$$= i \frac{1}{W(0)} \int \mathcal{D}[\vec{\phi}]\ \underset{\sim}{\phi}_i(x)\ \underset{\sim}{\phi}_j(y)\ \exp\left\{ i \int d^4x\, \mathcal{L}(x) \right\} =$$

$$= i \frac{\langle \underset{\sim}{0} | T(\underset{\sim}{\phi}_i(x)\ \underset{\sim}{\phi}_j(y)) | \underset{\sim}{0} \rangle}{\langle \underset{\sim}{0} | \underset{\sim}{0} \rangle} = i\ \underset{\sim}{G}_{c\,ij}(x,y) \qquad (1)$$

where $\underset{\sim}{G}$ means the green function for the $\vec{\underset{\sim}{\phi}}$ fields. More generally we have

$$\left. \frac{\delta^m Z[\vec{J}]}{\delta J_{i_1}(x_1) \cdots\cdots \delta J_{i_m}(x_m)} \right|_{\vec{J}=0} = i^{m-1}\ \underset{\sim}{G}_{c\,i_1 \dots i_m}(x_1, \dots, x_m) \qquad (2)$$

Taking into account (3.1) this is equivalent to

$$G_{c\,i_1 \dots i_m}(x_1, \dots x_m) = \underset{\sim}{G}_{c\,i_1 \dots i_m}(x_1, \dots, x_m) \qquad (3)$$

which can be shown by induction. Let us check the case $m=4$

$$\underset{\sim}{G}_{c\,abcd}(x_1 x_2 x_3 x_4) = \frac{\langle \underset{\sim}{0} | T(\underset{\sim}{\phi}_a(x_1)\ \underset{\sim}{\phi}_b(x_2)\ \underset{\sim}{\phi}_c(x_3)\ \underset{\sim}{\phi}_c(x_4)) | \underset{\sim}{0} \rangle^c}{\langle \underset{\sim}{0} | \underset{\sim}{0} \rangle} =$$

$$= \frac{\langle \underset{\sim}{0} | T(\underset{\sim}{\phi}_a(x_1)\ \underset{\sim}{\phi}_b(x_2)\ \underset{\sim}{\phi}_c(x_3)\ \underset{\sim}{\phi}_d(x_4)) | \underset{\sim}{0} \rangle}{\langle \underset{\sim}{0} | \underset{\sim}{0} \rangle} - \frac{\langle \underset{\sim}{0} | T(\underset{\sim}{\phi}_a(x_1)\ \underset{\sim}{\phi}_b(x_2)) | \underset{\sim}{0} \rangle}{\langle \underset{\sim}{0} | \underset{\sim}{0} \rangle} \frac{\langle \underset{\sim}{0} | T(\underset{\sim}{\phi}_c(x_3)\ \phi_d(x_4)) | \underset{\sim}{0} \rangle}{\langle \underset{\sim}{0} | \underset{\sim}{0} \rangle}$$

$$+ \text{permut} = \frac{\langle \underset{\sim}{0} | T(\phi_a(x_1)\ \phi_b(x_2)\ \phi_c(x_3)\ \phi_d(x_4)) | \underset{\sim}{0} \rangle}{\langle \underset{\sim}{0} | \underset{\sim}{0} \rangle} - v_a \frac{\langle \underset{\sim}{0} | T(\phi_b(x_2)\ \phi_c(x_3)\ \phi_d(x_4)) | \underset{\sim}{0} \rangle}{\langle \underset{\sim}{0} | \underset{\sim}{0} \rangle}$$

$$+ \text{perm.} + v_a v_b \frac{\langle \underset{\sim}{0} | T(\phi_c(x_3)\ \phi_d(x_4)) | \underset{\sim}{0} \rangle}{\langle \underset{\sim}{0} | \underset{\sim}{0} \rangle} - 4 v_a v_b v_c v_d + v_a v_b v_c v_d$$

$$- \frac{\langle \underset{\sim}{0} | T(\phi_a(x_1)\ \phi_b(x_2)) | \underset{\sim}{0} \rangle}{\langle \underset{\sim}{0} | \underset{\sim}{0} \rangle} \frac{\langle \underset{\sim}{0} | T(\phi_c(x_3)\ \phi_d(x_4)) | \underset{\sim}{0} \rangle}{\langle \underset{\sim}{0} | \underset{\sim}{0} \rangle} + \text{perm} +$$

$$+ v_a v_b \frac{\langle \underset{\sim}{0} | T(\phi_c(x_3)\ \phi_d(x_4) | \underset{\sim}{0} \rangle}{\langle \underset{\sim}{0} | \underset{\sim}{0} \rangle} + \text{perm.} - 3 v_1 v_2 v_3 v_4 =$$

$$= G_{c\,abcd}(x_1, x_2, x_3, x_4).$$

We shall now define the Legendre transform $\Gamma(\vec{\Phi})$ of $Z[\vec{J}]$. It is defined as

$$\Gamma[\vec{\Phi}] = Z[\vec{J}] - \int d^4x \; \vec{J}(x) \cdot \vec{\Phi}(x). \tag{1}$$

Notice

$$\exp\left\{ i \left[\Gamma[\vec{\Phi}] + \int d^4x \; \vec{J}(x) \cdot \vec{\Phi}(x) \right] \right\} = \exp\left\{ i Z[\vec{J}] \right\} =$$

$$= W[\vec{J}] = \int [\mathcal{D}\vec{\phi}] \exp\left\{ i \int d^4x \left[\mathcal{L}(x) + \vec{J}(x) \cdot \vec{\phi}(x) \right] \right\} \tag{2}$$

For no more that quadratic Lagrangians this shows that $\Gamma[\vec{\Phi}]$ is the effective action calculated along the classical path : $\vec{\Phi}(x)$ is the classical field.

Furthermore in (1) $\vec{J}(x)$ must be expressed as a functional of $\vec{\Phi}(x)$ from inverting eq (9.3) :

$$\frac{\delta Z[\vec{J}]}{\delta J_i(x)} = \Phi_i(x) \tag{3}$$

The Legendre transform (1) is a functional version of the well-known transformation familiar in classical mechanics and thermodynamics. By differentiating (1) with respect to $\Phi_i(x)$

$$\frac{\delta \Gamma[\vec{\Phi}]}{\delta \Phi_i(x)} = \sum_j \int d^4y \; \frac{\delta Z[\vec{J}]}{\delta J_j(y)} \frac{\delta J_j(y)}{\delta \Phi_i(x)} - J_i(x) - \sum_j \int d^4z \; \frac{\delta J_j(z)}{\delta \Phi_i(x)} \Phi_j(z)$$

and using (3) we obtain

$$\frac{\delta \Gamma[\vec{\Phi}]}{\delta \Phi_i(x)} = - J_i(x) \tag{4}$$

which is the dual of (3). The classical field and the classical source are duals. Equation (4) which expresses $\vec{\Phi}(x)$ in terms of $\vec{J}(x)$ is the inverse of eq (3) which expresses $\vec{J}(x)$ in terms of $\vec{\Phi}(x)$. This in particular means that eq (9.4) can be written as

$$\left. \frac{\delta \Gamma[\vec{\Phi}]}{\delta \Phi_i(x)} \right|_{\vec{\Phi} = \vec{v}} = 0 \tag{5}$$

i.e. when $\vec{J} = 0$, $\vec{\Phi}$ takes the value $\vec{v}$ and viceversa. Eq. (5) is very important. It expresses the vacuum expectation value $\vec{v}$ of the field $\vec{\phi}$ as the solution of a variational problem : $\vec{v}$ is the value of $\vec{\Phi}$ which extremizes $\Gamma[\vec{\Phi}]$.

What is the physical significance of Γ? To streamline our discussion, let us agree on the following convention : We will denote by subscripts $i, j, \ldots$ any labels that $\vec{J}$ or $\vec{\Phi}$ carry, including the space-time variable x. We will adopt the convention that summations and integrations are always to be carried out over repeated indices. Differentiating eq (11.3) with respect to Φ we obtain

$$\frac{\delta^2 Z[\vec{J}]}{\delta J_j \, \delta J_i} \, \frac{\delta J_j}{\delta \Phi_k} = \delta_{ik} \tag{1}$$

From eq. (11.4) we learn that

$$\frac{\delta J_i}{\delta \Phi_j} = - \frac{\delta^2 \Gamma[\vec{\Phi}]}{\delta \Phi_j \, \delta \Phi_i} \tag{2}$$

Define

$$\{ \Sigma^{-1} [\vec{J}] \}_{ij} \equiv - \frac{\delta^2 Z[\vec{J}]}{\delta J_i \, \delta J_j} \tag{3}$$

and

$$\{ \Sigma [\vec{\Phi}] \}_{ij} \equiv \frac{\delta^2 \Gamma[\vec{\Phi}]}{\delta \Phi_i \, \delta \Phi_j} \tag{5}$$

Then (1) means, using (2), that

$$\{ \Sigma^{-1} [\vec{J}] \}_{ij} \{ \Sigma [\vec{\Phi}] \}_{jk} = \delta_{ik} \tag{6}$$

Since

$$\{ \Sigma^{-1} [\vec{J} = 0] \}_{ij} = - \left. \frac{\delta^2 Z[\vec{J}]}{\delta J_i \, \delta J_j} \right|_{\vec{J}=0} = - i \, \underset{\sim}{G}_{c\,ij} \equiv [\underset{\sim}{\Delta}_F]_{ij} \tag{7}$$

is the full propagator for the field $\underset{\sim}{\phi}$, and as $\vec{J} = 0$ implies $\vec{\Phi} = \vec{v}$, it follows that

$$\{ \Sigma [\vec{\Phi} = \vec{v}] \}_{ij} = \left. \frac{\delta^2 \Gamma[\Phi]}{\delta \Phi_i \, \delta \Phi_j} \right|_{\vec{\Phi} = \vec{v}} = [\underset{\sim}{\Delta}_F^{-1}]_{ij} \tag{8}$$

is the inverse of the full propogator.

Next differentiate eq (1) with respect to J_ℓ. We obtain

$$\frac{\delta^3 Z[\vec{J}]}{\delta J_i \delta J_j \delta J_e} \frac{\delta J_j}{\delta \Phi_k} + \frac{\delta^2 Z[\vec{J}]}{\delta J_i \delta J_j} \frac{\delta^2 J_j}{\delta \Phi_k \delta \Phi_r} \frac{\delta \Phi_r}{\delta J_e} = 0$$

and using the above given equations

$$- \frac{\delta^3 Z[\vec{J}]}{\delta J_i \delta J_j \delta J_e} \Sigma_{jk} - [\Sigma^{-1}]_{ij} \frac{\delta^3 \Gamma[\vec{\Phi}]}{\delta \Phi_j \delta \Phi_k \delta \Phi_r} [\Sigma^{-1}]_{er} = 0$$

or

$$\frac{1}{i^2} \frac{\delta^3 Z[\vec{J}]}{\delta J_i \delta J_j \delta J_k} = [i \Sigma^{-1}]_{ie} [i \Sigma^{-1}]_{jr} [i \Sigma^{-1}]_{ks} \left[i \frac{\delta^3 \Gamma[\vec{\Phi}]}{\delta \Phi_e \delta \Phi_r \delta \Phi_s} \right] \tag{1}$$

and taking now the limit $\vec{J} = 0$ and $\vec{\Phi} = \vec{v}$ we obtain, since $\Sigma^{-1}[\vec{J} = 0]$ is the full propapoda, that

$$\left. \frac{\delta^3 \Gamma[\vec{\Phi}]}{\delta \Phi_i \delta \Phi_j \delta \Phi_k} \right|_{\vec{\Phi} = \vec{v}} = \Gamma^{(3)}_{ijk} \tag{2}$$

is the three point proper vertex. A proper vertex (or one-particle irreducible vertex) is a green's function which cannot be made disconnected by cutting a single internal propagator, and from which (by convention) full propagators corresponding to external lines are removed. The three point function has no such disconnected graphs except corrections to the propagators, which are explicitly removed in (1). In general, the n th derivative of Γ at $\vec{\Phi} = \vec{v}$ is the m-point proper vertex

$$\left. \frac{\delta^m \Gamma[\vec{\Phi}]}{\delta \Phi_{i_1} \delta \Phi_{i_2} \cdots \delta \Phi_{i_m}} \right|_{\Phi = \vec{v}} = \Gamma^{(m)}_{i_1 i_2 \cdots i_m} \tag{3}$$

The proof of this statement proceeds inductively. Assume that $\delta^m Z[\vec{J}] / \delta J_{i_1} \cdots \delta J_{i_m}$ can be expressed as a sum of tree diagrams, each diagram consisting of proper vertices corresponding to $\delta^m \Gamma[\vec{\Phi}] / \delta \Phi_{i_1} \cdots \delta \Phi_{i_m} |_{\vec{\Phi} = \vec{v}}$, internal lines corresponding to $\underset{\sim}{\Delta}_F$ connecting pairs of proper vertices, and external lines. In particular

$$\frac{1}{i^{m-1}} \frac{\delta^m Z[\vec{J}]}{\delta J_{i_1} \cdots \delta J_{i_m}} = [i \Sigma^{-1}]_{i_1 j_1} \cdots [i \Sigma^{-1}]_{i_m j_m} \left[i \frac{\delta^m \Gamma[\vec{\Phi}]}{\delta \Phi_{j_1} \cdots \delta \Phi_{j_m}} \right] +$$

$$+ \text{ one particle reducible terms} \tag{4}$$

Now differentiate the last equation with respect to J_k. Recall that

$$\frac{\delta}{\delta J_k} = \frac{\delta \Phi_r}{\delta J_k} \frac{\delta}{\delta \Phi_r} = \frac{\delta^2 Z[\vec{J}]}{\delta J_k \, \delta J_r} \frac{\delta}{\delta \Phi_r} = - [\, \Sigma^{-1}]_{kr} \frac{\delta}{\delta \Phi_r} \tag{1}$$

The differential operator $\delta / \delta \Phi_r$ when applied to the right-hand side of (13.4) can act either on some Σ^{-1} or on some $\delta^m \Gamma[\vec{\Phi}] / \delta \Phi_{i_1} \cdots \delta \Phi_{i_m}$. In the former case, we have (see (13.1))

$$\frac{1}{i} \frac{\delta}{\delta J_i} [\, i \Sigma^{-1}]_{k\ell} = [\, i \Sigma^{-1}]_{km} [\, i \Sigma^{-1}]_{\ell n} [\, i \Sigma^{-1}]_{ij} \left[i \frac{\delta^3 \Gamma [\vec{\Phi}]}{\delta \Phi_m \, \delta \Phi_n \, \delta \Phi_j} \right] \tag{2}$$

which amounts to adding a new external line to a newly created three point vertex, and in the latter

$$\frac{1}{i} \frac{\delta}{\delta J_i} \frac{\delta^m \Gamma[\vec{\Phi}]}{\delta \Phi_{i_1} \cdots \delta \Phi_{i_m}} = [\, i \Sigma^{-1}]_{ij} \frac{\delta^{m+1} \Gamma[\Phi]}{\delta \Phi_j \, \delta \Phi_{i_1} \cdots \delta \Phi_{i_m}}$$

which amounts to adding a new external line to what used to be an m-point proper vertex. In any case, when the differential operator of (1) is applied to (13.4) we generate all tree diagrams in the $(n+1)$-point Green's function, and

$$\frac{1}{i^m} \frac{\delta^{m+1} Z[\vec{J}]}{\delta J_k \, \delta J_{i_1} \cdots \delta J_{i_m}} = [\, i \Sigma^{-1}]_{kr} [\, i \Sigma^{-1}]_{i_1 j_1} \cdots [\, i \Sigma^{-1}]_{i_m j_m} \left[i \frac{\delta^{m+1} \Gamma [\vec{\Phi}]}{\delta \Phi_r \, \delta \Phi_{j_1} \cdots \delta \Phi_{j_m}} \right] +$$

$$+ \text{ one-particle reducible terms}$$

Therefore, in the limit $\vec{J} = 0$, $\vec{\Phi} = \vec{v}$ we recover (13.3) with $n \to n+1$. Q.E.D.

The generating functional of proper vertice $\Gamma[\vec{\Phi}]$ has the representation

$$\Gamma[\vec{\Phi}] = \sum_{n=2}^{\infty} \frac{1}{n!} \Gamma^{(n)}_{i_1 \cdots i_n} (\Phi - v)_{i_1} \cdots (\Phi - v)_{i_n} \tag{3}$$

with

$$\Gamma^{(2)}_{i_1 i_2} = [\, \underset{\sim}{\Delta}_F^{-1}]_{i_1 i_2} \tag{4}$$

Let us revert to the standard notation

$$\Gamma^{(n)}_{i_1 \cdots i_n} \equiv \Gamma^{(n)}_{i_1 \cdots i_n} (x_1, x_2, \cdots, x_n) \tag{5}$$

Because of the translational invariance $\Gamma^{(m)}$ depends only on $(m-1)$ differences $x_i - x_j$, so that its Fourier transform $\hat{\Gamma}^{(m)}$ is defined as

$$\hat{\Gamma}^{(m)}_{i_1 i_2 \cdots i_m}(p_1, p_2, \cdots, p_m)\,(2\pi)^4\,\delta(p_1 + p_2 + \cdots + p_m) =$$

$$= \int \prod_{i=1}^{m} d^4x_i\, e^{i p_i \cdot x_i}\, \Gamma^{(m)}_{i_1 i_2 \cdots i_m}(x_1, x_2, \cdots, x_m) \tag{1}$$

This means that four-momentum must be conserved at all vertices.

In discussing the implications of all that it is convenient to consider the case in which the source is translation invariant. Since it is a classical object this implies that it is a constant and $\vec{\Phi}$ is also constant. Let us call it $\vec{\Phi}_c$. Let us now define the effective potential or superpotential $\mathcal{V}(\vec{\Phi}_c)$ by

$$\Gamma[\vec{\Phi} = \vec{\Phi}_c] \equiv -(2\pi)^4\,\delta^{(4)}(0)\,\mathcal{V}(\vec{\Phi}_c) \tag{2}$$

which is a function and not a functional. Recalling that (14.3) means

$$\Gamma[\vec{\Phi}] = \sum_{m=2}^{\infty} \frac{1}{m!} \sum_{i_1} \cdots \sum_{i_m} \int d^4x_1 \cdots d^4x_m\, \Gamma^{(m)}_{i_1 \cdots i_m}(x_1 \cdots x_m).\,(\Phi_{i_1}(x_1) - v_{i_1}) \cdots (\Phi_{i_m}(x_m) - v_{i_m}) \tag{3}$$

then

$$\Gamma[\vec{\Phi}_c] = \sum_{m=2}^{\infty} \frac{1}{m!} \sum_{i_1} \cdots \sum_{i_m} \int d^4x_1 \cdots d^4x_m\, \Gamma^{(m)}_{i_1 \cdots i_m}(x_1 \cdots x_m)\,(\Phi_c - v)_{i_1} \cdots (\Phi_c - v)_{i_m} \tag{4}$$

and from equation (1)

$$\mathcal{V}(\vec{\Phi}_c) = -\sum_{m=2}^{\infty} \frac{1}{m!}\, \hat{\Gamma}^{(m)}_{i_1 \cdots i_m}(0, \cdots, 0)\,(\Phi_c - v)_{i_1} \cdots (\Phi_c - v)_{i_m} \tag{5}$$

where summation over repeated indices must be understood. Then

$$\left.\frac{d^m \mathcal{V}(\vec{\Phi}_c)}{d\Phi_{c i_1} \cdots d\Phi_{c i_m}}\right|_{\vec{\Phi}_c = \vec{v}} = -\hat{\Gamma}^{(m)}_{i_1 \cdots i_m}(0, \cdots, 0) \tag{6}$$

La condición (11.5) translates into

$$\left.\frac{d\,\mathcal{V}(\Phi_c)}{d\,\Phi_{ci}}\right|_{\vec{\Phi}_c = \vec{v}} = 0 \tag{7}$$

Furthermore

$$\left.\frac{d^2 \mathcal{V}(\vec{\Phi}_c)}{d\Phi_{ci}\, d\Phi_{cj}}\right|_{\vec{\Phi}_c = \vec{v}} = - \left[\hat{\Delta}_F^{-1}(0) \right]_{ij} \qquad (1)$$

is positive semi-definite, since $\hat{\Delta}_F^{-1}$ behaves like $(p^2 - m^2)$ near $p^2 \simeq m^2$ and it cannot have any other zero for $p^2 < m^2$. Thus the vacuum expectation value $\vec{\Phi}_c = \vec{v}$ must be the value of $\vec{\Phi}_c$ which minimizes $\mathcal{V}(\vec{\Phi}_c)$.

When $\mathcal{L}$ is invariant under the infinitesimal transformation

$$\phi_i \longrightarrow \phi_i - i\delta\theta^\alpha L^\alpha_{ij} \phi_j \qquad (2)$$

it follows from the structure of (9.1) that $Z[J]$ is invariant under

$$J_i \longrightarrow J_i - i\delta\theta^\alpha L^\alpha_{ij} J_j \qquad (3)$$

(because the antisymmetry of L^α_{ij}) and so on, and finally the superpotential $\mathcal{V}(\vec{\Phi}_c)$ is an invariant function of $\vec{\Phi}_c$ under the above transformation.

The analysis of spontaneous symmetry breaking and Goldstone theorem which is always performed on a classical level (zeroth order in Q.F.T.) can now be taken over verbatim to all orders of perturbation theory in Q.F.T. The old potential V is now the superpotential $\mathcal{V}$. $\mathcal{V}(\vec{\Phi}_c)$ is symmetric, but $\vec{v}$ transforms exactly like $\vec{\phi}$. However, if the vacuum is invariant under the symmetry (unique vacuum) the group generator has to annihilate the vacuum and $\vec{v}$ is invariant. This is thus only possible if $\vec{v} = 0$

$$\text{Invariant vacuum} \Longleftrightarrow \vec{v} = 0$$

We can construct $Z[\vec{J}]$, $\Gamma[\vec{\Phi}]$ and $\mathcal{V}(\vec{\Phi}_c)$ in perturbation theory. Of course not weak coupling perturbation theory (expansion in $\mathcal{L}_I$) since we showed that then $\vec{v} = 0$. For simplicity we shall consider the case of a single-component field. An effective way of expanding these quantities in a series is to write (9.1) with a fictitious parameter a:

$$\exp\{ i Z[\vec{J}] \} = \int [\mathcal{D}\phi] \exp\left\{ i \int d^4x \left[\frac{1}{a} \mathcal{L}(x) + J(x)\phi(x) \right] \right\} \sim$$

$$\sim \exp\left\{ i \int d^4x \frac{1}{a} \mathcal{L}_I \left[\frac{1}{i} \frac{\delta}{\delta J(x)} \right] \right\} \exp\left\{ \frac{1}{2} \int d^4x\, d^4y\, a\, J(x) \Delta_F(x-y) J(y) \right\} \qquad (4)$$

and expand $Z[J]$ in powers of a and let $a=1$ afterwards. Since each propagator is multiplied by a and each vertex by $1/a$ a Feynman diagram with E external lines, I internal lines and V vertices is multiplied by a factor a^{E+I-V}. There is a topological relation that holds for any Feynman diagram. It is

$$L = I - V + 1 \qquad (1)$$

where L is the number of loops (i.e. the number of independent fourmomentum integrations) in the diagram. Therefore the expansion in this fictitious parameter a corresponds to expanding a Green's function in the number of loops in the Feynman diagram a^{E+L-1}. The reason this expansion is preferable over the expansion in powers of some coupling constant is that in the former any symmetry of the Lagrangian is preserved in each order of perturbation theory since, effectively a multiplies the whole Lagrangian. Notice that all graphs of n loops or less include all graphs of n th order or less in the coupling constant. Since $1/a$ multiplies the whole Lagrangian this expansion is unaffected by shifts of fields and by redefinitions of free and interacting Lagrangians associated with these fields. It enables one to survey all possible vacuum states at once, and to compute higher order corrections before deciding which vacuum the theory finally picks.

In the following we shall discuss explicit constructions of Z, Γ and V in the first two order of loop expansion for a simple model

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \phi)(\partial_\mu \phi) - \frac{1}{2}\mu_0^2 \phi^2 + \frac{1}{4!}\lambda_0 \phi^4 \qquad (2)$$

The method can be generalized easily to other models. Our discussion will not show that our construction is in fact the expansion in the number of loops, but the interested student can convince himself of this fact by first referring to Nambu's paper (Y. Nambu Phys. Lett. 26B, 626 (1966)) which shows that the loop expansion is also an expansion in the Planck constant $\hbar$, and then noting that our method is an asymptotic evaluation of these quantities in $\hbar$.

Imagine that eq. (9.1) is written in the Euclidean space as explained before. Since the exponent in the r.h.s. is bounded from above in this case, we are tempted to evaluate the functional integral by the method of steepest descent. We shall keep the Minkowski notation for simplicity, but the ultimate justification of this method lies in the Euclidicity postulate.

We shall expand the exponent of the right-hand side of (9.1)

$$S[\phi] + \int d^4x \; J(x)\,\phi(x) = \int d^4x \; [\, \mathcal{L}(x) + J(x)\,\phi(x) \,] \tag{1}$$

about a point $\phi(x) = \phi_0(x)$

$$S[\phi] + \int d^4x \; J(x)\,\phi(x) = S[\phi_0] + \int d^4x \; J(x)\,\phi_0(x) +$$

$$+ \int d^4x \left[\frac{\delta S[\phi_0]}{\delta \phi_0(x)} + J(x) \right] [\,\phi(x) - \phi_0(x)\,] +$$

$$+ \frac{1}{2!} \int d^4x \, d^4y \; \frac{\delta^2 S[\phi_0]}{\delta\phi_0(x)\,\delta\phi_0(y)} \; [\,\phi(x) - \phi_0(x)\,]\,[\,\phi(y) - \phi_0(y)\,] + \cdots \tag{2}$$

and we choose $\phi_0(x)$ so that the linear term in $[\phi(x) - \phi_0(x)]$ is missing from the expansion (2). This means that

$$\frac{\delta S[\phi_0]}{\delta \phi_0(x)} = - J(x) \tag{3}$$

which means that ϕ_0 is the solution of the classical (non-quantized) field equation in the presence of an external source $J(x)$. In our case (3) can be written

$$[\, \partial_\mu \partial^\mu + \mu_0^2 \,]\,\phi_0(x) + \frac{1}{6}\,\lambda_0\,\phi_0^3(x) = J(x) \tag{4}$$

In any case $\phi_0(x)$ is obtained from (3) as a functional of the external source. When (2) is substituted in (9.1) we obtain.

$$\exp(\,i\,Z[J]\,) = \exp\left\{ i\,S[\phi_0] + i \int d^4x \; J(x)\,\phi_0(x) \right\} \cdot$$

$$\cdot \int \mathcal{D}[\phi] \, \exp\left\{ i \left[\int d^4x \, d^4y \; \frac{1}{2!} \, \frac{\delta^2 S[\phi_0]}{\delta\phi_0(x)\,\delta\phi_0(y)} \, (\phi(x) - \phi_0(x))\,(\phi(y) - \phi_0(y)) + \cdots \right] \right\} \tag{5}$$

The lowest order approximation (which is one order lower that the steepest decent approximation) is obtained if we ignore the functional integral over $\phi(x)$ altogether and set

$$Z[J] \simeq S[\phi_0] + \int d^4x \; J(x)\,\phi_0(x) \equiv Z^0[J] \tag{6}$$

which is a functional of J only, because ϕ_0 is a functional of J. We can evaluate $Z^0[J]$ by solving eq (18.4) and then substituting ϕ_0 in (18.6). Eq (18.4) can be solved in powers of $\lambda \equiv \lambda_0$ (classical eq ≡ no renormalization)

$$\phi_0(x) = -\int d^4y\, \Delta_F(x-y;\mu^2)\left[J(y) - \frac{1}{6}\lambda\left(\int d^4z\, \Delta_F(x-z)\, J(z)\right)^3 + \cdots\right] \tag{1}$$

where the use of Δ_F is dictated by the Euclidicity postulate. When eq (1) is substituted in (18.6), one finds that $Z^0[J]$ is the generating functional of Green's functions in the tree (i.e. no loop) approximation

$$Z^0[J] = -\frac{1}{2}\int d^4x\, d^4y\, J(x)\, \Delta_F(x-y;\mu^2)\, J(y) +$$

$$+ \frac{\lambda}{4!}\int d^4u\left[\int d^4x\, J(x)\, \Delta_F(x-u;\mu^2)\right]^4 + \cdots \tag{2}$$

We can see more readily that Z^0 is the tree approximation to Z if we compute $\Gamma[\Phi]$ in this approximation. Since

$$\Phi(x) = \frac{\delta Z[J]}{\delta J(x)} \simeq \frac{\delta Z^0[J]}{\delta J(x)} = \int d^4y \left\{ \frac{\delta S[\phi_0]}{\delta \phi_0(y)} \frac{\delta \phi_0(y)}{\delta J(x)} + J(y)\frac{\delta \phi_0(y)}{\delta J(x)} \right\} + \phi_0(x)$$

and to this order (use (18.3)) we obtain

$$\Phi(x) = \phi_0(x) \tag{3}$$

Therefore

$$\Gamma^0[\Phi] = Z^0[J] - \int d^4x\, J(x)\, \phi_0(x) =$$

$$= \left\{ S[\Phi] + \int d^4x\, J(x)\, \Phi(x) \right\} - \int d^4x\, J(x)\, \phi_0(x) = S[\phi] \tag{5}$$

So, to this order, proper vertices are generated by the Lagrangian itself and Green's functions are built up of these unmodified vertices by the rules of tree graphs. The superpotential $\mathcal{V}$ is, to this order,

$$\mathcal{V}(\Phi_c) = -\frac{1}{(2\pi)^4\, \delta^{(4)}(0)}\, S(\Phi_c) = V(\Phi_c) \tag{5}$$

where V is the negative of the part of the Lagrangian which is independent of the derivative fields. That is, $V(\phi)$ is the potential of the field ϕ. This justifies the name of "superpotential" for $\mathcal{V}$.

We can proceed further by applying the steepest descent method to the functional integral in (18.5). This consist in neglecting term higher than quadratic in $(\phi - \phi_0)$ in the exponent of the integral and performing the functional gaussian integration. In this way we obtain

$$\int \mathcal{D}[\phi] \exp\left\{ i \int d^4x\, d^4y\, \frac{1}{2} \frac{\delta^2 S[\phi_0]}{\delta\phi_0(x)\,\delta\phi_0(y)} (\phi(x) - \phi_0(x))(\phi(y) - \phi_0(y)) \right\} \simeq$$

$$\simeq \left[\det \frac{\delta^2 S[\phi_0]}{\delta\phi_0(x)\,\delta\phi_0(y)} \right]^{-1/2} = \exp\left\{ -\frac{1}{2} \operatorname{Tr} \ln \frac{\delta^2 S[\phi_0]}{\delta\phi_0(x)\,\delta\phi_0(y)} \right\} \qquad (1)$$

where we have used

$$\det(1 + A) = \exp\{ \operatorname{Tr} \ln (1 + A) \} \qquad (2)$$

This is equivalent to $\ln \det (1+A) = \operatorname{Tr}(1+A)$ and this is true as it can be proved by series expansion. So that

$$Z[J] \simeq Z^0[J] + \frac{i}{2} \operatorname{Tr} \ln \frac{\delta^2 S[\phi_0]}{\delta\phi_0(x)\,\delta\phi_0(y)} \equiv Z^1[J] \qquad (3)$$

For the Lagrangian that we are working with, we have

$$\frac{\delta^2 S[\phi]}{\delta\phi(x)\,\delta\phi(y)} = \left[-\partial_\mu \partial^\mu - \mu^2 - \frac{1}{2} \lambda \phi^2(x) \right] \delta^{(4)}(x-y) \qquad (4)$$

so that, taking into account that only the ϕ-dependent part is interesting

$$\frac{i}{2} \operatorname{Tr} \ln \frac{\delta^2 S[\phi]}{\delta\phi(x)\,\delta\phi(y)} \simeq \frac{i}{2} \operatorname{Tr} \ln \left| 1 - \frac{1}{2} \lambda \frac{1}{-\partial_\mu \partial^\mu - \mu^2 + i\epsilon} \phi^2(x) \right| =$$

$$= -\frac{i}{2} \sum_{n=1} \left(\frac{\lambda}{2} \right)^n \frac{1}{n} \int d^4x_1 \cdots d^4x_n\, \Delta_F(x_1 - x_2)\, \phi^2(x_2)\, \Delta_F(x_2 - x_3) \cdots \Delta_F(x_n - x_1)\, \phi^2(x_1) \qquad (5)$$

Let us now construct $\Gamma[\Phi]$ to this order

$$\Gamma^1[\Phi] \equiv Z^1[J] - \int d^4x\, J(x)\, \Phi(x) \qquad (6$$

where

$$\Phi(x) = \frac{\delta Z^1[J]}{\delta J(x)} \equiv \phi_0(x) + \epsilon(x) \tag{1}$$

and $\epsilon(x)$ is given by

$$\epsilon(x) = \frac{\delta}{\delta J(x)} \frac{i}{2} \operatorname{Tr} \ln \frac{\delta^2 S[\phi_0]}{\delta\phi_0(\xi)\,\delta\phi_0(\eta)} \tag{2}$$

Fortunately, it is not necessary to know the form of $\epsilon(x)$ to construct $\Gamma(\Phi)$ to first order in $\epsilon(x)$. Indeed, note that

$$Z^0[J] = S[\phi_0] + \int d^4x\; J(x)\,\phi_0(x) =$$

$$= S[\Phi] + \int d^4x\; J(x)\,\Phi(x) - \int d^4y \left\{ \frac{\delta S[\phi_0]}{\delta\phi_0(y)} + J(y) \right\} \epsilon(y) + O(\epsilon^2)$$

$$= S[\Phi] + \int d^4x\; J(x)\,\Phi(x) + O(\epsilon^2) \tag{3}$$

So that to order ϵ we have

$$\Gamma^1[\Phi] = S[\Phi] + \frac{i}{2} \operatorname{Tr} \ln \frac{\delta^2 S[\Phi]}{\delta\Phi(\xi)\,\delta\Phi(\eta)} \tag{4}$$

where the second term is the one loop correction to the generating functional of proper vertices. Quadratic quantum corrections are of the same order as two loops corrections and we thus neglected them.

. The superpotential $\mathcal{V}$ can then be evaluated explicitly using (4) and (20.5)

$$\mathcal{V}(\Phi_c) = \frac{1}{2}\mu_0^2\,\Phi_c^2 + \frac{1}{4!}\lambda_0\,\Phi_c^4 + \frac{i}{2}\int \frac{d^4k}{(2\pi)^4} \sum_{n=1}^{\infty} \frac{1}{n} \left(-\frac{1}{2}\lambda \frac{\Phi_c^2}{k^2-\mu^2+i\epsilon} \right)^n \tag{5}$$

The term for $n = 1$ and 2 are divergent. However these terms are proportional to Φ_c^2 and Φ_c^4 and the divergences can be amalgamated with μ_0^2 and λ_0. We may write

$$\mathcal{V}(\Phi_c) = \frac{1}{2}\mu^2\,\Phi_c^2 + \frac{1}{4!}\lambda\,\Phi_c^4 + J(\Phi_c)$$

$$J(\Phi_c) = \frac{i}{2}\int \frac{d^4k}{(2\pi)^4} \sum_{n=3}^{\infty} \frac{1}{n} \left(-\frac{1}{2}\lambda \frac{\Phi_c^2}{k^2-\mu^2+i\epsilon} \right)^n \tag{6}$$

and μ^2 and λ are defined as the value of the two and four point vertices at the point where all external momenta vanish.

The idea of using generating functionals for Green's functions and proper vertexes was originated by J.J. Schwinger Proc. Nat. Acad. Sci 37, 452, 455 (1951)

The following paper contains the first explicit construction of the generating functional for proper vertices by the Legendre transformation method

G. Jona-Lasinio Nuov. Cim. 34, 1790 (1965)

This paper also contains the derivation of the Goldstone theorem by this technique.

Recent reviews of this method may be found in

B.W. Lee. Chiral Dynamics (Gordon and Breach N.Y. 1972)

H.M. Fried. Functional Methods and Models in Quantum Field Theory (MIT Press Cambridge 1972)

B. Zumino in Lectures on Elementary particles and Q.F.T. (MIT. Press Cambridge 1970) Eds Deser, Grisaru and Pendleton.

The observation that the expansion in the number of loops is equivalent to the expansion in $\hbar$ is due to

Y. Nambu. Phys. Letters 26B, 626 (1966)

L.S. Brown and D. Boulware Phys. Rev. 172, 1628 (1968)

The evaluation of the one-loop corrections by the steepest descent approximation is discussed in

B.W. Lee and J. Zinn-Justin Phys. Rev. D5, 3121 (1972)

3 THE COLEMAN-WEINBERG THEORY

S. Coleman and E. Weinberg [Phys. Rev. D7, 1888 (1973)] have shown that in massless theories higher order corrections break the symmetry of the vacuum providing thus a dynamical mechanism to give masses to the massless fields.

G. Jona-Lasinio [Nuo Cim. 34, 1790 (1964)] has extended the functional methods in Q.F.T to the study of spontaneous symmetry breaking. It is common to study spontaneous symmetry breakdown in the so called semiclassical approximation, that is to say, to search for minima of the potential, the negative sum of all the nonderivative terms in the Lagrange density. The method of Jona-Lasinio enables us to define a function, called the effective potential, such that the minima of the effective potential give, without any approximation, the true vacuum states of the theory. Furthermore, there exists a diagramattic expansion for the effective potential, such that the first term in this expansion reproduces the semiclassical approximation. From our point of view, the virtue of this method is that it enables us to compute higher order corrections while still retaining a great advantage of the semiclassical approximation: the ability to survey all possible vacua simultaneously

Let us recall that the effective action $\Gamma[\Phi]$ has an expansion for $\langle \underset{\sim}{0}|\phi(x)|\underset{\sim}{0}\rangle = v = 0$ (15.3)

$$\Gamma[\Phi] = \sum_{m=2}^{\infty} \frac{1}{m} \int d^4x_1 \dots d^4x_m \, \Gamma^{(m)}(x_1, \dots, x_m)\, \Phi(x_1) \dots \Phi(x_m) \tag{1}$$

$\Gamma^{(m)}$ being the proper m-point or green's function. There is an alternative way to expand the effective action: instead of expanding in powers of $\Phi(x)$, we can expand in powers of momentum (about the point where all external momenta vanish). In position space such an expansion looks like

$$\Gamma[\Phi] = \int d^4x \, \left[-\mathcal{V}(\Phi) + \frac{1}{2}(\partial_\mu \Phi)(\partial^\mu \Phi)\, Z[\Phi) + \dots \right] \tag{2}$$

and from (15.2) we see that $\mathcal{V}(\Phi)$ is the effective potential. It is clear from above that

$$\left.\frac{\delta^2 \Gamma[\Phi]}{\delta\Phi(x_1)\,\delta\Phi(x_2)}\right|_{\Phi=0} = \Gamma^{(2)}(x_1, x_2)$$

$$\int d^4x_1\, d^4x_2\ \Gamma^{(2)}(x_1, x_2) = (2\pi)^4\, \delta^{(4)}(0)\ \hat{\Gamma}^{(2)}(0,0)$$

$$\left.\frac{\delta^2 \Gamma[\Phi]}{\delta\Phi(x_1)\,\delta\Phi(x_2)}\right|_{\Phi=0} = -\left.\frac{d^2\Upsilon(\Phi)}{d\Phi(x_1)\, d\Phi(x_2)}\right|_{\Phi=0} \delta(x_1 - x_2)$$

$$\Rightarrow \quad (2\pi)^4\, \delta^{(4)}(0)\ \hat{\Gamma}^{(2)}(0,0) = -\int d^4x \left.\frac{d^2\Upsilon(\Phi)}{d^2\Phi(x)}\right|_{\Phi=0}$$

$$\Rightarrow \quad \left.\frac{d^2\Upsilon(\Phi)}{d\Phi^2}\right|_{\Phi=0} = -\hat{\Gamma}^{(2)}(0,0) \qquad (1)$$

Then it is easy to show that the n-derivative of the superpotential for $\Phi=0$ is just minus the Fourier transform of the proper n-point function for all external momenta equal to zero.

The usual renormalization conditions of perturbation theory can be expressed in terms of the functions that occur in (23.2)

$$\left.\frac{d^2\Upsilon(\Phi)}{d\Phi^2}\right|_{\Phi=0} = \mu^2 \qquad (2)$$

$$\left.\frac{d^4\Upsilon(\Phi)}{d\Phi^4}\right|_{\Phi=0} = \lambda \qquad (3)$$

Similarly, the standard condition for the normalization of the field becomes

$$Z(\Phi=0) = 1 \qquad (4)$$

When SSB occurs similar formulae hold. One can redefine the field $\phi'(x) = \phi(x) - \langle 0|\phi(x)|0\rangle$ and $\Phi'(x) = \Phi(x) - \langle 0|\phi(x)|0\rangle$ and all the equations (2) (3) and (4) still hold.

We will now present a one-loop computation in the simplest possible case.

$$\mathcal{L}(x) = \frac{1}{2}(\partial_\mu\phi(x))(\partial^\mu\phi(x)) - \frac{\lambda}{4!}\phi^4(x) - \frac{1}{2}A\,(\partial_\mu\phi(x))(\partial^\mu\phi(x)) +$$

$$+ \frac{1}{2}B\,\phi^2(x) + \frac{1}{4!}C\,\phi^4(x) \qquad (5)$$

where A, B, and C are, respectively, the wave function, mass and coupling constant renormalization counterterm.

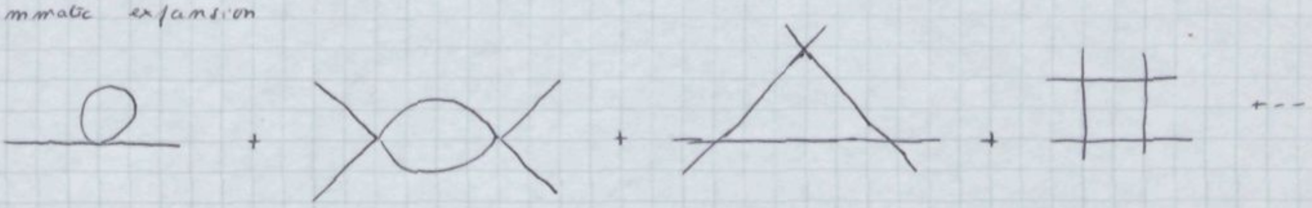
To lowest order (tree approximation) only one graph contribute and we obtain

$$\Gamma(\Phi) = V(\Phi) = \frac{\lambda}{4!}\,\Phi^4 \qquad (1)$$

To next order (one-loop approximation) we obtain (21.6)

$$\Gamma(\Phi) = \frac{\lambda}{4!}\,\Phi^4 - \frac{B}{2}\,\Phi^2 - \frac{C}{4!}\,\Phi^4 + i\int \frac{d^4k}{(2\pi)^4}\sum_{n=1}^{\infty}\frac{1}{2n}\left(+\frac{\lambda}{2}\,\frac{\Phi^2}{k^2+i\epsilon}\right)^n \qquad (2)$$

where B and C are, as usual in renormalization theory, only to be evaluated to lowest order in our expansion parameter, in this case the loop-counting parameter a. This result can also be obtained directly taking into account the diagrammatic expansion



(3)

using the Feynman rules

i) A factor i coming from $W[J] = e^{iZ[J]}$

ii) A factor $\lambda/2$ where the $1/2$ is a Bose statistics factor; interchange of the two external lines at the same vertex does not lead to a new graph, and therefore the $1/4!$ in the definition of the coupling constant is incompletely cancelled

iii) The $1/2n$ is a combinatorial factor; rotation or reflexion of the n-sided polygon does not lead to a new contraction in the Wick expansion and therefore the $1/n!$ in Dyson's formula is incompletely cancelled.

At first glance, the expression (2) seems hideously infrared divergent; each term in the sum is worse than the one before. However considerable improvement is obtained if we sum the series.

$$\Gamma(\Phi) = \frac{\lambda}{4!}\,\Phi^4 - \frac{B}{2}\,\Phi^2 - \frac{C}{4!}\,\Phi^4 + \frac{1}{2}\int \frac{d^4k_E}{(2\pi)^4}\,\ln\left[1 + \frac{\lambda}{2}\,\frac{\Phi^2}{k_E^2}\right] \qquad (4)$$

where we have rotated the integral into Euclidean space. As we seen, the apparent infrared divergence has been turned into a logarithmic singularity at $\Phi = 0$, which can be avoided staying away from vanishing Φ.

Of course this is still ultraviolet-divergent; to evaluate it, we cut off the integral at $k_E^2 = \Lambda^2$ and obtain

$$\Upsilon(\Phi) = \frac{\lambda}{4!}\Phi^2 + \frac{B}{2}\Phi^2 - \frac{C}{4!}\Phi^4 + \frac{\lambda\Lambda^2}{64\pi^2}\Phi^2 + \frac{\lambda^2\Phi^4}{256\pi^2}\left(\ln\frac{\lambda\Phi^2}{2\Lambda^2} - \frac{1}{2}\right) \tag{1}$$

where we have thrown away terms that vanish as $\Lambda^2 \to \infty$. We can now determine the value of the renormalization counterterm by imposing the definition of the renormalized mass and coupling constant.

We want the renormalized mass to vanish:

$$\left.\frac{d^2\Upsilon(\Phi)}{d\Phi^2}\right|_{\Phi=0} = 0 \tag{2}$$

This implies

$$B = \frac{\lambda\Lambda^2}{32\pi^2} \tag{3}$$

Unfortunately, we cannot do the same for the coupling constant due to the logarithmic infrared singularity at the origin. One therefore defines it

$$\left.\frac{d^4\Upsilon(\Phi)}{d\Phi^4}\right|_{\Phi=M} = \lambda \tag{4}$$

where M is a number with the dimensions of a mass. We emphasize that M is completely arbitrary; different choices for M will lead to different definitions of the coupling constant but any nonzero M is as good as any other. The same happens with the field renormalization constant and we put

$$Z(\Phi = M) = 1 \tag{5}$$

Imposing (4) we obtain

$$\lambda - C + \frac{3\lambda^2}{32\pi^2}\ln\left(\frac{\lambda M^2}{2\Lambda^2} - \frac{1}{2}\right) + \frac{\lambda}{64\pi^2}(48 - 36 + 16 - 3) = \lambda$$

$$C = \frac{3\lambda^2}{32\pi^2}\left(\ln\frac{\lambda M^2}{2\Lambda^2} + \frac{11}{3}\right) \tag{6}$$

so that finally

$$\Upsilon(\Phi) = \frac{\lambda}{4!}\Phi^4 + \frac{\lambda^2\Phi^4}{256\pi^2}\left(\ln\frac{\Phi^2}{M^2} - \frac{25}{6}\right) \tag{7}$$

Comments

i) This is a renormalizable theory; so we should expect all dependence of Λ to disappear from our final expression for $\mathcal{V}$; this is indeed the case.

ii) As we remarked earlier, the violent infrared singularities in the individual diagrams have become a singularity at $\Phi = 0$. This is true to all orders in the loop expansion.

iii) More surprisingly the logarithmic dependence on the coupling constant has also disappeared. It is possible to prove that the n-loop contribution to $\mathcal{V}$ is simply proportional to λ^{n+1}!

iv) It is easy to check explicitly that the renormalization mass, M, is indeed an arbitrary parameter, with no effect on the physics of the problem. If we pick a different mass M', then

$$\lambda' = \left.\frac{d^4 \mathcal{V}}{d\Phi^4}\right|_{\Phi = M'} = \lambda + \frac{3\lambda^2}{32\pi^2} \ln \frac{M'^2}{M^2}$$

and

$$\mathcal{V}(\Phi) = \frac{\lambda'}{4!}\Phi^4 + \frac{\lambda'^2 \Phi^4}{256\pi^2}\left(\ln \frac{\Phi^2}{M'^2} - \frac{25}{6}\right) + O(\lambda^3)$$

This is simply a reparametrization of the same function, to the order in which are working; it is a change of definitions, not a change of physics.

v) Since the logarithm of a small number is negative, it appears as though the one-loop corrections have turned the minimum at the origin into a maximum, and caused a new minimum to appear away from the origin — that is to say, that one-loop corrections have generated spontaneous symmetry breaking. Alas, appearances are deceptive: The apparent new minimum occurs at a value of Φ determined by

$$\lambda \ln \frac{\langle \Phi \rangle^2}{M^2} = -\frac{32}{3}\pi^2 + O(\lambda) \qquad (1)$$

Since we expect higher order corrections to bring in higher powers of $\lambda \ln \Phi^2/M^2$, the new minimum lies very far from the expected range of validity of the one-loop approximation, even for an arbitrarily small coupling constant, and must be rejected as an artifact of our approximation.

Let us consider massless scalar electrodynamics

$$L(x) = -\frac{1}{4} F_{\mu\nu}(x) F^{\mu\nu}(x) + \frac{1}{2}(\partial_\mu \varphi_1 - e A_\mu \varphi_2)^2 + \frac{1}{2}(\partial_\mu \varphi_2 + e A_\mu \varphi_1)^2$$

$$- \frac{\lambda}{4!}(\varphi_1^2 + \varphi_2^2)^2 \qquad F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu \tag{1}$$

or with

$$\varphi = \varphi_1 - i\varphi_2 \qquad \varphi^+ = \varphi_1 + i\varphi_2$$

$$\mathcal{L}(x) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2}\left[(\partial_\mu + ieA_\mu)\varphi\right]\left[(\partial^\mu - ieA^\mu)\varphi^+\right] - \frac{\lambda}{4!}\varphi^+\varphi \tag{2}$$

A calculation only slightly more complicated that the previous leads to

$$\Upsilon(\Phi) = \frac{\lambda}{4!}\Phi^4 + \left(\frac{5\lambda^2}{1152\pi^2} + \frac{3e^4}{64\pi^2}\right)\Phi^4\left(\ln\frac{\Phi^2}{M^2} - \frac{25}{6}\right) \tag{3}$$

($\Phi^2 = \varphi_{1\,clas}^2 + \varphi_{2\,clas}^2$) with minimum away from $\Phi = 0$. However now the situation is different. Consider that e^4 is of order λ, then to the order we are computing.

$$\Upsilon(\Phi) = \frac{\lambda}{4!}\Phi^4 + \frac{3e^4}{64\pi^2}\Phi^4\left(\ln\frac{\Phi^2}{M^2} - \frac{25}{6}\right) \tag{4}$$

and since M is arbitrary we can, as a computational trick, choose it to be the actual location of the minimum v. Then

$$\lambda = \frac{33}{8\pi^2}e^4 \tag{5}$$

In fact using the renormalization group one can proof that this is true without assumptions, if λ and e are small. If λ is not of the order of e^4 one can always make it that way by changing the renormalization mass. Notice that one has not lost parameters, but one has traded e and λ for e and v, so that the dimensionless λ is substituted by the v with dimension of mass. This is the dimensional transmutation. It is nothing but a reflection that for a fixed theory a change in the arbitrary renormalization mass leads to a change in the numerical value of the dimensionless coupling constant. Thus we have

$$\Upsilon(\Phi) = \frac{3e^4}{64\pi^2}\Phi^4\left(\ln\frac{\Phi^2}{v^2} - \frac{1}{2}\right) \tag{6}$$

From this point on the analysis is the same as in the Abelian Higgs model. The mass of the Higgs particle is

$$m^2(S) = V''(v) = \frac{3e^4}{8\pi^2} v^2$$

and the mass of the photon is

$$m^2(V) = e^2 v^2$$

so that

$$\frac{m^2(S)}{m^2(V)} = \frac{3}{2\pi} \frac{e^2}{4\pi}$$

Both when $\Phi \to \infty$ or $\Phi \to 0$ our one loop approximation breaks down, so that the maximum at $\Phi = 0$ is not reliable, although $V(0) = 0$ is reliable. Also

$$V(v) = -\frac{B\, e^4 v^4}{128\pi^2}$$

is a reliable result and it will remain a local minimum lower than that of the origin if this becomes again a minimum.

1/N EXPANSION IN FIELD THEORY

S. Coleman Erice 1979.

I. Introduction

There are field theories with symmetry group SO(N) or SU(N) which become simpler as N becomes large. More precisely, the solutions to these theories possess an expansion in powers of 1/N. We want to study this expansion

a) because it allows to study model field theories. All we know about field theories comes from perturbation theory and from the study of a handful of soluble models. The 1/N expansion allows to enlarge this set. We will study in Sot II the 1/N expansion for ϕ^4 and apply it to two-dimensional models with similar combinatoric structures, the Gross-Neveau model and the CP^{N-1} model. They display in the leading 1/N approximation phenomena like asymptotic freedom, dynamical symmetry breaking, dimensional transmutation and non-perturbative confinement. The reason why the 1/N expansion exhibits so much more dynamics than the normal expansion is that is really an expansion in the whole Lagrangian and preserves therefore much more nonlinear structure of the theory.

b) because it may be successfully applied to Q.C.D with N the number of colours.

Unfortunately one does not know how to compute even the first term in the expansion in closed form. But one can argue that this first term has many properties shared by the real world, and which are otherwise underived from field theory

II. Vector representations or soluble models

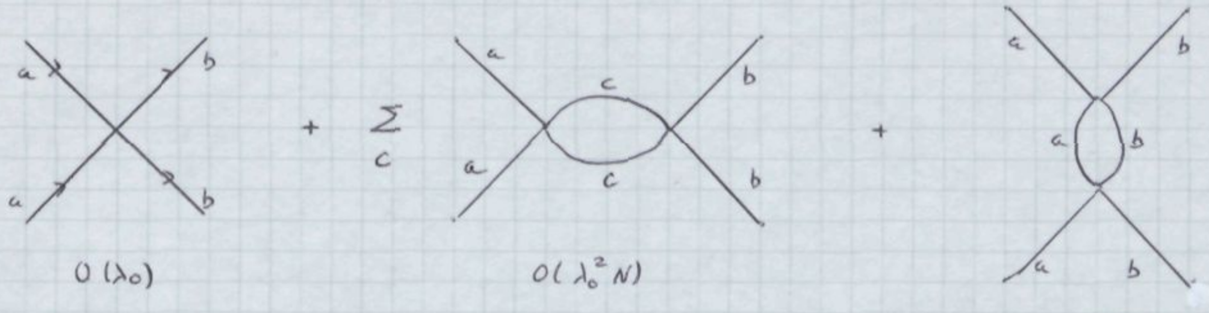
II.1. ϕ^4 - Theory (Coleman Jackiw, Politzer P.R. D10, 2491 (1974))

Let us consider $O(N)$ ϕ^4 theory

ϕ^a , $a = 1, 2, \dots N$

$$L(x) = \frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi^a - \frac{1}{2} \mu_0^2 \phi^a \phi^a - \frac{1}{8} \lambda_0 (\phi^a \phi^a)(\phi^b \phi^b) \quad (1)$$

Let us consider $a + a \to b + b$ scattering in ordinary perturbation theory.



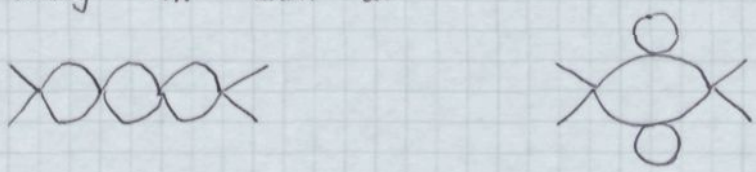
$O(\lambda_0)$ $\quad$ $O(\lambda_0^2 N)$ $\quad$ $O(\lambda_0^2)$

and the large N limit is non-sensical. Let us define

$$g_0 \equiv \lambda_0 N \tag{1}$$

and take the large N limit with g_0 fixed. Then the three diagrams go like $O(g_0/N)$, $O(g_0^2/N)$ and $O(g_0^2/N^2)$ and the last one is negligible in the large N limit.

This is the first step if one wants to construct a $1/N$ expansion: to decide which parameters should be held fixed. The wrong choice leads either to a trivial theory (only the Born term survives) or to no $1/N$ expansion (there are graphs proportional to positive powers of N). ϕ^4 has infinitely many $1/N$ term as



Let us now prove that the $1/N$ expansion exists. One can simplify the combinatorics by adding an auxiliary field σ and changing the Lagrange density

$$\mathcal{L} \to \mathcal{L} + \frac{1}{2}\frac{N}{g_0}\left(\sigma - \frac{1}{2}\frac{g_0}{N}\phi^a\phi^a\right)^2 \tag{2}$$

This added term has no effects on the dynamics of the theory. This is easily seen in the language of functional integration as the functional integral over the field σ is a trivial Gaussian integral; its only effect is to multiply the generating functional of the theory by an irrelevant constant. From the viewpoint of canonical quantization the Euler-Lagrange eq. for σ is

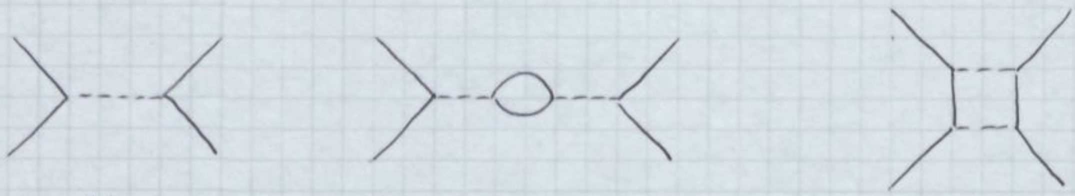
$$\sigma = \frac{1}{2}\frac{g_0}{N}\phi^a\phi^a \tag{3}$$

which has no time derivatives and is thus not an equation of motion but an equation of constraint exactly as the gauge fixing term in Q.E.D. The equations of motion for ϕ^a are unchanged if one uses the above constraint.

The dynamics are the same but not the Feynman rules:

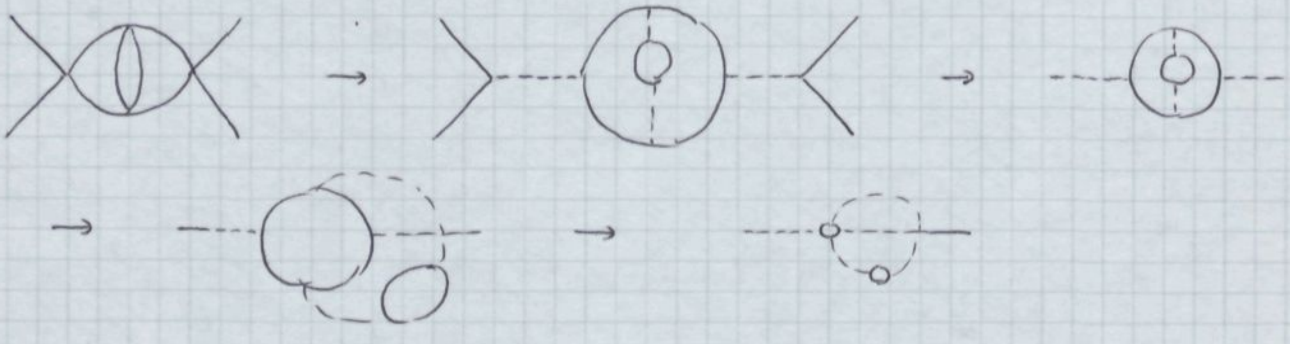
$$L(x) = \frac{1}{2}\partial_\mu \phi^a \partial^\mu \phi^a - \frac{1}{2}\mu_0^2 \phi^a \phi^a + \frac{1}{2}\frac{N}{g_0}\sigma^2 - \frac{1}{2}\sigma \phi^a \phi^a \qquad (1)$$

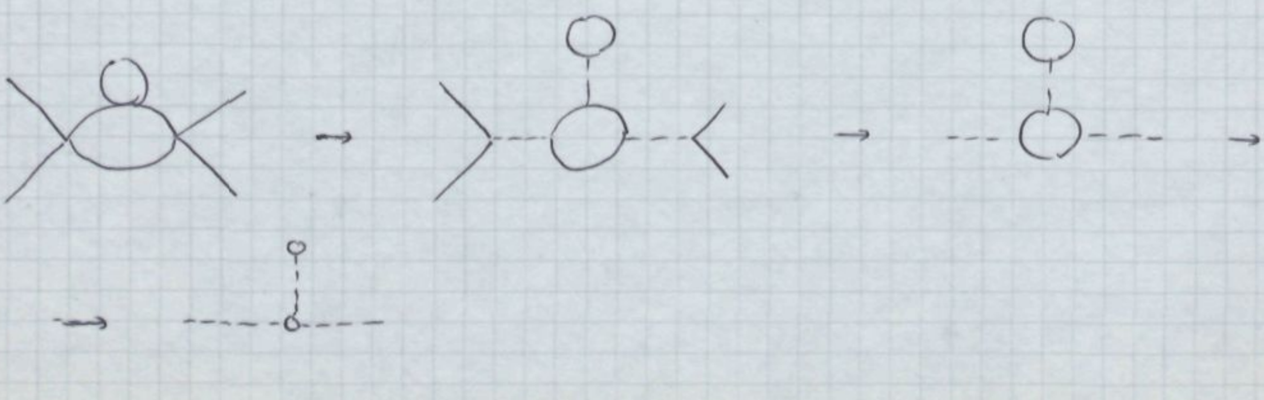
and the only nontrivial coupling is now $\phi\phi\sigma$. All factors $1/N$ come now from the σ propagator ig_0/N and every ϕ loop leads to a factor N. The above diagrams in the new formalism are



$1/N$ $\qquad$ $1/N$ $\qquad$ $1/N^2$

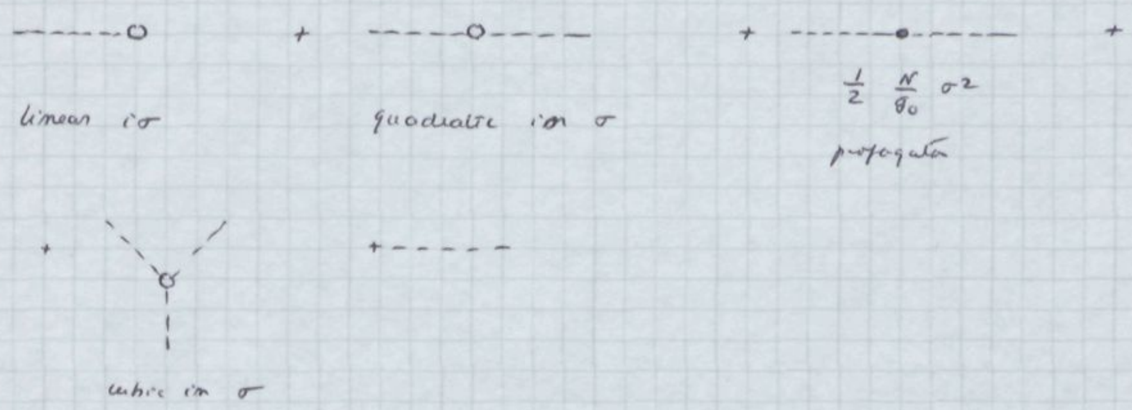
and counting $1/N$'s is much simpler. Now let us consider a general diagram and let us strip away all the ϕ lines which are external (not part of a loop). Then let us do all the momentum integrals over ϕ loops. Every ϕ loops becomes thus a non-local interaction between σ-fields. We have thus only σ fields in each graph.





It can be thought as a graph in an effective field theory where Feynman rules are derived from an effective action $S_{eff}(\sigma)$. S_{eff} can be described in terms of Feynman graphs or in terms of functional integrals

Graphic description



Functional integrals: The quantum theory is defined by integrating the exponential of iS, the classical action, over all configurations of all fields in the theory. The effective action is

$$e^{iS_{eff}(\sigma)} = \int \prod_a [\mathcal{D}\phi^a]\, e^{iS(\phi^a,\sigma)} \tag{1}$$

Since S is quadratic in ϕ the integrals are Gaussians and can be done in a closed form, which means that the answer is a functional determinant which can be evaluated only by doing the Feynman graphs given above. Whichever way one prefers, one thing is obvious, each term of $S_{eff}(\sigma)$ is proportional to N

$$S_{eff}(\sigma, N) = N\, S_{eff}(\sigma, 1) \tag{2}$$

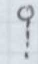
This makes the counting powers of N very easy. Consider a graph in the effective field theory with E external lines, I internal lines, V vertices and L independent loop integrations. For a connected graph

$$L = I - V + 1 \tag{3}$$

The power of N associated with a graph can be expressed in terms of these quantities. Each propagator carries a $1/N$. Since the external σ-lines are also propagators and since each vertex carries an N we have

$$N^{V-I-E} = N^{1-E-L} = \frac{1}{N},\ \frac{1}{N^2},\ \frac{1}{N^3},\ \ldots \tag{3}$$

Thus the smallest power of $1/N$ corresponds to tree graphs and with the minimal number of external lines to connect the external ϕ lines. If the diagram is not connected one repeats the reasoning for each connected subdiagram. Connected diagrams dominate.

This allows us to compute meson-meson scattering in closed form in the $O(1/N)$. However one first has to eliminate the term linear in σ in $S_{eff}(\sigma)$ since this leads to an infinite number of tree graphs with two external lines, by just adding  as many as one wishes which do not change the power of N. The cure is to define a new field

$$\sigma' \equiv \sigma - \sigma_0 \tag{1}$$

with σ_0 chosen as

$$\left.\frac{\delta S_{eff}}{\delta \sigma'}\right|_{\sigma'=0} = 0 \tag{2}$$

In terms of σ' there are not linear vertices. In order to construct $S_{eff}(\sigma')$ let us express $\mathcal{L}$ in terms of σ':

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi^a \partial^\mu \phi^a - \frac{1}{2}\mu_1^2 \phi^a\phi^a + \frac{1}{2}\frac{N}{g_0}\sigma'^2 - \frac{1}{2}\sigma'\phi^a\phi^a + \frac{N}{g_0}\sigma_0\sigma' + \text{cte.} \tag{3}$$

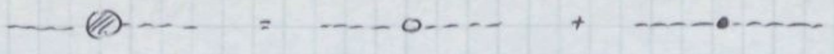
with

$$\mu_1^2 = \mu_0^2 + \sigma_0 \tag{4}$$

μ_1 being the ϕ mass to leading order $(1/N)^0$. The graphs of $S_{eff}(\sigma')$ are as the ones of $S_{eff}(\sigma)$ but with a new one which cancels $---\circ$:

One can now compute ϕ-ϕ scattering to leading order. There are only three diagrams which contribute

and the shaded ~~blobs~~ blobs reminds you



In momentum space

$$D^{-1}(p) = -N\left[\, i g_0^{-1} + \int \frac{d^D k}{(2\pi)^D}\, \frac{1}{k^2-\mu_1^2+i\epsilon}\, \frac{1}{(p+k)^2-\mu_1^2+i\epsilon} \right]$$

for $d=4$ we have a logarithmic divergence which may be absorbed by the bare coupling g_0. One can now study properties of the scattering amplitude, higher-order corrections, take $\mu_1^2<0$ and study spontaneous symmetry breaking but we will not do it. See original article.