

## Holography and Cosmology

W. FISCHLER<sup>1</sup>

<sup>1</sup>*Theory Group, Department of Physics, University of Texas  
Austin, TX 78712*

L. SUSSKIND<sup>2</sup>

<sup>2</sup>*Department of Physics, Stanford University  
Stanford, CA 94305-4060*

A cosmological version of the holographic principle is proposed. Various consequences are discussed including bounds on equation of state and the requirement that the universe be infinite.

## 1. Introduction

The recent revolution coming from string theory and black hole theory has taught us many unexpected things about the nature of spacetime and its relation to matter, energy and entropy. Such a conceptual paradigm shift must eventually have serious implications for cosmology.

Perhaps, the most radical modification of standard concepts required by the new theory is the holographic principle [1] [2] [3] [4]. The holographic principle requires that the degrees of freedom of a spatial region reside not in the interior as in an ordinary quantum field theory but on the surface of the region. Furthermore it requires the number of degrees of freedom per unit area to be no greater than 1 per Planck area. As a consequence, the entropy of a region must not exceed its area in Planck units. Thus far the holographic principle has not found its way into cosmology. In this paper we will try to correct this situation with some preliminary thoughts on how to formulate the holographic principle in cosmology and discuss some of its consequences.

## 2. Flat Universe

For most of this paper it will be assumed that spacetime is described by the usual Robertson-Walker cosmology with space being flat. The metric has the usual form.

$$ds^2 = dt^2 - a^2(t)dx^i dx^i \quad (2.1)$$

The number of spatial dimensions will be kept general so that  $i$  runs from 1 to  $d$ . The cosmological assumptions are the usual ones including homogeneity, isotropy and in the late universe constant entropy density in comoving coordinates.

Let us begin with a proposal that we will very quickly rule out; The entropy in any region of coordinate size  $\Delta x \sim R$  never exceeds the area. Since our assumptions require the entropy density to be constant, the entropy in a region is proportional to its coordinate volume  $R^d$ . Furthermore in flat space the surface area of the region grows with  $R$  like  $[Ra(t)]^{d-1}$ . Obviously when  $R$  becomes large enough, the entropy exceeds the area and the principle is violated.



A more sophisticated version goes as follows. Consider a spherical spatial region  $\Gamma$  of coordinate size  $R$  with boundary  $B$ . Now consider the light-like surface  $L$  formed by past light rays from  $B$  toward the center of  $\Gamma$ . There are three situations. If  $R$  is smaller than the coordinate distance to the cosmological horizon  $R_H$  then the surface  $L$  forms a light cone with its tip in the future of the singularity at  $t = 0$ . If  $R$  is the size of the horizon then  $L$  is still a cone but with its tip at  $t = 0$ . However if  $R > R_H$  the surface is a truncated cone. Now consider all the entropy (particles) which pass through  $L$ . For the first two cases this is the same as the entropy in the interior of  $\Gamma$  at the instant  $t$ . But in the last case the entropy within  $\Gamma$  exceeds the entropy passing through  $L$ . The proposed holographic principle is that the entropy passing through  $L$  never exceeds the area of the bounding surface  $B$ . It is not difficult to see that for the homogeneous case, this reduces to a single condition:

The entropy contained within a volume of coordinate size  $R_H$  should not exceed the area of the horizon in Planck units. In terms of the (constant) comoving entropy density  $\sigma$

$$\sigma R_H^d < [a R_H]^{d-1} \quad (2.2)$$

with every thing measured in Planck units. Note that both  $R_H$  and  $a$  are functions of time and that (2.2) must be true at all time.

Let us first determine whether (2.2) is true today. The entropy of the observable universe is of order  $10^{86}$  and the horizon size (age) of the universe is of order  $10^{60}$ . Therefore the ratio of entropy to area is much smaller than 1. Now consider whether it will continue to be true in the future. Assume that  $a(t) \sim t^p$ . The horizon size is determined by

$$R_H(t) = \int_0^t \frac{dt}{a(t)} \sim t^{1-p} \quad (2.3)$$

Thus in order for (2.2) to continue to be true into the remote future we must satisfy

$$p > \frac{1}{d} \quad (2.4)$$

In other words there is a lower bound on the expansion rate.

The bound on the expansion rate is easily translated to a bound on the equation of state. Assume that the equation of state has the usual form

$$P = \gamma \epsilon \quad (2.5)$$

where  $P$  and  $\epsilon$  are pressure and energy density. Standard methods yield a solution of the Einstein equations

$$a(t) \sim t^{\frac{2}{d(1+\gamma)}} \quad (2.6)$$

Thus  $p = \frac{2}{d(1+\gamma)}$  and the inequality (2.4) becomes

$$\gamma < 1 \quad (2.7)$$

The bound (2.7) is well known and follows from entirely different considerations. It describes the most incompressible fluid that is consistent with special relativity. A violation of the bound would mean that the velocity of sound exceeds the velocity of light. We will take this agreement as evidence that our formulation of the cosmological holographic principle is on the right track.

Although violating the bound (2.7) is impossible, saturating it is easy. We will give two examples. In the first example the energy density of the universe is dominated by a homogeneous minimally coupled scalar field. It is well known that in this case the pressure and energy density are equal.

Another example involves flat but anisotropic universes, for example the anisotropic toroidal (Kasner [5] [6]) universes with metric:

$$ds^2 = dt^2 - \sum_i t^{2p_i} dx_i^2 \quad (2.8)$$

The ratio entropy/area,  $S/A$ , for this case is easily evaluated to be:

$$S/A = \prod_i R_{H,i} / [\prod_j t^{p_j} t^{1-p_j}]^{d-1/d} \quad (2.9)$$

where  $R_{H,i} = t^{1-p_i}$  is the coordinate size of the horizon in direction  $i$ . The denominator in equation (2.9) is the proper area of the horizon.



Equation (2.9) gives:

$$S/A = t^{1-\sum_i p_i} \quad (2.10)$$

One of the two conditions on the exponents of the Kasner solutions is obtained by using the equation of state for the moduli of the torus,  $P = \epsilon$ . The exponents satisfy the following equation:

$$\sum_i p_i = 1 \quad (2.11)$$

it then follows from (2.10) and (2.11) that  $S/A$  for these flat anisotropic universes is constant in time. Thus depending on the boundary condition the holographic principle may be saturated by these universes.

The other equation for the  $p_i$  is  $\sum_i p_i^2 = 1$  and is obtained by using the Hubble equation.

Having established that the holographic principle will not be violated in the future we turn to the past. First consider the entropy-area ratio ( $\rho = S/A$ ) at the time of decoupling. Standard estimates give a ratio which is about  $10^6$  times bigger than today's ratio. Thus at decoupling

$$\rho(t_d) \sim 10^{-28} \quad (2.12)$$

During a radiation dominated era it is easily seen that  $\rho$  is proportional to  $t^{-\frac{1}{2}}$ .

$$\rho = 10^{-28} \left[ \frac{t_d}{t} \right]^{\frac{1}{2}} \quad (2.13)$$

Remarkably, in Planck units  $t_d^{\frac{1}{2}} \sim 10^{28}$  so that  $\rho < 1$  for all times later than the Planck time. The entropy in the universe is as large as it can be without the holographic principle having been violated in the early universe!

### 3. Non Flat Universe

Let us now turn to non flat universes and consider first the case of a closed universe where for simplicity we will restrict the discussion to  $3 + 1$  dimensions. The metric has the form

$$ds^2 = dt^2 - a^2(t)(d\chi^2 + \sin^2 \chi d\Omega^2) \quad (3.1)$$

where  $\chi$  and  $\Omega$  parametrize the three-sphere,  $S_3$ . The azimuthal angle of  $S_3$  is denoted by  $\chi$  and  $\Omega$  is the solid angle parametrizing the two-sphere at fixed  $\chi$ . We can then proceed in calculating the entropy/area,  $S/A$ , by following the procedure outlined in the flat case.

$$S/A = \frac{\chi_H - \sin 2\chi_H}{2a^2(\chi_H) \sin^2(\chi_H)} \quad (3.2)$$

where  $\chi_H = \int dt/a(t)$  is the coordinate size of the horizon.

As the universe recollapses after having reached its maximum size which is attained when  $\chi_H = \frac{\pi}{2}$  it will inevitably during recollapse saturate and then threaten to violate the holographic bound. This can be easily seen by considering an equation of state such that the energy density scales like  $a^{-2K}$ , with  $K > 1$ . Solving the Hubble equation gives

$$a(\chi_H)^{K-1} \sim \sin \chi_H \quad (3.3)$$

The holographic bound  $S/A < 1$  implies:

$$a^2(\chi_H) \geq \frac{\chi_H - \sin 2\chi_H}{2 \sin^2(\chi_H)} \quad (3.4)$$

So as  $\chi_H$  approaches  $\pi$ , the two previous equations (3.3) and (3.4) cannot be consistent. This seems to indicate that positively curved closed universes are inconsistent with the holographic principle. We do not know what new behaviour sets in to accomodate the holographic principle or if this violation of the holographic principle just excludes these universes as inconsistent?

The case of negatively curved open universes can be studied in a similar manner. At early times, both open and closed universes look flat as long as the energy density scales



like  $a^{-2K}$  and  $K > 1$ . During that period the holographic bound implies that the speed of sound  $< c$ . At later times the holographic bound for open universes does not put a strong bound on the equation of state. Since for "late" times the area and the volume grow in fixed proportion, there is then no bound on how slow the expansion has to be.

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