theta^{\dot\beta}v^b_\mu(y)\,\theta_\alpha\bar\lambda^c_{\dot\delta}(y)$$

$$\mp i\,[\theta\theta]\,\bar\lambda^{b}{}_{\dot\delta}(y)(\sigma^\mu)_{\alpha\dot\alpha}\,\bar\theta^{\dot\alpha}\, v_\mu{}^{c}(y) + i\,[\theta\theta]\,\bar\theta_{\dot\beta}\,\bar\lambda^{\dot\beta}{}_{b}(y)\,(\sigma^\mu)_{\alpha\dot\delta}\, v_\mu{}^{c} \Big\} =$$

$$= \theta^\beta (\sigma^\mu)_\beta{}^{\dot\gamma}\, v_\mu{}^{b}(y)\,(\sigma^\nu)_{\alpha\dot\gamma}\, v_\nu{}^{c}(y) - \theta^\beta(\sigma^\mu)_{\beta\dot\gamma}\, v_\mu{}^{b}(y)\,(\sigma^\nu)_\alpha{}^{\dot\gamma}\, v_\nu{}^{c}(y) + 2i\,\theta^\beta(\sigma^\mu)_\beta{}^{\dot\gamma}\, v_\mu{}^{b}(y)\,\theta_\alpha\,\bar\lambda^{c}_{\dot\gamma}(y)$$

$$+ 2i\,\theta^\beta(\sigma^\mu)_{\beta\dot\gamma}\, v_\mu{}^{b}(y)\,\theta_\alpha\,\bar\lambda_c{}^{\dot\gamma}(y) + i\,[\theta\theta]\,\bar\lambda^{\dot\gamma}_{b}(y)\,(\sigma^\mu)_{\alpha\dot\gamma}\, v_\mu{}^{c}(y) \mp i\,[\theta\theta]\,\bar\lambda^{b}{}_{\dot\gamma}(y)\,(\sigma^\mu)_\alpha{}^{\dot\gamma}\, v_\mu{}^{c}(y)\Big\} \Rightarrow$$

$$\bar D\bar D\, V_b\, D_\alpha V_c = -2\theta^\beta\,[\sigma^\mu]_{\beta\dot\gamma}\,[\tilde\sigma^\nu]^{\dot\gamma}{}_\alpha\, v_\mu{}^{b}(y)\, v_\nu{}^{c}(y) -$$

$$+ 4i\,[\theta\sigma^\mu\bar\lambda^{c}(y)]\,\theta_\alpha\, v_\mu{}^{b}(y) \mp 2i\,[\theta\theta]\,\bar\lambda^{b}_{\dot\alpha}(y)\,[\tilde\sigma^\mu]^{\dot\alpha}{}_\alpha\, v_\mu{}^{c}(y)$$

(1)

$$D D\, V_b\, \bar D_{\dot\alpha} V_c = -2\,\bar\theta_{\dot\beta}\,[\tilde\sigma^\mu]^{\dot\beta\gamma}\,[\sigma^\nu]_{\gamma\dot\alpha}\, v_\mu{}^{b}(y^+)\, v_\nu{}^{c}(y^+)$$

$$\mp 4i\,[\bar\theta\tilde\sigma^\mu\lambda_c(y^+)]\,\bar\theta_{\dot\alpha}\, v_\mu{}^{b}(y^+) + 2i\,[\bar\theta\bar\theta]\,\lambda^{b}{}_\alpha(y^+)\,[\sigma^\mu]^{\alpha}{}_{\dot\alpha}\, v_\mu{}^{c}(y^+)$$

Now we have

$$\frac{1}{16kg^2}\,Tr\,[W^\alpha W_\alpha] = \frac{1}{64}\Big\{\bar D\bar D D^\alpha V_a + ig\, f_{abc}\,\bar D\bar D\, V_b\, D^\alpha V_c\Big\}\cdot$$

$$\cdot\Big\{\bar D\bar D D_\alpha V_a + ig\, f_{ade}\,\bar D\bar D\, V_d\, D_\alpha V_e\Big\} \Rightarrow$$

$$\frac{1}{16kg^2}\,Tr\,[W^\alpha W_\alpha] = \frac{1}{64}\Big\{[\bar D\bar D D^\alpha V_a][\bar D\bar D D_\alpha V_a]$$

$$+ 2ig\, f_{abc}\,[\bar D\bar D D^\alpha V_a][\bar D\bar D\, V_b\, D_\alpha V_c] - g^2 f_{abc} f_{ade}\,[\bar D\bar D\, V_b\, D^\alpha V_c][\bar D\bar D\, V_d\, D_\alpha V_e]\Big\}$$

(2)

$$\frac{1}{16kg^2}\,Tr\,[\bar W_{\dot\alpha}\bar W^{\dot\alpha}] = \frac{1}{64}\Big\{[DD\bar D_{\dot\alpha} V_a][DD\bar D^{\dot\alpha} V_a]$$

$$- 2ig\, f_{abc}\,[DD\bar D_{\dot\alpha} V_a][DD\, V_b\,\bar D^{\dot\alpha} V_c] - g^2 f_{abc} f_{ade}\,[DD\, V_b\,\bar D_{\dot\alpha} V_c][DD\, V_d\,\bar D^{\dot\alpha} V_e]\Big\}$$

From (25.2)

$$\frac{1}{64}\Big\{[\bar D\bar D D^\alpha V_a][\bar D\bar D D_\alpha V_a]\Big|_{[\theta\theta]} + [DD\bar D_{\dot\alpha} V_a][DD\bar D^{\dot\alpha} V_a]\Big|_{[\bar\theta\bar\theta]}\Big\} =$$

$$= \frac{1}{2}\, D_a^2(x) - \frac{1}{4}\,[\partial^\mu v^\nu_a(x) - \partial^\nu v^\mu_a(x)][\partial_\mu v_\nu^a(x) - \partial_\nu v_\mu^a(x)]$$

$$+ i\,[\lambda_a(x)\,\sigma^\mu\partial_\mu\bar\lambda_a(x)]$$

(3)

Furthermore

$$\frac{i}{32} g f_{abc} [\bar{D}\bar{D} D^\alpha V_a][\bar{D}\bar{D} V_b D_\alpha V_c]\Big|_{[\theta\theta]} = \frac{i}{32} g f_{abc} \Big\{ 4i \lambda^\alpha_a(y) - 4\theta^\alpha D_a(y) +$$

$$+ 2i (\sigma^\mu)^\alpha{}_{\dot\alpha} (\tilde\sigma^\nu)^{\dot\alpha\beta} \theta_\beta [\partial_\mu v^a_\nu(y) - \partial_\nu v^a_\mu(y)] + 4[\theta\theta][\sigma^\mu]^\alpha{}_{\dot\alpha} \partial_\mu \bar\lambda^{\dot\alpha}_a(y) \Big\}$$

$$\Big\{ -2\theta^\gamma [\sigma^\rho]_{\gamma\dot\gamma} [\tilde\sigma^\tau]^{\dot\gamma}{}_\alpha v^b_\rho(y) v^c_\tau(y) + 4i[\theta\sigma^\rho \bar\lambda^c(y)]\theta_\alpha v^b_\rho(y) + 2i[\theta\theta]\bar\lambda^b_{\dot\gamma}(y)(\tilde\sigma^\rho)^{\dot\gamma}{}_\alpha v^c_\rho(y) \Big\}\Big|_{[\theta\theta]}$$

$$= \frac{i}{32} g f_{abc} \Big\{ -16 \lambda^\alpha_a(y) [\theta\sigma^\rho \bar\lambda^c(y)] \theta_\alpha v^b_\rho(y) + 8 \lambda^\alpha_a(y) [\theta\theta] \bar\lambda^b_{\dot\gamma}(y) [\tilde\sigma^\rho]^{\dot\gamma}{}_\alpha v^c_\rho(y) +$$

$$+ 8\theta^\alpha D_a \theta^\gamma [\sigma^\rho]_{\gamma\dot\gamma} [\tilde\sigma^\tau]^{\dot\gamma}{}_\alpha v^b_\rho(y) v^c_\tau(y) - 4i[\sigma^\mu]^\alpha{}_{\dot\alpha} [\tilde\sigma^\nu]^{\dot\alpha\beta} \theta_\beta [\partial_\mu v^a_\nu(y) - \partial_\nu v^a_\mu(y)] \cdot$$

$$\cdot \theta^\gamma [\sigma^\rho]_{\gamma\dot\gamma} [\tilde\sigma^\tau]^{\dot\gamma}{}_\alpha v^b_\rho v^c_\tau(y) \Big\}\Big|_{[\theta\theta]} = \frac{i}{32} g f_{abc} \Big\{ +8[\lambda_a(y)\sigma^\mu \bar\lambda_c(y)][\theta\theta] v^b_\mu(y)$$

$$+8[\lambda_a(y)\sigma^\rho \bar\lambda_b(y)][\theta\theta] v^c_\rho(y) + \frac{1}{2}[\theta\theta] D_a(y) v^\mu_b(y) v_{c\mu}(y) + 8i[\partial_\mu v^a_\nu(y) - \partial_\nu v^a_\mu(y)] \cdot$$

$$\cdot v^\mu_b(y) v^\nu_c(y) [\theta\theta] \Big\}\Big|_{[\theta\theta]} \Rightarrow$$

$$\frac{i}{32} g f_{abc} [\bar{D}\bar{D} D^\alpha V_a][\bar{D}\bar{D} V_b D_\alpha V_c]\Big|_{\theta\theta} = -\frac{i}{2} g f_{abc} [\lambda_a(x)\sigma^\mu \bar\lambda_b(x)] v^c_\mu(x)$$

$$- \frac{1}{4} g f_{abc} [\partial_\mu v^a_\nu(x) - \partial_\nu v^a_\mu(x)] v^\mu_b(x) v^\nu_c(x) \tag{1}$$

$$-\frac{i}{32} g f_{abc} [DD\bar{D}_{\dot\alpha} V_a][DD V_b \bar{D}^{\dot\alpha} V_c]\Big|_{[\bar\theta\bar\theta]} = -\frac{i}{2} g f_{abc} [\lambda_a(x)\sigma^\mu \bar\lambda_b(x)] v^c_\mu(x)$$

$$- \frac{1}{4} g f_{abc} [\partial_\mu v^a_\nu(x) - \partial_\nu v^a_\mu(x)] v^\mu_b(x) v^\nu_c(x)$$

Finally

$$-\frac{1}{64} g^2 f_{abc} f_{ade} [\bar{D}\bar{D} V_b D^\alpha V_c][\bar{D}\bar{D} V_d D_\alpha V_e]\Big|_{[\theta\theta]} =$$

$$= -\frac{1}{8} g^2 f_{abc} f_{ade} v^b_\mu(x) v^c_\nu(x) v^\mu_d(x) v^\nu_e(x) \tag{2}$$

$$-\frac{1}{64} g^2 f_{abc} f_{ade} [DD V_b \bar{D}_{\dot\alpha} V_c][DD V_d \bar{D}^{\dot\alpha} V_e]\Big|_{[\bar\theta\bar\theta]} =$$

$$= -\frac{1}{8} g^2 f_{abc} f_{ade} v^b_\mu(x) v^c_\nu(x) v^\mu_d(x) v^\nu_e(x)$$

From all that the Lagrangian density (72.4) can be written as

$$L(x) = L_1(x) + L_2(x) + L_3(x) \tag{1}$$

$$L_1(x) = \frac{1}{2} D_a^2(x) - \frac{1}{4} [\partial^\mu v^\nu_a(x) - \partial^\nu v^\mu_a(x)][\partial_\mu v_\nu^a(x) - \partial_\nu v_\mu^a(x)]$$

$$+ i [\lambda_a(x) \sigma^\mu \partial_\mu \bar{\lambda}_a(x)] - ig f_{abc} [\lambda_a(x) \sigma^\mu \bar{\lambda}_b(x)] v_\mu^c(x)$$

$$- \frac{1}{2} g f_{abc} [\partial_\mu v_\nu^a(x) - \partial_\nu v_\mu^a(x)] v_b^\mu(x) v_c^\nu(x)$$

$$- \frac{1}{4} g^2 f_{abc} f_{ade} v_\mu^b(x) v_\nu^c(x) v^\mu_d(x) v^\nu_e(x) \tag{2}$$

$$L_2(x) = \partial_\mu A_i^*(x) \partial^\mu A_i(x) + F_i^*(x) F_i(x) + i \bar{\psi}_i(x) \tilde{\sigma}^\mu \partial_\mu \psi_i(x)$$

$$- g (T_a)_{ij} A_i^*(x) A_j(x) D_a(x) - i\sqrt{2} g (T_a)_{ij} A_i^*(x) [\psi_j(x) \lambda_a(x)]$$

$$+ i\sqrt{2} g (T_a)_{ij} A_j(x) [\bar{\psi}_i(x) \bar{\lambda}_a(x)] + g (T_a)_{ij} [\bar{\psi}_i(x) \tilde{\sigma}_\mu \psi_j(x)] v^\mu_a(x)$$

$$+ ig (T_a)_{ij} [A_i^*(x) \partial_\mu A_j(x) - \partial_\mu A_i^*(x) A_j(x)] v^\mu_a(x)$$

$$+ g^2 (T_a T_b)_{ij} A_i^*(x) A_j(x) v_a^\mu(x) v_\mu^b(x) \tag{3}$$

$$L_3(x) = m_{ij} A_i(x) F_j(x) - \frac{1}{2} m_{ij} \psi_i(x) \psi_j(x) + g_{ijk} A_i(x) A_j(x) F_k(x)$$

$$- g_{ijk} \psi_i(x) \psi_j(x) A_k(x) + \text{h.c.} \tag{4}$$

Notice that $D_a(x)$ and $F_i(x)$ play the role of auxiliar fields. From their equations of motion

$$F_i(x) = - m^*_{ji} A_j^*(x) - g^*_{jki} A_j^*(x) A_k(x)$$

$$F_i^*(x) = - m_{ji} A_j(x) - g_{jki} A_j(x) A_k(x) \tag{5}$$

$$D_a(x) = g (T_a)_{ij} A_i^*(x) A_j(x)$$

Let us now introduce the superpotential

$$W[A_i] = \frac{1}{2} m_{ij} A_i(x) A_j(x) + \frac{1}{3} g_{ijk} A_i(x) A_j(x) A_k(x)$$

$$W^*[A_i^*] = \frac{1}{2} m_{ij}^* A_i^*(x) A_j^*(x) + \frac{1}{3} g_{ijk}^* A_i^*(x) A_j^*(x) A_k^*(x) \qquad (1)$$

Then our Lagrangian density can be written as

$$L(x) = L_1(x) + L_2(x) + L_3(x) + L_4(x) \qquad (2)$$

$$L_1(x) = -\frac{1}{4} [\partial^\mu v_a^\nu(x) - \partial^\nu v_a^\mu(x)][\partial_\mu v_\nu^a(x) - \partial_\nu v_\mu^a(x)]$$

$$+ i [\lambda_a(x) \sigma^\mu \partial_\mu \bar{\lambda}_a(x)] + \partial_\mu A_i^*(x) \partial^\mu A_i(x)$$

$$+ i \bar{\psi}_i(x) \tilde{\sigma}^\mu \partial_\mu \psi_i(x) \qquad (3)$$

$$L_2(x) = -\frac{1}{2} g f_{abc} [\partial_\mu v_\nu^a(x) - \partial_\nu v_\mu^a(x)] v_b^\mu(x) v_c^\nu(x)$$

$$- \frac{1}{4} g^2 f_{abc} f_{ade} v_\mu^b(x) v_\nu^c(x) v_d^\mu(x) v_e^\nu(x)$$

$$- i g f_{abc} [\lambda_a(x) \sigma^\mu \bar{\lambda}_b(x)] v_\mu^c(x) \qquad (4)$$

$$L_3(x) = g (T_a)_{ij} [\bar{\psi}_i(x) \tilde{\sigma}_\mu \psi_j(x)] v_a^\mu(x)$$

$$+ i g (T_a)_{ij} [A_i^*(x) \partial_\mu A_j(x) - \partial_\mu A_i^*(x) A_j(x)] v_a^\mu(x)$$

$$- i \sqrt{2} g (T_a)_{ij} A_i^*(x) [\lambda_a(x) \psi_j(x)]$$

$$+ i \sqrt{2} g (T_a)_{ij} A_j(x) [\bar{\lambda}_a(x) \bar{\psi}_i(x)]$$

$$+ g^2 (T_a T_b)_{ij} A_i^*(x) A_j(x) v_a^\mu(x) v_{b\mu}(x) \qquad (5)$$

$$L_4(x) = -\frac{1}{2} D_a^2(x) - F_i^*(x) F_i(x)$$

$$- \frac{1}{2} \frac{\partial^2 W}{\partial A_i \partial A_j} \psi_i(x) \psi_j(x) - \frac{1}{2} \frac{\partial^2 W^*}{\partial A_i^* \partial A_j^*} \bar{\psi}_i(x) \bar{\psi}_j(x) \qquad (6)$$

being

$$F_i(x) = -\frac{\partial W}{\partial A_i} \quad , \quad F_i^*(x) = -\frac{\partial W}{\partial A_i^*} \tag{1}$$

$$D_a(x) = g\,(T_a)_{ij}\,A_i^*(x)\,A_j(x)$$

Notice that the theory is completely specified when we fix the superpotential $W[A_i]$ which must be some cubic gauge-invariant function of the scalar matter fields $A_i(x)$.

It is easy to generalize these formulae to groups G that contain a U(1) factor. We will denote by $v_\mu'(x)$ the gauge field and by $\lambda'(x)$ its corresponding gaugino. Then we must add the following pieces

$$L_1'(x) = -\frac{1}{4}\,[\partial^\mu v'^\nu(x) - \partial^\nu v'^\mu(x)]\,[\partial_\mu v'_\nu(x) - \partial_\nu v'_\mu(x)]$$

$$+\, i\,[\lambda'(x)\,\sigma^\mu\,\partial_\mu\,\bar{\lambda}'(x)] \tag{2}$$

$$L_2'(x) = 0 \tag{3}$$

$$L_3'(x) = \frac{1}{2}\,g'\,Y_i\,[\bar{\psi}_i(x)\,\tilde{\sigma}_\mu\,\psi_i(x)]\,v'^\mu(x)$$

$$+\frac{i}{2}\,g'\,Y_i\,[A_i^*(x)\,\partial_\mu A_i(x) - \partial_\mu A_i^*(x)\,A_i(x)]\,v'^\mu(x)$$

$$-\frac{i}{\sqrt{2}}\,g'\,Y_i\,A_i^*(x)\,[\lambda'(x)\,\psi_i(x)] + \frac{i}{\sqrt{2}}\,g'\,Y_i\,A_i(x)\,[\bar{\lambda}'(x)\,\bar{\psi}_i(x)]$$

$$+\frac{1}{2}\,g\,g'\,Y_j\,(T_a)_{ij}\,A_i^*(x)\,A_j(x)\,v^\mu_a(x)\,v'_\mu(x)$$

$$+\frac{1}{2}\,g\,g'\,Y_i\,(T_a)_{ij}\,A_i^*(x)\,A_j(x)\,v^\mu_a(x)\,v'_\mu(x)$$

$$+\frac{1}{4}\,g'^2\,Y_i^2\,A_i^*(x)\,A_i(x)\,v'^\mu(x)\,v'_\mu(x) \tag{4}$$

$$L_4'(x) = -\frac{1}{2}\left[\frac{1}{2}\,g'\,Y_i\,A_i^*(x)\,A_i(x)\right]^2$$

where Y_i is the U(1) quantum number of the fields $\psi_i(x)$, $A_i(x)$.

Example I – Q.E.D.

Let us now assume that the gauge group is $U(1)$. We will consider two chiral superfields Φ_+ and Φ_- with $U(1)$ charges $y_\pm = \pm 1$. We must put $g_{ijk} \equiv 0$ and

$$m_{++} = m_{--} = 0 \qquad\qquad m_{+-} = m_{-m} \equiv m \tag{1}$$

We need clearly only one vector field V and furthermore we will take $g^1 \equiv -2e$. Then

$$L(x) = -\frac{1}{4}\left[\partial_\mu v_\nu(x) - \partial_\nu v_\mu(x)\right]\left[\partial^\mu v^\nu(x) - \partial^\nu v^\mu(x)\right] + i\left[\lambda(x)\sigma^\mu\partial_\mu\bar\lambda(x)\right]$$

$$+ \partial_\mu A_+^*(x)\partial^\mu A_+(x) + \partial_\mu A_-^*(x)\partial^\mu A_-(x) + i\bar\psi_+(x)\tilde\sigma^\mu\partial_\mu\psi_+(x) +$$

$$+ i\bar\psi_-(x)\tilde\sigma^\mu\partial_\mu\psi_-(x) - e\left[\bar\psi_+(x)\tilde\sigma_\mu\psi_+(x)\right]v^\mu(x) + e\left[\bar\psi_-(x)\tilde\sigma_\mu\psi_-(x)\right]v^\mu(x)$$

$$-ie\left[A_+^*(x)\partial_\mu A_+(x) - \partial_\mu A_+^*(x)A_+(x)\right]v^\mu(x) + ie\left[A_-^*(x)\partial_\mu A_-(x) - \partial_\mu A_-^*(x)A_-(x)\right]v^\mu(x)$$

$$+ i\sqrt{2}e\,A_+^*(x)\left[\lambda(x)\psi_+(x)\right] - i\sqrt{2}e\,A_+(x)\left[\bar\lambda(x)\bar\psi_+(x)\right] - i\sqrt{2}e\,A_-^*(x)\left[\lambda(x)\psi_-(x)\right]$$

$$+ i\sqrt{2}e\,A_-(x)\left[\bar\lambda(x)\bar\psi_-(x)\right] + e^2\left[A_+^*(x)A_+(x) + A_-^*(x)A_-(x)\right]v_\mu(x)v^\mu(x)$$

$$-\frac{1}{2}e^2\left[A_+^*(x)A_+(x) - A_-^*(x)A_-(x)\right]^2 - m^2A_+^*(x)A_+(x) - m^2A_-^*(x)A_-(x)$$

$$- m\,\psi_+(x)\psi_-(x) - m\,\bar\psi_+(x)\bar\psi_-(x)$$

Now let us introduce

$$\bar\Psi(x) = \left|\ \psi_+^\alpha(x) \quad \bar\psi_{-\dot\alpha}(x)\ \right| \qquad \Psi(x) = \begin{vmatrix}\psi_{-\alpha}(x)\\ \bar\psi_+^{\dot\alpha}(x)\end{vmatrix}$$ Electron Dirac Field

$$\bar\Lambda(x) \equiv \left|\ \lambda^\alpha(x),\ \bar\lambda_{\dot\alpha}(x)\ \right| \qquad \Lambda(x) = \begin{vmatrix}\lambda_\alpha(x)\\ \bar\lambda^{\dot\alpha}(x)\end{vmatrix}$$ Photino Majorana Field

$$A_\mu(x) \equiv v_\mu(x) \quad , \quad F_{\mu\nu}(x) \equiv \partial_\mu v_\nu(x) - \partial_\nu v_\mu(x)$$ Photon gauge field

$$A(x) \equiv A_+(x) \qquad\qquad B(x) \equiv A_-^*(x)$$ S-electrons scalar field

Then

$$L(x) = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) + \frac{i}{2} \bar{\lambda}(x) \gamma^\mu \partial_\mu \lambda(x) + i \bar{\psi}(x) \gamma^\mu \partial_\mu \psi(x) - m \bar{\psi}(x) \psi(x)$$

$$+ \partial^\mu A^+(x) \partial_\mu A(x) - m^2 A^+(x) A(x) + \partial^\mu B^+(x) \partial_\mu B(x) - m^2 B^+(x) B(x)$$

$$+ e \bar{\psi}(x) \gamma^\mu \psi(x) A_\mu(x) - ie [A^+(x) \partial_\mu A(x) - \partial_\mu A^+(x) A(x)] A^\mu(x)$$

$$- ie [B^+(x) \partial_\mu B(x) - \partial_\mu B^+(x) B(x)] A^\mu(x)$$

$$+ \frac{ie}{\sqrt{2}} A^+(x) \bar{\psi}(x) (1+\gamma_5) \Lambda(x) - \frac{ie}{\sqrt{2}} A(x) \bar{\Lambda}(x) (1-\gamma_5) \psi(x)$$

$$+ \frac{ie}{\sqrt{2}} B^+(x) \bar{\psi}(x) (1-\gamma_5) \Lambda(x) - \frac{ie}{\sqrt{2}} B(x) \bar{\Lambda}(x) (1+\gamma_5) \psi(x)$$

$$+ e^2 [A^+(x) A(x) + B^+(x) B(x)] A_\mu(x) A^\mu(x)$$

$$- \frac{e^2}{2} [A^+(x) A(x) - B^+(x) B(x)]^2 \qquad (1)$$

Example 2 - Q.C.D.

Let us now assume that the gauge group is $SU(N)$. Then we will consider the superfields $\Phi_{i\pm}(x, \theta, \bar{\theta})$, $i = 1, 2, \dots N$. Let us denote by T_a the generators in the fundamental representation N, then $-T_a^T$ are the matrices corresponding to $\bar{N}$. We choose

$$\Phi_+(x, \theta, \bar{\theta}) \longrightarrow \Phi_+'(x, \theta, \bar{\theta}) = e^{-2ig T_a \Lambda_a(x, \theta, \bar{\theta})} \Phi_+(x, \theta, \bar{\theta})$$

$$\Phi_-(x, \theta, \bar{\theta}) \longrightarrow \Phi_+'(x, \theta, \bar{\theta}) = e^{+2ig T_a^T \Lambda_a(x, \theta, \bar{\theta})} \Phi_-(x, \theta, \bar{\theta}) \qquad (2)$$

We are considering only one flavour but the generalization to N_f flavours is immediate. We will need N^2-1 vector fields $V_a(x, \theta, \bar{\theta})$, $a = 1, 2, \dots, N^2-1$

Now let us consider the possible terms of the superpotential. It is clear that gauge invariance implies

$$m_{+i,+j} = m_{-i,-j} = 0 \qquad (3)$$

and without loosing generality we can choose

$$m_{+i,-j} = m_{-j,+i} = m\,\delta_{ij} \qquad (1)$$

Let us consider $g_{ijk}\,\Phi_i\,\Phi_j\,\Phi_k$. Since $3\otimes 3\otimes 3 = 1\oplus 8\oplus 8\oplus 10$ it is possible to construct such combination but since 1 is fully antisymmetric the term cannot exists. Then

$$L(x) = -\frac{1}{4}\,[\partial^\mu v^\nu_a(x) - \partial^\nu v^\mu_a(x)]\,[\partial_\mu v_\nu^a(x) - \partial_\nu v_\mu^a(x)] + i\,[\lambda_a(x)\,\sigma^\mu\,\partial_\mu\,\bar\lambda_a(x)]$$

$$+\,\partial_\mu A^*_{+i}(x)\,\partial^\mu A_{+i}(x) + \partial_\mu A^*_{-i}(x)\,\partial^\mu A_{-i}(x) + i\,\bar\psi_{+i}(x)\,\tilde\sigma^\mu\,\partial_\mu\,\psi_{+i}(x)$$

$$+\,i\,\bar\psi_{-i}(x)\,\tilde\sigma^\mu\,\partial_\mu\,\psi_{-i}(x) - \frac{1}{2}\,g\,f_{abc}\,[\partial_\mu v_\nu^a(x) - \partial_\nu v_\mu^a(x)]\,v_b^\mu(x)\,v_c^\nu(x)$$

$$-\,\frac{1}{4}\,g^2\,f_{abc}\,f_{ade}\,v_\mu^b(x)\,v_\nu^c(x)\,v^\mu_d(x)\,v^\nu_e(x) - i\,g\,f_{abc}\,[\lambda_a(x)\,\sigma^\mu\,\bar\lambda_b(x)]\,v_\mu^c(x)$$

$$+\,g\,(T_a)_{ij}\,[\bar\psi_{+i}(x)\,\tilde\sigma_\mu\,\psi_{+j}(x)]\,v^\mu_a(x) - g\,(T_a)_{ji}\,[\bar\psi_{-i}(x)\,\tilde\sigma_\mu\,\psi_{-j}(x)]\,v^\mu_a(x)$$

$$+\,i\,g\,(T_a)_{ij}\,[A^*_{+i}(x)\,\partial_\mu A_{+j}(x) - \partial_\mu A^*_{+i}(x)\,A_{+j}(x)]\,v^\mu_a(x)$$

$$-\,i\,g\,(T_a)_{ji}\,[A^*_{-i}(x)\,\partial_\mu A_{-j}(x) - \partial_\mu A^*_{-i}(x)\,A_{-j}(x)]\,v^\mu_a(x)$$

$$-\,i\sqrt{2}\,g\,(T_a)_{ij}\,A^*_{+i}(x)\,[\lambda_a(x)\,\psi_{+j}(x)] + i\sqrt{2}\,g\,(T_a)_{ij}\,A_{+j}(x)\,[\bar\lambda_a(x)\,\bar\psi_{+i}(x)]$$

$$+\,i\sqrt{2}\,g\,(T_a)_{ji}\,A^*_{-i}(x)\,[\lambda_a(x)\,\psi_{-j}(x)] - i\sqrt{2}\,g\,(T_a)_{ji}\,A_{-j}(x)\,[\bar\lambda_a(x)\,\bar\psi_{-i}(x)]$$

$$+\,g^2\,(T_a T_b)_{ij}\,A^*_{+i}(x)\,A_{+j}(x)\,v_a^\mu(x)\,v_{b\mu}(x) + g^2\,(T_b T_a)_{ji}\,A^*_{-i}(x)\,A_{+j}(x)\,v_a^\mu(x)\,v_{b\mu}(x)$$

$$-\,\frac{1}{2}\,g^2\,[(T_a)_{ij}\,A^*_{+i}(x)\,A_{+j}(x) - (T_a)_{ji}\,A^*_{-i}(x)\,A_{-j}(x)]^2 - m^2\,A^*_{+i}(x)\,A_{+i}(x)$$

$$-\,m^2\,A^*_{-i}(x)\,A_{-i}(x) - m\,\psi_{+i}(x)\,\psi_{-i}(x) - m\,\bar\psi_{+i}(x)\,\bar\psi_{-i}(x) \qquad (1)$$

Let us now introduce

$$q_i(x) \equiv \begin{vmatrix} \psi_{+i\alpha}(x) \\ \bar\psi^{\dot\alpha}_{-i}(x) \end{vmatrix}, \qquad \bar q_i(x) \equiv |\,\psi^{\alpha}_{-i}(x) \quad \bar\psi_{+i\dot\alpha}(x)\,| \qquad \text{Quark Dirac fields}$$

$$\Lambda_a(x) = \begin{vmatrix} \lambda_{a\alpha}(x) \\ \bar{\lambda}_a^{\dot\alpha}(x) \end{vmatrix} \qquad \bar{\Lambda}_a(x) = | \lambda^\alpha_a(x) , \bar{\lambda}_{a\dot\alpha}(x) | \qquad \text{Gluino fields}$$

$$B_a^\mu(x) \equiv v_a^\mu(x) \qquad \text{Gluon fields}$$

$$A_i(x) \equiv A_{+i}(x) \quad , \quad B_i(x) \equiv A^*_{-i}(x) \qquad \text{S-quarks fields}$$

Then

$$L(x) = -\frac{1}{4} [\partial^\mu B_a^\nu(x) - \partial^\nu B^\mu_a(x)][\partial_\mu B_\nu^a(x) - \partial_\nu B_\mu^a(x)] + \frac{i}{2} \bar{\Lambda}_a(x) \gamma^\mu \partial_\mu \Lambda_a(x)$$

$$+ i \bar{q}(x) \gamma^\mu \partial_\mu q(x) - m \bar{q}(x) q(x) + \partial_\mu A^+(x) \partial^\mu A(x) - m^2 A^+(x) A(x)$$

$$+ \partial_\mu B^+(x) \partial^\mu B(x) - m^2 B^+(x) B(x) - \frac{1}{2} g f_{abc} [\partial_\mu B_\nu^a(x) - \partial_\nu B_\mu^a(x)] B_b^\mu(x) B_c^\nu(x)$$

$$- \frac{1}{4} g^2 f_{abc} f_{ade} B_\mu^b(x) B_\nu^c(x) B_d^\mu(x) B_e^\nu(x) - \frac{i}{2} g f_{abc} \Lambda_a(x) \gamma^\mu \Lambda_b(x) B_\mu^c(x)$$

$$+ g \bar{q}(x) \gamma^\mu T_a q(x) B_\mu^a(x) + i g [A^+(x) T_a \partial_\mu A(x) - \partial_\mu A^+(x) T_a A(x)] B_a^\mu(x)$$

$$+ i g [B^+(x) T_a \partial_\mu B(x) - \partial_\mu B^+(x) T_a B(x)] B_a^\mu(x)$$

$$- \frac{i}{\sqrt{2}} g A^+(x) T_a [\bar{\Lambda}_a(x) (1+\gamma_5) q(x)] + \frac{i}{\sqrt{2}} g [\bar{q}(x) (1-\gamma_5) \Lambda_a(x)] T_a A(x)$$

$$- \frac{i}{\sqrt{2}} g B^+(x) T_a [\bar{\Lambda}_a(x) (1-\gamma_5) q(x)] + \frac{i}{\sqrt{2}} g [\bar{q}(x) (1+\gamma_5) \Lambda_a(x)] T_a B(x)$$

$$+ g^2 A^+(x) T_a T_b A(x) B_a^\mu(x) B_\mu^b(x) + g^2 B^+(x) T_a T_b B(x) B_\mu^a(x) B_b^\mu(x)$$

$$- \frac{1}{2} g^2 [A^+(x) T_a A(x) - B^+(x) T_a B(x)]^2 \qquad (1)$$

which is the desired result.

Example 3 - Chiral Theory

Let us consider now a general theory with chiral superfields all of them transforming according to the representation N of SU(N), then

$$m_{ij} \equiv 0 \quad , \quad g_{ijk} \equiv 0 \quad \Rightarrow \quad W = 0 \qquad (2)$$

Then we can introduce the Majorana fields

$$\Lambda_a(x) \equiv \begin{vmatrix} \lambda_{a\alpha}(x) \\ \bar{\lambda}^{\dot{\alpha}}_a(x) \end{vmatrix} \quad , \quad \bar{\Lambda}_a(x) \equiv | \lambda^{\alpha}_a(x) \ , \ \bar{\lambda}_{a\dot{\alpha}}(x) | \tag{1}$$

as well the fields

$$q_{Li}(x) \equiv \begin{vmatrix} \psi_{i\alpha}(x) \\ 0 \end{vmatrix} \qquad \overline{q_{Li}(x)} = | 0 \ , \ \bar{\psi}_{i\dot{\alpha}}(x) | \tag{2}$$

$$q_{Li}(x) = \frac{1}{2}(1+\gamma_5)\, q_{Li}(x) \quad , \quad \overline{q_{Li}(x)} = \overline{q_{Li}(x)}\, \frac{1}{2}(1-\gamma_5) \tag{3}$$

Then

$$L(x) = -\frac{1}{4}[\partial^\mu v^\nu_a(x) - \partial^\nu v^\mu_a(x)][\partial_\mu v^a_\nu(x) - \partial_\nu v^a_\mu(x)] + \frac{i}{2}\bar{\Lambda}_a(x)\gamma^\mu\partial_\mu\Lambda_a(x)$$

$$+ \partial_\mu A^+(x)\,\partial^\mu A(x) + i\,\overline{q_L(x)}\,\gamma^\mu\partial_\mu q_L(x)$$

$$- \frac{1}{2} g f_{abc} [\partial_\mu v^a_\nu(x) - \partial_\nu v^a_\mu(x)]\, v^\mu_b(x)\, v^\nu_c(x)$$

$$- \frac{1}{4} g^2 f_{abc} f_{ade}\, v^b_\mu(x)\, v^c_\nu(x)\, v^\mu_d(x)\, v^\nu_e(x)$$

$$- \frac{i}{2} g f_{abc} [\overline{\Lambda_a}(x)\gamma^\mu\Lambda_b(x)]\, v^c_\mu(x)$$

$$+ g[\overline{q_L(x)}\,\gamma_\mu T_a\, q_L(x)]\, v^\mu_a(x)$$

$$+ ig[A^+(x) T_a \partial_\mu A(x) - \partial_\mu A^+(x) T_a A(x)]\, v^\mu_a(x)$$

$$- i\sqrt{2} g A^+(x) T_a [\bar{\Lambda}_a(x) q_L(x)] + i\sqrt{2} g [\overline{q_L(x)}\Lambda_a(x)] T_a A(x)$$

$$+ g^2 A^+(x) T_a T_b A(x)\, v^\mu_a(x)\, v^a_\mu(x)$$

$$- \frac{1}{2} g^2 [A^+(x) T_a A(x)]^2 \tag{4}$$

VI - I - MAJORANA PARTICLES

A Majorana particle is a particle of spin 1/2 that coincides with its own antiparticle. The free Majorana field can be written as

$$\psi(x) = \frac{\phi}{(2\pi)^3} \int \frac{d^3 p}{2E(\vec{p})} \sum_\lambda \left\{ u(\vec{p},\lambda)\, a(\vec{p},\lambda)\, e^{-ip.x} + \varphi\, u^c(\vec{p},\lambda)\, a^+(\vec{p},\lambda)\, e^{+ip.x} \right\} \qquad (1)$$

where ϕ and φ are arbitrary phases and

$$u^c(\vec{p},\lambda) \equiv \mathcal{C}\, [\overline{u(\vec{p},\lambda)}]^T =$$

$$= -\gamma^0\, \mathcal{C}\, u^*(\vec{p},\lambda) = i\lambda\, v(\vec{p},\lambda) \qquad (2)$$

Notice that from (1) we obtain

$$\psi(x) = -[\phi^2 \varphi]\, \gamma^0\, \mathcal{C}\, \psi^*(x) \qquad (3)$$

or equivalently

$$\psi(x) = [\phi^2\varphi]\, \mathcal{C}\, [\overline{\psi(x)}]^T$$

When we carry out perturbation theory we have two kinds of propagators

$$S(p) \equiv -i \int d^4x\, e^{ip.x} \langle 0 | T(\psi(x)\overline{\psi(0)}) | 0 \rangle = \frac{\not p + m}{p^2 - m^2 + i\epsilon} \qquad (4)$$

$$\tilde{S}(p) \equiv -i \int d^4x\, e^{ip.x} \langle 0 | T(\psi(x)[\psi(0)]^T) | 0 \rangle = -[\phi^2\varphi]\, \frac{m\, \mathcal{C}}{p^2 - m^2 + i\epsilon} \qquad (5)$$

The transformation properties of particles and fields under C, P and T are

i) Parity

$$\mathcal{P}\, a^+(\vec{p},\lambda)\, |0\rangle = -\epsilon_p\, a^+(-\vec{p},-\lambda)\, |0\rangle$$

$$\mathcal{P}\, \psi(x)\, \mathcal{P}^{-1} = i\epsilon_p\, \gamma^0\, \psi(\mathcal{P}x)$$

ii) Charge Conjugation

$$C\, a^+(\vec{p}, \lambda)\,|0\rangle = \epsilon_c\, a^+(\vec{p}, \lambda)\,|0\rangle$$

(1)

$$C\,\psi(x)\,C^{-1} = -\epsilon_c\,[\phi^2\varphi]\,\gamma^0\,\mathcal{C}\,[\psi(x)]^* = \epsilon_c\,\psi(x)$$

iii) Time reversal

$$T\, a^+(\vec{p}, \lambda)\,|0\rangle = \epsilon_T\,\lambda\,\varphi\, a^+(-\vec{p}, \lambda)\,|0\rangle$$

(2)

$$T\,\psi(x)\,T^{-1} = +\epsilon_T\,[\phi^2\varphi]^*\,\mathcal{T}\,\psi(Tx)$$

iv) CP

$$CP\, a^+(\vec{p}, \lambda)\,|0\rangle = -\epsilon_c\epsilon_p\, a^+(-\vec{p}, -\lambda)\,|0\rangle$$

(3)

$$CP\,\psi(x)\,(CP)^{-1} = -i\,\epsilon_c\epsilon_p\,[\phi^2\varphi]\,\mathcal{C}\,[\psi(\mathcal{P}x)]^* = i\,\epsilon_c\,\epsilon_p\,\gamma^0\,\psi(\mathcal{P}x)$$

v) CTP

$$CTP\, a^+(\vec{p}, \lambda)\,|0\rangle = -\epsilon_W\,\lambda\,\varphi\, a^+(\vec{p}, -\lambda)\,|0\rangle$$

(4)

$$CTP\,\psi(x)\,(CTP)^{-1} = i\,\epsilon_W\,\mathcal{T}\mathcal{C}\,[\psi(-x)]^* = -i\,\epsilon_W\,[\phi^2\varphi]\,\mathcal{T}\,\gamma^0\,\psi(-x)$$

The quantities ϵ_P, ϵ_C, ϵ_T must be real and equal to ± 1. The quantity $\epsilon_W \equiv \epsilon_P\,\epsilon_T\,\epsilon_C$ can always be taken equal to $+1$.

VI - J - SUPERSYMMETRIC QUANTUM MECHANICS

Ref. E. WITTEN Nucl. Phys. B188, 513 (1981)

Definition: A supersymmetric quantum mechanical system is one in which there are operators Q_i that commute with the Hamiltonian

$$[Q_i, H] = 0 \qquad i = 1, 2, \ldots N \tag{1}$$

and satisfy the algebra

$$\{Q_i, Q_j\} = \delta_{ij} H \tag{2}$$

Model: $N = 2$

$$Q_1 \equiv \frac{1}{2}[\sigma_1 p + \sigma_2 W(x)] \qquad Q_2 \equiv \frac{1}{2}[\sigma_2 p - \sigma_1 W(x)]$$

$$p \equiv -i\hbar \frac{d}{dx}, \qquad W(x) = \text{arbitrary derivable real function} \tag{3}$$

$$H = \frac{1}{2}\left[-\hbar^2 \frac{d^2}{dx^2} + W^2(x) + \hbar\sigma_3 \frac{dW(x)}{dx}\right]$$

We will assume $|W(x)| \to \infty$ as $|x| \to \infty$ so that the spectrum of the Hamiltonian is discrete.

In this model

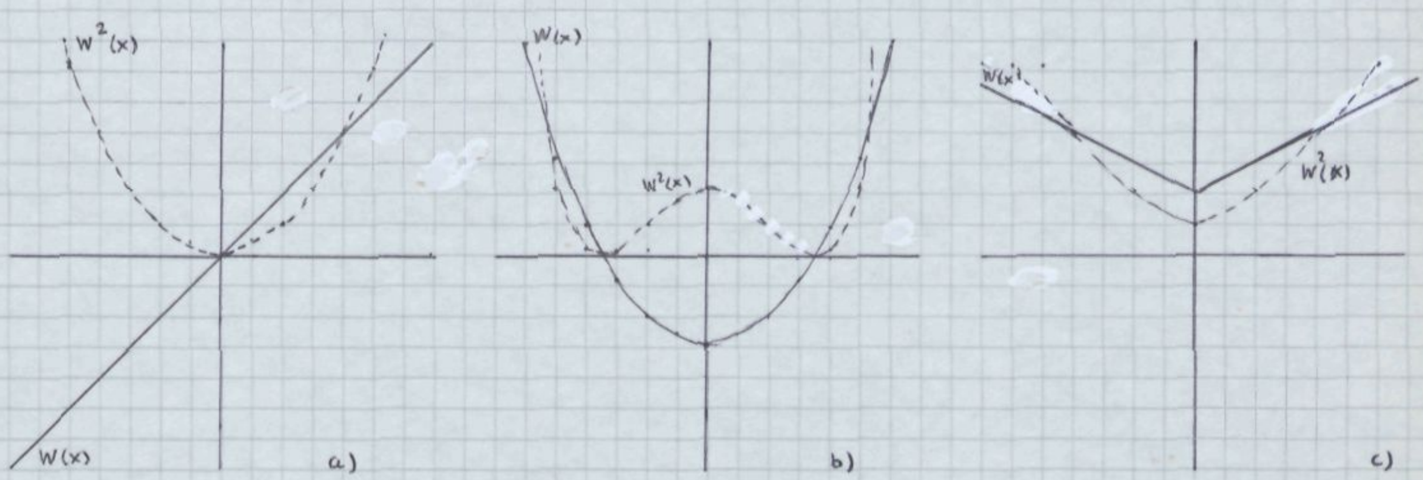
$$H = Q_1^2 + Q_2^2 \tag{4}$$

and since $Q_i^+ = Q_i$ we obtain that for any arbitrary state $|\Psi\rangle$

$$\langle\Psi| H |\Psi\rangle \geq 0 \tag{5}$$

A state can have zero energy only if it is annihilated by each of the Q_i, since if $H|0\rangle = 0$, then $0 = \langle 0|H|0\rangle = \sum_i \langle 0|Q_i^2|0\rangle = \sum_i \| Q_i|0\rangle\|^2$ which is only possible if $Q_i|0\rangle = 0$. If conversely, a state is annihilated by all Q_i, then its energy is zero because $Q_i|0\rangle = 0 \;\forall i$, implies $H|0\rangle = \sum_i Q_i^2|0\rangle = 0$. If there exists a supersymmetric invariant state - that is, a state annihilated by all the Q_i - then it is automatically the true vacuum state, since it has zero energy and any state that is not invariant under supersymmetry has positive energy. Thus, if a supersymmetric state exists, it is the ground state and supersymmetry is not spontaneously broken. Only if there does not exists a state invariant under supersymmetry is supersymmetry spontaneously broken. In this case the ground state energy is positive. Therefore a necessary and suffient condition for supersymmetry to be spontaneously broken is $\langle 0|H|0\rangle > 0$.

Now let us consider under what conditions supersymmetry is spontaneously broken. In the weak coupling (small $\hbar$) limit, one ignores the zero-point motion and one ignores the last term of (1.3), which is explicitly proportional to $\hbar$. At the tree level, the ground-state energy is therefore simply the minimum of $W^2(x)$. The number of supersymmetrically invariant, zero-energy states is therefore, at the tree level, equal to the number of solutions of the equation $W(x) = 0$. Let us consider some examples



a) $W(x)$ has a single zero, so the potential $W^2(x)$ has a single supersymmetric minimum at zero energy

b) $W(x)$ has two zeros, so that there, at the tree level, two supesymmetrically invariant states.

c) $W(x)$ has no zeroes; the minimum of the potential is at non-zero energy and supersymmetry is spontaneously broken at the tree level.

Let us now consider, in the exact theory, the existence of a supersymmetric state. It must be a normalizable state which must satisfy

$$Q_1 \psi(x) = Q_2 \psi(x) = 0 \qquad (1)$$

In our model $Q_2 \equiv -i\sigma_3 Q_1$ and hence it is enough to impose $Q_1 \psi(x) = 0$ i.e. $\sigma_1 p \psi(x) = -\sigma_2 W(x) \psi(x) \Rightarrow$

$$\frac{d\psi(x)}{dx} = \frac{1}{\hbar} \sigma_3 W(x) \psi(x) \qquad (2)$$

and hence

$$\psi(x) = \exp\left\{ \frac{1}{\hbar} \int_0^x dy\, W(y)\, \sigma_3 \right\} \psi(0) \qquad (3)$$

Writing

$$\psi(x) = \begin{vmatrix} \psi_1(x) \\ \psi_2(x) \end{vmatrix} \qquad (4)$$

$$\psi_1(x) = \psi_1(0) \exp\left\{ \frac{1}{\hbar} \int_0^x dy\, W(y) \right\}, \quad \psi_2(x) = \psi_2(0) \exp\left\{ -\frac{1}{\hbar} \int_0^x dy\, W(y) \right\}$$

and the norm is $\| \psi \|^2 = \| \psi_1 \|^2 + \| \psi_2 \|^2$. Let us now consider the two possible cases:

i) $W(x)$ has an odd number of zeroes

In this case the sign of $W(x)$ for $x \to \infty$ is opposite from the sign of $W(x)$ for $x \to -\infty$. If $W(x) > 0$ (<0) for $x \to \infty$, then the normalizable supersymmetric state is obtained taking $\psi_1(0) \equiv 0$ ($\psi_2(0) \equiv 0$). Therefore in this case there is always one supersymmetry state and only one.

ii) $W(x)$ has an even number of zeroes

In this case the sign of $W(x)$ for $x \to \infty$ is the same as the sign of $W(x)$ for $x \to -\infty$. Therefore there is no possibility to obtain a normalizable state and therefore the supersymmetry is broken.

We have found: while at the tree level the number of supersymmetric states equals the number of zeroes of $W(x)$, in the exact spectrum the number of supersymmetric states is equal to one if $W(x)$ has an odd number of zeroes, and it is equal to zero if $W(x)$ has an even number of zeroes.

Therefore in the case a) the exact ground state is supersymmetric, and in case c) supersymmetry is really spontaneously broken. But in case b) non-perturbative effects dynamically break the supersymmetry, which is unbroken at the tree level. We must still see that the breaking is non perturbative.

Let us assume

$$W(x) = x + \sum_{n=2}^{\infty} A_n x^n \equiv x + \tilde{W}(x)$$

$$W^2(x) = x^2 + 2x\,\tilde{W}(x) + \tilde{W}^2(x) \tag{1}$$

$$\frac{dW(x)}{dx} = 1 + \frac{d\tilde{W}(x)}{dx}$$

If we put $\psi_1(x) \equiv 0$ we obtain trivially that our problem reduces to consider

$$H = H_0 + H_I \tag{2}$$

$$H_0 \equiv \frac{1}{2}\left\{-\hbar^2\frac{d^2}{dx^2} + x^2 - \hbar\right\}, \quad H_I \equiv \frac{1}{2}\left\{2x\,\tilde{W}(x) + \tilde{W}^2(x) - \hbar\frac{d\tilde{W}(x)}{dx}\right\}$$

Since we can choose $\tilde{W}(x) \equiv 0$ it is evident that the harmonic is a supersymmetric quantum mechanical system. Let us return to the general case and let us write $y \equiv x/\sqrt{\hbar}$, then

$$H_0 = \hbar\,\tilde{H}_0 = \hbar\,\frac{1}{2}\left\{-\frac{d^2}{dy^2} + y^2 - 1\right\}$$

$$H_I = \hbar\,\tilde{H}_I = \hbar\sum_{m=1}^{\infty}\tilde{H}_m\,\hbar^{m/2} \tag{3}$$

$$\tilde{H}_m = A_{m+1}\left\{y^{m+2} - \frac{1}{2}(m+1)\,y^m\right\} + \frac{1}{2}\sum_{k=2}^{m} A_k A_{m+2-k}\,y^{m+2}$$

Let us introduce $\lambda \equiv \sqrt{\hbar}$. We will denote the eigenvectors of $\tilde{H}_0$ by

$$|\psi_m(\lambda=0)\rangle = |\psi_m^{(0)}\rangle \equiv |m\rangle \qquad (1)$$

which are those of the harmonic oscillator. The corresponding eigenvalues are

$$E_m(\lambda \equiv 0) = E_m^{(0)} = m \qquad (2)$$

The eigenvectors and eigenvalues of the full Hamiltonian $\tilde{H} = \tilde{H}_0 + \tilde{H}_I$ are

$$|\tilde{\psi}_i(\lambda)\rangle \equiv \sum_{m=0}^{\infty} \lambda^m |\tilde{\psi}_i^{(m)}\rangle \quad , \quad \tilde{E}_i(\lambda) \equiv \sum_{m=0}^{\infty} \lambda^m \tilde{E}_i^{(m)} \qquad (3)$$

o) Zero order

$$\tilde{E}_0^{(0)} = 0 \qquad (4)$$

i) First order

$$\tilde{E}_0^{(1)} = \langle 0|\tilde{H}_1|0\rangle \qquad (5)$$

and parity implies

$$\tilde{E}_0^{(1)} = 0 \qquad (6)$$

ii) Second order

$$\tilde{E}_0^{(2)} = \langle 0|\tilde{H}_2|0\rangle + \langle 0|\tilde{H}_1 \, S \, \tilde{H}_1|0\rangle \qquad S \equiv \frac{Q_0^{(0)}}{0 - \tilde{H}_0}$$

$$\Rightarrow \quad \tilde{E}_0^{(2)} = \langle 0|\tilde{H}_2|0\rangle - \sum_{k=1}^{\infty} \frac{1}{k} \langle 0|\tilde{H}_1|k\rangle\langle k|\tilde{H}_1|0\rangle$$

Introducing

$$\langle n|y^k|m\rangle \equiv I^k_{nm}$$

$$\tilde{E}_0^{(2)} = A_3 \left[I^4_{00} - \frac{3}{2} I^2_{00} \right] +$$

$$+ A_2^2 \left\{ \frac{1}{2} I^4_{00} - \left[I^3_{01} - I^1_{01} \right]^2 - \frac{1}{3} \left[I^3_{03} \right]^2 \right\}$$

Since

$$I_{00}^{2m} = \frac{(2m-1)!!}{2^m} \qquad I_{10}^{2m+1} = \sqrt{2}\,\frac{(2m+1)!!}{2^{m+1}} \tag{1}$$

$$I_{20}^{2m} = \sqrt{2}\; m\; \frac{(2m-1)!!}{2^m} \qquad I_{30}^{2m+1} = \frac{1}{\sqrt{3}}\; m\; \frac{(2m+1)!!}{2^m}$$

we get

$$\tilde{E}_0^{(2)} = 0 \tag{2}$$

(iii) Third order

$$\tilde{E}_0^{(3)} = \langle 0 | \tilde{H}_3 | 0 \rangle + \sum_{(3)} \langle 0 | \tilde{H}_{p_1} S \tilde{H}_{p_2} | 0 \rangle + \langle 0 | \tilde{H}_1 S \tilde{H}_1 S \tilde{H}_1 | 0 \rangle \tag{3}$$

where $p_1 + p_2 = 3$, $p_i \geq 1$. Parity implies

$$\tilde{E}_0^{(3)} = 0. \tag{4}$$

(iv) Fourth order

$$\tilde{E}_0^{(4)} = \langle 0 | \tilde{H}_4 | 0 \rangle + \sum_{(4)} \langle 0 | \tilde{H}_{p_1} S \tilde{H}_{p_2} | 0 \rangle + \sum_{(4)} \langle 0 | \tilde{H}_{p_1} S \tilde{H}_{p_2} S \tilde{H}_{p_3} | 0 \rangle + \langle 0 | \tilde{H}_1 S \tilde{H}_1 S \tilde{H}_1 S \tilde{H}_1 | 0 \rangle \tag{5}$$

or equivalently

$$\tilde{E}_0^{(4)} = \langle 0 | \tilde{H}_4 | 0 \rangle - 2 \sum_{k=1}^{\infty} \frac{1}{k} \langle 0 | \tilde{H}_1 | k \rangle \langle k | \tilde{H}_3 | 0 \rangle - \sum_{k=1}^{\infty} \frac{1}{k} \langle 0 | \tilde{H}_2 | k \rangle \langle k | \tilde{H}_2 | 0 \rangle$$

$$+ \sum_{k,l=1}^{\infty} \frac{1}{kl} \langle 0 | \tilde{H}_1 | k \rangle \langle k | \tilde{H}_2 | l \rangle \langle l | \tilde{H}_1 | 0 \rangle + 2 \sum_{k,l=1}^{\infty} \frac{1}{kl} \langle 0 | \tilde{H}_1 | k \rangle \langle k | \tilde{H}_1 | l \rangle \langle l | \tilde{H}_2 | 0 \rangle -$$

$$- \sum_{klj=1}^{\infty} \frac{1}{k.l.j} \langle 0 | \tilde{H}_1 | k \rangle \langle k | \tilde{H}_1 | l \rangle \langle l | \tilde{H}_1 | j \rangle \langle j | \tilde{H}_1 | 0 \rangle$$

Using results *

$$\tilde{E}_0^{(4)} = 0 \tag{6}$$

Hence we have seen that

$$E_0 = \hbar \left\{ 0 + O(\hbar^{5/2}) \right\} \tag{7}$$

If $E_0^{(i)} = 0$ $\quad i = 0, 1, \ldots (n-1)$ then

$\langle 0 | \tilde{H}_4 | 0 \rangle = \frac{15}{8} A_2 A_4 + \frac{15}{16} A_3^2$, $\langle 1 | \tilde{H}_3 | 0 \rangle = \frac{3\sqrt{2}}{8} A_4 + \frac{15\sqrt{2}}{8} A_2 A_3$

$\langle 3 | \tilde{H}_3 | 0 \rangle = \frac{3\sqrt{3}}{2} A_4 + \frac{5\sqrt{3}}{2} A_2 A_3$, $\langle 2 | \tilde{H}_2 | 0 \rangle = \frac{3\sqrt{2}}{4} A_3 + \frac{3\sqrt{2}}{4} A_2^2$

$\langle 4 | \tilde{H}_2 | 0 \rangle = \frac{\sqrt{6}}{2} A_3 + \frac{\sqrt{6}}{4} A_2^2$, $\langle 1 | \tilde{H}_2 | 1 \rangle = \frac{3}{2} A_3 + \frac{15}{8} A_2^2$

$\langle 3 | \tilde{H}_2 | 0 \rangle = \frac{7\sqrt{6}}{4} A_3 + \frac{5\sqrt{6}}{4} A_2^2$, $\langle 3 | \tilde{H}_2 | 3 \rangle = \frac{27}{2} A_3 + \frac{75}{8} A_2^2$

$\langle 1 | \tilde{H}_1 | 0 \rangle = \frac{\sqrt{2}}{4} A_2$, $\langle 3 | \tilde{H}_1 | 0 \rangle = \frac{\sqrt{3}}{2} A_2$, $\langle 2 | \tilde{H}_1 | 1 \rangle = 2 A_2$

$\langle 4 | \tilde{H}_1 | 1 \rangle = \sqrt{3} A_2$, $\langle 3 | \tilde{H}_1 | 2 \rangle = \frac{7\sqrt{6}}{4} A_2$, $\langle 5 | \tilde{H}_1 | 2 \rangle = \frac{\sqrt{30}}{2} A_2$

$\langle 4 | \tilde{H}_1 | 3 \rangle = 5\sqrt{2} A_2$, $\langle 6 | \tilde{H}_1 | 3 \rangle = \sqrt{15} A_2$

$$\tilde{E}_0^{(n)} = \langle 0 | \tilde{H}_n | 0 \rangle + \sum_{(n)} \langle 0 | \tilde{H}_{p_1} S \tilde{H}_{p_2} | 0 \rangle + \sum_{(n)} \langle 0 | \tilde{H}_{p_1} S \tilde{H}_{p_2} S \tilde{H}_{p_3} | 0 \rangle +$$

$$+ \sum_{(n)} \langle 0 | \tilde{H}_{p_1} S \tilde{H}_{p_2} S \tilde{H}_{p_3} S \tilde{H}_{p_4} | 0 \rangle + \dots + \langle 0 | \tilde{H}_1 S \tilde{H}_1 S \tilde{H}_1 S \dots S \tilde{H}_1 | 0 \rangle \qquad (1)$$

where there are n factors $\tilde{H}_1$ in the last term. It is clear that $\tilde{E}_0^{(2n+1)} = 0$, but it is non trivial to prove that also $\tilde{E}_0^{(2n)} = 0$, but I believe that this is so and therefore the supersymmetry breaking in this model is a non-perturbative phenomenon.

In supersymmetric quantum field theory there is an analogous result. If scalar fields are massless in the classical Lagrangian (tree level), they will stay massless to any finite order of perturbation theory.

Let us give a general proof of the perturbative result (derived by A. Galindo)

$$H = H_0 + H_I \qquad H_0 = \frac{1}{2} \left\{ - \hbar^2 \frac{d^2}{dx^2} + x^2 - \hbar \right\}$$

$$H_I = \frac{1}{2} \left\{ 2x \tilde{W}(x) + \tilde{W}^2(x) - \hbar \frac{d\tilde{W}(x)}{dx} \right\}$$

Let us write

$$E_0 = E_0^{(0)} + \sum_{n=1}^{\infty} E_0^{(n)} \qquad \psi_0(x) = \psi_0^{(0)}(x) + \sum_{n=1}^{\infty} \psi_0^{(n)}(x)$$

$$E_0^{(0)} = 0 \qquad \psi_0^{(0)}(x) = (\pi \hbar)^{-1/4} e^{-x^2/2\hbar}$$

$$E_0^{(n)} \sim O(\tilde{W}^n) \qquad \psi_0^{(n)}(x) \sim O(\tilde{W}^n)$$

Then

$$H_0 \psi_0^{(n)}(x) + \frac{1}{2} \left[2x \tilde{W}(x) - \hbar \frac{d\tilde{W}(x)}{dx} \right] \psi_0^{(n-1)}(x) + \frac{1}{2} \tilde{W}^2(x) \psi_0^{(n-2)}(x) =$$

$$= E_0^{(0)} \psi_0^{(n)} + E_0^{(1)} \psi_0^{(n-1)} + \dots + E_0^{(n)} \psi_0^{(0)}$$

and hence

$$E_0^{(n)} = \langle \psi_0^{(0)} | x \tilde{W}(x) - \frac{\hbar}{2} \frac{d\tilde{W}(x)}{dx} | \psi_0^{(n-1)} \rangle + \langle \psi_0^{(0)} | \frac{1}{2} \tilde{W}^2(x) | \psi_0^{(n-2)}(x) \rangle -$$

$$- E_0^{(1)} \langle \psi_0^{(0)} | \psi_0^{(n-1)} \rangle - \dots - E_0^{(n-1)} \langle \psi_0^{(0)} | \psi_0^{(1)} \rangle$$

We would like to prove that $E_0^{(n)} = 0$, $\forall n$. We have $E_0^{(0)} = 0$ and

$$E_0^{(1)} = \langle \psi_0^{(0)} | x \tilde{W}(x) - \frac{\hbar}{2} \frac{d\tilde{W}(x)}{dx} | \psi_0^{(0)} \rangle = \frac{1}{\sqrt{\pi \hbar}} \int_{-\infty}^{+\infty} dx \, e^{-x^2/\hbar} \left\{ x \tilde{W}(x) - \frac{\hbar}{2} \frac{d\tilde{W}(x)}{dx} \right\} =$$

$$= \frac{1}{\sqrt{\pi\hbar}}\left\{\int_{-\infty}^{+\infty} dx\, e^{-x^2/\hbar}\, x\,\tilde{W}(x) - \frac{\hbar}{2}\tilde{W} e^{-x^2/\hbar}\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} dx\, e^{-x^2/\hbar}\, x\,\tilde{W}(x)\right\} = 0$$

if we assume $\tilde{W}(x)\, e^{-x^2/\hbar} \to 0 \quad |x| \to \infty$

Let us now assume $E_0^{(j)} = 0 \quad j = 1, 2, \ldots k-1$ and let us check that $E_0^{(k)} = 0$.

$$E_0^{(k)} = \langle \psi_0^{(0)} |\, x\,\tilde{W}(x) - \frac{\hbar}{2}\frac{d\tilde{W}(x)}{dx}\, | \psi_0^{(k-1)} \rangle + \langle \psi_0^{(0)} |\, \frac{1}{2}\tilde{W}^2(x)\, | \psi_0^{(k-2)} \rangle$$

Let us write

$$\psi_0(x) = e^{-\int_0^x dx' \frac{\tilde{W}(x')}{\hbar}}\, \psi_0^{(0)}(x) \equiv e^{-D^{-1}\tilde{W}(x)}\, \psi_0^{(0)}(x)$$

which we know is true. Then

$$\psi_0^{(n)} \equiv (-1)^n \frac{1}{n!}\left[D^{-1} W(x)\right]^n \psi_0^{(0)}(x)$$

Then

$$E_0^{(k)} = (-1)^{k-2}\frac{1}{(k-2)!}\left\{ -\langle \psi_0^{(0)} | \left[x\,\tilde{W}(x) - \frac{\hbar}{2}\frac{d\tilde{W}}{dx}\right]\left[D^{-1}\tilde{W}(x)\right]^{k-1} | \psi_0^{(0)} \rangle \frac{1}{(k-1)} + \right.$$

$$\left. + \langle \psi_0^{(0)} |\, \frac{1}{2}\tilde{W}^2(x)\left[D^{-1}\tilde{W}(x)\right]^{k-2} | \psi_0^{(0)} \rangle \right\}$$

But

$$-\frac{1}{(k-1)}\langle \psi_0^{(0)} | \left[x\,\tilde{W}(x) - \frac{\hbar}{2}\frac{d\tilde{W}(x)}{dx}\right]\left[D^{-1}\tilde{W}(x)\right]^{k-1} | \psi_0^{(0)} \rangle =$$

$$= -\frac{1}{(k-1)}\frac{1}{\sqrt{\pi\hbar}}\int_{-\infty}^{+\infty} dx\, e^{-x^2/\hbar}\left[x\,\tilde{W}(x) - \frac{\hbar}{2}\frac{d\tilde{W}(x)}{dx}\right]\left[D^{-1}\tilde{W}(x)\right]^{k-1}$$

$$= \frac{1}{(k-1)}\frac{1}{\sqrt{\pi\hbar}}\int_{-\infty}^{+\infty} dx\left\{\frac{\hbar}{2}\frac{d}{dx}\, e^{-x^2/\hbar}\,\tilde{W}(x)\right\}\left[D^{-1}\tilde{W}(x)\right]^{k-1} =$$

$$= \frac{1}{(k-1)}\frac{1}{\sqrt{\pi\hbar}}\left\{\frac{\hbar}{2} e^{-x^2/\hbar}\,\tilde{W}(x)\left(\frac{1}{\hbar}\int_0^x dx'\,\tilde{W}(x')\right)^{k-1} - \frac{1}{2}\int_{-\infty}^{+\infty} dx\, e^{-x^2/\hbar}\,\tilde{W}(x)^2\,(k-1)\left[D^{-1}\tilde{W}(x)\right]^{k-2}\right\}$$

$$= -\langle \psi_0^{(0)} |\, \frac{1}{2}\tilde{W}(x)^2\left[D^{-1}\tilde{W}(x)\right]^{k-2} | \psi_0^{(0)} \rangle$$

and hence $E_0^{(k)} = 0$. Q.E.D.

VI. K - FOLDY-WOUTHUYSEN TRANSFORMATION

L. L. Foldy and S. A. Wouthuysen Phys. Rev. 78, 29 (1950)

In order to investigate the non-relativistic limit of the Dirac equation, in a systematic way, we will use the Foldy-Wouthuysen transformation. We want to find a unitary transformation

$$\psi(x) = e^{-iS} \psi'(x) \tag{1}$$

which decouples the small and large components of $\psi(x)$. and where S can be time-dependent. Up to this end let us write the Dirac equation in Hamiltonian form

$$i\partial_0 \psi(x) = H\psi(x) \tag{2}$$

Since $\psi'(x)$ satisfies the equation

$$i\partial_0 \psi'(x) = \left\{ e^{iS} H e^{-iS} - i e^{iS} (\partial_0 e^{-iS}) \right\} \psi'(x) \equiv H'\psi'(x) \tag{3}$$

our problem is to find S so as to get rid of odd operators in H' at a given order in 1/m. We will work in the Dirac representation and we say that an operator is odd if it connects the large and small components of the $\psi(x)$; otherwise is even.

In the free case

$$H = \vec{\alpha}\cdot\vec{p} + \beta m$$

$$\gamma^0 = \begin{vmatrix} I & 0 \\ 0 & -I \end{vmatrix} , \qquad \gamma^k = \begin{vmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{vmatrix} \tag{4}$$

$$\alpha^k \equiv \gamma^0\gamma^k = \begin{vmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{vmatrix} , \qquad \beta = \gamma^0 = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$$

and we can construct S exactly. It is time independent and can be chosen as

$$S \equiv -i\beta \frac{\vec{\alpha}\cdot\vec{p}}{|\vec{p}|}\,\theta = -i\,(\vec{\gamma}\cdot\hat{p})\,\theta \tag{5}$$

Since $(\vec{\gamma}\cdot\hat{p})^2 = -I$ we can write

$$e^{iS} = \cos\theta + \beta\,(\vec{\alpha}\cdot\hat{p})\,\sin\theta \tag{1}$$

Then

$$H' = (\vec{\alpha}\cdot\vec{p})\left\{\cos 2\theta - \frac{m}{|\vec{p}|}\sin 2\theta\right\} + \beta\left\{m\cos 2\theta + |\vec{p}|\sin 2\theta\right\} \tag{2}$$

If we choose

$$\sin 2\theta = \frac{|\vec{p}|}{E}, \quad \cos 2\theta = \frac{m}{E} \quad \Rightarrow \quad S = -\frac{i}{2}\,\beta\,\frac{\vec{\alpha}\cdot\vec{p}}{|\vec{p}|}\,\operatorname{artg}\frac{|\vec{p}|}{m} \tag{3}$$

we obtain

$$H' = \beta\,\frac{1}{E}\,(m^2 + \vec{p}^{\,2}) = \beta\sqrt{m^2 + \vec{p}^{\,2}} \tag{4}$$

as we would expect. In other words, we have decomposed H into a direct sum of two non-local operators $\pm(m^2 + \vec{p}^{\,2})^{1/2}$

In the interacting case we can not give a close expression for H'. We have

$$H' = H + i[S,H] + \frac{i^2}{2!}[S,[S,H]] + \frac{i^3}{3!}[S,[S,[S,H]]] + \cdots$$

$$+\, i\left\{ i\dot{S} + \frac{i^2}{2!}[S,\dot{S}] + \frac{i^3}{3!}[S,[S,\dot{S}]] + \cdots \right\} \tag{5}$$

Let us now consider a particle in an electromagnetic field

$$H = \beta m + \mathcal{O} + \mathcal{E} \tag{6}$$

$$\mathcal{O} \equiv \vec{\alpha}\cdot(\vec{p} - e\vec{A}), \qquad \mathcal{E} \equiv e A^0$$

where $\mathcal{O}$ ($\mathcal{E}$) is odd (even). Notice that

$$cp \sim mc^2\,\frac{v}{c} \tag{7}$$

En el límite no relativista, y de acuerdo con el teorema del virial, se tiene que

$$e\vec{A} \sim eA^0 \sim mc^2\,\frac{v^2}{c^2} \tag{8}$$

y por tanto

$$\mathcal{O} \sim mc^2\,\frac{v}{c}, \qquad \mathcal{E} \sim mc^2\,\frac{v^2}{c^2}.$$

We are going to keep terms up to $(v/c)^4$. Let us begin considering

$$S \equiv -\frac{i}{2m}\beta\mathcal{O} \qquad (1)$$

which is suggested by the free case. Notice $S \sim v/c$. We have

$$[S, \beta m] = i\mathcal{O} \qquad (\sim mc^2 (v/c))$$

$$[S,[S,\beta m]] = \frac{1}{m}\beta\mathcal{O}^2 \qquad (\sim mc^2 (v/c)^2)$$

$$[S,[S,[S,\beta m]]] = \frac{i}{m^2}\mathcal{O}^3 \qquad (\sim mc^2 (v/c)^3)$$

$$[S,[S,[S,[S,\beta m]]]] = \frac{1}{m^3}\beta\mathcal{O}^4 \qquad (\sim mc^2 (v/c)^4)$$

$$[S,\mathcal{E}] = -\frac{i}{2m}\beta[\mathcal{O},\mathcal{E}] \qquad (\sim mc^2 (v/c)^3)$$

$$[S,[S,\mathcal{E}]] = \frac{1}{4m^2}[\mathcal{O},[\mathcal{O},\mathcal{E}]] \qquad (\sim mc^2 (v/c)^4)$$

$$[S,\mathcal{O}] = -\frac{i}{m}\beta\mathcal{O}^2$$

$$[S,[S,\mathcal{O}]] = \frac{1}{m^2}\mathcal{O}^3$$

$$[S,[S,[S,\mathcal{O}]]] = -\frac{i}{m^3}\beta\mathcal{O}^4$$

$$\dot{S} = -\frac{i}{2m}\beta\dot{\mathcal{O}} \qquad (\sim mc^2 (v/c)^2)$$

$$[S,\dot{S}] = \frac{1}{4m^2}[\mathcal{O},\dot{\mathcal{O}}] \qquad (\sim mc^2 (v/c)^3) \qquad (2)$$

Then

$$H' = \beta m + \mathcal{O}' + \mathcal{E}'$$

$$\mathcal{O}' = \frac{1}{2m}\beta[\mathcal{O},\mathcal{E}] - \frac{1}{3m^2}\mathcal{O}^3 + \frac{i}{2m}\beta\dot{\mathcal{O}} \qquad (3)$$

$$\mathcal{E}' = \mathcal{E} + \frac{1}{2m}\beta\mathcal{O}^2 - \frac{1}{8m^3}\beta\mathcal{O}^4 - \frac{1}{8m^2}[\mathcal{O},[\mathcal{O},\mathcal{E}]] - \frac{i}{8m^2}[\mathcal{O},\dot{\mathcal{O}}]$$

Notice $\mathcal{O}' \sim (mc^2)(v/c)^2$. Let us now carry out another transformation with

$$S' = -\frac{i}{2m}\beta\mathcal{O}' \qquad (4)$$

which is of order $S' \sim (v/c)^2$. Then

$$[S', m\beta] = i\,\theta' \qquad (mc^2 (v/c)^2)$$

$$[S', [S', m\beta]] = \frac{1}{m}\,\beta\,\theta'^2 \qquad (mc^2 (v/c)^4)$$

$$[S', \theta'] = -\frac{i}{m}\,\beta\,\theta'^2 \qquad (mc^2 (v/c)^4)$$

$$\dot{S}' = -\frac{i}{2m}\,\beta\,\dot{\theta}' \qquad (mc^2 (v/c)^3) \qquad (1)$$

Hence

$$H'' = \beta m + \theta'' + \mathcal{E}''$$

(2)

$$\theta'' = \frac{i}{2m}\,\beta\,\dot{\theta}' \quad , \quad \mathcal{E}'' = \mathcal{E}' + \frac{1}{2m}\,\beta\,\theta'^2$$

Notice $\theta'' \sim mc^2 (v/c)^3$ and we must carry out another transformation with

$$S'' = -\frac{i}{2m}\,\beta\,\theta'' \qquad (3)$$

where $S'' \sim (v/c)^3$ Then

$$[S'', \beta m] = i\,\theta'' \qquad (mc^2 (v/c)^3)$$

and therefore

$$H''' = \beta m + \mathcal{E}'' \qquad (4)$$

Then we obtain

$$H''' = \beta m + \mathcal{E}' + \frac{1}{2m}\,\beta\,\theta'^2 =$$

$$= \beta m + \mathcal{E} + \frac{1}{2m}\,\beta\,\theta^2 - \frac{1}{8m^3}\,\beta\,\theta^4 - \frac{1}{8m^2}\,[\theta, [\theta, \mathcal{E}]] - \frac{i}{8m^2}\,[\theta, \dot{\theta}]$$

$$+ \frac{1}{8m^3}\,\beta\,\dot{\theta}^2 \qquad \Rightarrow$$

$$H''' = \beta \left\{ m + \frac{1}{2m}\,\theta^2 - \frac{1}{8m^3}\,\beta\,\theta^4 \right\} + \mathcal{E} - \frac{1}{8m^2}\,[\theta, [\theta, \mathcal{E}]]$$

$$- \frac{i}{8m^2}\,[\theta, \dot{\theta}] + \frac{1}{8m^3}\,\beta\,\dot{\theta}^2 + O\left(mc^2 \left(\frac{v}{c}\right)^5\right) \qquad (5)$$

Then up to desired order

$$\Theta^2 = \alpha^k \alpha^\ell (p^k - eA^k)(p^\ell - eA^\ell) = (\vec{p} - e\vec{A})^2 - e\vec{\Sigma}\cdot(\vec{\nabla}\times\vec{A}) =$$

$$= (\vec{p} - e\vec{A})^2 - e\vec{\Sigma}\cdot\vec{B} \qquad (1)$$

$$\Theta^4 = \vec{p}^{\,4} \qquad (2)$$

$$[\Theta, [\Theta, \mathcal{E}]] = e\alpha^k\alpha^\ell (p^k p^\ell A^0) + e\alpha^k\alpha^\ell [(p^\ell A^0)p^k - (p^k A^0)p^\ell]$$

$$= -e[\vec{\nabla}\cdot(\vec{\nabla}A^0)] - 2e\,\vec{\Sigma}\cdot[(\vec{\nabla}A^0)\times\vec{p}] \qquad (3)$$

$$[\Theta, \dot{\Theta}] = -e\alpha^k\alpha^\ell (p^k \dot{A}^\ell) - e\alpha^k\alpha^\ell [\dot{A}^\ell p^k - \dot{A}^k p^\ell] =$$

$$= ie(\vec{\nabla}\cdot\dot{\vec{A}}) + 2ie\,\vec{\Sigma}\cdot(\dot{\vec{A}}\times\vec{p}) - e\vec{\Sigma}\cdot(\vec{\nabla}\times\dot{\vec{A}}) \qquad (4)$$

Hence

$$H''' = \beta\left\{m + \frac{1}{2m}(\vec{p} - e\vec{A})^2 - \frac{1}{8m^3}\vec{p}^{\,4}\right\} + eA^0 - \frac{e}{2m}\beta\vec{\Sigma}\cdot\vec{B}$$

$$-\frac{e}{8m^2}\vec{\nabla}\cdot\vec{E} - \frac{e}{4m^2}\vec{\Sigma}\cdot(\vec{E}\times\vec{p}) - \frac{ie}{8m^2}\vec{\Sigma}\cdot(\vec{\nabla}\times\vec{E}) \qquad (5)$$

Therefore we can write

$$H''' = \beta\left\{m + \frac{1}{2m}(\vec{p} - e\vec{A})^2 - \frac{1}{8m^3}\vec{p}^{\,4}\right\} + eA^0 - \frac{e}{2m}\beta\vec{\Sigma}\cdot\vec{B}$$

$$+\left\{-\frac{ie}{8m^2}\vec{\Sigma}\cdot(\vec{\nabla}\times\vec{E}) - \frac{e}{4m^2}\vec{\Sigma}\cdot(\vec{E}\times\vec{p})\right\} - \frac{e}{8m^2}\vec{\nabla}\cdot\vec{E} \qquad (6)$$

The first term in the bracket is the expansion of $[(\vec{p} - e\vec{A})^2 + m^2]^{1/2}$ to the required order. The second term eA^0 is the electrostatic energy of a point-like charge. The third term represent the interaction energy of a magnetic dipole with $g = 2$. The term in the parentheses may be seen to correspond to a spin-orbit interaction. Indeed, for a static spherically symmetric potential. $\vec{A} = 0$ $A^0 = A^0(r)$ and therefore $(\vec{\nabla}\times\vec{E}) = 0$ and

$$\vec{\Sigma}\cdot(\vec{E}\times\vec{p}) = -\vec{\Sigma}[\vec{\nabla}A^0\times\vec{p}] = -\vec{\Sigma}\frac{1}{r}\frac{dA^0}{dr}[\vec{x}\times\vec{p}] = -\frac{1}{r}\frac{dA^0}{dr}\vec{\Sigma}\cdot\vec{L}$$

Finally the last term is the so-called Darwin term.

VII- PARTICULAS VECTORIALES CON MASA

Consideremos partículas de spin 1, con carga y con masa $m \neq 0$, las cuales vienen descritas por un campo que, en el formalismo de la teoría cuántica de campos, se transforma como

$$U(a,\Lambda)\,\phi^{\mu}(x)\,U^{-1}(a,\Lambda) = (\Lambda^{-1})^{\mu}{}_{\nu}\,\phi^{\nu}(\Lambda x + a) \tag{1}$$

La densidad Lagrangiana viene dada por

$$L(x) = -\frac{1}{2} : [\partial_\mu \phi_\nu^+(x) - \partial_\nu \phi_\mu^+(x)][\partial^\mu \phi^\nu(x) - \partial^\nu \phi^\mu(x)] : + m^2 : \phi_\mu^+(x)\phi^\mu(x) : \tag{2}$$

Las ecuaciones del movimiento son pues

$$\partial_\mu \partial^\mu \phi^\nu(x) - \partial^\nu \partial_\mu \phi^\mu(x) + m^2 \phi^\nu(x) = 0 \tag{3}$$

Derivando esta ecuación con respecto a x^ν se obtiene que al ser $m \neq 0$ se debe cumplir $\partial_\nu \phi^\nu(x) = 0$ y por tanto (3) es equivalente a

$$\left.\begin{aligned} &[\partial_\mu \partial^\mu + m^2]\,\phi^\nu(x) = 0 \\ &\partial_\mu \phi^\mu(x) = 0 \end{aligned}\right\} \tag{4}$$

es decir que cada una de las componentes de $\phi^\mu(x)$ satisface una ecuación de Klein-Gordon y todas ellas están relacionadas por la condición subsidiaria dada en (4). Su necesidad es evidente pues $\phi^\mu(x)$ tiene cuatro grados de libertad y para partículas de spin 1 y $m \neq 0$ sólo hay tres helicidades independientes.

Como en los casos anteriores podemos ahora buscar la solución más general de (4) y proceder a su cuantificación. En analogía con la anterior tra solución más general de la ecuación de Klein-Gordon es

$$\psi(x) = \frac{1}{(2\pi)^3}\int \frac{d^3p}{2E(\vec{p})} \sum_{\lambda=0}^{3} \left[\epsilon^\mu(\vec{p},\lambda)\, a(\vec{p},\lambda)\, e^{-ip.x} + \epsilon^{\mu *}(\vec{p},\lambda)\, b^+(\vec{p},\lambda)\, e^{+ip.x} \right] \tag{5}$$

donde, en principio y para cada valor de $\vec{p}$, los cuatro cuadrivectores $\epsilon^\mu(\vec{p};\lambda)$ forman una base. Sin embargo (5) debe cumplir la condición subsidiaria y esto implica que

$$p_\mu . \epsilon^\mu(\vec{p},\lambda) = 0 \tag{6}$$

y por tanto sólo hay tres vectores $\epsilon^\mu(\vec{p},\lambda)$ linealmente independientes y podemos restringir la suma que aparece en (1.5) a tres índices únicamente. De acuerdo con (VI. 13. 1) se puede elegir

$$\epsilon^\mu(\vec{p}, i) \equiv S^\mu(i) \qquad i = 1, 2, 3 \tag{1}$$

o explícitamente

$$\begin{aligned}
\epsilon^\mu(\vec{p}, 1) &\equiv (0, \cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta) \\
\epsilon^\mu(\vec{p}, 2) &\equiv (0, -\sin\phi, \cos\phi, 0) \\
\epsilon^\mu(\vec{p}, 3) &\equiv \left(\frac{|\vec{p}|}{m}, \frac{E}{m}\sin\theta\cos\phi, \frac{E}{m}\sin\theta\sin\phi, \frac{E}{m}\cos\theta\right)
\end{aligned} \tag{2}$$

Notemos que

$$\begin{aligned}
&p_\mu \cdot \epsilon^\mu(\vec{p}, i) = 0 \qquad i = 1, 2, 3 \\
&\epsilon^*_\mu(\vec{p}, i) = \epsilon_\mu(\vec{p}, i) \\
&\epsilon_\mu(\vec{p}, i) \cdot \epsilon^\mu(\vec{p}, j) = -\delta_{ij} \\
&\sum_{i=1}^{3} \epsilon^\mu(\vec{p}, i)\, \epsilon^\nu(\vec{p}, i) = -g^{\mu\nu} + \frac{1}{m^2} p^\mu p^\nu \\
&\vec{\epsilon}(\vec{p}, 1) \times \vec{\epsilon}(\vec{p}, 2) = \hat{p} \\
&\epsilon^\mu(-\vec{p}, 1) = \epsilon^\mu(\vec{p}, 1) \quad , \quad \epsilon^\mu(-\vec{p}, 2) = -\epsilon^\mu(\vec{p}, 2)
\end{aligned} \tag{3}$$

Más interesante desde el punto de vista físico es elegir

$$\epsilon^\mu(\vec{p}, \pm 1) = \mp \frac{1}{\sqrt{2}} [S^\mu(1) \pm i S^\mu(2)] \quad , \quad \epsilon^\mu(\vec{p}, 0) \equiv S^\mu(3) \tag{4}$$

que no son más que las componentes tensoriales del trivector $\vec{S}^\mu$. De aquí se obtiene inmediatamente que $(\lambda = \pm 1, 0)$

$$\begin{aligned}
&p^\mu \epsilon_\mu(\vec{p}, \lambda) = 0 \\
&\epsilon^{\mu *}(\vec{p}, \lambda) = (-1)^\lambda \epsilon^\mu(\vec{p}, -\lambda) \\
&\epsilon^*_\mu(\vec{p}, \lambda)\, \epsilon^\mu(\vec{p}, \lambda') = -\delta_{\lambda\lambda'} \\
&\sum_{\lambda=0,\pm1} \epsilon^\mu(\vec{p}, \lambda)\, \epsilon^{\nu *}(\vec{p}, \lambda) = -g^{\mu\nu} + \frac{1}{m^2} p^\mu p^\nu
\end{aligned}$$

y usando las leyes de transformación se puede ver que λ es la helicidad de la partícula

Escribiremos entonces el campo en la forma

$$\phi^\mu(x) = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2E(\vec{p})} \sum_{\lambda=0,\pm1} [\epsilon^\mu(\vec{p},\lambda)\, a(\vec{p},\lambda)\, e^{-ip.x} + \epsilon^{\mu *}(\vec{p},\lambda)\, b^+(\vec{p},\lambda)\, e^{+ip.x}] \quad (1)$$

con las reglas de conmutación

$$[a(\vec{p},\lambda)\, ,\, a^+(\vec{p}\,',\lambda')] = (2\pi)^3\, 2E(\vec{p})\, \delta_{\lambda\lambda'}\, \delta(\vec{p}-\vec{p}\,')$$
$$(2)$$
$$[b(\vec{p},\lambda)\, ,\, b^+(\vec{p}\,',\lambda')] = (2\pi)^3\, 2E(\vec{p})\, \delta_{\lambda\lambda'}\, \delta(\vec{p}-\vec{p}\,')$$

siendo nulos los restantes conmutadores.

Para las reglas de conmutación de los campos

$$[\phi^\mu(x)\, ,\, \phi^{\nu +}(y)] = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2E} \frac{d^3p'}{2E'} \sum_{\lambda\lambda'} [\epsilon^\mu(\vec{p},\lambda)\, \epsilon^{\nu *}(\vec{p}\,',\lambda')\, e^{-ip.x+ip'.y}\, 2E\, \delta_{\lambda\lambda'}\, \delta(\vec{p}-\vec{p}\,') -$$

$$- \epsilon^{\mu *}(\vec{p},\lambda)\, \epsilon^\nu(\vec{p}\,',\lambda')\, e^{ipx-ip'y}\, 2E\, \delta_{\lambda\lambda'}\, \delta(\vec{p}-\vec{p}\,')]$$

$$= \frac{1}{(2\pi)^3} \int \frac{d^3p}{2E} \left\{ e^{-ip.(x-y)} \sum_\lambda \epsilon^\mu(\vec{p},\lambda)\, \epsilon^{\nu *}(\vec{p},\lambda) - e^{+ip.(x-y)} \sum_\lambda \epsilon^{\mu *}(\vec{p},\lambda)\, \epsilon^\nu(\vec{p},\lambda) \right\} =$$

$$= \frac{1}{(2\pi)^3} \int \frac{d^3p}{2E} [-g^{\mu\nu} + \frac{1}{m^2} p^\mu p^\nu] [e^{-ip.(x-y)} - e^{+ip.(x-y)}] =$$

$$= [-g^{\mu\nu} - \frac{1}{m^2} \partial_x^\mu \partial_x^\nu] \frac{1}{(2\pi)^3} \int \frac{d^3p}{2E} [e^{-ip.(x-y)} - e^{+ip.(x-y)}]$$

de donde

$$[\phi^\mu(x)\, ,\, \phi^{\nu +}(y)] = -i\, (g^{\mu\nu} + \frac{1}{m^2} \partial^\mu \partial^\nu)\, \Delta(x-y\, ;\, m^2) \quad (3)$$

Siendo nulos los restantes conmutadores.

Por otra parte de (1.2) se tiene que el momento conjugado al campo $\phi^\mu(x)$ es

$$\pi_\mu(x) = \frac{\partial L(x)}{\partial(\partial_0 \phi^\mu(x))} = \partial_\mu \phi_0^+(x) - \partial_0 \phi_\mu^+(x) \quad (4)$$

En principio parece claramente que $\pi_0(x) \equiv 0$, pero en realidad esto no es importante pues $\phi^0(x)$ puede escribirse como

$$\phi^0(x) = -\frac{1}{m^2} \partial^k \pi_k^+(x) \quad (5)$$

como consecuencia de las ecuaciones del movimiento y de la condición subsidia ría. Esto implica que los únicos campos realmente independientes son los $\phi^i(x)$ y sus momentos conjugados son los $\Pi_i(x)$. Para estos se tiene como consecuencia de (3.3) que

$$\delta(x^0 - y^0)\,[\,\phi^i(x)\,,\,\Pi_j(y)\,] = i\,\delta_{ij}\,\delta^{(4)}(x-y) \tag{1}$$

$$\delta(x^0-y^0)\,[\,\phi^i(x)\,,\,\phi^j(y)\,] = \delta(x^0-y^0)\,[\,\Pi_i(x)\,,\,\Pi_j(y)\,] = 0$$

Si pretendemos estudiar campos reales con masa $\phi^\mu(x) \equiv \phi^{\mu +}(x)$, entonces se procede de forma totalmente análoga partiendo de la densidad Lagrangiana

$$L(x) = -\frac{1}{4} : [\partial_\mu \phi_\nu(x) - \partial_\nu \phi_\mu(x)][\partial^\mu \phi^\nu(x) - \partial^\nu \phi^\mu(x)] : + \frac{1}{2} m^2 : \phi^\mu(x)\,\phi_\mu(x) : \tag{2}$$

Es evidente que el hecho de que $m \neq 0$ juega un papel fundamental en todo este desarrollo y que el límite $m \to 0$ no puede realizarse. Es necesario tratar los campos vectoriales con $m = 0$ de forma totalmente distinta.

VII.- PARTICULAS VECTORIALES SIN MASA

Vamos ahora a considerar el caso de partículas de spin 1 y masa $m = 0$; supondremos además que su carga es nula, pues así sucede en el caso del campo electromagnético que es el que nos interesa. La densidad Lagrangiana, por analogía con el caso anterior, es

$$L(x) = -\frac{1}{4} : F_{\mu\nu}(x) F^{\mu\nu}(x) :$$

$$F_{\mu\nu}(x) \equiv \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) \tag{1}$$

donde $F_{\mu\nu}(x)$ son las intensidades del campo y se hallan relacionadas con los campos eléctricos y magnéticos por las ecuaciones

$$E^i(x) \equiv F^{i0}(x) \quad , \quad B^i(x) \equiv -\frac{1}{2} \epsilon_{ijk} F^{jk}(x) \quad *$$

$$F_{\mu\nu}(x) = \begin{vmatrix} 0 & +E^1 & +E^2 & +E^3 \\ -E^1 & 0 & -B^3 & +B^2 \\ -E^2 & +B^3 & 0 & -B^1 \\ -E^3 & -B^2 & +B^1 & 0 \end{vmatrix} \tag{2}$$

$$A^\mu(x) \equiv (\phi(x), \vec{A}(x))$$

$$\vec{E}(x) = -\vec{\nabla}\phi(x) - \frac{\partial \vec{A}(x)}{\partial t} \quad , \quad \vec{B}(x) \equiv \vec{\nabla} \times \vec{A}(x)$$

Evidentemente

$$\partial_\mu F_{\nu\lambda}(x) + \partial_\nu F_{\lambda\mu}(x) + \partial_\lambda F_{\mu\nu}(x) = 0 \tag{3}$$

que no son más que las dos primeras ecuaciones de Maxwell

$$\vec{\nabla} \cdot \vec{B}(x) = 0 \quad , \quad \vec{\nabla} \times \vec{E}(x) + \frac{\partial \vec{B}(x)}{\partial t} = 0 \tag{4}$$

De (1) se deduce que las ecuaciones del movimiento son

$$\partial_\nu \partial^\nu A^\mu(x) - \partial^\mu [\partial_\nu A^\nu(x)] = 0 \tag{5}$$

A diferencia de lo que sucede con $m \neq 0$, no podemos ahora deducir de la ecuación anterior que $\partial_\mu A^\mu(x) = 0$. Esto conduce a dificultades, pues en particular continua siendo cierto que $\Pi^0(x) \equiv 0$, pero ahora nos

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* $F_{ij} = - \epsilon_{ijk} B^k$

es posible expresar $A^0(x)$ en términos de las $A^i(x)$. Fermi propuso sustituir el $L(x)$ dado antes por *

$$L(x) = -\frac{1}{2} : \partial_\mu A_\nu(x)\, \partial^\mu A^\nu(x) : \tag{1}$$

que conduce a

$$\partial_\nu \partial^\nu A^\mu(x) = 0 \tag{2}$$

Entonces la solución general es

$$A^\mu(x) = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2E(\vec{p})} \sum_{\lambda=0}^{3} \left[\epsilon^\mu(\vec{p},\lambda)\, a(\vec{p},\lambda)\, e^{-ip.x} + \epsilon^{\mu *}(\vec{p},\lambda)\, a^+(\vec{p},\lambda)\, e^{+ip.x} \right] \tag{3}$$

donde los cuatro vectores polarización independientes se pueden elegir como

$$\epsilon^\mu(\vec{p},0) \equiv (1,0) \equiv n^\mu \quad , \quad \epsilon^\mu(\vec{p},1) \equiv (0, \vec{A}(1)) \equiv s^\mu(1)$$

$$\epsilon^\mu(\vec{p},2) \equiv (0, \vec{A}(2)) \equiv s^\mu(2) \quad , \quad \epsilon^\mu(\vec{p},3) \equiv (0, \hat{p}) \equiv (0, \vec{A}(3)) \tag{4}$$

$\hat{p} = \hat{A}(3) = (\sin\theta\cos\phi, \sin\theta\sin\phi, 0)$
$\hat{A}(1) = (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\phi)$
$\hat{A}(2) = (-\sin\phi, \cos\phi, 0)$

en la notación de (VI-13). Se cumple

$$\epsilon_\mu(\vec{p},\lambda)\, \epsilon^\mu(\vec{p},\lambda') = g_{\lambda\lambda'}$$

$$\sum_{\lambda=0}^{3} \epsilon^\mu(\vec{p},\lambda)\, \epsilon^\nu(\vec{p},\lambda) = \delta^{\mu\nu} \tag{5}$$

Para las reglas de conmutación se postula

$$[a(\vec{p},\lambda), a^+(\vec{p}',\lambda')] = -(2\pi)^3 g_{\lambda\lambda'}\, 2E(\vec{p})\, \delta(\vec{p}-\vec{p}') \tag{6}$$

siendo nulos los restantes conmutadores. Entonces

$$[A^\mu(x), A^\nu(y)] = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2E} \frac{d^3p'}{2E'} \sum_{\lambda\lambda'} 2E\, g_{\lambda\lambda'}\, \delta(\vec{p}-\vec{p}') \Big\{ -\epsilon^\mu(p,\lambda)\, \epsilon^{\nu *}(p'\lambda')\, e^{-ipx+ip'y} + \epsilon^{\mu *}(\vec{p}\lambda)\, \epsilon^\nu(\vec{p}'\lambda')\, e^{ipx-ip'y} \Big\}$$

$$= \frac{1}{(2\pi)^3} \int \frac{d^3p}{2E} \Big\{ e^{-ip.(x-y)} \sum_{\lambda\lambda'} \epsilon^\mu(\vec{p},\lambda)(-g_{\lambda\lambda'})\, \epsilon^{\nu *}(\vec{p},\lambda') - e^{+ip(x-y)} \sum_{\lambda\lambda'} \epsilon^{\mu *}(\vec{p},\lambda)(-g_{\lambda\lambda'})\, \epsilon^\nu(\vec{p},\lambda') \Big\} =$$

$$= -g^{\mu\nu} \frac{1}{(2\pi)^3} \int \frac{d^3p}{2E} \Big\{ e^{-ip.(x-y)} - e^{+ip.(x-y)} \Big\}$$

esto es

$$[A^\mu(x), A^\nu(y)] = -g^{\mu\nu}\, i\, D(x-y) \tag{7}$$

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$$L_1 - L_2 \equiv -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} \partial^\mu A^\nu \partial_\mu A_\nu = + \frac{1}{2} (\partial_\mu A_\nu)(\partial^\nu A^\mu) \Rightarrow$$

$$L_2 = L_1 - \frac{1}{2} (\partial_\mu A_\nu)(\partial^\nu A^\mu) = L_1 - \frac{1}{2} (\partial_\mu A^\mu)^2 - \frac{1}{2} \partial_\mu \left[A^\nu \partial_\nu A^\mu - A^\mu \partial_\nu A^\nu \right]$$

Then

(up to a total divergence)

$$-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} (\partial_\mu A^\mu)^2 \doteq -\frac{1}{2} \partial^\mu A^\nu \partial_\mu A_\nu$$

Usually we write $-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2a} (\partial_\mu A^\mu)^2$

The Fermi Lagrangian corresponds to $a = 1$. (Fermi gauge).

y son nulos los restantes conmutadores a tiempos iguales

Las reglas de conmutación para los fotones transversales ($\lambda = 1, 2$) son las usuales y lo mismo sucede con los fotones longitudinales ($\lambda = 3$), sin embargo los fotones temporales ($\lambda = 0$) satisfacen una regla de conmutación de signo opuesto a la usual; esto es altamente desagradable pues entonces estados tales como $a^+(\vec{p}, 0)|0\rangle$ tienen norma negativa. Precisamente es la condición de Lorentz la que nos resolverá la dificultad. Es evidente que no podemos imponer la condición $\partial_\mu A^\mu(x) = 0$, pues no es compatible con las leyes de conmutación de los campos. Gupta - Bleuber consideraron que para todo estado físicamente realizable $|\Phi\rangle$ se debía cumplir

$$\partial_\mu A^{(+)\mu}(x) \, |\Phi\rangle = 0 \tag{1}$$

donde $A^{(+)\mu}(x)$ es la parte de frecuencias positivas (la que contiene los operadores destrucción) de $A^\mu(x)$. Entonces para cualquier par de estados físicos $|\Phi\rangle$ y $|\Phi'\rangle$

$$\langle \Phi' | \partial_\mu A^\mu(x) | \Phi \rangle = 0 \tag{2}$$

Veamos las consecuencias. De (2.3)

$$\partial_\mu A^{(+)\mu}(x) = -\frac{i}{(2\pi)^3}\int \frac{d^3p}{2E(\vec{p})} \sum_{\lambda=0}^{3} p_\mu \epsilon^\mu(\vec{p},\lambda)\, a(\vec{p},\lambda)\, e^{-ip.x} =$$

$$= -\frac{i}{(2\pi)^3}\int \frac{d^3p}{2E(\vec{p})} [\, a(\vec{p},0) - a(\vec{p},3)\,]\, E(\vec{p})\, e^{-ip.x}$$

y por tanto los estados físicos son aquellos que cumplen

$$L(\vec{p})\,|\Phi\rangle = 0 \qquad , \qquad L(\vec{p}) \equiv a(\vec{p};0) - a(\vec{p},3) \tag{3}$$

Es fácil darse cuenta que si $|\Phi_0\rangle$ es un estado físico que contiene sólo fotones transversales y que por tanto cumple (3), el estado más general posible con este contenido en fotones transversales y que sea físico es

$$|\Phi_f\rangle \equiv N_f \, R_f \, |\Phi_0\rangle$$

(4)

$$R_f \equiv 1 + \sum_{n=1}^{\infty} \int \prod_{j=1}^{n} \frac{d^3p_j}{(2\pi)^3\, 2E_j} \, f(\vec{p}_1, \ldots \vec{p}_j, \ldots \vec{p}_n)\, L^+(\vec{p}_1) \ldots L^+(\vec{p}_n)$$

Notar que

$$[\,L(\vec{p})\,,\,L^{+}(\vec{p}')\,] = 0 \qquad (1)$$

En (3.4) N_f es una constante de normalización que garantiza que $\langle \Phi_f | \Phi_f \rangle = 1$, si $\langle \Phi_0 | \Phi_0 \rangle = 1$. Se puede ver fácilmente que los estados $|\Phi_f\rangle = N_f R_f |\Phi_0\rangle$, $|\Psi_g\rangle = N_g R_g |\Psi_0\rangle$ que son físicos y están construidos a partir de los estados $|\Phi_0\rangle$ y $|\Psi_0\rangle$ que contienen sólo fotones transversales cumplen

$$\langle \Psi_g | \Phi_f \rangle = \langle \Psi_0 | \Phi_0 \rangle \qquad (2)$$

Como todas las cantidades observables vienen dadas en términos de elementos de matriz, concluimos que las contribuciones de los fotones transversales y longitudinales al valor esperado de cualquier observable se compensan, de forma que estos grados de libertad adicionales son inobservables contribuyendo solo los fotones transversales ($\lambda = 1, 2$) al cálculo de las magnitudes físicas. Podemos pues sin más escribir

$$A^{\mu}(x) = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2E(\vec{p})} \sum_{\lambda=\pm 1} [\, \epsilon^{\mu}(\vec{p},\lambda)\, a(\vec{p},\lambda)\, e^{-ip.x} + \epsilon^{\mu *}(\vec{p},\lambda)\, a^{+}(\vec{p},\lambda)\, e^{+ip.x} \,]$$

$$[\, a(\vec{p},\lambda)\,,\, a^{+}(\vec{p}',\lambda')\,] = 2E(\vec{p})\, \delta_{\lambda\lambda'}\, (2\pi)^3\, \delta(\vec{p}-\vec{p}') \qquad (3)$$

$$\epsilon^{\mu}(\vec{p},\pm 1) = \left(0\,,\, \mp \frac{1}{\sqrt{2}} [\, \vec{A}(1) \pm i\, \vec{A}(2)\,] \right)$$

y se cumple

$$p^{\mu}\, \epsilon_{\mu}(\vec{p},\lambda) = 0$$

$$\epsilon^{*}_{\mu}(\vec{p},\lambda) = (-1)\, \epsilon_{\mu}(\vec{p},-\lambda)$$

$$\epsilon^{*}_{\mu}(\vec{p},\lambda)\, \epsilon^{\mu}(\vec{p},\lambda') = \delta_{\lambda\lambda'}$$

$$\sum_{\lambda=\pm 1} \epsilon^{\mu}(\vec{p},\lambda)\, \epsilon^{\nu *}(\vec{p},\lambda) = -g^{\mu\nu} + \frac{m^{\mu}p^{\nu} + m^{\nu}p^{\mu}}{(m.p)} - \frac{p^{\mu}p^{\nu}}{(m.p)^2} \qquad (4)$$

Notemos, finalmente, que el tensor energía-momento viene dado por

$$\tilde{T}^{\mu\nu} = \frac{1}{2}\, \partial_{\lambda} A_{\rho}(x)\, \partial^{\lambda} A^{\rho}(x)\, g^{\mu\nu} - \frac{1}{2}\, \partial_{\lambda} A_{\rho}(x)\, \partial^{\rho} A^{\lambda}(x)\, g^{\mu\nu} \qquad (5)$$
$$- \partial^{\mu} A^{\rho}(x)\, \partial^{\nu} A_{\rho}(x) + \partial^{\rho} A^{\mu}(x)\, \partial^{\nu} A_{\rho}(x)$$

donde hemos partido de (1.1) que es equivalente al Lagrangiano de Fermi sobre estados físicos. Entonces una forma equivalente de escribir la expresión anterior es

$$\tilde{T}^{\mu\nu} = \frac{1}{4} g^{\mu\nu} F^{\rho\lambda}(x) F_{\rho\lambda}(x) + F^{\mu\rho}(x) \partial^{\nu} A_{\rho}(x) \tag{1}$$

y en particular sobre las soluciones de las ecuaciones se obtienen los bien conocidos resultados

$$\tilde{T}^{00} = \frac{1}{2} [\vec{E}^2(x) + \vec{B}^2(x)] \tag{2}$$

$$\tilde{T}^{0i} = [\vec{E}(x) \times \vec{B}(x)]_i$$

VIII-A- PARTICULA EN UN CAMPO ELECTROMAGNETICO

En la teoría clásica el Lagrangiano de interacción de una partícula de carga e y masa m en el seno de un campo electromagnético viene dado por

$$L = m\sqrt{1-\vec{v}^2} + e\left[\vec{A}(x)\cdot\vec{v} - A^0(x)\right] \tag{1}$$

y por tanto el momento es

$$\vec{p} \equiv \frac{\partial L}{\partial \vec{v}} = \frac{m\vec{v}}{\sqrt{1-\vec{v}^2}} + e\vec{A}(x) \tag{2}$$

de donde

$$1-\vec{v}^2 = m^2\left[m^2 + (\vec{p} - e\vec{A}(x))^2\right]^{-1} \tag{3}$$

El Hamiltoniano es

$$H \equiv \vec{v}\cdot\frac{\partial L}{\partial \vec{v}} - L = \frac{m}{\sqrt{1-\vec{v}^2}} + eA^0(x) \tag{4}$$

y usando (3)

$$H = \left[m^2 + (\vec{p} - e\vec{A}(x))^2\right]^{1/2} + eA^0(x) \tag{5}$$

Consideremos ahora las ecuaciones del movimiento

$$\frac{\partial L}{\partial v_i} = \frac{m v^i}{\sqrt{1-\vec{v}^2}} + eA^i(x)$$

$$\frac{\partial L}{\partial x_i} = e\vec{v}\cdot\frac{\partial \vec{A}}{\partial x_i} - e\frac{\partial A^0}{\partial x_i}$$

$$\frac{d}{dt}\frac{m v^i}{\sqrt{1-\vec{v}^2}} + e\frac{\partial A^i(x)}{\partial t} + e\frac{\partial A^i(x)}{\partial x^j}v^j = e v^j\frac{\partial A^j(x)}{\partial x_i} - e\frac{\partial A^0(x)}{\partial x_i} \Rightarrow$$

$$\frac{d}{dt}\frac{m v^i}{\sqrt{1-\vec{v}^2}} = -e\frac{\partial A^i(x)}{\partial t} - e\frac{\partial A^0(x)}{\partial x_i} + e v^j\frac{\partial A^j(x)}{\partial x_i} - e\frac{\partial A^i(x)}{\partial x_j}v^j \Rightarrow$$

$$\frac{d}{dt}\frac{m\vec{v}}{\sqrt{1-\vec{v}^2}} = e(\vec{E} + \vec{v}\times\vec{B}) \tag{6}$$

que no es más que la ley de fuerza de Lorentz.

Si indicamos por $\mathcal{E}$ la energía

$$\frac{d\mathcal{E}}{dt} = \frac{d}{dt}\left(\frac{m}{\sqrt{1-\vec{v}^2}}\right) = \vec{v}\cdot\frac{d}{dt}\frac{m\vec{v}}{\sqrt{1-\vec{v}^2}} = e\,\vec{E}\cdot\vec{v} \quad (1)$$

que expresa el bien conocido hecho de que sólo los campos eléctricos realizan trabajo

Si introducimos la cuadrivelocidad $u^\mu \equiv dx^\mu/d\tau$ entonces (1.6) y (1) son equivalentes a

$$m\frac{du^\mu}{d\tau} = e\,F^{\mu\nu}(x)\,u_\nu \quad (2)$$

1) Movimiento de una partícula en un campo constante y uniforme

Vamos a resolver este problema usando el método spinorial. A cada vector A^μ le asociaremos una matriz A definida como

$$A \equiv A^0 I + \vec{A}\cdot\vec{\sigma} \quad (3)$$

Entonces

$$\frac{du}{d\tau} = \frac{e}{m}\left[F^{0\nu}u_\nu I + F^{i\nu}u_\nu\sigma_i\right] =$$

$$= \frac{e}{m}\left[\vec{E}\cdot\vec{u}\,I + \vec{E}\cdot\vec{\sigma}\,u^0 - (\vec{B}\times\vec{u})\cdot\vec{\sigma}\right]$$

$$= \frac{e}{2m}\left[(\vec{E}\cdot\vec{\sigma})(\vec{u}\cdot\sigma) + (\vec{u}\cdot\vec{\sigma})(\vec{E}\cdot\sigma) + 2(\vec{E}\cdot\vec{\sigma})(u^0 I)\right.$$

$$\left. + i(\vec{B}\cdot\vec{\sigma})(\vec{u}\cdot\vec{\sigma}) - i(\vec{u}\cdot\vec{\sigma})(\vec{B}\cdot\vec{u})\right] =$$

$$= \frac{e}{2m}\left\{\left[(\vec{E}+i\vec{B})\cdot\vec{\sigma}\right]\left[u_0 I + \vec{u}\cdot\vec{\sigma}\right] + \left[u_0 I + \vec{u}\cdot\vec{\sigma}\right]\left[(\vec{E}-i\vec{B})\cdot\vec{\sigma}\right]\right\}$$

$\Rightarrow$

$$\frac{du}{d\tau} = \frac{e}{2m}\left\{\left[(\vec{E}+i\vec{B})\cdot\vec{\sigma}\right]u + u\left[(\vec{E}-i\vec{B})\cdot\vec{\sigma}\right]\right\} \quad (4)$$

La ecuación (4) es la misma que (2) en notación spinorial. Como $\vec{E}$ y $\vec{B}$ son independientes del espacio y del tiempo la ecuación (4) da

$$u(\tau) = \exp\left[\frac{e\tau}{2m}(\vec{E}+i\vec{B})\cdot\vec{\sigma}\right] u(0) \exp\left[\frac{e\tau}{2m}(\vec{E}-i\vec{B})\cdot\vec{\sigma}\right] \quad (1)$$

Si

$$\vec{m} \equiv (\vec{E}+i\vec{B}) \qquad a \equiv \frac{e}{2m}(\vec{m}\cdot\vec{m})^{1/2} \quad (2)$$

entonces, como

$$\exp\left[\frac{e\tau}{2m}\vec{m}\cdot\vec{\sigma}\right] = I\cosh(a\tau) + (\hat{m}\cdot\vec{\sigma})\sinh(a\tau) , \quad (3)$$

obtenemos para la cuadrivelocidad la expresión

$$u(\tau) = \left\{I\cosh(a\tau) + (\hat{m}\cdot\vec{\sigma})\sinh(a\tau)\right\} u(0) \left\{I\cosh(a^*\tau) + (\hat{m}^*\cdot\vec{\sigma})\sinh(a^*\tau)\right\} \quad (4)$$

Consideremos el caso en que $\vec{E}$ y $\vec{B}$ son perpendiculares y $\vec{E}^2-\vec{B}^2>0$. Entonces

$$\vec{E}\cdot\vec{B}=0 , \quad \vec{m}\cdot\vec{m} = \vec{m}^*\cdot\vec{m}^* = \vec{E}^2-\vec{B}^2>0 , \quad a=\frac{e}{2m}(\vec{E}^2-\vec{B}^2)>0$$

$$(\hat{m}\cdot\vec{\sigma})(\hat{m}^*\vec{\sigma}) = \frac{1}{\vec{m}\cdot\vec{m}}\left[\vec{m}\cdot\vec{m}^* + i(\vec{m}\times\vec{m}^*)\cdot\vec{\sigma}\right] = \frac{1}{\vec{m}^2}\left[(\vec{E}+\vec{B}^2) + 2(\vec{E}\times\vec{B})\cdot\vec{\sigma}\right. \quad (5)$$

Suponemos además $\vec{u}(0)=0 \Rightarrow u(0)=I$ Entonces

$$u(\tau) = \left[\cosh^2(a\tau) + \sinh^2(a\tau)\frac{\vec{E}^2+\vec{B}^2}{\vec{m}^2}\right] I +$$

$$+\left[2\sinh(a\tau)\cosh(a\tau)\frac{\vec{E}}{(\vec{m}^2)^{1/2}} + 2\sinh^2(a\tau)\frac{\vec{E}\times\vec{B}}{\vec{m}^2}\right]\cdot\vec{\sigma} \quad (6)$$

Teniendo en cuenta

$$\int d\tau \cosh^2(a\tau) = \frac{\sinh(2a\tau)}{4a} + \frac{\tau}{2} , \quad \int d\tau \sinh^2(a\tau) = \frac{\sinh(2a\tau)}{4a} - \frac{\tau}{2}$$

$$\int d\tau \sinh(a\tau)\cosh(a\tau) = \frac{\cosh(2a\tau)}{4a} - \frac{1}{4a} \quad (7)$$

y suponiendo $X(0)=0$ se obtiene

$$X(\tau) = \left[\tau + \left(\frac{\sinh(2a\tau)}{4a} - \frac{\tau}{2}\right)\left(1 + \frac{\vec{E}^2+\vec{B}^2}{\vec{m}^2}\right)\right] I +$$

$$+\left[\frac{\cosh(2a\tau)-1}{2a}\frac{\vec{E}}{(\vec{m}^2)^{1/2}} + \frac{\sinh(2a\tau)-2a\tau}{2a}\frac{\vec{E}\times\vec{B}}{\vec{m}^2}\right]\cdot\vec{\sigma} \quad (8)$$

En el caso límite $\vec{E}^2 = \vec{B}^2$ entonces $a \to 0$ y (3.6) se reduce a

$$u^0(\tau) = 1 + \frac{e^2 \vec{E}^2}{m^2} \frac{\tau^2}{2} \quad , \quad \vec{u}(\tau) = \tau \frac{e \vec{E}}{m} + \frac{e^2}{m^2} \frac{\tau^2}{2} (\vec{E} \times \vec{B}) \tag{1}$$

Notar que la velocidad crece más rápidamente en la dirección $\vec{E} \times \vec{B}$

2) Partículas con spin

Introduciremos el momento magnético intrínseco y la razón giromagnética en esta aproximación clásica. Esto es fuente de controversia si se exige una teoría consistente. Nosotros lo consideraremos como una situación límite útil de un tratamiento cuántico completo.

Recordemos que un circuito elemental de corriente equivale a un momento magnético $d\vec{\mu} = j \, d\vec{\Sigma}$, donde j es la corriente y $d\vec{\Sigma} = \vec{n} \, d\Sigma$ es un vector normal al plano del anillo de magnitud igual a su área. Cuando la corriente es generada por una carga en movimiento de giro no relativista $j = \frac{ev}{2\pi r}$ y

$$\vec{\mu} = \frac{e}{2m} \vec{L} \tag{2}$$

donde $\vec{L} = \vec{r} \times \vec{p}$ es el momento angular orbital. Por otra parte para un campo magnético externo homogéneo $\vec{B}$ la parte de interacción del Lagrangiano es

$$L_{int} = e \vec{A} . \vec{v} = \frac{e}{2} (\vec{B} \times \vec{r}) . \vec{v} = \vec{\mu} . \vec{B} \tag{3}$$

y de la ley de fuerza de Lorentz

$$\frac{d\vec{L}}{dt} = \frac{d}{dt} (\vec{r} \times \vec{p}) = \vec{r} \times \frac{d\vec{p}}{dt} = \vec{r} \times [e (\vec{v} \times \vec{B})] = \frac{e}{2m} \vec{L} \times \vec{B} \tag{4}$$

donde hemos tenido en cuenta que la partícula describe un movimiento de rotación en un plano perpendicular a $\vec{B}$. Vemos pues que tanto $\vec{L}$ como $\vec{\mu}$ precesan alrededor de $\vec{B}$ con la frecuencia clásica de Larmor

$$\omega = \frac{eB}{2m} \tag{5}$$

En 1926 Uhlenbeck y Goudsmit introdujeron la idea del spin del electrón y del momento magnético $\vec{\mu} = g \, (e/2m) \, \vec{S}$ donde $\vec{S}$ es el spin con $|\vec{S}| = \hbar/2$ y $g = 2$ es la llamada razon giromagnetica y con esto lograban dar una explicación adecuada del efecto Zeeman. Desafortunadamente este valor de $g = 2$ parecía dar para un electrón que se moviera en un potencial central un acoplamiento spin-orbita dos veces mayor que el necesario para explicar la estructura fina del hidrógeno. Supongamos que a semejanza de (3.4)

$$\frac{d\vec{S}}{dt} = \vec{\mu} \times \vec{B}_{rest} \tag{1}$$

vale en el sistema de referencia en el que el electrón se halla en reposo. Antes de continuar veamos las leyes de transformación de $\vec{E}$ y $\vec{B}$ bajo un boost caracterizado por una velocidad $\vec{v}$. Si introducimos $\vec{m} \equiv \vec{v}/|\vec{v}|$ entonces

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu}$$

$$\Lambda^{0}{}_{0} = \frac{1}{\sqrt{1-\vec{v}^{2}}} \quad , \quad \Lambda^{0}{}_{i} = \Lambda^{i}{}_{0} = \frac{|\vec{v}|\, m^{i}}{\sqrt{1-\vec{v}^{2}}} \tag{2}$$

$$\Lambda^{i}{}_{j} = \delta^{i}{}_{j} + \left[1 - \frac{1}{\sqrt{1-\vec{v}^{2}}} \right] m^{i} m_{j}$$

según se deduce de (I-9.3). La ley de transformación del tensor campo es

$$F^{\mu\nu\prime} = \Lambda^{\mu}{}_{\lambda}\, \Lambda^{\nu}{}_{\rho}\, F^{\lambda\rho} \tag{3}$$

de donde

$$E'^{i} = [\Lambda^{i}{}_{j}\Lambda^{0}{}_{0} - \Lambda^{i}{}_{0}\Lambda^{0}{}_{j}]\, E^{j} - \Lambda^{i}{}_{k}\Lambda^{0}{}_{e}\, \epsilon_{kes}\, B^{s}$$

$$B'^{i} = \frac{1}{2}\epsilon_{ijk}[\Lambda^{j}{}_{e}\Lambda^{k}{}_{0} - \Lambda^{j}{}_{0}\Lambda^{k}{}_{e}]\, E^{e} + \frac{1}{2}\epsilon_{ijk}\,\epsilon_{ems}\,\Lambda^{j}{}_{e}\Lambda^{k}{}_{m}\, B^{s} \tag{4}$$

de lo cual se deduce

$$\vec{E}' = (\vec{E}.\vec{m})\,\vec{m} + \frac{\vec{m}\times(\vec{E}\times\vec{m}) + \vec{v}\times\vec{B}}{\sqrt{1-\vec{v}^{2}}} \tag{5}$$

$$\vec{m} = \frac{\vec{v}}{|\vec{v}|}$$

$$\vec{B}' = (\vec{B}.\vec{m})\,\vec{m} + \frac{\vec{m}\times(\vec{B}\times\vec{m}) - \vec{v}\times\vec{E}}{\sqrt{1-\vec{v}^{2}}}$$

Volvamos a nuestro problema. Si $\vec{E}$ y $\vec{B}$ son los campos en el sistema laboratorio en el que el electrón tiene velocidad $\vec{v}$, entonces de (5) se deduce que

$$\vec{B}_{rest} = \vec{B} - \vec{v} \times \vec{E} + O(\vec{v}^2) \qquad (1)$$

Entonces, en la aproximación de pequeñas velocidades, la energía de interacción magnética debida al spin sería

$$U' = -\vec{\mu}\,(\vec{B} - \vec{v} \times \vec{E}) \qquad (2)$$

Si el campo eléctrico que actúa sobre el electrón se deriva de un potencial central medio

$$e\vec{E} = -\frac{\vec{r}}{r}\frac{dV(r)}{dr} \qquad (3)$$

entonces

$$U' = -\vec{\mu}\cdot\vec{B} - \vec{\mu}\cdot(\vec{v} \times \vec{r})\,\frac{1}{er}\,\frac{dV(r)}{dr} \quad \Rightarrow$$

$$U' = -\frac{ge}{2m}\,\vec{S}\cdot\vec{B} + \frac{g}{2m^2}\,\vec{S}\cdot\vec{L}\,\frac{1}{r}\,\frac{dV(r)}{dr} \qquad (4)$$

El gazapo, como demostró primero Thomas, proviene de un tratamiento incorrecto de las transformaciones de Lorentz debido al movimiento rotacional. En otras palabras, un boost puro desde el sistema inercial del laboratorio conduce a un sistema que gira con una velocidad angular ω_T y por tanto la energía correcta es

$$U = U' - \vec{S}\cdot\vec{\omega}_T \qquad (5)$$

Este es un efecto típicamente relativista que puede obtenerse considerando el producto de dos boosts de velocidades $-\vec{v}$ y $+\vec{v} + d\vec{v}$ donde $d\vec{v} = \dot{\vec{v}}\,dt$. Se tiene que el efecto neto es

$$x'^{\mu} = B^{\mu}{}_{\nu}(\vec{v} + d\vec{v})\,B^{\nu}{}_{\lambda}(-\vec{v})\,x^{\lambda} \equiv \hat{B}^{\mu}{}_{\nu}\,\tilde{B}^{\nu}{}_{\lambda}\,x^{\lambda} \equiv \Lambda^{\mu}{}_{\lambda}\,x^{\lambda} \qquad (6)$$

Recordemos además que (6) puede siempre escribirse como el producto de un un boost de velocidad $\Delta\vec{v}$ y una rotación $\Delta\vec{\omega}$

$$x'^{\mu} = R^{\mu}{}_{\nu}(\Delta\vec{\omega})\,B^{\nu}{}_{\lambda}(\Delta\vec{v})\,x^{\lambda} \equiv R^{\mu}{}_{\nu}\,B^{\nu}{}_{\lambda}\,x^{\lambda} \qquad (7)$$

Teniendo en cuenta (I-9.3) y (I-7.5)

$$x'^0 = [\hat{B}^0{}_0 \tilde{B}^0{}_0 + \hat{B}^0{}_j \tilde{B}^j{}_0] x^0 + [\hat{B}^0{}_0 \tilde{B}^0{}_i + \hat{B}^0{}_j \tilde{B}^j{}_i] x^i$$

$$= B^0{}_0 x^0 + B^0{}_i x^i \qquad (1)$$

$$x'^i = [\hat{B}^i{}_0 \tilde{B}^0{}_0 + \hat{B}^i{}_j \tilde{B}^j{}_0] x^0 + [\hat{B}^i{}_0 \tilde{B}^0{}_k + \hat{B}^i{}_j \tilde{B}^j{}_k] x^k =$$

$$= R^i{}_j B^j{}_0 x^0 + R^i{}_j B^j{}_k x^k \qquad (2)$$

Entonces

$$\Delta v^i = \frac{B^0{}_i}{B^0{}_0}$$

de donde

$$\Delta \vec{v} = \frac{1}{\sqrt{1-\vec{v}^2}} \left\{ \delta\vec{v} + \left[\frac{1}{\sqrt{1-\vec{v}^2}} - 1 \right] \frac{\vec{v}\cdot\delta\vec{v}}{\vec{v}^2} \vec{v} \right\} \qquad (3)$$

Ya determinado el boost se tiene que la rotación es $R = \Lambda B^{-1} = \hat{B}\tilde{B}B^{-1}$

$$R^i{}_j = \hat{B}^i{}_0 \tilde{B}^0{}_0 B(-\Delta\vec{v})^0{}_j + \hat{B}^i{}_0 \tilde{B}^0{}_k B(-\Delta\vec{v})^k{}_j$$

$$+ \hat{B}^i{}_k \tilde{B}^k{}_0 B(-\Delta\vec{v})^0{}_j + \hat{B}^i{}_\ell \tilde{B}^\ell{}_k B(-\Delta\vec{v})^k{}_j = [I - i\Delta\vec{\omega}\cdot\vec{J}]^i{}_j$$

de donde

$$\Delta\vec{\omega} = \vec{\omega}_T \delta t = \left(\frac{1}{\sqrt{1-v^2}} - 1 \right) \frac{\vec{v}\times\delta\vec{v}}{\vec{v}^2} \simeq \frac{1}{2} \vec{v}\times\dot{\vec{v}}\, \delta t \qquad (5)$$

En nuestro caso

$$\dot{\vec{v}} \simeq -\frac{\vec{r}}{mr}\frac{dV(r)}{dr} \quad , \quad \vec{\omega}_T = \frac{\vec{L}}{2m^2}\frac{1}{r}\frac{dV(r)}{dr} \qquad (5)$$

y por tanto la energía de interacción es

$$U = -\frac{ge}{2m}\vec{S}\cdot\vec{B} + \frac{g-1}{2m^2}\vec{S}\cdot\vec{L}\frac{1}{r}\frac{dV(r)}{dr} \qquad (6)$$

Si $g = 2$ este efecto reduce por un factor dos el acoplamiento spin-órbita de acuerdo con la situación experimental

Vamos a continuación a exponer el método de Bargmann, Michel y Telegdi para la descripción relativística del movimiento clásico del spin en campos electromagnéticos externos que varían lentamente. Representaremos los grados de libertad de spin por un vector $\vec{S}$ en el sistema en el que la partícula está en reposo. En notación covariante viene descrito por un cuadrivector S^μ que es ortogonal a la cuadrivelocidad u^μ. En el sistema en que la partícula está en reposo la ecuación de movimiento de $\vec{S}$ es

$$\frac{d\vec{S}}{dt} = g\,\frac{e}{2m}\,\vec{S}\times\vec{B} \tag{1}$$

que debemos generalizar a cualquier sistema de referencia. Para mantener la condición $S.u=0$ se debe cumplir $\dot{S}.u + S.\dot{u}=0$. En el sistema de referencia en reposo instantáneo $u^\mu \equiv (1,0)$ y por tanto

$$\dot{S}^0 = \vec{S}.\dot{\vec{u}} \quad , \quad \dot{S}^\mu = \left[\,\vec{S}.\dot{\vec{u}}\,,\ \frac{ge}{2m}\,\vec{S}\times\vec{B}\,\right] \tag{2}$$

Observemos que $F^\mu{}_\nu S^\nu$ se reduce en el sistema de referencia instantáneo a

$$\begin{aligned} F^0{}_i S^i &= F^{i0} S^i = \vec{E}.\vec{S} \\ F^i{}_j S^j &= +\epsilon_{ijk} B^k S^j = (\vec{S}\times\vec{B})^i \end{aligned} \tag{3}$$

Entonces la generalización de (1) es

$$\dot{S}^\mu = \frac{g}{2}\,\frac{e}{m}\,F^\mu{}_\nu S^\nu + \frac{e}{m}\left(\frac{g}{2}-1\right) u^\mu \left[S^\alpha F_{\alpha\beta} u^\beta\right] \tag{4}$$

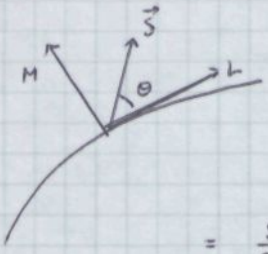
Notar que si $g=2$ entonces comparando (4) con (2.2) se ve que S y u se mueven rígidamente. Esto no sucede cuando $g\neq 2$ y esto nos proporciona un método para medir la anomalía magnética $(g/2-1)$.

Definamos en el plano laboratorio con el eje de tiempos $n^\mu \equiv (1,0)$ un cuadrivector L^μ con $\vec{L}$ paralelo a $\vec{u}$ y $L.u=0$ y tal que $L^2=-1$. Entonces si introducimos $u^0 \equiv \cosh\varphi$ se tiene que $\vec{u}^2 = \sinh^2\varphi$ y

$$L^\mu = \frac{1}{\sinh\varphi}\left[\cosh\varphi\, u^\mu - n^\mu\right] \tag{5}$$

Además, sea M^μ un cuadrivector unitario en el plano (S^μ, L^μ) ortogonal a

u^μ y L^μ. Escribamos $S^\mu \equiv L^\mu \cos\theta + M^\mu \sin\theta$ y determinemos a partir de (8.4) la ecuación del movimiento para θ



$$\dot{S}^\mu = \dot{\theta}\left[-L^\mu \sin\theta + M^\mu \cos\theta\right] + \dot{L}^\mu \cos\theta + \dot{M}^\mu \sin\theta$$

$$= \frac{g}{2}\frac{e}{m} F^\mu{}_\nu \left[L^\mu \cos\theta + M^\mu \sin\theta\right] +$$

$$+ \frac{e}{m}\left(\frac{g}{2} - 1\right) u^\mu \left[L^\alpha \cos\theta + M^\alpha \sin\theta\right] F_{\alpha\beta} u^\beta \quad (1)$$

Multipliquemos esta ecuación escalarmente por M_μ y tengamos en cuenta que $M.L = u.L = u.M = 0$ y $M_\mu M^\mu = -1 \Rightarrow M.\dot{M} = 0$

$$-\dot{\theta}\cos\theta + M_\mu \dot{L}^\mu \cos\theta = \frac{g}{2}\frac{e}{m} F^{\mu\nu} M_\mu L_\nu \cos\theta$$

$$\Rightarrow \quad -\dot{\theta} + M\cdot\dot{L} = \frac{g}{2}\frac{e}{m} M_\mu F^{\mu\nu} L_\nu \quad (2)$$

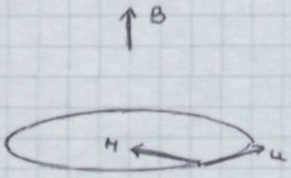
Ahora bien $M_\mu \dot{L}^\mu = \frac{\cosh\varphi}{\sinh\varphi} M_\mu \dot{u}^\mu = \coth\varphi\, M_\mu \dot{u}^\mu = \frac{e}{m}\coth\varphi\, M_\mu F^{\mu\nu} u_\nu$ y por tanto

$$\dot{\theta} = \frac{e}{m}\left\{\coth\varphi\, M_\mu F^{\mu\nu} u_\nu - \frac{g}{2} M_\mu F^{\mu\nu} L_\nu\right\} \quad (3)$$

o explícitamente

$$\dot{\theta} = \frac{e}{m}\left\{\left[\frac{g}{2}\sinh^2\varphi - \cosh^2\varphi\right]\frac{\vec{E}.\vec{M}}{\sinh\varphi} + \left[\frac{g}{2} - 1\right]\coth\varphi\, (\vec{M}\times\vec{B}).\vec{u}\right\} \quad (4)$$

En el caso de un campo magnético puro con la partícula en movimiento circular a la frecuencia de Larmor (propia) $\omega = eB/m$, entonces de (3.4)



$\vec{E} = 0 \quad \vec{m} = i\vec{B} \quad \vec{m}.\vec{m} = -B^2 \quad a = i\frac{\omega}{2} \quad \omega = \frac{eB}{m}$

$\cosh a\tau = \cos(\omega\tau/2) \qquad \sinh a\tau = i\sin(\omega\tau/2)$

$u^\mu(0) = (\cosh\varphi, 0, \sinh\varphi, 0) \qquad \hat{m} = (0, 0, 1)$

$u(\tau) = \cosh\varphi\, I + \sin\omega\tau \sinh\varphi\, \sigma_1 + \cos\omega\tau \sinh\varphi\, \sigma_2$

$u^\mu(\tau) = (\cosh\varphi, \sin\omega\tau\sinh\varphi, \cos\omega\tau\sinh\varphi, 0)$

$L^\mu(\tau) = (\sinh\varphi, \sin\omega\tau\cosh\varphi, \cos\omega\tau\cosh\varphi, 0)$

$M^\mu(\tau) = (0, -\cos\omega\tau, +\sin\omega\tau, 0)$

De donde $(\vec{M} \times \vec{B}).\vec{u} = B \sinh\varphi$, con lo que

$$\dot{\theta} = \left(\frac{g}{2} - 1\right) \omega \cosh\varphi$$

$$\Rightarrow \quad \theta - \theta_0 = \left(\frac{g}{2} - 1\right) \omega \tau \cosh\varphi$$

Para un periodo $\Delta\tau = 2\pi/\omega$

$$\Delta\theta\Big|_{\text{period}} = \left(\frac{g}{2} - 1\right) 2\pi \cosh\varphi = \left(\frac{g}{2} - 1\right) 2\pi \frac{\mathcal{E}}{mc^2} \qquad (1)$$

que es la expresión deseada.

VIII - B - THE ENERGY-MOMENTUM TENSOR

Let us consider different possibilities for the Lagrangian density

$$L_I(x) = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) = -\frac{1}{2}[\partial_\mu A_\nu(x)][\partial^\mu A^\nu(x)] + \frac{1}{2}[\partial_\mu A_\nu(x)][\partial^\nu A^\mu(x)]$$

$$L_{II}(x) = -\frac{1}{2}[\partial_\mu A_\nu(x)][\partial^\mu A^\nu(x)] \qquad (1)$$

$$L_{III}(x) = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) - \frac{1}{2a}[\partial_\lambda A^\lambda(x)]^2$$

Hence

$$\frac{\partial L_I(x)}{\partial(\partial_\mu A_\nu(x))} = -\partial^\mu A^\nu(x) + \partial^\nu A^\mu(x) \quad , \quad \frac{\partial L_I(x)}{\partial A_\nu(x)} = 0$$

$$\frac{\partial L_{II}(x)}{\partial(\partial_\mu A_\nu(x))} = -\partial^\mu A^\nu(x) \quad , \quad \frac{\partial L_{II}(x)}{\partial A_\nu(x)} = 0 \qquad (2)$$

$$\frac{\partial L_{III}(x)}{\partial(\partial_\mu A_\nu(x))} = -\partial^\mu A^\nu(x) + \partial^\nu A^\mu(x) - \frac{1}{a} g^{\mu\nu}[\partial^\lambda A_\lambda(x)] \quad , \quad \frac{\partial L_{III}(x)}{\partial A_\nu(x)} = 0$$

The equations of motion are therefore

$$\text{I)} \quad \partial_\mu \partial^\mu A^\nu(x) = \partial^\nu \partial_\mu A^\mu(x)$$

$$\text{II)} \quad \partial_\mu \partial^\mu A^\nu(x) = 0 \qquad (3)$$

$$\text{III)} \quad \partial_\mu \partial^\mu A^\nu(x) = \partial^\nu \partial_\mu A^\mu(x) - \frac{1}{a}\partial^\nu \partial_\mu A^\mu(x)$$

The canonical energy-momentum tensor is

$$\tilde{T}_I^{\mu\nu}(x) = -L_I(x) g^{\mu\nu} - [\partial^\mu A^\lambda(x)][\partial^\nu A_\lambda(x)] + [\partial^\lambda A^\mu(x)][\partial^\nu A_\lambda(x)]$$

$$\tilde{T}_{II}^{\mu\nu}(x) = -L_{II}(x) g^{\mu\nu} - [\partial^\mu A^\lambda(x)][\partial^\nu A_\lambda(x)] \qquad (4)$$

$$\tilde{T}_{III}^{\mu\nu}(x) = -L_{III}(x) g^{\mu\nu} - [\partial^\mu A^\lambda(x)][\partial^\nu A_\lambda(x)] + [\partial^\lambda A^\mu(x)][\partial^\nu A_\lambda(x)] - \frac{1}{a}[\partial_\lambda A^\lambda(x)][\partial^\nu A^\mu(x)]$$

All of them satisfy

$$\partial_\mu \tilde{T}^{\mu\nu}(x) = 0 \tag{1}$$

Let us remember that under infinitesimal Lorentz transformations

$$A_\lambda(x) \longrightarrow A'_\lambda(x') = A_\lambda(x) + \frac{1}{2}[S_{\mu\nu}]_\lambda{}^\rho\, \delta\omega^{\mu\nu} A_\rho(x) \; ; \quad [S_{\mu\nu}]_\lambda{}^\rho = g_{\mu\lambda} g^\rho{}_\nu - g_\mu{}^\rho g_{\nu\lambda} \tag{2}$$

Then the angular momentum density tensor is

$$\tilde{M}_i{}^\mu{}_{\nu\sigma}(x) = \tilde{T}_i{}^\mu{}_\nu(x)\, x_\sigma - \tilde{T}_i{}^\mu{}_\sigma(x)\, x_\nu + \tilde{M}_i^{(S)\mu}{}_{\nu\sigma}(x) \qquad i = \mathrm{I}, \mathrm{II}, \mathrm{III}$$

$$\tilde{M}_{\mathrm{I}}^{(S)\mu}{}_{\nu\sigma}(x) = [\partial^\mu A_\nu(x)] A_\sigma(x) - [\partial^\mu A_\sigma(x)] A_\nu(x) - [\partial_\nu A^\mu(x)] A_\sigma(x) + [\partial_\sigma A^\mu(x)] A_\nu(x)$$

$$\tilde{M}_{\mathrm{II}}^{(S)\mu}{}_{\nu\sigma}(x) = [\partial^\mu A_\nu(x)] A_\sigma(x) - [\partial^\mu A_\sigma(x)] A_\nu(x) \tag{3}$$

$$\tilde{M}_{\mathrm{III}}^{(S)\mu}{}_{\nu s}(x) = [\partial^\mu A_\nu(x)] A_\sigma(x) - [\partial^\mu A_\sigma(x)] A_\nu(x) - [\partial_\nu A^\mu(x)] A_\sigma(x) + [\partial_\sigma A^\mu(x)] A_\nu(x)$$

$$+ \frac{1}{a}[g^\mu{}_\nu A_\sigma(x) - g^\mu{}_\sigma A_\nu(x)][\partial_\lambda A^\lambda(x)]$$

Hence for all of them

$$\partial_\mu \tilde{M}^\mu{}_{\nu\sigma}(x) = 0 \quad , \quad \tilde{M}^\mu{}_{\nu\sigma}(x) = -\tilde{M}^\mu{}_{\sigma\nu}(x) \tag{4}$$

Furthermore

$$G_{\mathrm{I}}^{\sigma\mu\nu}(x) = -[\partial^\sigma A^\mu(x)] A^\nu(x) + [\partial^\mu A^\sigma(x)] A^\nu(x)$$

$$G_{\mathrm{II}}^{\sigma\mu\nu}(x) = -\frac{1}{2}\Big\{ [\partial^\sigma A^\mu(x)] A^\nu(x) - [\partial^\sigma A^\nu(x)] A^\mu(x) + [\partial^\mu A^\nu(x)] A^\sigma(x) -$$

$$- [\partial^\mu A^\sigma(x)] A^\nu(x) + [\partial^\nu A^\mu(x)] A^\sigma(x) - [\partial^\nu A^\sigma(x)] A^\mu(x) \Big\} \tag{5}$$

$$G_{\mathrm{III}}^{\sigma\mu\nu}(x) = -[\partial^\sigma A^\mu(x)] A^\nu(x) + [\partial^\mu A^\sigma(x)] A^\nu(x) - \frac{1}{a}[\partial_\lambda A^\lambda(x)][g^{\mu\nu} A^\sigma(x) - g^{\sigma\nu} A^\mu(x)]$$

and for all of them

$$G^{\sigma\mu\nu}(x) = -G^{\mu\sigma\nu}(x) \tag{6}$$

There the Belinfante energy-momentum density tensor is

$$T_i^{\mu\nu}(x) \equiv \tilde{T}_i^{\mu\nu}(x) + \partial_\sigma G^{\sigma\mu\nu}_i(x) \qquad i = \mathrm{I}, \mathrm{II}, \mathrm{III} \tag{1}$$

and since (using the eqs. of motion)

$$\partial_\sigma G^{\sigma\mu\nu}_{\mathrm{I}}(x) = -[\partial^\sigma A^\mu(x)][\partial_\sigma A^\nu(x)] + [\partial^\mu A^\sigma(x)][\partial_\sigma A^\nu(x)]$$

$$\partial_\sigma G^{\sigma\mu\nu}_{\mathrm{II}}(x) = -\frac{1}{2}\{ [\partial_\sigma\partial^\mu A^\nu(x)]A^\sigma(x) + [\partial_\sigma\partial^\nu A^\mu(x)]A^\sigma(x) + [\partial^\mu A^\nu(x)][\partial_\sigma A^\sigma(x)] +$$

$$+ [\partial^\nu A^\mu(x)][\partial_\sigma A^\sigma(x)] - [\partial^\mu\partial^\sigma A_\sigma(x)]A^\nu(x) - [\partial^\nu\partial^\sigma A_\sigma(x)]A^\mu(x) - [\partial^\mu A^\sigma(x)][\partial_\sigma A^\nu(x)] - [\partial^\nu A^\sigma(x)][\partial_\sigma A^\mu(x)]\}$$

$$\partial_\sigma G^{\sigma\mu\nu}_{\mathrm{III}}(x) = -[\partial_\sigma A^\mu(x)][\partial^\sigma A^\nu(x)] + [\partial^\mu A^\sigma(x)][\partial_\sigma A^\nu(x)] - \frac{1}{a} g^{\mu\nu}[\partial_\sigma\partial_\lambda A^\lambda(x)]A^\sigma(x)$$

$$-\frac{1}{a} g^{\mu\nu}[\partial_\lambda A^\lambda(x)]^2 + \frac{1}{a}[\partial^\mu\partial_\sigma A^\sigma(x)]A^\nu(x) + \frac{1}{a}[\partial^\nu\partial_\sigma A^\sigma(x)]A^\mu(x) + \frac{1}{a}[\partial_\lambda A^\lambda(x)][\partial^\nu A^\mu(x)])$$

we obtain

$$T_{\mathrm{I}}^{\mu\nu}(x) = -g^{\mu\nu} L_{\mathrm{I}}(x) - F^{\mu\lambda}(x) F^\nu{}_\lambda(x)$$

$$T_{\mathrm{II}}^{\mu\nu}(x) = -g^{\mu\nu} L_{\mathrm{II}}(x) - [\partial^\mu A^\lambda(x)][\partial^\nu A_\lambda(x)] - \frac{1}{2}[\partial^\mu A^\nu(x)][\partial^\lambda A_\lambda(x)]$$

$$-\frac{1}{2}[\partial^\nu A^\mu(x)][\partial^\lambda A_\lambda(x)] + \frac{1}{2}[\partial^\mu A^\lambda(x)][\partial_\lambda A^\nu(x)] + \frac{1}{2}[\partial^\nu A^\lambda(x)][\partial_\lambda A^\mu(x)]$$

$$-\frac{1}{2}[\partial_\lambda\partial^\mu A^\nu(x)]A^\lambda(x) - \frac{1}{2}[\partial_\lambda\partial^\nu A^\mu(x)]A^\lambda(x) + \frac{1}{2}[\partial^\mu\partial^\lambda A_\lambda(x)]A^\nu(x) \tag{2}$$

$$+\frac{1}{2}[\partial^\nu\partial^\lambda A_\lambda(x)]A^\mu(x)$$

$$T_{\mathrm{III}}^{\mu\nu}(x) = -g^{\mu\nu} L_{\mathrm{III}}(x) - F^{\mu\lambda}(x) F^\nu{}_\lambda(x) - \frac{1}{a} g^{\mu\nu}\partial_\sigma[(\partial_\lambda A^\lambda(x))A^\sigma(x)]$$

$$+\frac{1}{a}[(\partial^\mu\partial^\sigma A_\sigma(x))A^\nu(x) + (\partial^\nu\partial^\sigma A_\sigma(x))A^\mu(x)]$$

and all of them satisfy

$$\partial_\mu T^{\mu\nu}(x) = 0 \qquad\qquad T^{\mu\nu}(x) = T^{\nu\mu}(x)$$

The Belinfante angular-momentum density tensor is

$$M^{\mu}{}_{\nu\sigma}(x) = T^{\mu}{}_{\nu}(x)\, x_{\sigma} - T^{\mu}{}_{\sigma}(x)\, x_{\nu} \tag{1}$$

Notice

$$T_{I}{}^{\mu}{}_{\mu}(x) = 0$$

$$T_{II}{}^{\mu}{}_{\mu}(x) = -[\partial^{\mu} A_{\mu}(x)]^2 + [\partial^{\mu} A^{\nu}(x)][\partial_{\mu} A_{\nu}(x)] + [\partial^{\mu} A^{\nu}(x)][\partial_{\nu} A_{\mu}(x)] \tag{2}$$

$$T_{III}{}^{\mu}{}_{\mu}(x) = -\frac{2}{a}\, \partial_{\mu} \left\{ [\partial_{\nu} A^{\nu}(x)]\, A^{\mu}(x) \right\}$$

Let us now try to construct the improved energy momentum tensor.

I) $V_{I}{}^{\mu}(x) = 0$

Hence the Belinfante coincides with the improved

II) $$V_{II}{}^{\mu}(x) = -[\partial^{\mu} A^{\lambda}(x)]\, A_{\lambda}(x) - [\partial^{\lambda} A^{\mu}(x)]\, A_{\lambda}(x) + [\partial^{\lambda} A_{\lambda}(x)]\, A^{\mu}(x) \tag{3}$$

III) $$V_{III}{}^{\mu}(x) = \frac{2}{a}\, [\partial^{\nu} A_{\nu}(x)]\, A^{\mu}(x) \tag{5}$$

IX.- LAS INTERACCIONES

En el formalismo Lagrangiano la interacción entre campos se expresa añadiendo a la densidad Lagrangiana libre $L_0(x)$, que es la suma de las densidades Lagrangianas libres de cada uno de los campos que se toman en consideración, la llamada densidad Lagrangiana de interacción $L_I(x)$ que depende de los distintos campos en consideración, de forma que

$$L(x) = L_0(x) + L_I(x) \tag{1}$$

Consideraremos interacciones locales, es decir $L_I(x)$ está formada por una suma algebraica finita de términos cada uno de los cuales contiene los distintos campos o/y derivadas de orden finito de los mismos en un mismo punto del espacio tiempo. Como ejemplo simple podemos considerar un campo escalar real en autointeracción

$$L(x) = \frac{1}{2} :\partial_\mu \phi(x) \partial^\mu \phi(x): - \frac{1}{2} m^2 :\phi^2(x): - \frac{1}{4!} g :\phi^4(x): \tag{2}$$

donde g es una constante adimensional que recibe el nombre de constante de acoplamiento y que mide, de alguna forma, la intensidad de la interacción.

En un caso general al deducir de (1) las ecuaciones del campo se encuentran ecuaciones para los distintos campos que ya no son separables, sino que en ellas los campos quedan acoplados por la interacción. Esto hace que las ecuaciones del campo se compliquen extraordinariamente y que en general sepamos muy poco de las soluciones exactas. Así por ejemplo la ecuación de $\phi(x)$ correspondiente a (2) es

$$[\partial_\mu \partial^\mu + m^2]\, \phi(x) = - \frac{1}{3!} g :\phi^3(x): \tag{3}$$

cuya solución exacta se desconoce

Las ecuaciones de los campos en interacción son, en general, demasiado complicadas para admitir soluciones exactas y se debe recurrir a métodos aproximados para obtener información útil. Una forma conveniente de hacer esto es mediante la imagen de interacción, en la que los operadores campo satisfacen las ecuaciones de los campos libres, y la interacción afecta sólo la evolución temporal de los estados. Veamos como llevar a cabo esto. Para no complicar la notación consideraremos un campo escalar en autointeracción, pero la demostración es completamente general.

Los campos juegan el papel de variables dinámicas y hasta aquí hemos considerado la imagen de Heisenberg. En esta imagen y de acuerdo con las leyes de transformación de los campos

$$i\,\dot{\phi}(x) = [\phi(x), H] \quad , \quad i\,\dot{\Pi}(x) = [\Pi(x), H] \tag{1}$$

donde el Hamiltoniano total H es una constante del movimiento si el sistema es invariante Poincaré. Entonces

$$\phi(t,\vec{x}) = e^{iHt}\,\phi(0,\vec{x})\,e^{-iHt} \tag{2}$$

$$\Pi(t,\vec{x}) = e^{iHt}\,\Pi(0,\vec{x})\,e^{-iHt}$$

Por otra parte los vectores estado $|\Psi\rangle$ son independientes del tiempo.

Podríamos también describir la evolución temporal de nuestro sistema en la imagen de Schrödinger

$$\phi_S(x) \equiv e^{-iHt}\,\phi(x)\,e^{+iHt} \quad , \quad \Pi_S(x) \equiv e^{-iHt}\,\Pi(x)\,e^{+iHt} \tag{3}$$

$$|\Psi_S(t)\rangle = e^{-iHt}\,|\Psi\rangle$$

Es inmediato probar entonces que

$$\phi_S(t,\vec{x}) = \phi_S(0,\vec{x}) = \phi(0,\vec{x})$$

$$\Pi_S(t,\vec{x}) = \Pi_S(0,\vec{x}) = \Pi(0,\vec{x}) \tag{4}$$

$$i\frac{d}{dt}|\Psi_S(t)\rangle = H_S(t)\,|\Psi_S(t)\rangle \quad , \quad H_S(t) = H_S(0) = H$$

Como es bien sabido es útil para desarrollar la teoría de perturbaciones introducir la imagen de interacción. Teniendo presente que $H = H_0 + H_I$, se definen los operadores campo como

$$\phi_{int}(x) \equiv e^{iH_{0s}t}\,\phi_S(x)\,e^{-iH_{0s}t} \quad , \quad \Pi_{int}(x) \equiv e^{iH_{0s}t}\,\Pi_S(x)\,e^{-iH_{0s}t} \tag{5}$$

$$|\Psi_{int}(t)\rangle \equiv e^{iH_{0s}t}\,|\Psi_S(t)\rangle$$

donde $H_{0s} = H_0(t=0)$. Se obtiene inmediatamente que

$$i \frac{d}{dt} |\Psi_{int}(t)\rangle = H_{I,int}(t) |\Psi_{int}(t)\rangle \tag{1}$$

$$H_{I,int}(t) \equiv e^{iH_{0s}t} H_{I,s}(t) e^{-iH_{0s}t}$$

y

$$i\dot{\phi}_{int}(x) = [\phi_{int}(x), H_{0,int}(t)] \quad , \quad i\dot{\Pi}_{int}(x) = [\Pi_{int}(x), H_{0,int}(t)] \tag{2}$$

$$H_{0,int}(t) = e^{iH_{0s}t} H_{0s} e^{-iH_{0s}t} = H_{0,s}$$

Puesto que $\phi_{int}(x)$ y $\Pi_{int}(x)$ están relacionados con $\phi(x)$ y $\Pi(x)$ mediante una transformación unitaria, deberán obedecer las mismas reglas de conmutación que $\phi(x)$ y $\Pi(x)$. Entonces, tanto las ecuaciones del movimiento como las reglas de conmutación a tiempos iguales para $\phi_{int}(x)$ y $\Pi_{int}(x)$ son las mismas que las del caso libre. Por tanto podemos expresar estos campos mediante sus desarrollos en ondas planas

$$\phi_{int}(x) = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2E(\vec{p})} [a(\vec{p}) e^{-ip.x} + a^+(\vec{p}) e^{+ip.x}] \tag{3}$$

válida para todo instante de t.

Veamos ahora la evolución temporal de los estados dada en (1)

$$|\Psi_{int}(t)\rangle = U_{int}(t, t_0) |\Psi_{int}(t_0)\rangle \tag{4}$$

donde $U_{int}(t, t_0)$ es el operador de evolución unitario solución de

$$U_{int}(t, t_0) = I - i \int_{t_0}^{t} dt' \, H_{I,int}(t') U_{int}(t', t_0) \tag{5}$$

cuya solución formal es

$$U_{int}(t, t_0) = T \left(e^{-i \int_{t_0}^{t} dt' \, H_{I\,int}(t')} \right) \tag{6}$$

donde T es el operador de ordenación temporal. Teniendo en cuenta que

$$H_{I,int}(t) = \int d^3x \, \mathcal{H}_{I,int}(t, \vec{x}) \tag{7}$$

se obtiene

$$U_{int}(t, t_0) = T \exp\left(-i\int_{t_0}^{t} d^4x \, \mathcal{H}_{I\,int}(x)\right) \qquad (1)$$

donde la integración se halla extendida a todo el espacio

Nosotros estamos interesados en calcular la amplitud de transición desde un estado inicial $t=-\infty$ a un estado final $t=+\infty$. Supondremos que la interacción se introduce adiabáticamente de forma que tanto el estado inicial $|i\rangle$ como el final $|f\rangle$ son propios del Hamiltoniano libre de forma que la amplitud deseada es $\langle f|S|i\rangle$ donde la matriz S es

$$S = T\left\{\exp\left[-i\int d^4x \, \mathcal{H}_I(x)\right]\right\} \qquad (2)$$

donde hemos eliminado el subíndice "int" pues siempre trabajaremos en esta imagen. Notar que para acoplamientos no derivativos $\mathcal{H}_I(x) = -L_I(x)$ y por tanto la matriz S viene dada por

$$S = T\left\{\exp\left(i\int d^4x \, L_I(x)\right)\right\} \qquad (3)$$

Recordar

$$T\left(L_I(x_1) \dots L_I(x_m)\right) = L_I(x_{i_1}) \dots L_I(x_{i_m})$$
$$x^0_{i_1} \geq x^0_{i_2} \geq \dots \geq x^0_{i_m} \qquad (4)$$

la ambigüedad que pueda surgir por el hecho de que dos variables temporales sean iguales no importa pues

$$[L_I(x), L_I(y)] = 0 \qquad (x-y)^2 < 0 \qquad (5)$$

es consecuencia de las relaciones de conmutación de los campos y del hecho de que el número de campos fermiónicos en $L_I(x)$ es siempre par.

Se puede probar que es precisamente (3) la expresión, manifiestamente covariante, de la matriz S para todo tipo de acoplamientos. y ésta será la que usaremos en el futuro. Notar que al ser $L_I(x)$ autoadjunta la matriz S es unitaria

Notemos finalmente que si se usan las típicas notaciones de estados entrantes (incoming) y salientes (outgoing) se tiene

$$\lim_{x^0 \to -\infty} \phi(x) = \phi_{in}(x) \quad , \quad \lim_{x^0 \to +\infty} \phi(x) = \phi_{out}(x)$$

$$\phi_{out}(x) = S^{-1} \phi_{in}(x) S \tag{1}$$

$$|in\rangle = S\,|out\rangle \quad , \quad |out\rangle = S^{-1}|in\rangle = S^{+}|in\rangle$$

IX- A- TECNICAS DE REDUCCIÓN

De acuerdo con el Cap III un campo escalar o pseudoescalar neutro de masa m satisface, si es libre, la ecuación de Klein-Gordon

$$(\Box + m^2)\ \phi(x) = 0 \tag{1}$$

En lo que sigue manejaremos frecuentemente $K \equiv \Box + m^2$. Sabemos que el campo $\phi(x)$ (representación de Heisenberg) satisface

$$[\ \phi(x)\ ,\ \phi(y)\ ] = i\ \Delta(x-y;\ m^2) \tag{2}$$

Introduzcamos como en (III-5.2) las funciones

$$f_{\vec{p}}(x) \equiv e^{-i\,p.x} \qquad p \in \Omega_+(m) \tag{3}$$

y recordemos los valores de los productos escalares (III.-5.3)

$$(f_{\vec{p}'}\ ,\ f_{\vec{p}}) = -(f^*_{\vec{p}'}\ ,\ f^*_{\vec{p}}) = (2\pi)^3\ 2E(\vec{p})\ \delta(\vec{p}-\vec{p}')$$
$$(f_{\vec{p}'},\ f^*_{\vec{p}}) = (f^*_{\vec{p}'}\ ,\ f_{\vec{p}}) = 0 \tag{4}$$

y además (III.-5.4)

$$\phi(x) = \frac{1}{(2\pi)^3}\int \frac{d^3p}{2E(\vec{p})}\ [\ f_{\vec{p}}(x)\ a(\vec{p}) + f^*_{\vec{p}}(x)\ a^+(\vec{p})\ ]$$

$$a(\vec{p}) = (\ f_{\vec{p}}\ ,\ \phi\ )\ , \qquad a^+(\vec{p}) = -(\ f^*_{\vec{p}},\ \phi) \tag{5}$$

Si $\phi(x)$ no es solución de la ecuación de Klein-Gordon la descomposición (5) es aún válida pero con $a(\vec{p})$ sustituido por $a(\vec{p};\ x^0)$ que depende explícitamente del tiempo

Los campos cuánticos entrantes y salientes satisfacen la ecuación

$$(\Box + m^2)\ \phi_{\substack{in\\out}}(x) = 0 \tag{6}$$

y por tanto

$$\phi_{\substack{in\\out}}(x) = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2E(\vec{p})} \left[f_{\vec{p}}(x)\, a_{\substack{in\\out}}(\vec{p}) + f^*_{\vec{p}}(x)\, a^+_{\substack{in\\out}}(\vec{p}) \right] \tag{1}$$

El campo interpolante $\phi(x)$ satisface la ecuación

$$(\Box + m^2)\, \phi(x) = J(x) \tag{2}$$

donde $J(x)$ es el llamado operador corriente, con las condiciones

$$\lim_{x^0 \to \mp\infty} \phi(x) = \phi_{\substack{in\\out}}(x) \tag{3}$$

Los operadores $\phi_{in}(x)$ y $\phi_{out}(x)$ satisfacen la misma ecuación y además postulamos para ellos idénticas reglas de conmutación por lo cual deben estar relacionados por una transformación unitaria S tal que

$$\phi_{out}(x) = S^{-1}\, \phi_{in}(x)\, S \tag{5}$$

de donde

$$a_{in}(\vec{p}) = S\, a_{out}(\vec{p})\, S^{-1} \quad , \quad a_{out}(\vec{p}) = S^{-1}\, a_{in}(\vec{p})\, S \tag{5}$$

Esta matriz S relaciona los estados entrantes y salientes

$$|\Omega\rangle_{in} = S\, |\Omega\rangle_{out} \quad , \quad |\Omega\rangle_{out} = S^+\, |\Omega_{in}\rangle \tag{6}$$

Finalmente de (3) se deduce que en sentido debil

$$\lim_{x^0 \to \mp\infty} a(\vec{p}\,;\, x^0) = a_{\substack{in\\out}}(\vec{p}) \tag{7}$$

Sea $|\vec{p}\rangle$ el estado correspondiente a una partícula del campo cuya fuente es $J(x)$, entonces

$$\langle 0 | J(x) | \vec{p}\rangle = (\Box + m^2)\, \langle 0 | \phi(x) | \vec{p}\rangle = \langle 0 | \phi(0) | \vec{p}\rangle\, (\Box + m^2)\, e^{-ip.x} = 0$$

es decir

$$\langle 0 | J(x) | \vec{p}\rangle = 0 \tag{8}$$

Por otra parte de (IV - 8.3) se deduce

$$\langle 0 | \phi(x) | \vec{p} \rangle = \langle 0 | \phi_{in}(x) | \vec{p} \rangle = \langle 0 | \phi_{out}(x) | \vec{p} \rangle = e^{-ip.x} \qquad (1)$$

donde la última igualdad es consecuencia inmediata de (1.5).

Después de esta introducción veamos en que consisten las fórmulas de reducción. Empecemos considerando el proceso

$$A + \phi(\vec{k}) \longrightarrow B \qquad (2)$$

donde $\phi(\vec{k})$ es una partícula correspondiente al campo $\phi(x)$. La amplitud invariante para este proceso es

$$\mathfrak{S}\,(A + \phi(\vec{k}) \to B) \equiv {}_{out}\langle B \,|\, A\ \phi(\vec{k}) \rangle_{in} \qquad (3)$$

es decir

$$\mathfrak{S}\,(A + \phi(\vec{k}) \to B) = {}_{out}\langle B \,|\, a^{+}_{in}(\vec{k}) \,|\, A \rangle_{in} \qquad (4)$$

Omitiremos en los pasos intermedios los índices in y out de los estados por comodidad

$$\mathfrak{S}\,(A + \phi(\vec{k}) \longrightarrow B) = \lim_{x^0 \to -\infty} \langle B | a^{+}(\vec{k}; x^0) | A \rangle = - \lim_{x^0 \to -\infty} \langle B | (f^{*}_{\vec{k}}, \phi) | A \rangle$$

$$= - i \lim_{x^0 \to -\infty} \int d^3x\ f_{\vec{k}}(x) \overleftrightarrow{\partial_0} \langle B | \phi(x) | A \rangle = i \lim_{x^0 \to -\infty} \int d^3x\ \langle B | \phi(x) | A \rangle \overleftrightarrow{\partial_0} f_{\vec{k}}(x)$$

Ahora bien

$$\lim_{x^0 \to -\infty} F(x) = \lim_{x^0 \to +\infty} F(x) - \int_{-\infty}^{+\infty} dx^0\ \frac{\partial F(x)}{\partial x^0} \qquad (5)$$

y por tanto

$$\mathfrak{S}\,(A + \phi(\vec{k}) \longrightarrow B) = i \lim_{x^0 \to +\infty} \int d^3x\ \langle B | \phi(x) | A \rangle \overleftrightarrow{\partial_0} f_{\vec{k}}(x) - i \int d^4x\ \partial^0 \left[\langle B | \phi(x) | A \rangle \overleftrightarrow{\partial_0} f_{\vec{k}}(x) \right]$$

El primer termino es nulo salvo para colisiones elásticas hacia adelante y lo podemos omitir obteniendo

$$\mathfrak{S}\,(A + \phi(\vec{k}) \to B) = - i \int d^4x\ \partial^0 \left[\langle B | \phi(x) | A \rangle \overleftrightarrow{\partial_0} f_{\vec{k}}(x) \right] \qquad (6)$$

Ahora bien para todo operador $A(x)$

$$\int d^4x \frac{\partial}{\partial x^0}\left[A(x)\overleftrightarrow{\partial}_0 f_{\vec{k}}(x)\right] = \int d^4x \frac{\partial}{\partial x^0}\left\{ A(x)\frac{\partial f_{\vec{k}}(x)}{\partial x^0} - \frac{\partial A(x)}{\partial x^0} f_{\vec{k}}(x)\right] =$$

$$= \int d^4x \left\{ A(x)\frac{\partial^2 f_{\vec{k}}(x)}{\partial x_0^2} - \frac{\partial^2 A(x)}{\partial x_0^2} f_{\vec{k}}(x)\right] =$$

$$= \int d^4x \left\{ A(x)\left[\vec{\nabla}^2 - m^2\right] f_{\vec{k}}(x) - \frac{\partial^2 A(x)}{\partial x_0^2} f_{\vec{k}}(x)\right\} = -\int d^4x\, f_{\vec{k}}(x)\,(\Box + m^2)\,A(x) \quad (1)$$

y por tanto

$$S\,(A + \phi(\vec{k}) \longrightarrow B) = i\int d^4x\, f_{\vec{k}}(x)\, \langle B|(\Box + m)\,\phi(x)\,|A\rangle \quad (2)$$

Un análisis análogo puede hacerse para el proceso $A \to B + \phi(\vec{k})$ y así podemos escribir

$$S\,(A + \phi(\vec{k}) \longrightarrow B) = i\int d^4x\, e^{-ik.x}\,(\Box + m^2)\;{}_{out}\langle B|\,\phi(x)\,|A\rangle_{in}$$

$$S\,(A \longrightarrow B + \phi(\vec{k})) = i\int d^4x\, e^{+ik.x}\,(\Box + m^2)\;{}_{out}\langle B|\,\phi(x)\,|A\rangle_{in} \quad (3)$$

que son ejemplos de las fórmulas de reducción deseadas.

Consideremos ahora

$$\phi(\vec{k}) + A \to B + \phi(\vec{k}') \quad (4)$$

La amplitud invariante de transición es

$$S \equiv {}_{out}\langle B\,\phi(\vec{k}')\,|\,A\,\phi(\vec{k})\rangle_{in} = S\,(\phi(\vec{k}) + A \to B + \phi(\vec{k}')) \quad (5)$$

y usando la primera de (3)

$$S = i\int d^4x\, e^{-ik.x}\,(\Box + m^2)\;{}_{out}\langle B\,\phi(\vec{k}')\,|\,\phi(x)\,|A\rangle_{in} \quad (6)$$

Ahora bien

$$_{out}\langle B\,\phi(\vec{k}')\,|\,\phi(x)\,|A\rangle_{in} = {}_{out}\langle B\,|\,a_{out}(\vec{k}')\,\phi(x)\,|\,A\rangle_{in} = \lim_{y^0\to+\infty}\ {}_{out}\langle B\,|\,a(k';y^0)\,\phi(x)\,|A\rangle_{in}$$

$$= i\int d^3y \lim_{y^0\to\infty} f^*_{\vec{k}'}(y)\,\overleftrightarrow{\partial}_{y_0}\ {}_{out}\langle B\,|\,\phi(y)\,\phi(x)\,|A\rangle_{in} \qquad (1)$$

Si definimos como es usual la ordenación temporal como

$$T\,(\phi(x_1)\dots\phi(x_n)) \equiv \sum_P \theta(x^0_{P1}-x^0_{P2})\dots\theta(x^0_{P(n-1)}-x^0_{Pn})\,\phi(x_{P1})\dots\phi(x_{Pn}) \qquad (2)$$

donde la suma se extiende a todas las permutaciones de las n coordenadas, obtenemos

$$_{out}\langle B\,\phi(\vec{k}')\,|\,\phi(x)\,|\,A\rangle_{in} = i\lim_{y^0\to\infty}\int d^3y\, f^*_{\vec{k}'}(y)\,\frac{\overleftrightarrow{\partial}}{\partial y^0}\ {}_{out}\langle B|\,T\,(\phi(y)\,\phi(x))\,|A\rangle_{in}$$

Usando la (3.5) y despreciando los términos que sólo contribuyen a las colisiones elásticas hacia adelante, así como (4.1)

$$_{out}\langle B\,\phi(\vec{k}')\,|\,\phi(x)\,|\,A\rangle_{in} = i\int d^4y\,\frac{\partial}{\partial y^0}\left\{ f^*_{\vec{k}'}(y)\,\frac{\overleftrightarrow{\partial}}{\partial y^0}\ {}_{out}\langle B\,|\,T\,(\phi(y)\,\phi(x))\,|\,A\rangle_{in}\right\} =$$

$$= i\int d^4y\, f^*_{\vec{k}'}(y)\,(\Box_y + m^2)\ {}_{out}\langle B\,|\,T\,(\phi(x)\,\phi(y))\,|A\rangle_{in} \qquad (3)$$

y así obtenemos

$$S\,(\phi(\vec{k}) + A \to B + \phi(\vec{k}')) = i^2\int d^4x\,d^4x'\,e^{-ik.x}\,e^{+ik'.x'}$$

$$.\,(\Box_x + m^2)\,(\Box_{x'} + m^2)\ {}_{out}\langle B\,|\,T\,(\phi(x')\,\phi(x))\,|\,A\rangle_{in} \qquad (4)$$

que es otra de las fórmulas de reducción deseadas

En ocasiones es útil una fórmula alternativa de (4). Introduzcamos

$$R\,(\phi(x)\,\phi(x_1)\dots\phi(x_n)) \equiv \sum_P \theta(x-x_{P_1})\dots\theta(x_{P_{n-1}}-x_{P_n})\,[[\dots[[\phi(x),\phi(x_{P_1})],\phi(x_{P_2})]\dots]\,\phi(x_{P_n})] \qquad (5)$$

y en particular

$$R\,(\phi(x)\,\phi(y)) = \theta(x-y)\,[\phi(x),\phi(y)]$$

Entonces de (5.1) obtenemos por ser $y^0 > x^0$

$${}_{out}\langle B\ \phi(\vec{k}')\ |\ \phi(x)\ |\ A\rangle_{in} = i \lim_{y^0 \to +\infty} \int d^3y\ f^{+}_{\vec{k}'}(y)\ \overleftrightarrow{\frac{\partial}{\partial y^0}}\ {}_{out}\langle B\ |\ R\ (\phi(y)\ \phi(x))\ |A\rangle_{in}$$

y por el mismo procedimiento de antes

$$\mathcal{J}\ (\phi(\vec{k}) + A \longrightarrow B + \phi(\vec{k}')) = i^2 \int d^4x\ d^4x'\ e^{-ik.x}\ e^{+ik'.x'}$$

$$(\Box_x + m^2)\ (\Box_{x'} + m^2)\ {}_{out}\langle B\ |\ R\ (\phi(x')\ \phi(x))\ |A\rangle_{in} \tag{1}$$

También por procedimientos analogos se obtiene

$$\mathcal{J}\ (A \longrightarrow B + \phi(\vec{k}) + \phi(\vec{k}')) = i^2 \int d^4x\ d^4x'\ e^{ik.x}\ e^{ik'.x'}$$

$$(\Box_x + m^2)\ (\Box_{x'} + m^2)\ {}_{out}\langle B\ |\ T\ (\phi(x')\ \phi(x))\ |A\rangle_{in} \tag{2}$$

y esto constituye las formulas de reducción mas usuales.

En el estudio de las interacciones debiles o electromagneticas es util

$${}_{out}\langle B\ \phi(\vec{k})\ |\ d(0)\ |\ A\rangle_{in} = i \int d^4x\ e^{ik.x}\ (\Box_x + m^2)\ {}_{out}\langle B\ |\ T\ (\phi(x)\ d(0))\ |A\rangle_{in} \tag{3}$$

donde $d(x)$ es un operador que causa la transición.

Formulas útiles relacionadas con las técnicas de reducción son por ejemplo

$$\partial_\mu\ T\ (A(x)\ B(x')) = T\ (\ [\partial_\mu A(x)]\ B(x')\ ) + \delta(x_0 - x_0')\ [\ A(x)\ ,\ B(x')\ ]\ g_{\mu 0} \tag{4}$$

cuya demostración es inmediata

$$\partial_\mu\ T\ (A(x)\ B(x')) = \partial_\mu \left\{ \theta(x^0 - x'^0)\ A(x)\ B(x') + \theta(x'^0 - x^0)\ B(x')\ A(x) \right\} =$$

$$= T\ (\ [\partial_\mu\ A(x)]\ B(x')\ ) + g_{\mu 0}\ \delta(x^0 - x'^0)\ A(x)\ B(x') - g_{\mu 0}\ \delta(x^0 - x'^0)\ B(x')\ A(x)$$

que es la llamada igualdad de Ward - Takahashi y que se generaliza a

$$\partial_\mu T\,[\,O(x)\,A(x_1)\,B(x_2)\,C(x_3)\,\ldots] = T\,[\,\partial_\mu O(x)\,A(x_1)\,B(x_2)\,C(x_3)\,\ldots\,] +$$

$$+\, g_{\mu 0}\,\delta(x^0 - x_1^0)\,T\,[\,[\,O(x)\,,\,A(x_1)\,]\,B(x_2)\,C(x_3)\,\ldots\,] + g_{\mu 0}\,\delta(x^0 - x_2^0)\,T\,[\,A(x_1)\,[\,O,\,B(x_2)\,]\,C(x_3)\,\ldots\,]$$

$$+\, g_{\mu 0}\,\delta(x^0 - x_3^0)\,T\,[\,A(x_1)\,B(x_2)\,[\,O(x)\,,\,C(x_3)\,]\,\ldots\,] + \ldots \qquad (1)$$

que se demuestra de forma análoga a la anterior.

IX-B- LA INTERACCIÓN PION-NUCLEÓN

c) Nucleones

En el formalismo de isospin el protón y el neutrón se transforman de acuerdo a la representación $\mathcal{D}^{(1/2)}$ de $SU_T(2)$ y se dice que el nucleón tiene isospin $T = 1/2$. Se suele hacer la identificación

$$|p\rangle \equiv |1/2, 1/2\rangle \quad , \quad |n\rangle \equiv |1/2, -1/2\rangle \tag{1}$$

Está identificación es totalmente arbitraria y en los libros de Física Nuclear, se hace, con frecuencia, precisamente la opuesta. Surge ahora la pregunta ¿Cómo asignar los estados de antipartícula $|\bar{p}\rangle$ y $|\bar{n}\rangle$ con los estados base de $\mathcal{D}^{(1/2)}$? Se definió la conjugación de carga o conjugación partícula-antipartícula como una correspondencia biyectiva de un sistema físico a otro sistema físico en el que cada partícula es sustituida por la antipartícula correspondiente. Sea U_C el operador que implementa esta simetría. Sabemos que este operador es unitario y como la nueva aplicación del mismo nos reproduce el sistema original se puede escribir

$$U_c|\text{partícula}\rangle = |\text{antipartícula}\rangle \equiv |\overline{\text{partícula}}\rangle$$
$$U_C = U_C^+ \quad , \quad U_C^2 = I \tag{2}$$

En particular

$$|\bar{p}\rangle = U_c|p\rangle \quad , \quad |\bar{n}\rangle = U_c|n\rangle \tag{3}$$

Sea $\mathcal{R}$ una rotación de $SU_T(2)$, entonces bajo ella

$$|T\,T_3\rangle \longrightarrow R\,|T\,T_3\rangle = \sum_{T_3'} D^{(T)}(\mathcal{R})_{T_3' T_3}\,|T\,T_3'\rangle \tag{4}$$

donde R es el operador unitario que implementa $\mathcal{R}$. Si deseamos leyes simples para la transformación de los campos se debe cumplir

$$|\overline{T\,T_3}\rangle \longrightarrow R\,|\overline{T\,T_3}\rangle = \sum_{T_3'} D^{(T)}(\mathcal{R})^*_{T_3' T_3}\,|\overline{T\,T_3'}\rangle \tag{5}$$

es decir que partículas y antipartículas se transforman las unas como las duales de las otras.

Este es un hecho general bajo simetrías internas. En efecto consideremos un grupo de transformaciones G ($SU_T(2)$ en nuestro caso) y abreviemos por α ($T T_3$ en nuestro caso) el conjunto de números cuánticos necesarios para clasificar las partículas según las representaciones irreducibles de G. Consideremos un conjunto de campos $\phi_\alpha(x)$ que supondremos escalares para simplificar al máximo la notación. Entonces

$$\phi_\alpha(x) = \frac{1}{(2\pi)^3}\int \frac{d^3p}{2E(\vec{p})}\left[e^{-ip.x}\, a_{|\alpha;\vec{p}>} + e^{+ip.x}\, b^+_{|\bar{\alpha};\vec{p}>}\right] \qquad (1)$$

Entonces bajo un elemento de G se tiene

$$\phi_\alpha(x) \longrightarrow \phi'_\alpha(x) = \mathcal{U}\,\phi_\alpha(x)\,\mathcal{U}^{-1} =$$

$$= \frac{1}{(2\pi)^3}\int \frac{d^3p}{2E(\vec{p})}\left\{ e^{-ip.x}\, a_{U|\alpha;\vec{p}>} + e^{+ip.x}\, b^+_{U|\bar{\alpha};\vec{p}>}\right\} =$$

$$= \frac{1}{(2\pi)^3}\int \frac{d^3p}{2E(\vec{p})}\left\{ e^{-ip.x}\, a_{\sum_\beta <\beta|U|\alpha>\,|\beta;\vec{p}>} + e^{+ip.x}\, b^+_{\sum_\beta <\bar{\beta}|U|\bar{\alpha}>\,|\bar{\beta};\vec{p}>}\right\} =$$

$$= \frac{1}{(2\pi)^3}\int \frac{d^3p}{2E(\vec{p})}\sum_\beta \left\{ <\beta|U|\alpha>^* e^{-ip.x}\, a_{|\beta;\vec{p}>} + <\bar{\beta}|U|\bar{\alpha}>\, e^{ip.x}\, b^+_{|\bar{\beta};\vec{p}>}\right\}$$

y por tanto para obtener una ley simple de transformación de los campos debe cumplirse

$$<\bar{\beta}|U|\bar{\alpha}> = <\beta|U|\alpha>^* \qquad (2)$$

y entonces

$$\phi_\alpha(x) \longrightarrow \phi'_\alpha(x) = \mathcal{U}\,\phi_\alpha(x)\,\mathcal{U}^{-1} = \sum_\beta <\alpha|U^+|\beta>\,\phi_\beta(x) \qquad (3)$$

o equivalentemente

$$\phi(x) \longrightarrow \phi'(x) = \mathcal{U}\,\phi(x)\,\mathcal{U}^{-1} = U^+\,\phi(x) \qquad (5)$$

En nuestro caso (2.2) es

$$\langle T T_3' \mid R \mid T T_3 \rangle = \langle \overline{T T_3'} \mid R \mid \overline{T T_3} \rangle^* \quad (1)$$

Como $R = exp(-i\hat{m}\,\theta . \vec{T})$ se tiene

$$\langle T T_3' \mid T T_3 \rangle = \langle \overline{T T_3'} \mid \overline{T T_3} \rangle^*$$
$$\quad (2)$$
$$\langle T T_3' \mid \vec{T} \mid T T_3 \rangle = - \langle \overline{T T_3'} \mid \vec{T} \mid \overline{T T_3} \rangle^*$$

y usando (1.2)

$$\vec{T}^* = - U_c \vec{T} U_c \quad (3)$$

Esto es

$$T_1 = - U_c T_1 U_c \quad , \quad T_2 = + U_c T_2 U_c \quad , \quad T_3 = - U_c T_3 U_c \quad (4)$$

la última de las cuales nos indica que el T_3 de la partícula y de la antipartícula son siempre opuestos. Usando (2) con $\vec{T} = \vec{\tau}/2$ es fácil ver que, salvo fase no esencial, las asignaciones correctas de los antinucleones son

$$\mid \bar{n} \rangle = + \mid 1/2 \;\; 1/2 \rangle \quad , \quad \mid \bar{p} \rangle = - \mid 1/2 \;\; -1/2 \rangle \quad (5)$$

Además para nucleones la ecuación (2.4) es

$$\psi(x) \longrightarrow \psi'(x) = \mathcal{U} \psi(x) \mathcal{U}^{-1} = e^{+ i \theta \hat{m} . \frac{\vec{\tau}}{2}} \psi(x) \quad (6)$$

ii) <u>Piones</u>

Puesto que los piones tienen únicamente carga de tipo eléctrico el π^0 coincide con su antipartícula y el $\pi^+ - \pi^-$ forman un sistema partícula antipartícula. Entonces

$$U_c \mid \pi^{\pm} \rangle = \mid \pi^{\mp} \rangle \quad , \quad U_c \mid \pi^0 \rangle = \mid \pi^0 \rangle \quad (7)$$

De las ecuaciones (2) se tiene

$$\langle T T_3' | T T_3 \rangle = \langle \overline{T T_3'} | \overline{T T_3} \rangle^* \tag{1}$$

$$\langle T T_3' | T_\pm | T T_3 \rangle = - \langle \overline{T T_3'} | T_\mp | \overline{T T_3} \rangle^* \quad , \quad \langle T T_3' | T_3 | T T_3 \rangle = - \langle \overline{T T_3'} | T_3 | \overline{T T_3} \rangle^*$$

De la primera y la última se ve que se puede tomar

$$|\pi^+\rangle = \alpha |1\,1\rangle \quad , \quad |\pi^0\rangle = \beta |1\,0\rangle \quad , \quad |\pi^-\rangle = \gamma |1\,-1\rangle \tag{2}$$

donde α, β y γ son fases arbitrarias. De la segunda ecuación se obtiene como ecuaciones, a lo sumo, independientes

$$\langle \pi^0 | T_+ | \pi^- \rangle = - \langle \pi^0 | T_- | \pi^+ \rangle^* \quad , \quad \langle \pi^+ | T_+ | \pi^0 \rangle = - \langle \pi^- | T_- | \pi^0 \rangle^* \tag{3}$$

y si queremos que

$$T_\pm | T T_3 \rangle = + \left[(T \mp T_3)(T \pm T_3 + 1) \right]^{1/2} | T\, T_3 \pm 1 \rangle \tag{4}$$

entonces la única ecuación que se debe satisfacer es $\alpha\beta^* = -\beta\gamma^*$, que se satisface con la elección

$$|\pi^+\rangle = - |1, 1\rangle \quad , \quad |\pi^0\rangle = + |1, 0\rangle \quad , \quad |\pi^-\rangle = + |1, -1\rangle \tag{5}$$

Indiquemos por ϕ_{π^+} (ϕ_{π^0}) el campo que destruye π^+ (π^0). Evidentemente $\phi_{\pi^-}(x) = \phi_{\pi^+}^+(x)$ y $\phi_{\pi^0}(x) = \phi_{\pi^0}^+(x)$. Introduzcamos los campos

$$\tilde{\phi}_1(x) = - \phi_{\pi^+}(x) \quad , \quad \tilde{\phi}_2(x) = \phi_{\pi^0}(x) \quad , \quad \tilde{\phi}_3(x) = \phi_{\pi^-}(x) \tag{6}$$

De acuerdo con el convenio de fases elegido

$$\tilde{\phi}_\alpha(x) \longrightarrow \tilde{\phi}'_\alpha(x) = \mathcal{U}\, \phi_\alpha(x)\, \mathcal{U}^+ = \sum_\beta \langle \tilde{\alpha} | \tilde{U}^+ | \tilde{\beta} \rangle\, \tilde{\phi}_\beta(x) \tag{7}$$

donde

$$\tilde{U} = \exp\left\{ -i\, \theta\, \hat{m} \cdot \vec{\tilde{T}} \right\} \tag{8}$$

$$\tilde{T}_1 = \frac{1}{\sqrt{2}} \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} \qquad \tilde{T}_2 = \frac{i}{\sqrt{2}} \begin{vmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix} \qquad \tilde{T}_3 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix}$$

Es conveniente, trabajando con piones, introducir la base matemática definida por

$$\phi_1(x) \equiv \frac{1}{\sqrt{2}}\left[\phi_{\pi^+}(x) + \phi_{\pi^-}(x)\right] \quad , \quad \phi_2(x) \equiv \frac{i}{\sqrt{2}}\left[\phi_{\pi^+}(x) - \phi_{\pi^-}(x)\right] \quad , \quad \phi_3(x) \equiv \phi_{\pi^0}(x) \qquad (1)$$

que son todos ellos autoadjuntos y en términos de los cuales tenemos

$$\phi_{\pi^+}(x) = \frac{1}{\sqrt{2}}\left[\phi_1(x) - i\,\phi_2(x)\right] \quad , \quad \phi_{\pi^-}(x) = \frac{1}{\sqrt{2}}\left[\phi_1(x) + i\,\phi_2(x)\right] \quad , \quad \phi_{\pi^0}(x) = \phi_3(x) \qquad (2)$$

Por otra parte usando las notaciones matriciales es evidente que

$$\phi(x) = A\,\tilde{\phi}(x) \quad , \quad A = \frac{1}{\sqrt{2}}\begin{vmatrix} -1 & 0 & +1 \\ -i & 0 & -i \\ 0 & \sqrt{2} & 0 \end{vmatrix} \quad , \quad A^{-1} = \frac{1}{\sqrt{2}}\begin{vmatrix} -1 & i & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & i & 0 \end{vmatrix} \qquad (3)$$

con lo cual la ley de transformación es ahora

$$\phi_\alpha(x) \longrightarrow \phi'_\alpha(x) = \mathcal{U}\,\phi_\alpha(x)\,\mathcal{U}^+ = \sum_\beta \langle\alpha|\,U^+\,|\beta\rangle\,\phi_\beta(x) \qquad (3)$$

con

$$U^+ \equiv A\,\tilde{U}^+\,A^{-1} \qquad (5)$$

y por tanto los generadores para la nueva base T_i vienen dados en función de los $\tilde{T}_i$ introducidos antes por

$$T_i = A\,\tilde{T}_i\,A^{-1} \qquad (6)$$

y es inmediato comprobar que

$$(T_i)_{jk} = -i\,\epsilon_{ijk} \qquad (7)$$

que es la expresión usual para los generadores en la representación adjunta.

iii) <u>La interacción</u>

Queremos ahora determinar la interacción pion - nucleón. Exigiendo la conservación de la carga y de la paridad el acoplamiento más general posible de tipo Yukawa es

$$L(x) = i g_1 \bar{\Psi}_p(x) \gamma_5 \Psi_p(x) \phi_{\pi^0}(x) + i g_2 \bar{\Psi}_n(x) \gamma_5 \Psi_n(x) \phi_{\pi^0}(x)$$
$$+ i g_3 \bar{\Psi}_p(x) \gamma_5 \Psi_n(x) \phi_{\pi^+}(x) + i g_3 \bar{\Psi}_n(x) \gamma_5 \Psi_p(x) \phi^+_{\pi^+}(x) \qquad (1)$$

donde si $g_i \equiv g_i^*$ el lagrangiano es autoadjunto. La imposición de esta condición ha obligado a que los dos últimos términos tengan la misma constante de acoplamiento

Introduciendo

$$\Psi_N(x) \equiv \begin{vmatrix} \Psi_p(x) \\ \Psi_n(x) \end{vmatrix} \qquad \vec{\phi}(x) \equiv (\phi_1(x), \phi_2(x), \phi_3(x)) \qquad (2)$$

se tiene que (1) equivale a

$$L(x) = \frac{i}{2}(g_1+g_2) \bar{\Psi}_N(x) \gamma_5 \Psi_N(x) \phi_3(x) + \frac{i}{2}(g_1-g_2) \bar{\Psi}_N(x) \gamma_5 \tau_3 \Psi_N(x) \phi_3(x)$$
$$+ \frac{i}{\sqrt{2}} g_3 \bar{\Psi}_N(x) \gamma_5 \tau_1 \Psi_N(x) \phi_1(x) + \frac{i}{\sqrt{2}} g_3 \bar{\Psi}_N(x) \gamma_5 \tau_2 \Psi_N(x) \phi_2(x) \qquad (3)$$

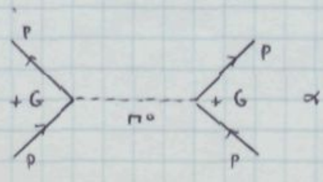
Si elegimos

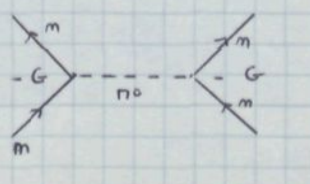
$$g_1 = -g_2 = \frac{1}{\sqrt{2}} g_3 = + G_{NN\pi} \qquad (5)$$

entonces (3) se puede escribir

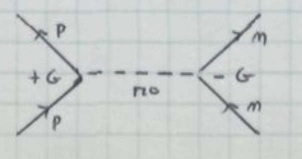
$$L(x) = i G_{NN\pi} \bar{\Psi}_N(x) \gamma_5 \vec{\tau} \Psi_N(x) \cdot \vec{\phi}(x) \qquad (5)$$

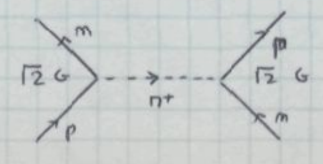
Intuitivamente se ve que esta interacción produce fuerzas iguales entre los nucleones. Basta considerar

pp : 

$\propto + G^2$; nn : 

$\propto + G^2$

pn : 

$+$ 

$\propto - G^2 + 2G^2 = + G^2.$

Veamos ahora que (6.5) es un escalar bajo $SU_T(2)$, es decir la interacción postulada es invariante bajo $SU_T(2)$. Consideremos pues como se transforma la parte esencial de (6.5) bajo $SU_T(2)$. De acuerdo con lo dicho antes

$$\bar{\psi}(x)\,\vec{\tau}\cdot\vec{\phi}(x)\,\psi(x) \longrightarrow \bar{\psi}'(x)\,\vec{\tau}\,\vec{\phi}'(x)\,\psi'(x) = \bar{\psi}(x)\,U\,\tau_K\,U^{+}\,\psi(x)\,(U^{+}\phi)_K(x). \tag{1}$$

Basta, evidentemente, considerar transformaciones infinitesimales alrededor de la dirección e

$$\bar{\psi}'(x)\,\vec{\tau}\,\vec{\phi}'(x)\,\psi'(x) = \bar{\psi}(x)\left[1 - i\delta\theta\frac{1}{2}\tau_e\right]\tau_K\left[1 + i\delta\theta\frac{1}{2}\tau_e\right]\psi(x)\left[\phi_K(x) + i\delta\theta\,(T_e)_{Ki}\,\phi_i(x)\right]$$

$$= \bar{\psi}(x)\left\{\tau_K + \frac{i\delta\theta}{2}\left[\tau_K, \tau_e\right]\right\}\psi(x)\left\{\phi_K(x) + \delta\theta\,\epsilon_{eKi}\,\phi_i(x)\right\} =$$

$$= \bar{\psi}(x)\left\{\tau_K - \delta\theta\,\epsilon_{Ker}\,\tau_r\right\}\psi(x)\left\{\phi_K(x) + \delta\theta\,\epsilon_{eKi}\,\phi_i(x)\right\} =$$

$$= \bar{\psi}(x)\,\tau_K\,\psi(x)\,\phi_K(x) + \delta\theta\,\epsilon_{eKi}\,\bar{\psi}(x)\,\tau_K\,\psi(x)\,\phi_i(x) - \delta\theta\,\epsilon_{Ker}\,\bar{\psi}(x)\,\tau_r\,\psi(x)\,\phi_K(x)$$

$$= \bar{\psi}(x)\,\tau_K\,\psi(x)\,\phi_K(x)$$

y por tanto la interacción pión-nucleón es escalar bajo $SU_T(2)$.

XI- C- CLASIFICACIÓN DE PARTICULAS EN SU(3)

Los quarks u, d, s se colocan en la representación fundamental de SU(3)

$$|u\rangle \equiv |q^1\rangle \equiv |3;\, 1/3\ 1/2\ 1/2\rangle \quad , \quad |d\rangle \equiv |q^2\rangle \equiv |3;\, 1/3\ 1/2\, -1/2\rangle$$
$$|s\rangle \equiv |q^3\rangle \equiv |3;\, -2/3\ 0\ 0\rangle \qquad (1)$$

y para usar las λ_i en la representación usual se debe tomar

$$|q^1\rangle \equiv \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} \quad , \quad |q^2\rangle \equiv \begin{vmatrix} 0 \\ 1 \\ 0 \end{vmatrix} \quad , \quad |q^3\rangle \equiv \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix} \qquad (2)$$

y por tanto los elementos de matriz en la base matemática vienen dados por

$$H_1 = \frac{1}{2\sqrt{3}}\left[\,|u\rangle\langle u| - |d\rangle\langle d|\,\right] \quad , \quad H_2 = \frac{1}{6}\left[\,|u\rangle\langle u| + |d\rangle\langle d| - 2|s\rangle\langle s|\,\right]$$
$$E_{+1} = \frac{1}{\sqrt{6}}\,|u\rangle\langle d| \quad , \quad E_{+2} = \frac{1}{\sqrt{6}}\,|u\rangle\langle s| \quad , \quad E_{+3} = \frac{1}{\sqrt{6}}\,|d\rangle\langle s| \quad , \quad E_{-\alpha} = E_\alpha^+ \qquad (3)$$

Notar que bajo un elemento $U(\alpha_i) \equiv \exp\left\{-\frac{i}{2}\alpha_i\lambda_i\right\}$ de SU(3) la ley de transformación de los estados es

$$|q^i\rangle \longrightarrow U|q^i\rangle = \sum_{j=1}^{3} D^{(3)}_{ji}(U)\,|q^j\rangle \qquad (4)$$

$$D^{(3)}_{ji}(U) \equiv \langle q^j|\exp\left\{-\frac{i}{2}\alpha_i\lambda_i\right\}|q^i\rangle$$

Pasemos ahora a estudiar la clasificación de los antiquarks. Es bien sabido que si una representación de SU(3) viene dada por las matrices unitarias $U = \exp\{-iA\}$ con $A = A^+$ y $Tr(A) = 0$, entonces la representación dual o contragradiente viene dada por las matrices $U^* = \exp\{-i(-A^T)\}$ y a diferencia de lo que sucede en SU(2) estas dos representaciones no son equivalentes.

La asignación de los antiquarks con los estados matemáticos se puede hacer ahora como

$$|\bar{u}\rangle \equiv |q_1\rangle \equiv \alpha_1 |3^*; -1/3\ 1/2\ -1/2\rangle \quad , \quad |\bar{d}\rangle \equiv |q_2\rangle \equiv \alpha_2 |3^*; -1/3\ 1/2\ 1/2\rangle \tag{1}$$

$$|\bar{s}\rangle \equiv |q_3\rangle \equiv \alpha_3 |3^*; +2/3\ 0\ 0\rangle$$

donde $|\alpha_i| = 1$. Si hacemos la identificación

$$|q_1\rangle \equiv \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} \quad , \quad |q_2\rangle \equiv \begin{vmatrix} 0 \\ 1 \\ 0 \end{vmatrix} \quad , \quad |q_3\rangle \equiv \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix} \tag{2}$$

entonces en la representación 3^* se tiene

$$H_1 = -\frac{1}{2\sqrt{3}} [\,|\bar{u}\rangle\langle\bar{u}| - |\bar{d}\rangle\langle\bar{d}|\,] \quad , \quad H_2 = -\frac{1}{6} [\,|\bar{u}\rangle\langle\bar{u}| + |\bar{d}\rangle\langle\bar{d}| - 2|s\rangle\langle\bar{s}|\,]$$

$$E_{+1} = -\frac{1}{\sqrt{6}} |\bar{d}\rangle\langle\bar{u}| \quad , \quad E_{+2} = -\frac{1}{\sqrt{6}} |\bar{s}\rangle\langle\bar{u}| \quad , \quad E_{+3} = -\frac{1}{\sqrt{6}} |\bar{s}\rangle\langle\bar{d}|, \quad E^{+}_{-\alpha} = E_{\alpha} \tag{3}$$

Si se eligen las fases α_i de forma que $\alpha_i = 1$ $(i = 1, 2, 3)$ entonces al expresar las cantidades anteriores se ve que el convenio de fases elegido para SU(3) (es decir que los elementos de matriz de $E_{\pm 1}$, $E_{\pm 2}$ en la base matemática sean no negativos) viene violado. La forma de evitar esto es exigiendo que $\alpha_1^* \alpha_2 = -1$ y $\alpha_1^* \alpha_3 = -1$, lo cual se cumple si elegimos $\alpha_1 = +1$, $\alpha_2 = \alpha_3 = -1$ y por tanto las relaciones de los estados antiquarks con la base matemática que cumpla el convenio de fases elegido es

$$|\bar{u}\rangle \equiv |q_1\rangle \equiv |3^*; -1/3\ 1/2\ -1/2\rangle \quad , \quad |\bar{d}\rangle \equiv |q_2\rangle \equiv -|3^*; -1/3\ 1/2\ 1/2\rangle \tag{4}$$

$$|\bar{s}\rangle \equiv |q_3\rangle \equiv -|3^*; 2/3, 0\ 0\rangle$$

Fijémonos que el hecho de que los quarks y antiquarks se transformen unos con la representación dual de los otros garantiza automáticamente que los campos tienen una ley de transformación simple. Además de (1.3) y (3) se deduce

$$H_1|u\rangle = \frac{1}{2\sqrt{3}}|u\rangle \quad , \quad H_1|d\rangle = -\frac{1}{2\sqrt{3}}|d\rangle$$

$$H_2|u\rangle = \frac{1}{6}|u\rangle \quad , \quad H_2|d\rangle = \frac{1}{6}|d\rangle \quad , \quad H_2|s\rangle = -\frac{1}{3}|s\rangle$$

$$E_{+1}|d\rangle = \frac{1}{\sqrt{6}}|u\rangle \quad , \quad E_{+2}|s\rangle = \frac{1}{\sqrt{6}}|u\rangle \quad , \quad E_{+3}|s\rangle = \frac{1}{\sqrt{6}}|d\rangle$$

$$E_{-1}|u\rangle = \frac{1}{\sqrt{6}}|d\rangle \quad , \quad E_{-2}|u\rangle = \frac{1}{\sqrt{6}}|s\rangle \quad , \quad E_{-3}|d\rangle = \frac{1}{\sqrt{6}}|s\rangle \tag{1}$$

$$H_1|\bar{u}\rangle = -\frac{1}{2\sqrt{3}}|\bar{u}\rangle \quad , \quad H_1|\bar{d}\rangle = +\frac{1}{2\sqrt{3}}|\bar{d}\rangle$$

$$H_2|\bar{u}\rangle = -\frac{1}{6}|\bar{u}\rangle \quad , \quad H_2|\bar{d}\rangle = -\frac{1}{6}|\bar{d}\rangle \quad , \quad H_2|\bar{s}\rangle = +\frac{1}{3}|\bar{s}\rangle$$

$$E_{+1}|\bar{u}\rangle = -\frac{1}{\sqrt{6}}|\bar{d}\rangle \quad , \quad E_{+2}|\bar{u}\rangle = -\frac{1}{\sqrt{6}}|\bar{s}\rangle \quad , \quad E_{+3}|\bar{d}\rangle = -\frac{1}{\sqrt{6}}|\bar{s}\rangle$$

$$E_{-1}|\bar{d}\rangle = -\frac{1}{\sqrt{6}}|\bar{u}\rangle \quad , \quad E_{-2}|\bar{s}\rangle = -\frac{1}{\sqrt{6}}|\bar{u}\rangle \quad , \quad E_{-3}|\bar{s}\rangle = -\frac{1}{\sqrt{6}}|\bar{d}\rangle$$

Los que no aparecen aquí significa que al aplicar el generador al estado quark o antiquark da cero.

Pasemos ahora a estudiar la representación 8. Al hacer el producto $3 \otimes 3^*$ los nueve estados a considerar son

$$|A_1\rangle \equiv |u\rangle|\bar{u}\rangle \quad (0,0) \quad ; \quad |A_2\rangle \equiv -|u\rangle|\bar{d}\rangle \quad (0,1) \quad ; \quad |A_3\rangle \equiv -|u\rangle|\bar{s}\rangle \quad (1,1/2)$$

$$|A_4\rangle \equiv |d\rangle|\bar{u}\rangle \quad (0,-1) \quad ; \quad |A_5\rangle \equiv -|d\rangle|\bar{d}\rangle \quad (0,0) \quad ; \quad |A_6\rangle \equiv -|d\rangle|\bar{s}\rangle \quad (1,-1/2) \tag{2}$$

$$|A_7\rangle \equiv |s\rangle|\bar{u}\rangle \quad (-1,-1/2) \quad ; \quad |A_8\rangle \equiv -|s\rangle|\bar{d}\rangle \quad (-1,1/2) \quad ; \quad |A_9\rangle \equiv -|s\rangle|\bar{s}\rangle \quad (0,0)$$

donde se han indicado (Y, T_3). Sabemos que $3 \otimes 3^* = 1 \oplus 8$. Entonces

$$|8;011\rangle = -|u\rangle|\bar{d}\rangle \tag{3}$$

Para obtener $|8;010\rangle$ basta aplicar T_- al anterior

$$|8;010\rangle = \frac{1}{\sqrt{2}} T_-|8;011\rangle = -\sqrt{3}\,E_{-1}|u\rangle|\bar{d}\rangle = \frac{1}{\sqrt{2}}\left[|u\rangle|\bar{u}\rangle - |d\rangle|\bar{d}\rangle\right] \tag{4}$$

Similarmente

$$|8;01-1\rangle = \frac{1}{\sqrt{2}} T_-|8;010\rangle = \sqrt{\frac{3}{2}}\,E_{-1}\left[|u\rangle|\bar{u}\rangle - |d\rangle|\bar{d}\rangle\right] = |d\rangle|\bar{u}\rangle \tag{5}$$

Procedamos ahora a aplicar E_{+2} al estado anterior

$$E_{+2}|8;\,0\,1\,{-1}\rangle = -\frac{1}{\sqrt{6}}\,|d\rangle|\bar{s}\rangle \quad \propto \quad |8;\,1\,1/2\,-1/2\rangle \tag{1}$$

y como estos estados deben ser normalizados y los elementos de matriz E_{+2} deben ser positivos.

$$|8;\,1\,1/2\,-1/2\rangle = -\,|d\rangle|\bar{s}\rangle \tag{2}$$

Similarmente

$$|8;\,1\,1/2\,1/2\rangle = -|u\rangle|\bar{s}\rangle \quad ; \quad |8;\,-1\,1/2\,-1/2\rangle = |s\rangle|\bar{u}\rangle \quad ; \quad |8;\,-1\,1/2\,1/2\rangle = -|s\rangle|\bar{d}\rangle \tag{3}$$

Nos falta ahora construir $|8;000\rangle$ que debe ser una combinación de $|u\rangle|\bar{u}\rangle$, $|d\rangle|\bar{d}\rangle$ y $|s\rangle|\bar{s}\rangle$. Este estado debe ser ortogonal al (3.4) y por tanto

$$|8;\,000\rangle \;\propto\; [\,|u\rangle|\bar{u}\rangle + |d\rangle|\bar{d}\rangle + \alpha|s\rangle|\bar{s}\rangle\,] \tag{4}$$

Aplicando E_{-2} al estado $|8;\,1\,1/2\,1/2\rangle$ se obtiene una combinación de $|8;010\rangle$ y $|8;000\rangle$ de donde

$$|8;\,000\rangle = \frac{1}{\sqrt{6}}\,[\,|u\rangle|\bar{u}\rangle + |d\rangle|\bar{d}\rangle - 2|s\rangle|\bar{s}\rangle\,] \tag{5}$$

y el $|1;000\rangle$ es ortogonal a $|8;010\rangle$ y $|8;000\rangle$. Resumiendo

$$|8;\,1\,1/2\,1/2\rangle = -|u\rangle|\bar{s}\rangle \quad ; \quad |8;\,1\,1/2\,-1/2\rangle = -|d\rangle|\bar{s}\rangle$$

$$|8;\,0\,1\,1\rangle = -|u\rangle|\bar{d}\rangle \quad ; \quad |8;010\rangle = \frac{1}{\sqrt{2}}\,[\,|u\rangle|\bar{u}\rangle - |d\rangle|\bar{d}\rangle\,] \quad ; \quad |8;\,0\,1\,{-1}\rangle = |d\rangle|\bar{u}\rangle$$

$$|8;\,000\rangle = \frac{1}{\sqrt{6}}\,[\,|u\rangle|\bar{u}\rangle + |d\rangle|\bar{d}\rangle - 2|s\rangle|\bar{s}\rangle\,] \tag{6}$$

$$|8;\,-1\,1/2\,1/2\rangle = -|s\rangle\langle\bar{d}| \quad ; \quad |8;\,-1\,1/2\,-1/2\rangle = |s\rangle|\bar{u}\rangle$$

$$|1;\,000\rangle = \frac{1}{\sqrt{3}}\,[\,|u\rangle|\bar{u}\rangle + |d\rangle|\bar{d}\rangle + |s\rangle|\bar{s}\rangle\,]$$

Entonces para el noneto de mesones 0^- se deben llevar a cabo las siguientes asignaciones

$|K^+\rangle \equiv |u\rangle|\bar{s}\rangle = -|8; 1\,\tfrac{1}{2}\,\tfrac{1}{2}\rangle \quad , \quad |K^0\rangle \equiv |d\rangle|\bar{s}\rangle = -|8; 1\,\tfrac{1}{2}\,-\tfrac{1}{2}\rangle$

$|\pi^+\rangle \equiv |u\rangle|\bar{d}\rangle = -|8; 0\,1\,1\rangle \quad , \quad |\pi^0\rangle \equiv \frac{1}{\sqrt{2}}\left[\,|u\rangle|\bar{u}\rangle - |d\rangle|\bar{d}\rangle\,\right] = +|8; 0\,1\,0\rangle$

$|\pi^-\rangle \equiv |d\rangle|\bar{u}\rangle = +|8; 0\,1\,-1\rangle$ (2)

$|\eta\rangle \equiv \frac{1}{\sqrt{6}}\left[\,|u\rangle|\bar{u}\rangle + |d\rangle|\bar{d}\rangle - 2\,|s\rangle|\bar{s}\rangle\,\right] = +|8; 0\,0\,0\rangle$

$|\bar{K}^0\rangle \equiv |s\rangle|\bar{d}\rangle = -|8; -1\,\tfrac{1}{2}\,\tfrac{1}{2}\rangle \quad , \quad |K^-\rangle \equiv |s\rangle|\bar{u}\rangle = +|8; -1\,\tfrac{1}{2}\,-\tfrac{1}{2}\rangle$

$|\eta'\rangle \equiv \frac{1}{\sqrt{3}}\left[\,|u\rangle|\bar{u}\rangle + |d\rangle|\bar{d}\rangle + |s\rangle|\bar{s}\rangle\,\right] = +|1; 0\,0\,0\rangle$

Evidentemente $8 = 8^*$ lo cual es consecuencia directa de que en la representación 8 de las F_i $[\,(F_i)_{jk} = -i f_{ijk}\,]$ se cumple $F_i^T = -F_i$

En ocasiones es útil introducir una notación tensorial. El tensor correspondiente a la 8 es

$$T_i^{\ j} \equiv y_i x^j - \frac{1}{3}\delta_i^{\ j} \sum_k y_k x^k \qquad (3)$$

y recordando $x^1 = u$, $x^2 = d$, $x^3 = s$, $y_1 = \bar{u}$, $y_2 = \bar{d}$ y $y_3 = \bar{s}$ se puede escribir

$$T = \begin{vmatrix} \frac{2}{3}u\bar{u} - \frac{1}{3}d\bar{d} - \frac{1}{3}s\bar{s} & u\bar{d} & u\bar{d} \\ d\bar{u} & -\frac{1}{3}u\bar{u} + \frac{2}{3}d\bar{d} - \frac{1}{3}s\bar{s} & d\bar{s} \\ s\bar{u} & s\bar{d} & -\frac{1}{3}u\bar{u} - \frac{1}{3}d\bar{d} + \frac{2}{3}s\bar{s} \end{vmatrix} =$$

$$= \begin{vmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \bar{K}^0 & -\frac{2}{\sqrt{6}}\eta \end{vmatrix} \equiv M \qquad (4)$$

Veamos ahora como hacer las asignaciones matemáticas de los bariones $1/2^+$. Tenemos plena libertad de elección de los estados partícula y por conveniencia los elegiremos análogos a (5.1), es decir

$$|p\rangle \equiv -|8;\,1\,\tfrac12\,\tfrac12\rangle \quad , \quad |n\rangle \equiv -|8;\,1\,\tfrac12\,-\tfrac12\rangle$$

$$|\Sigma^+\rangle \equiv -|8;\,0\,1\,1\rangle \quad , \quad |\Sigma^0\rangle \equiv +|8;\,0\,1\,0\rangle \quad , \quad |\Sigma^-\rangle \equiv +|8;\,0\,1\,-1\rangle \qquad (1)$$

$$|\Lambda\rangle \equiv +|8;\,0\,0\,0\rangle$$

$$|\Xi^0\rangle \equiv -|8;\,-1\,\tfrac12\,\tfrac12\rangle \quad , \quad |\Xi^-\rangle \equiv +|8;\,-1\,\tfrac12\,-\tfrac12\rangle$$

Recordemos que el contenido en quarks de cada uno de estos estados es

$$|p\rangle \equiv (u\,u\,d) \quad , \quad |n\rangle \equiv (d\,d\,u) \quad , \quad |\Sigma^+\rangle \equiv (u\,u\,s) \quad , \quad |\Sigma^0\rangle \equiv |\Lambda^0\rangle \equiv (u\,d\,s)$$

$$|\Sigma^-\rangle \equiv (d\,d\,s) \quad , \quad |\Xi^0\rangle = (u\,s\,s) \quad , \quad |\Xi^-\rangle \equiv (d\,s\,s) \qquad (2)$$

y por tanto las asignaciones de las antipartículas son

$$|\bar p\rangle \equiv +|8;-1\,\tfrac12\,-\tfrac12\rangle \quad , \quad |\bar n\rangle \equiv -|8;-1\,\tfrac12\,\tfrac12\rangle$$

$$|\bar\Sigma^+\rangle \equiv +|8;\,0\,1\,-1\rangle \quad , \quad |\bar\Sigma^0\rangle = +|8;\,0\,1\,0\rangle \quad , \quad |\bar\Sigma^-\rangle = -|8;\,0\,1\,1\rangle \qquad (3)$$

$$|\bar\Lambda\rangle \equiv +|8;\,0\,0\,0\rangle$$

$$|\bar\Xi^0\rangle \equiv -|8;\,1\,\tfrac12\,-\tfrac12\rangle \quad , \quad |\bar\Xi^-\rangle \equiv -|8;\,1\,\tfrac12\,\tfrac12\rangle$$

En notación matricial esto corresponde a

$$B \equiv \begin{vmatrix} \frac{1}{\sqrt2}\Sigma^0 + \frac{1}{\sqrt6}\Lambda & \Sigma^+ & p \\ \Sigma^- & -\frac{1}{\sqrt2}\Sigma^0 + \frac{1}{\sqrt6}\Lambda & n \\ \Xi^- & \Xi^0 & -\frac{2}{\sqrt6}\Lambda \end{vmatrix} \qquad (4)$$

Vamos a ver ahora lo que sucede en el marco del SU(3) exacto al intentar acoplar a lo Yukawa los octetes 0^- y $1/2^+$. Como en el primer multiplete tenemos 3 isomultipletes y en el segundo 4, si usamos únicamente $SU_T(2)$ habrá 12 constantes de acoplamiento arbitrarias. Al usar SU(3) como el producto 8 x 8 contiene solo dos veces la representación 8 solo pueden aparecer 2 constantes de acoplamiento independientes. Para ver esto de forma precisa es conveniente usar la representación matricial (5.4) y (6.4). El Lagrangiano debe ser el escalar más general que se pueda construir a partir de $\bar{B}$, B y M donde $\bar{B} \equiv B^+\gamma^0$ (olvidaremos la $i\gamma_5$ que no juega ningún papel). Esto es

$$L(x) = g_1 \, \mathrm{Tr}\,[\bar{B} B M] + g_2 \, \mathrm{Tr}\,[\bar{B} M B] \equiv$$

$$\equiv \sqrt{2}\, g_S \, \mathrm{Tr}\,[\bar{B} M B + \bar{B} B M] + \sqrt{2}\, g_A \, \mathrm{Tr}\,[\bar{B} M B - \bar{B} B M] \qquad (1)$$

donde

$$g_1 = \sqrt{2}\,(g_S - g_A) \qquad\qquad g_S = \frac{1}{2\sqrt{2}}\,(g_2 + g_1)$$

$$\Leftrightarrow \qquad (2)$$

$$g_2 = \sqrt{2}\,(g_S + g_A) \qquad\qquad g_A = \frac{1}{2\sqrt{2}}\,(g_2 - g_1)$$

Entonces

$$L(x) = \sqrt{2}\, g_S \Big\{ \frac{\pi^0}{\sqrt{2}} \Big[\bar{p}p - \bar{n}n + \frac{2}{\sqrt{3}}(\bar{\Lambda}\Sigma^0 + \bar{\Sigma}^0\Lambda) + \bar{\Xi}^-\Xi^- - \bar{\Xi}^0\Xi^0 \Big] +$$

$$+ \pi^- \Big[\frac{2}{\sqrt{6}}(\bar{\Lambda}\Sigma^+ + \bar{\Sigma}^-\Lambda) + \bar{n}p + \bar{\Xi}^-\Xi^0 \Big] + \pi^+ \Big[\frac{2}{\sqrt{6}}(\bar{\Sigma}^+\Lambda + \bar{\Lambda}\Sigma^-) + \bar{p}n + \bar{\Xi}^0\Xi^- \Big] +$$

$$+ \frac{\eta}{\sqrt{6}} \Big[2(\bar{\Sigma}^+\Sigma^+ + \bar{\Sigma}^0\Sigma^0 + \bar{\Sigma}^-\Sigma^-) - \bar{p}p - \bar{n}n - 2\bar{\Lambda}\Lambda - \bar{\Xi}^-\Xi^- - \bar{\Xi}^0\Xi^0 \Big] +$$

$$+ K^+ \Big[\frac{1}{\sqrt{2}}\bar{p}\Sigma^0 - \frac{1}{\sqrt{6}}\bar{p}\Lambda + \bar{n}\Sigma^- + \frac{1}{\sqrt{2}}\bar{\Sigma}^0\Xi^- + \bar{\Sigma}^+\Xi^0 - \frac{1}{\sqrt{6}}\bar{\Lambda}\Xi^- \Big] +$$

$$+ K^- \Big[\frac{1}{\sqrt{2}}\bar{\Sigma}^0 p - \frac{1}{\sqrt{6}}\bar{\Lambda}p + \bar{\Sigma}^- n + \frac{1}{\sqrt{2}}\bar{\Xi}^-\Sigma^0 + \bar{\Xi}^0\Sigma^+ - \frac{1}{\sqrt{6}}\bar{\Xi}^-\Lambda \Big] +$$

$$+ K^0 \Big[\bar{p}\Sigma^+ - \frac{1}{\sqrt{2}}\bar{n}\Sigma^0 - \frac{1}{\sqrt{6}}\bar{n}\Lambda - \frac{1}{\sqrt{2}}\bar{\Sigma}^0\Xi^0 - \frac{1}{\sqrt{6}}\bar{\Lambda}\Xi^0 + \bar{\Sigma}^-\Xi^- \Big]$$

$$+ \bar{K}^0 \Big[\bar{\Sigma}^+ p - \frac{1}{\sqrt{2}}\bar{\Sigma}^0 n - \frac{1}{\sqrt{6}}\bar{\Lambda}n - \frac{1}{\sqrt{2}}\bar{\Xi}^0\Sigma^0 - \frac{1}{\sqrt{6}}\bar{\Xi}^0\Lambda + \bar{\Xi}^-\Sigma^- \Big] \Big\}$$

$$-\sqrt{2}\, g_A \Big\{ \frac{\pi^0}{\sqrt{2}} [\, 2\bar\Sigma^-\Sigma^- - 2\bar\Sigma^+\Sigma^+ + \bar\Xi^-\Xi^- - \bar\Xi^0\Xi^0 - \bar p p + \bar n n \,] +$$

$$+\pi^-[\sqrt{2}\,(\bar\Sigma^0\Sigma^+ - \bar\Sigma^-\Sigma^0) + \bar\Xi^-\Xi^0 - \bar n p\,] + \pi^+[\sqrt{2}\,(\bar\Sigma^+\Sigma^0 - \bar\Sigma^0\Sigma^-) + \bar\Xi^0\Xi^- - \bar p n\,] +$$

$$+\frac{3\eta}{\sqrt{6}}[\,\bar\Xi^-\Xi^- + \bar\Xi^0\Xi^0 - \bar p p - \bar n n\,] +$$

$$+K^+[\,\tfrac{1}{\sqrt{2}}\bar p\Sigma^0 + \sqrt{\tfrac{3}{2}}\,\bar p\Lambda + \bar n\Sigma^- - \bar\Sigma^+\Xi^0 - \tfrac{1}{\sqrt{2}}\bar\Sigma^0\Xi^- - \sqrt{\tfrac{3}{2}}\,\bar\Lambda\Xi^-\,] +$$

$$+K^-[\,\tfrac{1}{\sqrt{2}}\bar\Sigma^0 p + \sqrt{\tfrac{3}{2}}\,\bar\Lambda p + \bar\Sigma^-\bar n - \bar\Xi^0\Sigma^+ - \tfrac{1}{\sqrt{2}}\bar\Xi^-\Sigma^0 - \sqrt{\tfrac{3}{2}}\,\bar\Xi^-\Lambda\,] +$$

$$+K^0[\,\bar p\Sigma^+ - \tfrac{1}{\sqrt{2}}\bar n\Sigma^0 + \sqrt{\tfrac{3}{2}}\,\bar n\Lambda - \bar\Sigma^-\Xi^- + \tfrac{1}{\sqrt{2}}\bar\Sigma^0\Xi^0 - \sqrt{\tfrac{3}{2}}\,\bar\Lambda\Xi^0\,] +$$

$$+\bar K^0[\,\bar\Sigma^+ p - \tfrac{1}{\sqrt{2}}\bar\Sigma^0 n + \sqrt{\tfrac{3}{2}}\,\bar\Lambda n - \bar\Xi^-\Sigma^- + \tfrac{1}{\sqrt{2}}\bar\Xi^0\Sigma^0 - \sqrt{\tfrac{3}{2}}\,\bar\Xi^0\Lambda\,]\Big\} \qquad (1)$$

De donde

$$L(x) = (g_s + g_A)\{\pi^0(\bar p p - \bar n n) + \sqrt{2}\,\pi^-\bar n p + \sqrt{2}\,\pi^+\bar p n\}$$

$$+\frac{2}{\sqrt{3}}\, g_s\{\pi^0(\bar\Lambda\Sigma^0 + \bar\Sigma^0\Lambda) + \pi^-(\bar\Lambda\Sigma^+ + \bar\Sigma^-\Lambda) + \pi^+(\bar\Sigma^+\Lambda + \bar\Lambda\Sigma^-)\} +$$

$$+(g_s - g_A)\{\pi^0(\bar\Xi^-\Xi^- - \bar\Xi^0\Xi^0) + \sqrt{2}\,\pi^-\bar\Xi^-\Xi^0 + \sqrt{2}\,\pi^+\bar\Xi^0\Xi^-\} +$$

$$-2g_A\{\pi^0(\bar\Sigma^-\Sigma^- - \bar\Sigma^+\Sigma^+) + \pi^-(\bar\Sigma^0\Sigma^+ - \bar\Sigma^-\Sigma^0) + \pi^+(\bar\Sigma^+\Sigma^0 - \bar\Sigma^0\Sigma^-)\}$$

$$+\frac{1}{\sqrt{3}}(3g_A - g_s)\,\eta\,(\bar p p + \bar n n) + \frac{2}{\sqrt{3}}\, g_s\,\eta\,(\bar\Sigma^+\Sigma^+ + \bar\Sigma^0\Sigma^0 + \bar\Sigma^-\Sigma^-)$$

$$-\frac{2}{\sqrt{3}}\, g_s\,\eta\,\bar\Lambda\Lambda - \frac{1}{\sqrt{3}}(g_s + 3g_A)\,\eta\,(\bar\Xi^-\Xi^- + \bar\Xi^0\Xi^0)$$

$$+(g_s - g_A)\{\sqrt{2}\,K^0\bar p\Sigma^+ + \sqrt{2}\,K^+\bar n\Sigma^- + K^+\bar p\Sigma^0 - K^0\bar n\Sigma^0 + \sqrt{2}\,\bar K^0\bar\Sigma^+ p + \sqrt{2}\,K^-\bar\Sigma^- n + K^-\bar\Sigma^0 p - K^0\bar\Sigma^0 n\}$$

$$-\frac{1}{\sqrt{3}}(g_s + 3g_A)\{K^+\bar p\Lambda + K^-\bar\Lambda p + K^0\bar n\Lambda + \bar K^0\bar\Lambda n\}$$

$$+(g_s + g_A)\{K^+\bar\Sigma^0\Xi^- - K^0\bar\Sigma^0\Xi^0 + K^-\bar\Xi^-\Sigma^0 - \bar K^0\bar\Xi^0\Sigma^0 + \sqrt{2}\,K^+\bar\Sigma^+\Xi^0 + \sqrt{2}\,K^-\bar\Xi^0\Sigma^+ + \sqrt{2}\,K^0\bar\Sigma^-\Xi^- + \sqrt{2}\,K^0\bar\Xi^-\Sigma^-\}$$

$$+\frac{1}{\sqrt{3}}(3g_A - g_s)\{K^+\bar\Lambda\Xi^- + K^-\bar\Xi^-\Lambda + K^0\bar\Lambda\Xi^0 + \bar K^0\bar\Xi^0\Lambda\}. \qquad (2)$$

Now let us introduce les camps

$$\vec{\pi} \equiv \left(\frac{1}{\sqrt{2}}(\pi^+ + \pi^-),\ \frac{i}{\sqrt{2}}(\pi^+ - \pi^-),\ \pi^0\right) \qquad N \equiv \begin{pmatrix} -p \\ -n \end{pmatrix} \qquad (3)$$

$$\vec{\Sigma} \equiv \left(\frac{1}{\sqrt{2}} (\Sigma^+ + \Sigma^-) \; , \; \frac{i}{\sqrt{2}} (\Sigma^+ - \Sigma^-) \; , \; \Sigma^0 \right) \qquad \Xi \equiv \begin{pmatrix} -\Xi^0 \\ +\Xi^- \end{pmatrix}$$

$$K \equiv \begin{pmatrix} -K^+ \\ -K^0 \end{pmatrix} \qquad K_c \equiv \begin{pmatrix} -\bar{K}^0 \\ K^- \end{pmatrix} \qquad \bar{K} \equiv (-K^-, -\bar{K}^0) \qquad \bar{K}_c \equiv (-K^0, K^+) \tag{1}$$

Entonces

$$\bar{N} \vec{\tau} N \vec{\pi} = \pi^0 (\bar{p} p - \bar{n} n) + \sqrt{2}\, \pi^- \bar{n} p + \sqrt{2}\, \pi^+ \bar{p} n$$

$$\bar{\Lambda} \vec{\Sigma} . \vec{\pi} + \vec{\bar{\Sigma}} \Lambda . \vec{\pi} = \pi^0 (\bar{\Lambda} \Sigma^0 + \bar{\Sigma}^0 \Lambda) + \pi^- (\bar{\Lambda} \Sigma^+ + \bar{\Sigma}^- \Lambda) + \pi^+ (\bar{\Sigma}^+ \Lambda + \bar{\Lambda} \Sigma^-)$$

$$\bar{\Xi} \vec{\tau} \Xi \vec{\pi} = \pi^0 (\bar{\Xi}^0 \Xi^0 - \bar{\Xi}^- \Xi^-) - \sqrt{2}\, \pi^- \bar{\Xi}^- \Xi^0 - \sqrt{2}\, \pi^+ \bar{\Xi}^- \Xi^0$$

$$i (\vec{\bar{\Sigma}} \times \vec{\Sigma}) . \vec{\pi} = \pi^0 (\bar{\Sigma}^- \Sigma^- - \bar{\Sigma}^+ \Sigma^+) + \pi^- (\bar{\Sigma}^0 \Sigma^+ - \bar{\Sigma}^- \Sigma^0) + \pi^+ (\bar{\Sigma}^+ \Sigma^0 - \bar{\Sigma}^0 \Sigma^-)$$

$$\bar{N} N \eta = \eta (\bar{p} p + \bar{n} n)$$

$$\bar{\Xi} \Xi \eta = \eta (\bar{\Xi}^- \Xi^- + \bar{\Xi}^0 \Xi^0)$$

$$\eta \vec{\bar{\Sigma}} . \vec{\Sigma} = \eta (\bar{\Sigma}^+ \Sigma^+ + \bar{\Sigma}^- \Sigma^- + \bar{\Sigma}^0 \Sigma^0)$$

$$\bar{N} K \Lambda + \bar{\Lambda} \bar{K} N = K^+ \bar{p} \Lambda + K^- \bar{\Lambda} p + K^0 \bar{n} \Lambda + \bar{K}^0 \bar{\Lambda} n$$

$$\bar{\Xi} K_c \vec{\tau} N + \bar{N} \vec{\tau} K \vec{\Sigma} = \sqrt{2}\, K^0 \bar{p} \Sigma^+ + \sqrt{2}\, K^+ \bar{n} \Sigma^- + K^+ \bar{p} \Sigma^0 - K^0 \bar{n} \Sigma^0$$

$$+ \sqrt{2}\, \bar{K}^0 \bar{\Sigma}^+ p + \sqrt{2}\, K^- \bar{\Sigma}^- n + K^- \bar{\Sigma}^0 p - \bar{K}^0 \Sigma^0 n$$

$$\bar{\Xi} K_c \Lambda + \bar{\Lambda} \bar{K}_c \Xi = K^+ \bar{\Lambda} \Xi^- + K^- \bar{\Xi}^- \Lambda + K^0 \bar{\Lambda} \Xi^0 + \bar{K}^0 \bar{\Xi}^0 \Lambda$$

$$\vec{\bar{\Sigma}} K_c \vec{\tau} \Xi + \bar{\Xi} \vec{\tau} K_c \vec{\Sigma} = - K^+ \bar{\Sigma}^0 \Xi^- + K^0 \bar{\Sigma}^0 \Xi^0 - K^- \bar{\Xi}^- \Sigma^0 + \bar{K}^0 \bar{\Xi}^0 \Sigma^0$$

$$- \sqrt{2}\, K^+ \bar{\Sigma}^+ \Xi^0 - \sqrt{2}\, K^- \bar{\Xi}^0 \Sigma^+ - \sqrt{2}\, K^0 \bar{\Sigma}^- \Xi^- - \sqrt{2}\, \bar{K}^0 \bar{\Xi}^- \Sigma^- \tag{2}$$

Entonces introduciendo

$$g_S + g_A \equiv g \qquad g_A / g \equiv \alpha \tag{3}$$

obtenemos

$$
\begin{aligned}
L(x) = \; & g \left[\bar{N} \vec{\tau} N \cdot \vec{\pi} - \bar{\vec{\Sigma}} \bar{K}_c \vec{\tau} \Xi - \bar{\Xi} \vec{\tau} K_c \cdot \vec{\Sigma} \right] \\
& + \frac{2}{\sqrt{3}} (1-\alpha) g \left[\bar{\vec{\Sigma}} \cdot \vec{\Sigma} \eta + \bar{\Lambda} \vec{\Sigma} \cdot \vec{\pi} + \bar{\vec{\Sigma}} \cdot \Lambda \vec{\pi} - \bar{\Lambda} \Lambda \eta \right] \\
& - \frac{1}{\sqrt{3}} (1-4\alpha) g \left[\bar{N} N \eta + \bar{\Xi} K_c \Lambda + \bar{\Lambda} \bar{K}_c \Xi \right] \\
& - \frac{1}{\sqrt{3}} (1+2\alpha) g \left[\bar{\Xi} \Xi \eta + \bar{N} K \Lambda + \bar{\Lambda} \bar{K} N \right] \\
& - (1-2\alpha) g \left[\bar{\Xi} \vec{\tau} \Xi \cdot \vec{\pi} - \bar{\vec{\Sigma}} \bar{K} \vec{\tau} N - \bar{N} \vec{\tau} K \cdot \vec{\Sigma} \right] \\
& - 2 i \alpha g \, (\bar{\vec{\Sigma}} \times \vec{\Sigma}) \cdot \vec{\pi} \qquad (1)
\end{aligned}
$$