REFERENCES

J. SCHERK, "An Introduction to the theory of dual models and strings" Rev. Mod. Phys 47, 123 - 164 (1975)

P. H. FRAMPTON. "Dual Resonance models". W.A. Benjamin Inc, Reading Massachusetts (1974)

S. FUBINI and G. VENEZIANO. "Duality in operator formalism" Nuovo Cim. 67A, 29 - 47 (1970)

S. FUBINI and G. VENEZIANO. "Algebraic Treatment of Subsidiary Conditions in Dual Resonance Models" Ann Phys. (NY) 63, 12 - 27 (1971)

J. H. SCHWARZ. "Dual Resonance Theory" Phys. Rep. 8C, 269 - 335 (1973)

L. BRINK and D. OLIVE. "The Physical State Projection Operator in Dual Resonance Models for the Critical Dimension of Space - time" Nucl. Phys. B56, 253 - 265 (1973).

D. AMATI, C. BOUCHIAT and J. L. GERVAIS. "On the Building of Dual Diagrams from Unitarity" Lett. Nuovo Cim 9, 399 (1969)

D. J. GROSS, A. NEVEU, J. SCHERK and J. H. SCHWARZ "Renormalization and Unitarity in the Dual - Resonance Model" Phys Rev. D2, 697 (1970).

J. H. SCHWARZ "Superstring Theory" Phys. Rep. 89C, 223 - 322 (1982)

M. B. GREEN and J. H. SCHWARZ. "Supersymmetrical Dual String Theory" Nucl. Phys. B181, 502 - 530 (1981)

D. J. GROSS, J. A. HARVEY, E. MARTINEC and R. ROHM. "Heterotic String Theory" Nucl. Phys B256, 253 - 284 (1985) and

R. SLANSKY. "Group Theory for Unified Model Building" Phys. Rep 79C, 1 - 128 (1981)

R. SLANSKY. "Symmetries of Theories in Higher Dimensions" Lectures of the 1985 Les Houches Summer School.

M. KATO and K. OGAWA "Covariant Quantization of the String Based on BRS Invariance". Nucl. Phys B212, 443 - 460 (1983).

INDEX

I - The Relativistic Point Particle 4

II - Classical Dynamics of a Free String 12

III - Covariant Solution of the Equations of Motion 6

IV - Non - Covariant Solutions of the Equations of Motion 9

V - Mass, Momentum and Angular Momentum of the String 6

VI - Hamiltonian Formalism: General Ideas 7

VII - String : Covariant Formalism and Quantization 27 + 6

VIII - Transverse Gauge and Quantization 16 +

IX - The Interacting String 21

X - Loop Diagrams 23

XI - Superstring Theory 35

XII - Commutation Relations 23

XIII - Spectrum of the Open String 8 + 11

XIV - Spectrum of the Closed String 4

1 - THE RELATIVISTIC POINT PARTICLE

The action for a free moving relativistic point particle must be Lorentz invariant and should be described by a first order differential expression integrated along the world-line of the particle. A simple choice for the action is given by the relativistic "length" of the world-line

$$S_2 \equiv -m \int_{\tau_1}^{\tau_2} d\tau \, [\, \dot{X}^\mu(\tau)\, \dot{X}_\mu(\tau)\,]^{1/2} \qquad \mu = 0, 1, \ldots, D-1 \tag{1}$$

where τ is an arbitrary parameter, which parametrizes the world-line $X^\mu(\tau)$, $\dot{X}^\mu \equiv dX^\mu/d\tau$ and m is a proportionality constant with dimensions of mass. We shall work in natural units $\hbar = c = 1$. We shall use the flat metric

$$\eta_{00} = +1 \quad , \quad \eta_{ii} = -1 \quad , \quad \eta_{\mu\nu} = 0 \quad \text{if} \quad \mu \neq \nu \tag{2}$$

for a D-dimensional Minkowski space.

The action (1) is obviously Lorentz invariant and translational invariant, i.e. Poincaré invariant. Furthermore it is also invariant under reparametrizations of the world-line given by

$$\tau' = f(\tau) \tag{3}$$

where f is a diffeomorfism which preserves the orientation ($\dot{f}(\tau) > 0$). This is the so-called reparametrization gauge invariance.

Imposing that the action S_2 should be stationary under small deformations of the world-line with fixed endpoints:

$$X^\mu(\tau) \longrightarrow X'^\mu(\tau) = X^\mu(\tau) + \delta X^\mu(\tau)$$

$$\delta X^\mu(\tau_1) = \delta X^\mu(\tau_2) = 0 \tag{4}$$

we obtain

$$0 = \delta S_2 = -m \int_{\tau_1}^{\tau_2} d\tau \, \frac{\dot{X}^\mu(\tau)}{[\, \dot{X}^\nu(\tau)\, \dot{X}_\nu(\tau)\,]^{1/2}} \, \frac{d}{d\tau} \, \delta X_\mu(\tau) =$$

$$= -m \left\{ \frac{\dot{X}^\mu(\tau)}{[\, \dot{X}^\nu(\tau)\, \dot{X}_\nu(\tau)\,]^{1/2}} \right\} \delta X_\mu(\tau) \Big|_{\tau_1}^{\tau_2} + m \int_{\tau_1}^{\tau_2} d\tau \, \delta X_\mu(\tau) \, \frac{d}{d\tau} \left\{ \frac{\dot{X}^\mu(\tau)}{[\, \dot{X}^\nu(\tau)\, \dot{X}_\nu(\tau)\,]^{1/2}} \right\} \tag{5}$$

which, taken into account (1.4), yields the equations of motion

$$\frac{d}{d\tau}\left\{\frac{\dot{X}^\mu(\tau)}{[\dot{X}^\nu(\tau)\,\dot{X}_\nu(\tau)]^{1/2}}\right\}=0 \tag{1}$$

Introducing the momenta

$$P^\mu(\tau)\equiv-\frac{\partial L_2}{\partial \dot{X}_\mu(\tau)}=m\,\frac{\dot{X}^\mu(\tau)}{[\dot{X}^\nu(\tau)\,\dot{X}_\nu(\tau)]^{1/2}} \tag{2}$$

equation (1) reads

$$\frac{d}{d\tau}\,P^\mu(\tau)=0 \tag{3}$$

showing that the momenta $P^\mu(\tau)$ are conserved quantities along a world-line corresponding to an actual motion of the particle. However, equation (2) implies that $P^\mu(\tau)$ are not independent objects because of the functional relation

$$P^\mu(\tau)\,P_\mu(\tau)=m^2 \tag{4}$$

So, equation (2) does not give $\dot{X}^\mu(\tau)$ as function of the momenta and therefore we cannot obtain the equation of the trajectory $X^\mu(\tau)$. This is due to the reparametrization gauge invariance of the world-line. Then, we must fix the gauge in order to obtain the equation of the trajectory. This is achieved, for instance, by identifying the parameter τ with the time X^0, in a choosen Lorentz frame,

$$X^0(\tau)\equiv t=\tau \tag{5}$$

From equations (2) and (3)

$$X^0(\tau)=X^0(0)+\frac{P^0}{m}\int_0^\tau d\tau'\,h(\tau')$$

$$h(\tau)\equiv[\dot{X}^\mu(\tau)\,\dot{X}_\mu(\tau)]^{1/2} \tag{6}$$

being $h(\tau)$ the "length" of the tangent vector, which is a gauge depent quantity. Now imposing (5) and choosing $X^0(0)=0$, we obtain

$$\tau=\frac{P^0}{m}\int_0^\tau d\tau'\,h(\tau') \tag{7}$$

Therefore the function $h(\tau)$ is fixed, in this gauge, to be

$$h(\tau) = \frac{m}{p^0} \tag{1}$$

and we get

$$x^0(\tau) = \tau \quad , \quad \vec{x}(\tau) = \vec{x}(0) + \vec{v}\,\tau \quad , \quad \vec{v} \equiv \vec{p}/p^0 \tag{2}$$

being $\vec{v}$ the velocity of the particle.

The formalism that we have just reviewed is the so-called second order formalism and it is not suitable for zero-mass particles, since the action S_2 vanishes when m goes to zero. Then, we need to introduce the first-order formalism for the study of both massive and massless particles. We start from an action functional

$$S_1 \equiv \frac{1}{2} \int_{\tau_1}^{\tau_2} d\tau \sqrt{g(\tau)} \left\{ - \frac{1}{g(\tau)} \dot{x}^\mu(\tau)\, \dot{x}_\mu(\tau) - m^2 \right\} \tag{3}$$

The action S_1 is similar to the one corresponding to a bunch of scalar fields $\{x^\mu(\tau)\}$ coupled to gravity in a one-dimensional space. The mass plays the role of a cosmological constant. This action is Poincaré and reparametrization invariant. From it varying the "metric" $g(\tau)$ and the $x^\mu(\tau)$ independently, we obtain the equations

$$\dot{x}^\mu(\tau)\, \dot{x}_\mu(\tau) = m^2\, g(\tau) \tag{4a}$$

$$\frac{d}{d\tau} \left\{ \frac{\dot{x}^\mu(\tau)}{\sqrt{g(\tau)}} \right\} = 0 \tag{4b}$$

For $m \neq 0$, we can obtain from (4.a) $g(\tau)$ and substituting in (3) we recover S_1. Also inserting $g(\tau)$ in (4b) we recover (2.1)

On the other hand if $m \equiv 0$, we have that $g(\tau)$ is an arbitrary function of τ and equations (4) can be written as

$$\dot{x}^\mu(\tau)\, \dot{x}_\mu(\tau) = 0 \tag{5a}$$

$$\frac{d}{d\tau} \left\{ \frac{\dot{x}^\mu(\tau)}{\sqrt{g(\tau)}} \right\} = 0 \tag{5b}$$

Introducing the momenta

$$p^\mu(\tau) = -\frac{\partial L_1}{\partial \dot{X}_\mu(\tau)} = \frac{\dot{X}^\mu(\tau)}{\sqrt{g(\tau)}} \tag{1}$$

equations (4.5) become

$$p^\mu(\tau)\, p_\mu(\tau) = 0 \,, \qquad \frac{d}{d\tau}\, p^\mu(\tau) = 0 \tag{2}$$

expressing that the momentum $p^\mu(\tau)$ is a constant light-like vector.

In order to write down the equation of the trajectory we fix the gauge by means of

$$n^\mu\, X_\mu(\tau) = p^0\, \tau \tag{3}$$

where n^μ is a time-like vector which in an appropiate Lorentz frame will be of the form $n^\mu \equiv (1, \vec{0})$. Then (3) identifies the time $X^0(\tau)$, in a Lorentz frame, with the parameter τ times the energy of the particle. From (1) and choosing $X^0(0) = 0$

$$X^0(\tau) = p^0 \int_0^\tau d\tau' \sqrt{g(\tau')} = p^0\, \tau \tag{4}$$

So the gauge condition (3) translates into

$$\sqrt{g(\tau)} = 1 \tag{5}$$

which says that the world-line is locally flat. In this gauge equation (1) integrates to

$$X^0(\tau) = p^0 \tau = t \,, \qquad \vec{X}(\tau) = \vec{X}(0) + \vec{p}\,\tau = \vec{X}(0) + \frac{\vec{p}}{p^0}\, t \tag{6}$$

where $\vec{p}\,/\,p^0$ is the velocity of the particle, which due to (2) has modulus one, i.e. the particle moves at the speed of light.

Notice that we can obtain gauge fixed expressions for the action of a point-like particle. Starting from (1.1) and (3.3) and using (2.5) and (4) we obtain

$$S_{g.f.,2} = -m \int_{\tau_1}^{\tau_2} d\tau \left[1 - \vec{v}^{\,2}(\tau)\right]^{1/2} \tag{7a}$$

$$S_{g.f.,1} = \frac{1}{2} \int_{\tau_1}^{\tau_2} d\tau\, \dot{X}^i(\tau)\, \dot{X}^i(\tau) + c\overline{c}$$

II - CLASSICAL DYNAMICS OF A FREE STRING

The simplest step beyond a point-like object is a one-dimensional object, i.e. a string. It is parametrized by two internal coordinates σ and τ. The first can be thought of as labelling the points along the string, while the second plays the role of the proper time. During its evolution, the string spans a two-dimensional surface in the D-dimensional spacetime given by $X^\mu = X^\mu(\sigma, \tau)$, this is the so called world-sheet. We adopt the convention that the σ parameter runs from 0 to π. We shall consider two types of strings: open and closed, that will give rise to two types of world sheets having the topology of a square and of a cylinder, respectively.

Both types of strings must satisfy the condition that their points do not move faster than light. We analyze now the meaning of this condition for the world-sheet. We consider first a particular parametrization of the world-sheet in which τ is identified with the time in a particular Lorentz frame $X^0(\sigma, \tau) \equiv t \equiv \tau$. At a given time t, the string is a curve in the $(D-1)$-dimensional space, parametrized by the functions $X^i = X^i(\sigma, t)$. We note that if we do not want the spring to propagate faster than light at any of its points, its motion must obey

$$\frac{\partial X^\mu}{\partial t}\,\frac{\partial X_\mu}{\partial t} = 1 - \frac{\partial X^i}{\partial t}\,\frac{\partial X^i}{\partial t} \geq 0 \qquad (1)$$

On the other hand, one also has

$$\frac{\partial X^\mu}{\partial \sigma}\,\frac{\partial X_\mu}{\partial \sigma} = -\frac{\partial X^i}{\partial \sigma}\,\frac{\partial X^i}{\partial \sigma} \leq 0 \qquad (2)$$

However, we would like to express the condition that the points of the string do not move faster than light, in a way which is independent of the parametrization chosen for the world-sheet. If we make a Lorentz transformation, in general X^0 will depend on σ and τ. If we give the two-dimensional surface $X^\mu = X^\mu(\sigma, \tau)$ the evolution of the string in any Lorentz frame is obtained by slicing the surface with the family of

planes $n^\mu X_\mu = \tau$, where n^μ is a time-like unit vector. In the particular frame we have previously discussed, at each point of the surface, there is a spacelike (1.2) and a timelike (1.1) tangent vectors. This has to be true in any frame and the necessary and sufficient condition for this is expressed by

$$\left\{ \frac{\partial X^\mu}{\partial \tau} \frac{\partial X_\mu}{\partial \sigma} \right\}^2 - \left\{ \frac{\partial X^\mu}{\partial \tau} \frac{\partial X_\mu}{\partial \tau} \right\} \left\{ \frac{\partial X^\mu}{\partial \sigma} \frac{\partial X_\mu}{\partial \sigma} \right\} \geq 0 \tag{1}$$

This is deduced by requiring that

$$\left\{ \frac{\partial X^\mu}{\partial \tau} + \lambda \frac{\partial X^\mu}{\partial \sigma} \right\} \left\{ \frac{\partial X_\mu}{\partial \tau} + \lambda \frac{\partial X_\mu}{\partial \sigma} \right\} \tag{2}$$

takes positive and negative values whom λ is varied.

In order to simplify the notations we shall use

$$\dot{X}^\mu \equiv \frac{\partial X^\mu}{\partial \tau} \quad , \quad X'^\mu \equiv \frac{\partial X^\mu}{\partial \sigma} \tag{3}$$

and (2) can be written as

$$(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2 \geq 0 \tag{4}$$

In analogy with the relativistic action of a free point particle, Y. Nambu [Lectures at the Copenhagen Symposium (1970)] suggested that the relativistic action for a free string be proportional to the area of the surface spanned in space-time by the evolution of the string. For a two-dimensional surface embedded in a space-time the area element is

$$d^2A = \left\{ (\dot{X} \cdot X')^2 - \dot{X}^2 X'^2 \right\}^{1/2} d\sigma\, d\tau \tag{5}$$

The quantity under the square-root is positive if the surface is of the kind previously discussed (4). So we postulate the following action for the string

$$S_{N.G.} \equiv -\frac{1}{2\pi\alpha'} \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma \left[(\dot{X} \cdot X')^2 - \dot{X}^2 X'^2 \right]^{1/2} \tag{6}$$

which is the so-called Nambu-Goto action. Notice that the dimensions of α' are $[\alpha'] = M^{-2}$.

. We can introduce also a more symmetric notation for the space of the two parameters σ and τ

$$(\xi^0, \xi^1) \equiv (\tau, \sigma) \tag{1}$$

The metric tensor for this two dimensional space is given by

$$ds^2 = [\, X^\mu(\xi + d\xi) - X^\mu(\xi)\,][\, X_\mu(\xi + d\xi) - X_\mu(\xi)\,] =$$

$$= \partial_\alpha X^\mu(\xi)\, \partial_\beta X_\mu(\xi)\, d\xi^\alpha d\xi^\beta \quad \Rightarrow$$

$$ds^2 = g_{\alpha\beta}(\xi)\, d\xi^\alpha d\xi^\beta \quad , \qquad g_{\alpha\beta}(\xi) \equiv \partial_\alpha X^\mu(\xi)\, \partial_\beta X_\mu(\xi) \tag{2}$$

Now it is immediate to check that

$$d^2A = [\, -\det g(\tau)\,]^{1/2}\, d\sigma\, d\tau \tag{3}$$

and therefore

$$S_{N.G.} = -\frac{1}{2\pi\alpha'} \int_{\tau_1}^{\tau_2} d\tau \int_0^\pi d\sigma\, [\, -\det g(\tau)\,]^{1/2} \tag{4}$$

Let us now consider the symmetries of this action

i) <u>Global Poincaré Invariance</u>

The so-called Poincaré transformations are

$$X^\mu(\sigma\tau) \longrightarrow \tilde{X}^\mu(\sigma\tau) = \Lambda^\mu{}_\nu\, X^\nu(\sigma\tau) + a^\mu \tag{5}$$

$$a^*_\mu = a_\mu \quad , \quad \Lambda^*_{\mu\nu} = \Lambda_{\mu\nu} \quad , \quad \Lambda^T \eta\, \Lambda = \eta$$

and the action is cleanly invariant under them. For the infinitesimal proper Lorentz transformations we will use

$$\Lambda_{\mu\nu} = \eta_{\mu\nu} + \omega_{\mu\nu} \quad , \qquad \omega^*_{\mu\nu} = \omega_{\mu\nu} = -\omega_{\nu\mu} \tag{6}$$

being $\omega_{\mu\nu}$, $D(D-1)/2$ arbitrary real infinitesimal parameters.

ii) <u>Reparametrization Invariance</u>

The geometrical meaning of the surface element is reflected in its invariance under any change of parametrization. If we set

$$\tilde{\xi}_\alpha = \tilde{\xi}_\alpha(\xi) \qquad \alpha = 0, 1 \tag{7}$$

Then we have, from (3.3)

$$g_{\alpha\beta}(\zeta) = \frac{\partial X^\mu(\zeta)}{\partial \zeta^\alpha}\frac{\partial X_\mu(\zeta)}{\partial \zeta^\beta} = \frac{\partial \tilde{X}^\mu(\tilde{\zeta})}{\partial \tilde{\zeta}^\gamma}\frac{\partial \tilde{\zeta}^\gamma}{\partial \zeta^\alpha}\frac{\partial \tilde{X}_\mu(\tilde{\zeta})}{\partial \tilde{\zeta}^\delta}\frac{\partial \tilde{\zeta}^\delta}{\partial \zeta^\beta} \tag{1}$$

where $\tilde{X}^\mu(\tilde{\zeta}) \equiv X^\mu[\zeta(\tilde{\zeta})]$. This can be written as

$$g_{\alpha\beta}(\zeta) = \frac{\partial \tilde{\zeta}^\gamma}{\partial \zeta^\alpha}\tilde{g}_{\gamma\delta}(\tilde{\zeta})\frac{\partial \tilde{\zeta}^\delta}{\partial \zeta^\beta} \tag{2}$$

Let us now introduce the matrix g with matrix elements $g_{\alpha\beta}$ and the matrix M with matrix elements $\partial\tilde{\zeta}^\delta/\partial\zeta^\beta$, then (2) can be written as

$$g(\zeta) = M^T(\zeta)\,\tilde{g}(\tilde{\zeta})\,M(\zeta) \tag{3}$$

and hence

$$-\det g(\zeta) = -\det \tilde{g}(\tilde{\zeta})\,[\det M(\zeta)]^2 \tag{4}$$

Since

$$d^2\zeta \equiv d\zeta^0 d\zeta^1 = d^2\tilde{\zeta}\,/\,|\det M(\zeta)| \tag{5}$$

we obtain immediately that the Nambu-Goto action is invariant under reparametrizations.

Let us consider the situation for infinitesimal transformations. From

$$\tilde{\zeta}^\alpha = \zeta^\alpha + \delta\zeta^\alpha(\zeta) \tag{6}$$

we get

$$X^\mu(\tilde{\zeta}) = X^\mu(\zeta) + \frac{\partial X^\mu(\zeta)}{\partial \zeta^\alpha}\delta\zeta^\alpha(\zeta) \tag{7}$$

and

$$g_{\alpha\beta}(\tilde{\zeta}) = \left\{\frac{\partial X^\mu(\zeta)}{\partial \zeta^\alpha} + \frac{\partial^2 X^\mu(\zeta)}{\partial\zeta^\alpha\partial\zeta^\gamma}\delta\zeta^\gamma(\zeta) + \frac{\partial X^\mu(\zeta)}{\partial\zeta^\gamma}\frac{\partial\delta\zeta^\gamma(\zeta)}{\partial\zeta^\alpha}\right\}\cdot$$

$$\cdot\left\{\frac{\partial X_\mu(\zeta)}{\partial\zeta^\beta} + \frac{\partial^2 X_\mu(\zeta)}{\partial\zeta^\beta\partial\zeta^\delta}d\zeta^\delta(\zeta) + \frac{\partial X_\mu(\zeta)}{\partial\zeta^\delta}\frac{\partial\delta\zeta^\delta(\zeta)}{\partial\zeta^\beta}\right\} \Rightarrow$$

$$g_{\alpha\beta}(\tilde{\zeta}) = g_{\alpha\beta}(\zeta) + \frac{\partial g_{\alpha\beta}(\zeta)}{\partial\zeta^\gamma}\delta\zeta^\gamma(\zeta) + g_{\alpha\gamma}(\zeta)\frac{\partial\delta\zeta^\gamma(\zeta)}{\partial\zeta^\beta} +$$

$$+ g_{\gamma\beta}(\zeta)\frac{\partial\delta\zeta^\gamma(\zeta)}{\partial\zeta^\alpha} \tag{8}$$

Equations of Motion

The equations of motion for the string are obtained, as usual, through the principle of least action. For arbitrarely infinitesimal variations of the world-sheet we have

$$\delta S_{N.G} = \int_{\tau_1}^{\tau_2} d\tau \int_0^{\pi} d\sigma \left\{ \frac{\partial L_{N.G}}{\partial \dot{X}_\mu} \frac{\partial}{\partial \tau} \delta X_\mu + \frac{\partial L_{N.G}}{\partial X'_\mu} \frac{\partial}{\partial \sigma} \delta X_\mu \right\} \tag{1}$$

and hence

$$\delta S_{N.G.} = \int_0^{\pi} d\sigma \frac{\partial L_{N.G}}{\partial \dot{X}_\mu} \delta X_\mu \Big|_{\tau_1}^{\tau_2} + \int_{\tau_1}^{\tau_2} d\tau \frac{\partial L_{N.G}}{\partial X'_\mu} \delta X_\mu \Big|_0^{\pi} -$$

$$- \int_{\tau_1}^{\tau_2} d\tau \int_0^{\pi} d\sigma \left\{ \frac{\partial}{\partial \tau} \frac{\partial L_{N.G}}{\partial \dot{X}_\mu} + \frac{\partial}{\partial \sigma} \frac{\partial L_{N.G}}{\partial X'_\mu} \right\} \delta X_\mu \tag{2}$$

In order to obtain the equations of motion we must impose $\delta S_{N.G} = 0$ for variations such that the initial and final position of the string are kept fixed, i.e.

$$\delta X^\mu(\sigma, \tau_1) = \delta X^\mu(\sigma, \tau_2) = 0 \tag{3}$$

Then we obtain

$$\frac{\partial}{\partial \tau} \frac{\partial L_{N.G.}}{\partial \dot{X}_\mu} + \frac{\partial}{\partial \sigma} \frac{\partial L_{N.G}}{\partial X'_\mu} = 0 \qquad \text{Equations of motion} \tag{4}$$

$$\frac{\partial L_{N.G.}}{\partial X'_\mu} = 0 \qquad \text{for} \quad \sigma = 0, \pi \qquad \text{Edge condition} \tag{5}$$

Notice that the edge condition only appears for open strings, since the second term in the r.h.s of (2) does not exists for closed strings (there is not such a boundary for the world-sheet).

Momentum and Angular Momentum

Let us now introduce the energy-momentum tensors

$$P^\mu_\tau \equiv - \frac{\partial L_{N.G}}{\partial \dot{X}_\mu} \quad , \quad P^\mu_\sigma \equiv - \frac{\partial L_{N.G}}{\partial X'_\mu} \tag{6}$$

Since

$$L_{N.G.} = - \frac{1}{2\pi\alpha'} \left[(\dot{X} X')^2 - \dot{X}^2 X'^2 \right]^{1/2} \tag{7}$$

We get

$$p^\mu_\tau = \frac{1}{2\pi\alpha'} \frac{(\dot X X')\, X'^\mu - X'^2 \dot X^\mu}{[(\dot X X')^2 - \dot X^2 X'^2]^{1/2}} \tag{10a}$$

$$p^\mu_\sigma = \frac{1}{2\pi\alpha'} \frac{(\dot X X')\, \dot X^\mu - \dot X^2 X'^\mu}{[(\dot X X')^2 - \dot X^2 X'^2]^{1/2}} \tag{10b}$$

Notice that (5.4) and (5.5) can be written

$$\frac{\partial}{\partial\tau} p^\mu_\tau + \frac{\partial}{\partial\sigma} p^\mu_\sigma = 0 \qquad \text{Equations of motion} \tag{2}$$

$$p^\mu_\sigma(0,\tau) = p^\mu_\sigma(\pi,\tau) = 0 \qquad \text{Edge condition} \tag{3}$$

From (1) we obtain the constrains

$$p_\tau \cdot X' = 0 \qquad p_\tau^2 + \frac{1}{(2\alpha'\pi)^2} X'^2 = 0 \tag{4}$$

$$p_\sigma \cdot \dot X = 0 \qquad p_\sigma^2 + \frac{1}{(2\alpha'\pi)^2} \dot X^2 = 0 \tag{5}$$

Furthermore from (5) and (3) we obtain

$$\dot X^2(0,\tau) = \dot X^2(\pi,\tau) = 0 \tag{6}$$

i.e. for the open string the end-points move at the speed of light.

The energy-momentum along any curve C on the world-sheet is given by

$$P^\mu(C) \equiv \int_C \{ d\sigma\, p^\mu_\tau(\sigma,\tau) + d\tau\, p^\mu_\sigma(\sigma,\tau) \} \tag{7}$$

Let us check that if C is a closed curve, in the sense that for continuum deformations can be reduced to a point, then $P^\mu(C) = 0$. Up to this end we can decompose the surface enclosed by C in elementary surfaces with sides $d\sigma$ and $d\tau$. The contribution to $P^\mu(\tau)$ of any one of them is

$$\left\{ \frac{\partial p^\mu_\tau}{\partial\tau} + \frac{\partial p^\mu_\sigma}{\partial\sigma} \right\} d\sigma\, d\tau \tag{8}$$

which is zero as a consequence of the equations of motion (2).

The total momentum of the string is

$$P^\mu = \int_0^\pi d\sigma \; P^\mu_2 (\sigma, \tau) \tag{1}$$

and this is a conserved quantity in the sense

$$\frac{d}{d\tau} P^\mu = \int_0^\pi d\sigma \; \frac{\partial P^\mu_2}{\partial \tau} = - \int_0^\pi d\sigma \; \frac{\partial P^\mu_\sigma}{\partial \sigma} = - P^\mu_\sigma (\sigma, \tau) \Big|_0^\pi = 0 \tag{2}$$

We can introduce the angular-momentum on the surface

$$M^{\mu\nu}_\alpha \equiv X^\mu P^\nu_\alpha - X^\nu P^\mu_\alpha \qquad \alpha = \sigma, \tau \tag{3}$$

Notice

$$\frac{\partial}{\partial \tau} M^{\mu\nu}_\tau + \frac{\partial}{\partial \sigma} M^{\mu\nu}_\sigma = 0 \tag{4}$$

Proof:

$$\frac{\partial}{\partial \tau} \left[P^\mu_2 X^\nu - P^\nu_2 X^\mu \right] + \frac{\partial}{\partial \sigma} \left[P^\mu_\sigma X^\nu - P^\nu_\sigma X^\mu \right] = \frac{\partial}{\partial \tau} \left[\frac{\partial \mathcal{L}}{\partial \dot X_\mu} \dot X^\nu - \frac{\partial \mathcal{L}}{\partial \dot X_\nu} \dot X^\mu \right] + \frac{\partial}{\partial \sigma} \left[\frac{\partial \mathcal{L}}{\partial X'_\mu} X^\nu - \frac{\partial \mathcal{L}}{\partial X'_\nu} X^\mu \right] =$$

$$= \left\{ \frac{\partial}{\partial \sigma} \frac{\partial \mathcal{L}}{\partial X'_\mu} + \frac{\partial}{\partial \tau} \frac{\partial \mathcal{L}}{\partial \dot X_\mu} \right\} X^\nu + \frac{\partial \mathcal{L}}{\partial X'_\mu} X'^\nu + \frac{\partial \mathcal{L}}{\partial \dot X_\mu} \dot X^\nu - \left\{ \frac{\partial}{\partial \sigma} \frac{\partial \mathcal{L}}{\partial X'_\nu} + \frac{\partial}{\partial \tau} \frac{\partial \mathcal{L}}{\partial \dot X_\nu} \right\} X^\mu - \frac{\partial \mathcal{L}}{\partial X'_\nu} X'^\mu - \frac{\partial \mathcal{L}}{\partial \dot X_\nu} \dot X^\mu =$$

$$= \dot X^\mu P^\nu_\sigma + X'^\mu P^\nu_\tau - \dot X^\nu P^\mu_\sigma - X'^\nu P^\mu_\tau = 0 \qquad \text{Q.E.D.}$$

where use has been made of the explicit expressions (6.1). The angular momentum of a curve C on the world-sheet is

$$M^{\mu\nu}(C) \equiv \int_C \left[d\sigma \; M^{\mu\nu}_{2\tau}(\sigma, \tau) + d\tau \; M^{\mu\nu}_\sigma (\sigma\tau) \right] \tag{5}$$

and if C is closed $M^{\mu\nu}(C) = 0$ as a consequence of (4).

Finally the total angular momentum of the string is

$$M^{\mu\nu} = \int_0^\pi d\sigma \; M^{\mu\nu}_{2\tau} (\sigma, \tau) \tag{6}$$

which is a conserved quantity

$$\frac{d}{d\tau} M^{\mu\nu} = \int_0^\pi d\sigma \; \frac{\partial M^{\mu\nu}_{2\tau}}{\partial \sigma} = - \int_0^\pi d\sigma \; \frac{\partial M^{\mu\nu}_\sigma}{\partial \sigma} = - M^{\mu\nu}_\sigma (\sigma, \tau) \Big|_0^\pi = 0 \tag{7}$$

Exercise

Let us consider the world - sheet in $D=4$

$$X^\mu(\sigma,\tau) \equiv \left(\tau,\ \frac{1}{\omega}\left(\frac{2\sigma}{\pi} - 1\right)\cos\omega\tau,\ \frac{1}{\omega}\left(\frac{2\sigma}{\pi} - 1\right)\sin\omega\tau,\ 0 \right) \tag{1}$$

being ω a constant. Hence

$$\dot{X}^\mu(\sigma,\tau) = \left(1,\ -\left(\frac{2\sigma}{\pi} - 1\right)\sin\omega\tau,\ \left(\frac{2\sigma}{\pi} - 1\right)\cos\omega\tau,\ 0 \right) \tag{2}$$

$$X'^\mu(\sigma,\tau) = \left(0,\ \frac{2}{\omega\pi}\cos\omega\tau,\ \frac{2}{\omega\pi}\sin\omega\tau,\ 0 \right) \tag{3}$$

The unit vector, $\hat{m}$, along the string is

$$\hat{m} \equiv (\cos\omega\tau,\ \sin\omega\tau,\ 0) \tag{4}$$

and the velocity $\vec{v}$ can be decomposed in a component $\vec{v}_\parallel$ ($\vec{v}_\perp$) along (transversal to) the string

$$\vec{v} \equiv \frac{d\vec{x}}{dt} = \vec{v}_\parallel + \vec{v}_\perp \tag{5}$$

$$\vec{v}_\parallel = \hat{m}\cdot(\hat{m}\cdot\vec{v}) = 0 \qquad , \qquad \vec{v}_\perp = \left(\frac{2\sigma}{\pi} - 1\right)(-\sin\omega\tau,\ \cos\omega\tau,\ 0)$$

and

$$1 - \vec{v}_\perp^{\,2} = \frac{4\sigma}{\pi}\left(1 - \frac{\sigma}{\pi}\right) \tag{6}$$

Notice

$$\dot{X}\,X' = 0 \qquad , \qquad \dot{X}^2 = 1 - \vec{v}_\perp^{\,2} = 4\,\frac{\sigma}{\pi}\left(1 - \frac{\sigma}{\pi}\right)$$

$$X'^2 = -\frac{4}{\omega^2\pi^2} \qquad , \qquad (\dot{X}X')^2 - \dot{X}^2 X'^2 = \left(\frac{2}{\omega\pi}\right)^2 (1 - v_\perp^2) \tag{7}$$

Then

$$P_\tau^\mu = \frac{1}{2\pi\alpha'}\,\frac{2}{\pi\omega}\,\frac{1}{\sqrt{1-\vec{v}_\perp^{\,2}}}\,(1,\ \vec{v}_\perp)$$

$$P_\sigma^\mu = -\frac{1}{2\pi\alpha'}\,\sqrt{1-\vec{v}_\perp^{\,2}}\,(0,\ \hat{m}) \tag{8}$$

or equivalently

$$p^\mu_\tau = \frac{1}{2\pi\alpha'}\,\frac{1}{\pi\omega}\left\{\frac{\sigma}{\pi}\left(1-\frac{\sigma}{\pi}\right)\right\}^{-1/2}\left(1,\ -\left(\frac{2\sigma}{\pi}-1\right)\sin\omega\tau,\ \left(\frac{2\sigma}{\pi}-1\right)\cos\omega\tau,\ 0\right)$$

$$p^\mu_\sigma = -\frac{\omega}{2\alpha'}\left\{\frac{\sigma}{\pi}\left(1-\frac{\sigma}{\pi}\right)\right\}^{1/2}\left(0,\ \frac{2}{\omega\pi}\cos\omega\tau,\ \frac{2}{\omega\tau}\sin\omega\tau,\ 0\right) \tag{1}$$

which satisfy the equations of motion and the edge condition and therefore it is an open string of the type considered here.

Furthermore the momentum of the string (7.1) is

$$p^\mu = \frac{1}{2\alpha'\pi^2\omega}\int_0^\pi d\sigma\left(\frac{\sigma}{\pi}-\frac{\sigma^2}{\pi^2}\right)^{-1/2}\left(1,\ -\left(\frac{2\sigma}{\pi}-1\right)\sin\omega\tau,\ \left(\frac{2\sigma}{\pi}-1\right)\cos\omega\tau,\ 0\right) =$$

$$= \frac{1}{2\alpha'\pi\omega}\int_{-1}^{+1} d\sigma\,\frac{1}{\sqrt{1-\sigma^2}}\left(1,\ -\sigma\sin\omega\tau,\ \sigma\cos\omega\tau,\ 0\right) \Rightarrow$$

$$p^\mu = \left(M \equiv \frac{1}{2\alpha'\omega},\ \vec{0}\right) \tag{2}$$

where M is the mass of the string. Similarly

$$M^{12} = \int_0^\pi d\sigma\,(x'p^2_\tau - x^2 p'_\tau) = \frac{1}{2\alpha'\pi^2\omega^2}\int_0^\pi d\sigma\left(\frac{2\sigma}{\pi}-1\right)^2\left[\frac{\sigma}{\pi}\left(1-\frac{\sigma}{\pi}\right)\right]^{-1/2}$$

$$= \frac{1}{2\alpha'\pi\omega^2}\int_{+1}^{+1} d\sigma\,\frac{\sigma^2}{\sqrt{1-\sigma^2}} = \frac{1}{4\alpha'\omega^2} \tag{3}$$

and

$$\vec{J} \equiv \left(M^{23}=0,\ M^{31}=0,\ M^{12} = \frac{1}{4\alpha'\omega^2} \equiv J\right) \tag{4}$$

From (2) and (4)

$$J = \alpha' M^2 \tag{5}$$

which is the typical relation between mass and spin for a Regge trajectory.

The Polyakov Action

As in the case of the point-like particle, we can write an action for the string in which the metric $g_{\alpha\beta}(\sigma,\tau)$ is independent of the embedding in the world-sheet in the Minkowski space. This is the so-called Polyakov action:

$$S_P = -\frac{1}{4\pi\alpha'}\int_{\tau_1}^{\tau_2} d\tau \int_0^{\pi} d\sigma\ \eta_{\mu\nu}\sqrt{-g(\sigma,\tau)}\ g^{\alpha\beta}(\sigma,\tau)\,\partial_\alpha X^\mu(\sigma,\tau)\,\partial_\beta X^\nu(\sigma,\tau) \qquad (1)$$

where $\eta_{\mu\nu}$ is given in (I-1.2) but we can use instead of $\eta_{\mu\nu}$ any arbitrary background metric, solution of Einstein's equations for a D-dimensional empty space.

We shall consider as independent fields $X^\mu(\sigma,\tau)$ and $g_{\alpha\beta}(\sigma,\tau)$

Notice

$$g_{\alpha\beta} \equiv \begin{vmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{vmatrix} \qquad g_{\alpha\beta}\,g^{\beta\gamma} = \delta_\alpha^{\ \gamma} \qquad (2)$$

$$g \equiv g_{00}g_{11} - g_{01}g_{10}\ , \qquad g^{\alpha\beta} = \frac{1}{g}\begin{vmatrix} g_{11} & -g_{01} \\ -g_{10} & g_{22} \end{vmatrix}$$

From this it is easy to check that

$$g_{\alpha\beta} \longrightarrow g_{\alpha\beta} + \delta g_{\alpha\beta} \qquad (3)$$

implies

$$g \longrightarrow g + g\,g^{\alpha\beta}\,\delta g_{\alpha\beta} \qquad (4)$$

$$g^{\alpha\beta} \longrightarrow g^{\alpha\beta} - g^{\alpha\gamma}g^{\beta\rho}\,\delta g_{\gamma\rho}$$

Then under (3) and

$$X^\mu \longrightarrow X^\mu + \delta X^\mu \qquad (5)$$

we get

$$\delta S_P = -\frac{1}{4\pi\alpha'}\int_{\tau_1}^{\tau_2} d\tau \int_0^{\pi} d\sigma\ \eta_{\mu\nu}\sqrt{-g}\ \Big\{ 2g^{\alpha\beta}\,\partial_\alpha X^\mu\,\partial_\beta(\delta X^\nu) \qquad (6)$$

$$+ \frac{1}{2}\,g^{\alpha\beta}\,\partial_\alpha X^\mu\,\partial_\beta X^\nu\,g^{\rho\gamma}\,g^{\sigma\delta}\,\delta g_{\gamma\delta} - \partial_\alpha X^\mu\,\partial_\beta X^\nu\,g^{\alpha\gamma}\,g^{\beta\rho}\,\delta g_{\gamma\rho}\Big\} \Rightarrow$$

$$\delta S_P = -\frac{1}{4\pi\alpha'}\int_0^{\pi} d\sigma\ \eta_{\mu\nu}\sqrt{-g}\ 2\,g^{\alpha\rho}\,\partial_\alpha X^\mu\,\delta X^\nu\Big|_{\tau_1}^{\tau_2}$$

$$-\frac{1}{4\pi\alpha'}\int_{\tau_1}^{\tau_2} d\tau\ \eta_{\mu\nu}\sqrt{-g}\ 2\,g^{\alpha\tau}\,\partial_\alpha X^\mu\,\delta X^\nu\Big|_0^{\pi}$$

$$-\frac{1}{4\pi\alpha'}\int_{\tau_1}^{\tau_2} d\tau \int_0^{\pi} d\sigma\ \Big\{ -\partial_\beta\big[\,2\,\eta_{\mu\nu}\sqrt{-g}\,g^{\alpha\beta}\,\partial_\alpha X^\mu\,\big]\,\delta X^\nu +$$

$$+ \frac{1}{2}\eta_{\mu\nu}\sqrt{-g}\,[\,g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu g^{\gamma\delta} - 2\,\partial_\alpha X^\mu \partial_\beta X^\nu g^{\alpha\gamma} g^{\beta\delta}\,]\,\delta g_{\gamma\delta}\,\Big\} \tag{1}$$

Since

$$\delta X^\nu(\sigma,\tau_1) = \delta X^\nu(\sigma,\tau_2) = 0 \tag{2}$$

we obtain as equations of motion

$$\partial_\alpha\{\sqrt{-g}\;g^{\alpha\beta}\partial_\beta X^\mu\} = 0 \tag{3}$$

$$\partial_\alpha X^\mu \partial_\beta X_\nu = \frac{1}{2} g_{\alpha\beta} g^{\gamma\delta}\partial_\gamma X^\mu \partial_\delta X_\mu \tag{4}$$

and for the open string the edge condition

$$\sqrt{-g}\;g^{1\alpha}\partial_\alpha X^\mu = 0 \qquad \text{for} \quad \sigma = 0,\pi \tag{5}$$

Notice that from equation (4)

$$[-\det(\partial_\alpha X^\mu \partial_\beta X_\mu)]^{1/2} = \frac{1}{2}\sqrt{-g}\;g^{\gamma\delta}\partial_\gamma X^\mu \partial_\delta X^\nu \eta_{\mu\nu} \tag{6}$$

and therefore the Polyakov action (10.1) reduces to the Nambu-Goto action (3.4), using only algebraic equations. This implies that the classical theories derived from both actions are equivalent.

It is clear that the induced metric (3.2)

$$g_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu \tag{7}$$

is a solution of (4), but it is not the unique one since

$$g_{\alpha\beta}(\sigma,\tau) = e^{\phi(\sigma,\tau)}\,\partial_\alpha X^\mu(\sigma,\tau)\,\partial_\beta X_\mu(\sigma,\tau) \tag{8}$$

is also a solution for any arbitrary world-sheet scalar $\phi(\sigma,\tau)$.

This is due to the Weyl (conformal) invariance of the Polyakov action (10.1)

$$\begin{aligned} X^\mu(\sigma,\tau) &\longrightarrow X^\mu(\sigma,\tau) \\ g_{\alpha\beta}(\sigma,\tau) &\longrightarrow e^{\phi(\sigma,\tau)}\,g_{\alpha\beta}(\sigma,\tau) \end{aligned} \tag{9}$$

and hence

$$\begin{aligned} g(\sigma,\tau) &\longrightarrow e^{d\,\phi(\sigma,\tau)}\,g(\sigma,\tau) \\ g^{\alpha\beta}(\sigma,\tau) &\longrightarrow e^{(1-d)\,\phi(\sigma,\tau)}\,g^{\alpha\beta}(\sigma,\tau) \end{aligned} \tag{10}$$

where d is the dimension of the space of the indexes α and β. Clearly the Polyakov action is conformal invariant only if $d=2$.

Furthermore inserting the solution (11.8) in equation (11.3) we recover the equations of motion (6.2).

III - COVARIANT SOLUTION OF THE EQUATIONS OF MOTION

Our next step will be to solve the equations of motion (II - 6.2).

$$\frac{\partial}{\partial \tau} P^{\mu}{}_{\tau} + \frac{\partial}{\partial \sigma} P^{\mu}{}_{\sigma} = 0 \qquad (1)$$

where P^{μ}_{τ} and P^{μ}_{σ} are explicitely given in (6.1). Since this equation is reparametrization gauge invariant, we cannot pretend that it will determine the functional form of $X^{\mu}(\sigma,\tau)$, because it will depend on the parametrization chosen. Then, we need to fix the gauge in order to solve (1) for $X^{\mu}(\sigma,\tau)$. In this section we will do that in a Lorentz covariant way

We recall a well known fact from the general theory of surfaces: every surface is locally conformally flat. This means that it is always possible to choose a parametrization in which the metric reads

$$g_{\alpha\beta} = e^{\phi} \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (2)$$

The signature $(+1,-1)$ account for the fact that we have a Minkowskian surface instead of an Euclidean one. If we compare (2) with the definition of the induced metric (II - 3.2) we conclude

$$\dot{X} \cdot X' = 0 \quad , \quad \dot{X}^2 + X'^2 = 0 \qquad (3)$$

or equivalently

$$(\dot{X} \pm X')^2 = 0 \qquad (4)$$

Equations (3) express that the parametrization that we are using for the world-sheet is orthonormal, i.e. the lines of constant τ and constant σ are orthonormal each other in the Minkowski sense. Then, equations (4) should be understood as gauge fixing conditions. This gauge is known as the "orthogonal gauge" and it provides a Lorentz covariant gauge fixing condition, in the same way as $\partial_\mu A^\mu(x) = 0$ does it for electromagnetism.

In the orthonormal gauge equations (II - 6.1) can be written as.

$$P^{\mu}_{C}(\sigma,\tau) = \frac{1}{2\alpha'\pi}\,\dot{X}^{\mu}(\sigma,\tau) \quad , \quad P^{\mu}_{O}(\sigma,\tau) = -\frac{1}{2\alpha'\pi}\,X'^{\mu}(\sigma,\tau) \tag{1}$$

so that the constraints (II-6.4) and (II-6.5) become identical to the gauge fixing conditions (1.3). On the other hand the edge condition (II-6.3) for the open string reads now :

$$X'^{\mu}(0,\tau) = X'^{\mu}(\pi,\tau) = 0 \tag{2}$$

Finally, inserting (1) in (1.1) we obtain the equations of motion in the orthogonal gauge

$$\ddot{X}^{\mu}(\sigma,\tau) - X''^{\mu}(\sigma,\tau) = 0 \tag{3}$$

The Open String

The most general solution of (3) satisfying the edge condition (2) is

$$X^{\mu}(\sigma,\tau) = \sum_{n=0}^{\infty} X^{\mu}_{n}(\tau)\cos n\sigma \tag{4}$$

where the Fourier coefficients $X^{\mu}_{n}(\tau)$ satisfy

$$\ddot{X}^{\mu}_{n}(\tau) + n^2 X^{\mu}_{n}(\tau) = 0 \quad , \quad n = 0, 1, 2, \ldots \tag{5}$$

In order to solve these equations for $n \neq 0$ we introduce

$$\alpha^{\mu}_{n}(\tau) \equiv \frac{1}{2}\left[\dot{X}^{\mu}_{n}(\tau) - in\,X^{\mu}_{n}(\tau)\right] \tag{6}$$

which satisfy

$$\dot{\alpha}^{\mu}_{n}(\tau) + in\,\alpha^{\mu}_{n}(\tau) = 0 \tag{7}$$

and therefore

$$\alpha^{\mu}_{n}(\tau) = \alpha^{\mu}_{n}\,e^{-in\tau} \quad , \quad \alpha^{\mu *}_{n}(\tau) = \alpha^{\mu *}_{n}\,e^{+in\tau} \tag{8}$$

$$\alpha^{\mu}_{n} \equiv \alpha^{\mu}_{n}(0)$$

Then

$$X^{\mu}_{n}(\tau) = \frac{i}{n}\left\{\alpha^{\mu}_{n}\,e^{-in\tau} - \alpha^{\mu *}_{n}\,e^{+in\tau}\right\} \quad , \quad n = 1, 2, \ldots \tag{9}$$

When $n=0$ the solution of (5) is

$$X_0^\mu(\tau) = q^\mu + \alpha_0^\mu \tau \qquad (1)$$

where q^μ and α_0^μ are constant vectors. Hence

$$X^\mu(\sigma,\tau) = q^\mu + \alpha_0^\mu \tau + i \sum_{m=1}^{\infty} \frac{1}{m} \left[\alpha_m^\mu e^{-im\tau} - \alpha_m^{\mu *} e^{+im\tau} \right] \cos m\sigma \qquad (2)$$

or equivalently

$$X^\mu(\sigma,\tau) = q^\mu + \alpha_0^\mu \tau + i \sum_{\substack{m=-\infty \\ m\neq 0}}^{+\infty} \frac{1}{m} \alpha_m^\mu e^{-im\tau} \cos m\sigma \qquad (3)$$

$$\alpha_{-m}^\mu \equiv \alpha_m^{\mu *}$$

Equation (3) gives the general solution of the equation of motion for an open string in terms of the constants q^μ and the Fourier coefficients $\{\alpha_m^\mu\}_{-\infty}^{+\infty}$. However, these Fourier coefficients are not all independent because the solution (3) must obey the constraints (1.4).

The easiest way to impose the constraints it is to notice that if we extend analytically $X^\mu(\sigma,\tau)$ from $0 \leq \sigma \leq \pi$ to $-\pi \leq \sigma \leq \pi$, we have

$$\dot{X}^\mu(\sigma,\tau) = \dot{X}^\mu(-\sigma,\tau) = \sum_{m=-\infty}^{+\infty} \alpha_m^\mu e^{-im\tau} \cos m\sigma \qquad (4)$$

$$X'^\mu(\sigma,\tau) = -X'^\mu(-\sigma,\tau) = -i \sum_{m=-\infty}^{+\infty} \alpha_m^\mu e^{-im\tau} \sin m\sigma$$

Then we can unify the constraints by requiring that

$$\left[\dot{X}(\sigma,\tau) + X'(\sigma,\tau) \right]^2 = 0 \qquad -\pi \leq \sigma \leq \pi \qquad (5)$$

From (4)

$$\dot{X}^\mu(\sigma,\tau) + X'^\mu(\sigma\tau) = \sum_{m=-\infty}^{+\infty} \alpha_m^\mu e^{-im(\tau+\sigma)} \qquad (6)$$

and hence

$$\left[\dot{X}(\sigma,\tau) + X'(\sigma,\tau) \right]^2 = \sum_{m=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \alpha_m^\mu \alpha_{m\mu} e^{-i(m+m)(\tau+\sigma)} = \qquad (7)$$

$$= \sum_{m=-\infty}^{+\infty} e^{-im(\tau+\sigma)} \sum_{m=-\infty}^{+\infty} \alpha_{m-m}^\mu \alpha_{m\mu}$$

If we introduce the Virasoro gauges

$$L_m \equiv -\frac{1}{4\alpha'} \sum_{m=-\infty}^{\infty} \alpha^{\mu}_{m-m}\, \alpha_{m\mu} \quad , \quad m \in \mathbb{Z} \tag{1}$$

we obtain

$$[\dot{X}(\sigma,\tau) + X'(\sigma,\tau)]^2 = -4\alpha' \sum_{m=-\infty}^{+\infty} L_m\, e^{-im(\tau+\sigma)} \tag{2}$$

We can therefore express the constraint (2.5) as

$$L_m = 0 \quad , \quad m \in \mathbb{Z} \tag{3}$$

Notice that L_m is a dimensionless quantity. Because the existence of constrains the variables α_m^{μ} and $\alpha_m^{\mu *}$ are not dinamically independent and obviously problems will appear when we try to quantize the theory.

The Closed String

For the closed string we must solve equation (2.3) with the boundary conditions

$$X^{\mu}(0,\tau) = X^{\mu}(\pi,\tau) \quad , \quad X'^{\mu}(0,\tau) = X'^{\mu}(\pi,\sigma) \tag{4}$$

The most general solution of (2.3) which satisfies these boundary conditions will be of the form

$$X^{\mu}(\sigma,\tau) = q^{\mu} + 2\alpha_0^{\mu}\tau + \sum_{m=1}^{\infty} i\left[\mathring{X}_m^{\mu}(\tau)\cos 2m\sigma + \hat{X}_m^{\mu}(\tau)\sin 2m\sigma\right] \tag{5}$$

$$= q^{\mu} + 2\alpha_0^{\mu}\tau + \sum_{m=1}^{\infty}\left\{\frac{1}{2}\left[\mathring{X}_m^{\mu}(\tau) - i\hat{X}_m^{\mu}(\tau)\right]e^{+i2m\sigma} + \frac{1}{2}\left[\mathring{X}_m^{\mu}(\tau) + i\hat{X}_m^{\mu}(\tau)\right]e^{-i2m\sigma}\right\}$$

Furthermore, if we introduce

$$X_m^{\mu}(\tau) \equiv \frac{1}{2}\left[\mathring{X}_m^{\mu}(\tau) - i\hat{X}_m^{\mu}(\tau)\right] \quad , \quad \tilde{X}_m^{\mu}(\tau) \equiv \frac{1}{2}\left[\mathring{X}_m^{\mu}(\tau) + i\hat{X}_m^{\mu}(\tau)\right] \tag{6}$$

in order that (5) be a solution of the equations of motion we must have

$$\ddot{X}_m^{\mu}(\tau) + 4m^2 X_m^{\mu}(\tau) = 0 \quad , \quad \ddot{\tilde{X}}_m^{\mu}(\tau) + 4m^2 \tilde{X}_m^{\mu}(\tau) = 0 \qquad m = 1, 2, \ldots \tag{7}$$

Introducing

$$\alpha_m^\mu(\tau) \equiv \frac{1}{2}\left[\dot{X}_m^\mu(\tau) - i2m X_m^\mu(\tau)\right], \quad \tilde{\alpha}_m^\mu(\tau) \equiv \frac{1}{2}\left[\dot{X}_m^\mu(\tau) - i2m \tilde{X}_m^\mu(\tau)\right] \tag{1}$$

we find

$$\dot{\alpha}_m^\mu(\tau) + 2im\,\alpha_m(\tau) = 0, \quad \dot{\tilde{\alpha}}_m(\tau) + 2im\,\alpha_m(\tau) = 0, \quad m = 1, 2, \ldots \tag{2}$$

Hence

$$\alpha_m^\mu(\tau) = \alpha_m^\mu e^{-2im\tau}, \quad \tilde{\alpha}_m^\mu(\tau) = \tilde{\alpha}_m^\mu e^{-2im\tau} \tag{3}$$

$$\alpha_m^\mu \equiv \alpha_m^\mu(0) \qquad \tilde{\alpha}_m^\mu \equiv \tilde{\alpha}_m^\mu(0)$$

From all that we see that the required general solution of the equations of motion is

$$X^\mu(\sigma,\tau) = q^\mu + 2\alpha_0^\mu\tau + \frac{i}{2}\sum_{m=1}^{\infty}\frac{1}{m}\left\{\left[\alpha_m^\mu e^{-2im\tau} - \alpha_m^{\mu *} e^{+2im\tau}\right]e^{+2im\sigma} + \left[\tilde{\alpha}_m^\mu e^{-2im\tau} - \tilde{\alpha}_m^{\mu *} e^{+2im\tau}\right]e^{-2im\sigma}\right\} \tag{4}$$

or equivalently

$$X^\mu(\sigma,\tau) = q^\mu + 2\alpha_0^\mu\tau + \frac{i}{2}\sum_{\substack{m=-\infty \\ m\neq 0}}^{+\infty}\frac{1}{m}\left\{\alpha_m^\mu e^{-2im(\tau-\sigma)} + \tilde{\alpha}_m^\mu e^{-2im(\tau+\sigma)}\right\} \tag{5}$$

$$\alpha_{-m}^\mu \equiv \alpha_m^{\mu *}, \qquad \tilde{\alpha}_{-m}^\mu \equiv \tilde{\alpha}_m^{\mu *}$$

From this equation we see that the Fourier coefficients α_m^μ ($\tilde{\alpha}_m^\mu$) correspond to the amplitudes of right (left) moving waves on the closed string. The coefficients α_m^μ and $\tilde{\alpha}_m^\mu$ are not independent because of the constraints (1.5). Let us analyze them

$$\dot{X}^\mu(\sigma,\tau) = \sum_{m=-\infty}^{+\infty}\left[\alpha_m^\mu e^{-2im(\tau-\sigma)} + \tilde{\alpha}_m^\mu e^{-2im(\tau+\sigma)}\right] \tag{6}$$

$$X'^\mu(\sigma,\tau) = \sum_{m=-\infty}^{+\infty}\left[-\alpha_m^\mu e^{-2im(\tau-\sigma)} + \tilde{\alpha}_m^\mu e^{-2im(\tau+\sigma)}\right]$$

$$\tilde{\alpha}_0^\mu \equiv \alpha_0^\mu$$

Hence

$$\dot{X}^\mu(\sigma,\tau) + X'^\mu(\sigma,\tau) = 2\sum_{m=-\infty}^{\infty}\tilde{\alpha}_m^\mu e^{-2im(\tau+\sigma)} \tag{7}$$

$$\dot{X}^\mu(\sigma,\tau) - X'^\mu(\sigma,\tau) = 2\sum_{m=-\infty}^{+\infty}\alpha_m^\mu e^{-2im(\tau-\sigma)}$$

If we introduce

$$L_m \equiv -\frac{1}{2\alpha'} \sum_{m=-\infty}^{+\infty} \alpha^{\mu}_{m-m} \alpha_{m\mu} \quad , \quad \tilde{L}_m \equiv -\frac{1}{2\alpha'} \sum_{m=-\infty}^{+\infty} \tilde{\alpha}^{\mu}_{m-m} \tilde{\alpha}_{m\mu} \tag{1}$$

we obtain as before

$$[\dot{X} + X']^2 = -8\alpha' \sum_{m=-\infty}^{+\infty} \tilde{L}_m \, e^{-2im(\tau+\sigma)}$$

$$[\dot{X} - X']^2 = -8\alpha' \sum_{m=-\infty}^{+\infty} L_m \, e^{-2im(\tau-\sigma)} \tag{2}$$

The constraints are now expressed by the vanishing of the Virasoro gauges

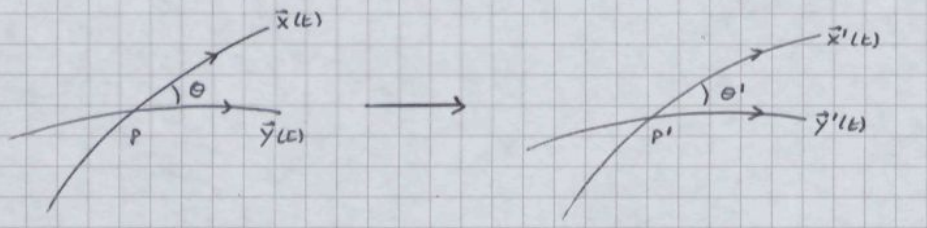
$$L_m = \tilde{L}_m = 0 \quad , \quad m \in \mathbb{Z} \tag{3}$$

A - III - CONFORMAL GROUP

Let us consider a D-dimensional Euclidean space; let us denote by $\{x_i\}$ $(i = 1, 2, \ldots D)$ the coordinates of a generic point in this space. Let us now consider the infinitesimal transformation

$$x_i \longrightarrow x_i' = x_i + \delta f_i(\vec{x}) \tag{1}$$

This transformation is called a conformal transformation if the angle between two arbitrary curves which intersect in the point P is invariant under (1). Let us parametrize the curves with a parameter t.



$$P \equiv \vec{Z} \equiv \vec{x}(t_0) \equiv \vec{y}(t_0) \qquad\qquad P' \equiv \vec{Z}' \equiv \vec{x}'(t_0') \equiv \vec{y}'(t_0')$$

The tangent vectors are

$$\vec{P} : \dot{\vec{x}}, \quad \dot{\vec{y}} \tag{2}$$

$$P' : \dot{\vec{x}}' = \dot{\vec{x}} + (\partial_j \delta \vec{f})\, \dot{x}_j, \quad \dot{\vec{y}}' = \dot{\vec{y}} + (\partial_j \delta \vec{f})\, \dot{y}_j$$

Let us now impose the condition that $\Theta' = \Theta$

$$(\dot{\vec{x}} \cdot \dot{\vec{y}})\, \frac{1}{\dot{x}\,\dot{y}} = (\dot{\vec{x}}' \cdot \dot{\vec{y}}')\, \frac{1}{\dot{x}'\,\dot{y}'} \tag{3}$$

Hence

$$\dot{x}\,\dot{y} \left\{ (\dot{\vec{x}} \cdot \dot{\vec{y}}) + \dot{x}_i (\partial_j \delta f_i)\, \dot{y}_j + \dot{y}_i (\partial_j \delta f_i)\, \dot{x}_j \right\} =$$

$$= (\dot{\vec{x}} \cdot \dot{\vec{y}}) \left\{ \dot{x} + \frac{1}{\dot{x}} \dot{x}_i (\partial_j \delta f_i) \dot{x}_j \right\} \left\{ \dot{y} + \frac{1}{\dot{y}} \dot{y}_i (\partial_j \delta f_i) \dot{y}_j \right\} \Rightarrow$$

$$\dot{x}_i (\partial_j \delta f_i) \dot{y}_j + \dot{y}_i (\partial_j \delta f_i) \dot{x}_j = (\dot{\vec{x}} \cdot \dot{\vec{y}}) \left\{ \frac{1}{\dot{x}^2} \dot{x}_i (\partial_j \delta f_i) \dot{x}_j + \frac{1}{\dot{y}^2} \dot{y}_i (\partial_j \delta f_i) \dot{y}_j \right\} \tag{4}$$

Let us now apply $\delta_{ke}\, \partial_k^x\, \partial_e^y$, then

$$2 (\partial_k \delta f_k) = D \left\{ \frac{1}{\dot{x}^2} \dot{x}_i (\partial_j \delta f_i) \dot{x}_j + \frac{1}{\dot{y}^2} \dot{y}_i (\partial_j \delta f_i) \dot{y}_j \right\} \tag{5}$$

and substituting (1.5) in (1.6) we get

$$\dot{\bar{x}}_i \left[\partial_j \delta f_i\right] \dot{\bar{y}}_j + \dot{\bar{y}}_i \left[\partial_j \delta f_i\right] \dot{\bar{x}}_j = \frac{2}{D} \left(\dot{\bar{x}}_i \dot{\bar{y}}_i\right) \left[\partial_k \delta f_k\right] \tag{1}$$

Since this equation must be valid for any values of $\dot{\bar{x}}$ and $\dot{\bar{y}}$ we get

$$\partial_j \delta f_i(\bar{x}) + \partial_i \delta f_j(\bar{x}) = \delta_{ij} \frac{2}{D} \left[\partial_k \delta f_k(\bar{x})\right] \tag{2}$$

which is the equation that determine $\delta \bar{f}(\bar{x})$.

This system can be solved exactly. In orden to simplify the notation let us write (2) as

$$\partial_j v_i + \partial_i v_j = \delta_{ij} \psi$$

$$v_i \equiv \delta f_i \quad , \quad \psi \equiv \frac{2}{D} \partial_k \delta f_k \tag{3}$$

From (3)

$$\partial_k \partial_j v_i + \partial_k \partial_i v_j = \delta_{ij} \partial_k \psi$$

$$\partial_j \partial_k v_i + \partial_j \partial_i v_k = \delta_{ik} \partial_j \psi \tag{4}$$

Subtracting these two relations and taking into account the integrability condition

$$\partial_k \partial_j v_i - \partial_j \partial_k v_i = 0 \tag{5}$$

we get

$$\partial_k \partial_i v_j - \partial_j \partial_i v_k = \delta_{ij} \partial_k \psi - \delta_{ik} \partial_j \psi \tag{6}$$

Hence

$$\partial_e \partial_k \partial_i v_j - \partial_e \partial_j \partial_i v_k = \delta_{ij} \partial_e \partial_k \psi - \delta_{ik} \partial_e \partial_j \psi$$

$$\partial_e \partial_k \partial_j v_i - \partial_e \partial_i \partial_j v_k = \delta_{ji} \partial_e \partial_k \psi - \delta_{jk} \partial_e \partial_i \psi \tag{7}$$

which implies

$$\partial_e \partial_k \left(\partial_i v_j - \partial_j v_i\right) - \partial_e \partial_j \partial_i v_k + \partial_e \partial_i \partial_j v_k = \delta_{jk} \partial_e \partial_i \psi - \delta_{ik} \partial_e \partial_j \psi \Rightarrow$$

$$\partial_k \partial_e \left(\partial_i v_j - \partial_j v_i\right) - \partial_k \partial_j \partial_i v_e + \partial_k \partial_i \partial_j v_e = \delta_{je} \partial_k \partial_i \psi - \delta_{ie} \partial_k \partial_j \psi \tag{8}$$

Subtracting these two equations and using the integrability conditions

$$\partial_e \partial_k \left(\partial_i v_j - \partial_j v_i\right) - \partial_k \partial_e \left(\partial_i v_j - \partial_j v_i\right) = 0$$

we get

$$\partial_e \partial_i \partial_j v_k - \partial_e \partial_j \partial_i v_k + \partial_k \partial_j \partial_i v_e - \partial_k \partial_i \partial_j v_e =$$

$$= \delta_{jk} \partial_e \partial_i \psi - \delta_{ik} \partial_e \partial_j \psi + \delta_{ie} \partial_k \partial_j \psi - \delta_{je} \partial_k \partial_i \psi \qquad (1)$$

Contracting this expression with δ_{ej}

$$\partial_j \partial_i \partial_j v_k - \partial_j \partial_j \partial_i v_k + \partial_k \partial_j \partial_i v_j - \partial_k \partial_i \partial_j v_j =$$

$$= \partial_k \partial_i \psi - \delta_{ik} \partial_j \partial_j \psi + \partial_k \partial_i \psi - D \partial_k \partial_i \psi \qquad (2)$$

and using (2.5)

$$(D-2) \partial_k \partial_i \psi + \delta_{ik} \partial_j \partial_j \psi = 0 \qquad (3)$$

Contracting with δ_{ki}

$$(D-1) \partial_i \partial_i \psi = 0 \qquad (4)$$

and since $D \geq 2$ we obtain

$$\partial_i \partial_i \psi = 0 \qquad (5)$$

and (3) implies

$$(D-2) \partial_k \partial_i \psi = 0 \qquad (6)$$

Let us now consider

$D \neq 2$

Then (6) implies

$$\partial_i \partial_j \psi = 0 \qquad (7)$$

This means that the Taylor expansion of ψ terminates at second order

$$\psi(\vec{x}) = 2b + 4 b_i x_i \qquad (8)$$

Substituting (8) in (2.6) we get, using (2.5),

$$\partial_i [\partial_k v_j - \partial_j v_k] = 4 (\delta_{ij} b_k - \delta_{ik} b_j) \qquad (9)$$

After integration with respect x_i

$$\partial_K v_j - \partial_j v_K = 4 (b_K x_j - b_j x_K) + 2 a_{jK} \quad , \quad a_{Kj} = - a_{jK} \tag{1}$$

Furthermore (2.3)

$$\partial_K v_j + \partial_j v_K = 2 \delta_{jK} b + 4 \delta_{jK} (b_i x_i) \tag{2}$$

Adding (1) and (2)

$$\partial_K v_j = a_{jK} + 2 b_K x_j - 2 b_j x_K + b \delta_{jK} + 2 \delta_{Kj} (b_i x_i) \tag{3}$$

and integration with respect to x_K gives

$$v_j = a_j + a_{ji} x_i + b x_j + 2 (b_i x_i) x_j - b_j (x_i x_i) \tag{4}$$

which is the general solution of (2.3). Then

$$\delta f_i (\bar{x}) = a_i + a_{ij} x_j + b x_i + 2 (b_j x_j) x_i - b_i (x_j x_j) \quad , \quad D \neq 2 \tag{5}$$

This solutions depends of

$$D + \frac{1}{2} D (D-1) + 1 + D = \frac{1}{2} (D+1) (D+2) \equiv N \tag{6}$$

arbitrary constants. This is the order of the group for $D \neq 2$. The different transformations are

$\delta f_i (\bar{x}) = a_i$	Translations	Order D
$\delta f_i (\bar{x}) = a_{ij} x_j$	Rotations	Order $\frac{1}{2} D (D-1)$
$\delta f_i (\bar{x}) = b x_i$	Dilatations	Order 1
$\delta f_i (\bar{x}) = 2 (b_j x_j) x_i - b_i (x_j x_j)$	Proper conformal Transformations	Order D

(7)

Notice that for $D = 4$, $N = 15$.

The N independent generators are

$$P_i = \partial_i \quad , \quad M_{ij} = x_i \partial_j - x_j \partial_i$$

$$d = x_i \partial_i \quad , \quad k_i = 2 x_i x_j \partial_j - (x_j x_j) \partial_i \tag{8}$$

with commutation relations

$$[P_i, P_j] = 0 \quad , \quad [M_{ij}, P_k] = \delta_{jk} P_i - \delta_{ik} P_j$$

$$[M_{ij}, M_{ke}] = \delta_{jk} M_{ie} + \delta_{ie} M_{jk} - \delta_{ik} M_{je} - \delta_{je} M_{ik}$$

$$[d, M_{ij}] = 0 \quad , \quad [d, P_i] = -P_i \quad , \quad [d, K_i] = K_i \qquad (1)$$

$$[M_{ij}, K_k] = \delta_{je} K_i - \delta_{ie} K_j \quad , \quad [P_i, K_j] = -M_{ij} + \delta_{ij} d$$

$$[K_i, K_j] = 0$$

In order to obtain the finite transformations generated by the different transformations we must solve the equation

$$\frac{d x_i(t)}{dt} = \delta f_i [x(t)] \qquad (2)$$

where δf_i is given by (4.5)., where we shall set all the constants of integration in (4.5) equal to zero except the ones that we are interested in.

Translations : $a_i \neq 0$

$$\frac{d x_i(t)}{dt} = a_i$$

$$\Rightarrow \quad x_i(t) = x_i(0) + a_i t \qquad (3)$$

Dilatations : $b \neq 0$

$$\frac{d x_i(t)}{dt} = b\, x_i(t)$$

$$\Rightarrow \quad x_i(t) = e^{bt} x_i(0) \qquad (4)$$

Rotations : $a_{ij} \neq 0$

$$\frac{d x_i(t)}{dt} = a_{ij} x_j(t) \quad \Rightarrow$$

$$x_i(t) = [\exp(at)]_{ij} x_j(0) \qquad (5)$$

Proper transformal transformations : $b_i \neq 0$

$$\frac{dx_i(t)}{dt} = 2\,[\,b_j\,x_j(t)\,]\,x_i(t) - b_i\,x^2(t) \quad , \quad x^2(t) \equiv x_i(t)\,x_i(t) \tag{1}$$

In order to solve this equation we shall rewrite the infinitesimal transformation as

$$x_i(t+\epsilon) = Q\,P_\epsilon\,Q\,x_i(t) \tag{2}$$

$$Q\,x_i(t) \equiv \frac{x_i(t)}{x^2(t)} \quad , \quad P_\epsilon\,x_i(t) \equiv x_i(t) + \epsilon\,b_i$$

Let us check

$$Q\,P_\epsilon\,Q\,x_i \equiv Q\,P_\epsilon\,\frac{x_i}{x^2} = Q\,\frac{x_i + \epsilon\,b_i}{(x+\epsilon b)^2} = Q\,[\,x_i + \epsilon b_i\,]\,\frac{1}{x^2 + 2\epsilon\,(\vec{b}.\vec{x})} =$$

$$= Q\,(x_i + \epsilon\,b_i)\,\frac{1}{x^2}\,\left(1 + 2\epsilon\,\frac{\vec{b}.\vec{x}}{x^2}\right) = Q\,\frac{1}{x^2}\,\left[\,x_i + \epsilon\,b_i + 2\epsilon\,\frac{(\vec{b}.\vec{x})\,x_i}{x^2}\right] =$$

$$= x^2\left(\frac{x_i}{x^2} + \epsilon\,b_i + 2\epsilon\,\frac{(\vec{b}\,\frac{\vec{x}}{x^2})\,\frac{x_i}{x^2}}{\frac{x^2}{x^4}}\right) = x_i + \epsilon\,b_i\,x^2 + 2\epsilon\,(\vec{b}.\vec{x})\,x_i \qquad \text{Q.E.D.}$$

Notice that $Q^2 = 1$. Then

$$x_i(t) = Q\,(P_{t/N})^N\,Q\,x_i(0) \tag{3}$$

Hence

$$x_i(t) = Q\,(P_{t/N})^N\,\frac{x_i(0)}{x^2(0)} = Q\,\frac{x_i(0) - b_i\,t}{(x(0) - b\,t)^2} = Q\,\frac{x_i(0) - b_i\,t}{x^2(0) - 2t\,(\vec{b}.\vec{x}(0)) + b^2\,t^2} =$$

$$= \frac{x^{-2}(0)\,x_i(0) - b_i\,t}{x^{-2}(0) - 2\,x^{-2}(0)\,t\,(b.\vec{x}(0)) + b^2t^2} \Rightarrow$$

$$x_i(t) = \frac{x_i(0) - b_i\,t\,x^2(0)}{1 - 2\,[\vec{b}.\vec{x}(0)]\,t + b^2\,t^2\,\vec{x}^2(0)} \tag{4}$$

which is the so called proper conformal transformation. Then

$$x_i' = x_i + a_i \qquad \text{Translations}$$

$$x_i' = e^b\,x_i \qquad \text{Dilatations} \tag{5}$$

$$x_i' = (e^a)_{ij}\,x_j \quad , \quad a^T = -a \qquad \text{Rotations}$$

$$x_i' = \frac{x_i - b_i\,x^2}{1 - 2\,(\vec{b}.\vec{x}) + b^2x^2} \qquad \text{Proper conformal transformation}$$

D = 2

In this case (3.6) is an identity, but (2.2) is equivalent to

$$\partial_1 \delta f_1 = \partial_2 \delta f_2 \quad , \quad \partial_1 \delta f_2 = -\partial_2 \delta f_1 \tag{1}$$

but this are no more that the Cauchy - Riemann conditions. Hence if

$$z \equiv x_1 + i x_2 \quad , \quad \delta f = \delta f_1 + i \delta f_2 \tag{3}$$

The transformations

$$z' = z + \delta f(z) \tag{4}$$

are conformal if $\delta f(z)$ is an analytic function. This is equivalent to write

$$z' = z + z\, \delta g(z) \tag{5}$$

where $\delta g(z)$ is an analytic function. Making a Laurent expansion of $\delta g(z)$

$$\delta g(z) = \sum_{m=-\infty}^{+\infty} \epsilon_{-m} z^m \tag{6}$$

we see that the group has an infinite number of parameters $\{\epsilon_m\}_{-\infty}^{+\infty}$ and its generators are

$$L_m = -z^{m+1} \frac{d}{dz} \quad , \quad m \in \mathbb{Z} \tag{7}$$

Their commutation relations are

$$[L_n, L_m] = (n-m) L_{n+m} \tag{8}$$

Let us now consider the central extensions of this algebra. This means to find an algebra such that

$$[L_n, L_m] = (n-m) L_{n+m} + C_{nmk} K_k \tag{9}$$

$$[L_n, K_k] = [K_k, K_m] = 0$$

being $C_{nmk} = -C_{mnk}$ constants.

$$C_{nmk} = -C_{mnk} \tag{10}$$

Furthermore

$$[[L_m, L_m], L_k] + [[L_m, L_k], L_m] + [[L_k, L_m], L_m] = 0 \qquad (1)$$

implies

$$(m-m)[L_{m+m}, L_k] + (m-k)[L_{m+k}, L_m] + (k-m)[L_{k+m}, L_m] = 0 \Rightarrow$$

$$(m-m)(m+m-k) L_{m+m+k} + (m-m) C_{m+m,k,e} K_e + (m-k)(m+k-m) L_{m+m+k} +$$

$$+ (m-k) C_{m+k,m,e} K_e + (k-m)(k+m-m) L_{k+m+m} + (k-m) C_{k+m,m,e} K_e = 0 \Rightarrow$$

$$(m-m) C_{m+m,k,e} + (m-k) C_{m+k,m,e} + (k-m) C_{k+m,m,e} = 0 \qquad (2)$$

or equivalently

$$(m-m) C_{e,m+m,k} + (m-e) C_{m,m+e,k} + (e-m) C_{m,e+m,k} = 0 \qquad (3)$$

In our case the only possibility is $K_k = \alpha k$ and then if

$$C_{m\,m\,k} K_k \equiv C_{m\,m} \qquad (4)$$

Then we have

$$C_{m,m} = -C_{m,m} \qquad (5)$$

$$(m-m) C_{e,m+m} + (m-e) C_{m,m+e} + (e-m) C_{m,e+m} = 0$$

Using the transformation

$$L_m \longrightarrow L_m + m C_{m,0} \equiv L'_m \quad m \neq 0$$

$$L_0 \longrightarrow L_0 + \frac{1}{2} C_{1,-1} \equiv L'_0 \qquad (6)$$

we have afterwards

$$[L'_m, L'_0] = m L'_m \quad , \quad [L'_0, L'_m] = -m L'_m$$

$$[L'_1, L'_{-1}] = 2 L'_0 \qquad (7)$$

and hence

$$C_{m,0} = C_{0,m} = C_{1,-1} = 0 \qquad (8)$$

Now let us put $\ell = 0$ in (8.5)

$$m\, C_{m,m} - m\, C_{m,m} = 0 \quad \Rightarrow \quad (m+m)\, C_{m,m} = 0 \quad \Rightarrow$$

$$C_{m,m} = f(m)\, \delta_{m+m,0} \tag{4}$$

Now let us put in (8.5) $n = 1$ and $\ell = -(m+1)$, then

$$(m-1)\, C_{-(m+1),\,m+1} + (m+2)\, C_{m,-m} = 0 \quad \Rightarrow$$

$$(m-1)\, C_{m+1,\,-(m+1)} = (m+2)\, C_{m,-m} \tag{2}$$

Then if

$$c = 2\, C_{2,-2} \tag{3}$$

we get

$$C_{m,m} = \frac{c}{12}\, m\,(m^2-1)\, \delta_{m+m,0} \tag{5}$$

and the extended Virasoro algebra is

$$[\,L_m,\, L_m\,] = (m-m)\, L_{m+m} + \frac{c}{12}\, m\,(m^2-1)\, \delta_{m+m,0} \tag{5}$$

where C is an arbitrary constant.

An important finite-dimensional subgroup of the conformal group consists of the Möbius transformations

$$z \longrightarrow z'(z) = \frac{a\,z + b}{b^*\,z + a^*}\,, \qquad |a|^2 - |b|^2 = 1 \tag{6}$$

It is immediate to check that these transformations form a group.

$$z_2 = \frac{a_2\, z_1 + b_2}{b_2^*\, z_1 + a_2^*}\,, \qquad z_1 = \frac{a_1\, z + b_1}{b_1^*\, z + a_1^*} \tag{7}$$

$$z_2 = \frac{(a_1 a_2 + b_1^* b_2)\, z + (b_1 a_2 + a_1^* b_2)}{(b_1 a_2 + a_1^* b_2)^*\, z + (a_1 a_2 + b_1^* b_2)^*}$$

$$|a_1 a_2 + b_1^* b_2|^2 - |b_1 a_2 + a_1^* b_2|^2 = 1$$

The map $z \longrightarrow z'$ covers S^1 once positively as z goes round S^1 positively. Indeed

$$z' = \frac{az + b}{b^* z + a^*} \tag{1}$$

is equivalent to

$$z' = \frac{a - w}{b^*} \quad , \quad w = \frac{1}{\zeta} \quad , \quad \zeta = b^* z + a^* \tag{2}$$

and the assertion follows.

Let us now consider the set of matrices

$$A = \begin{vmatrix} a & b \\ b^* & a^* \end{vmatrix} \quad , \quad |a|^2 - |b|^2 = 1 \tag{3}$$

which form the so called projective group $SU(1,1)$. The transformations

$$\begin{vmatrix} \zeta_1' \\ \zeta_2' \end{vmatrix} = \begin{vmatrix} a & b \\ b^* & a^* \end{vmatrix} \begin{vmatrix} \zeta_1 \\ \zeta_2 \end{vmatrix} \quad , \quad z = \frac{\zeta_1}{\zeta_2} \quad , \quad z' = \frac{\zeta_1'}{\zeta_2'} \tag{5}$$

is equivalent to (4)

This group has three parameters and we can choose as generators

$$L_0 = \frac{1}{2} \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \quad , \quad L_1 = \begin{vmatrix} 0 & 0 \\ -1 & 0 \end{vmatrix} \quad , \quad L_{-1} = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \tag{5}$$

Hence

$$[L_0, L_1] = -L_1 \quad , \quad [L_0, L_{-1}] = +L_{-1} \quad , \quad [L_1, L_{-1}] = 2L_0 \tag{6}$$

Which is precisely a subgroup $(L_0, L_{\pm 1})$ of the Virasoro algebra.

IV - NON-COVARIANT SOLUTIONS OF THE EQUATIONS OF MOTION

The orthonormal gauge does not fix completely the parametrization of the world-sheet, since there exists infinitely many orthonormal systems in a surface. Let us assume that (σ,τ) are such that

$$\dot{X}(\sigma,\tau)\cdot X'(\sigma,\tau)=0 \quad , \quad \dot{X}^2(\sigma,\tau)+X'^2(\sigma,\tau)=0 \tag{1}$$

Let us introduce now a new parametrization $\tilde{\sigma}=\tilde{\sigma}(\sigma,\tau)$, $\tilde{\tau}=\tilde{\tau}(\sigma,\tau)$. Notice

$$\frac{\partial\tau}{\partial\tilde{\sigma}}\frac{\partial\tilde{\sigma}}{\partial X^\mu}+\frac{\partial\tau}{\partial\tilde{\tau}}\frac{\partial\tilde{\tau}}{\partial X^\mu}=\frac{\partial\tau}{\partial X^\mu} \quad , \quad \frac{\partial\sigma}{\partial\tilde{\sigma}}\frac{\partial\tilde{\sigma}}{\partial X^\mu}+\frac{\partial\sigma}{\partial\tilde{\tau}}\frac{\partial\tilde{\tau}}{\partial X^\mu}=\frac{\partial\sigma}{\partial X^\mu} \tag{2}$$

If we require that $\partial\tau/\partial X^\mu$, $\partial\sigma/\partial X^\mu$ as well as $\partial\tilde{\tau}/\partial X^\mu$, $\partial\tilde{\sigma}/\partial X^\mu$ satisfy (1), we get from (2)

$$\frac{\partial\tilde{\tau}}{\partial X^\mu}\frac{\partial\tilde{\tau}}{\partial X^\nu}\left\{\frac{\partial\tau}{\partial\tilde{\sigma}}\frac{\partial\tau}{\partial\tilde{\tau}}-\frac{\partial\sigma}{\partial\tilde{\sigma}}\frac{\partial\sigma}{\partial\tilde{\tau}}\right\}=0 \tag{3}$$

$$\frac{\partial\tilde{\sigma}}{\partial X^\mu}\frac{\partial\tilde{\sigma}}{\partial X^\nu}\left\{\frac{\partial\tau}{\partial\tilde{\sigma}}\frac{\partial\tau}{\partial\tilde{\tau}}-\frac{\partial\tau}{\partial\tilde{\tau}}\frac{\partial\tau}{\partial\tilde{\sigma}}+\frac{\partial\sigma}{\partial\tilde{\tau}}\frac{\partial\sigma}{\partial\tilde{\sigma}}-\frac{\partial\sigma}{\partial\tilde{\sigma}}\frac{\partial\sigma}{\partial\tilde{\tau}}\right\}=0$$

which are satisfied if

$$\frac{\partial\tau}{\partial\tilde{\sigma}}=\lambda\frac{\partial\sigma}{\partial\tilde{\tau}} \quad , \quad \frac{\partial\sigma}{\partial\tilde{\sigma}}=\lambda\frac{\partial\tau}{\partial\tilde{\tau}} \quad , \quad \lambda=1 \text{ or } \lambda=-1 \tag{4}$$

Both solutions (4) lead to the integrability conditions

$$\frac{\partial^2\tau}{\partial\tilde{\tau}^2}=\frac{\partial^2\sigma}{\partial\tilde{\sigma}^2} \quad , \quad \frac{\partial^2\tau}{\partial\tilde{\sigma}^2}=\frac{\partial^2\sigma}{\partial\tilde{\tau}^2} \tag{5}$$

We see that the transformations $\tilde{\sigma}=\tilde{\sigma}(\sigma,\tau)$, $\tilde{\tau}=\tilde{\tau}(\sigma,\tau)$ are orthonormalization preserving if they are solutions of the D'Alambert equations (5).

Notice furthermore that the induced metrics

$$g_{\alpha\beta}\equiv\frac{\partial X^\mu}{\partial\xi^\alpha}\frac{\partial X_\mu}{\partial\xi^\beta} \qquad \tilde{g}_{\alpha\beta}\equiv\frac{\partial X^\mu}{\partial\tilde{\xi}^\alpha}\frac{\partial X_\mu}{\partial\tilde{\xi}^\beta} \tag{6}$$

are related by

$$\tilde{g}_{\alpha\beta}=\frac{\partial\xi^\rho}{\partial\tilde{\xi}^\alpha}\frac{\partial\xi^\delta}{\partial\tilde{\xi}^\beta}g_{\rho\delta} \tag{7}$$

But since for orthonormal matrices $g_{01} = g_{10} = 0$ $g_{11} = -g_{00}$ we get

$$\tilde{g}_{\alpha\beta} = \lambda \Delta \, g_{\alpha\beta} \tag{1}$$

$$\Delta \equiv \det \begin{pmatrix} \partial\tau/\partial\tilde{\tau} & \partial\tau/\partial\tilde{\sigma} \\ \partial\sigma/\partial\tilde{\tau} & \partial\sigma/\partial\tilde{\sigma} \end{pmatrix}$$

Therefore, the orthonormal gauge is preserved by conformal reparametrization of the world-sheet. If we take the convention that $\tilde{\tau}$ ($\tilde{\sigma}$) is again a time-like (space-like) coordinate in the sense that the metric still reads $\sim (\tau'_{-1})$ after reparametrization, then (1.4) with $\lambda = +1$ (-1) corresponds to conformal reparametrizations preserving (changing) orientation. Conformal reparametrizations, in which both metric and coordinates change, should not be confused with conformal Weyl transformations in which the metric does change but the coordinates do not. Notice that the factor $\exp\phi(\sigma,\tau)$ in conformal Weyl transformations is arbitrary, while Δ in (1) is not since τ and σ in (1) satisfy the wave equations (1.5). Also we point out that the conformal reparametrizations play a similar role to the gauge transformations $A_\mu \to A_\mu + \partial_\mu \Lambda$, with $\Box \Lambda = 0$, which preserves the Lorentz gauge $\partial_\mu A^\mu = 0$.

In order to fix completely the gauge, i.e. the parametrization of the world-sheet, we must impose more conditions in addition to the orthonormality ones

$$\dot{X} \cdot X' = 0 \quad , \quad \dot{X}^2 + X'^2 = 0 \tag{2}$$

Up to this end we choose an arbitrary vector n^μ such that $n^2 \geq 0$. We can choose as our new variable

$$n_\mu X^\mu(\sigma,\tau) = \rho\, \tau \tag{3}$$

This is compatible with the d'Alambert equation since $X^\mu(\sigma,\tau)$ is also a solution of such equation. Since $p^\mu_\tau = \dot{X}^\mu / 2\alpha' \pi$, (3) implies

$$n_\mu \, p^\mu_\tau(\sigma,\tau) = \frac{1}{2\pi\alpha'} \, n_\mu \dot{X}^\mu(\sigma\tau) = \frac{1}{2\pi\alpha'} \rho \tag{4}$$

and integrating over σ from 0 to π we obtain

$$\beta = 2\alpha' \cdot (m_\mu \cdot P^\mu) \qquad (1)$$

Where P^μ is the total momentum of the string. Hence (2.3) can be written as

$$m_\mu X^\mu(\sigma,\tau) = 2\alpha' (m_\mu P^\mu)\,\tau \qquad (2)$$

The meaning of this equation is simple: We intersect the surface $X^\mu(\sigma\tau)$ with the plane (2). In itself the function $X^\mu(\sigma,\tau)$ does not specify which parametrization is used, but once the surface is intersected by the plane (2), the τ variable is uniquely defined, as well as the lines of constant τ on the surface. Note how restricted this parametrization is since in general the lines of constant τ are not contained in a hypersurface. Once this choice for the τ has been done, we choose an orientation for the string and fix an equal $\sigma=0$ - line, orthogonal to the constant τ - lines, as origin for counting σ. Then we see, with the help of (1.4), that orthonormality (1.6) together with (2) fixes completely the parametrization of the world-sheet.

Let us discuss how the $\sigma=0$ - line is chosen. For open strings, the boundary conditions together with P^μ_σ given in (III-2.1) tell us that the constant $\sigma=0$ - line can be consistently chosen as the world line $(\tau, 0)$ of the first end of the string. However for the closed string there is no preferred point on it. So we must choose a point in the initial configuration and assign to it the coordinates $(\tau_1, \sigma=0)$. This point in turn determines a unique line C from $(\tau_1, 0)$ to the equal τ_2 - line, orthogonal to the constant τ - lines. Then we take the line C as the constant $\sigma=0$ - line. Notice that closed strings posses an extra invariance as compared to the open strings corresponding to the choice of a point out of a circle. This extra invariance does not exists for the open string due to the boundary conditions. For the closed string this invariance has been fixed by the choice of the C - line described above.

Now that the lines of constant τ are defined, the orthogonal family of the lines of constant σ is also well

defined and in principle we can deduce the σ parametrization from our choice of τ and the orthonormality condition. It is, however, easier to choose the σ parametrization in a specific way and show that together with the equations of motion it implies the orthonormality condition. We will define τ and σ by

$$\tau = \frac{1}{2\alpha' (n.\mathcal{P})} (n.X) \tag{1}$$

$$\sigma = \frac{\pi}{(n.\mathcal{P})} \int_0^\sigma d\sigma' \, [n.\mathcal{P}_\tau(\sigma'\tau)] \tag{2}$$

being (2) consistent with the definition (II-7.1). Hence, deriving (2) with respect to σ we obtain

$$n.\mathcal{P}_\tau(\sigma\tau) = \frac{1}{\pi} n.\mathcal{P} \tag{3}$$

So, in this parametrization, the projection of $\mathcal{P}_\tau^\mu(\sigma,\tau)$ on n_μ is constant over the world-sheet, and the parameter σ counts the fraction of the momentum component $(n.\mathcal{P})$ carried by the piece of the string between 0 and σ. Then multiplying the equation of motion by n_μ

$$\frac{\partial}{\partial\tau}(n.\mathcal{P}_\tau) + \frac{\partial}{\partial\sigma}(n.\mathcal{P}_\sigma) = 0 \tag{4}$$

and therefore

$$\frac{\partial}{\partial\sigma}(n.\mathcal{P}_\sigma) = 0 \tag{5}$$

Now for open strings the boundary conditions (II-6.3) implies

$$n.\mathcal{P}_\sigma(\sigma,\tau) = 0 \tag{6}$$

For closed strings $\dot{X}.X'$ vanishes on G, and using (1) and (II-6.16) we see that $(n.\mathcal{P}_\sigma)$ also vanishes on G. Therefore, from (5) follows again that (6) holds also for closed strings.

Now, in our parametrization, (1) implies

$$[n.X'(\sigma,\tau)] = 0 \tag{7}$$

and from this, (6) and the explicit expression of $\mathcal{P}_\sigma^\mu(\sigma,\tau)$ in

(II-6.1b) we obtain

$$(\dot{X} \cdot X') = 0 \tag{1}$$

and from $P_\tau^\mu(\sigma,\tau)$ given in (II-6.1a) and (4.3) we get

$$\dot{X}^2 + X'^2 = 0 \tag{2}$$

Hence our parametrization (4.1) and (4.2) is orthonormal.

Now we can introduce the light-cone gauge as the gauge where n^μ is a light-like vector

$$n^\mu \equiv \frac{1}{\sqrt{2}} (1, -1, 0, 0, \cdots) \tag{3}$$

It is then convenient to introduce the light-cone coordinates

$$u_\pm = \frac{1}{\sqrt{2}} (u^0 \pm u^1)$$

$$u^0 = \frac{1}{\sqrt{2}} (u_+ + u_-) \quad , \quad u^1 = \frac{1}{\sqrt{2}} (u_+ - u_-) \tag{4}$$

$$u \cdot v = u_+ v_- + u_- v_+ - \sum_{i=2}^{D-1} u^i v^i$$

In particular

$$n_+ = 0 \quad , \quad n_- = 1 \quad , \quad n^i = 0 \qquad i = 2, 3, \cdots D-1 \tag{5}$$

$$n \cdot v = v_+$$

Then equations (4.1) and (4.3) imply

$$P_{\tau +} = \frac{1}{\Pi} P_+ \quad , \quad X_+ = 2\alpha' P_+ \tau \tag{6}$$

Furthermore let us consider the constraints (II-6.4):

$$P_\tau^\mu X'_\mu = 0 \quad \Rightarrow \quad P_{\tau +} X'_- + P_{\tau -} X'_+ = P_\tau^i X'^i \overset{(2)}{\Rightarrow}$$

$$X'_- = \frac{\Pi}{P_+} P_\tau^i X'^i \qquad i = 2, 3, \cdots D-1 \tag{7}$$

Similarly

$$P_\tau^2 + \frac{1}{(2\alpha'\Pi)^2} X'^2 = 0 \quad \Rightarrow \quad 2 P_{\tau +} P_{\tau -} - (P_\tau^i)^2 + \frac{1}{(2\alpha'\Pi)^2} [2 X'_+ X'_- - (X'^i)^2] = 0 \Rightarrow$$

$$\frac{2}{\Pi} P_+ P_{\tau -} - (P_\tau^i)^2 = \frac{1}{(2\alpha'\Pi)^2} (X'^i)^2$$

Then all together

$$X_-' = \frac{\pi}{P_+} P_\tau^i X'^i \tag{1}$$

$$P_{\tau -} = \frac{1}{2\pi P_+} \left\{ \pi^2 (P_\tau^i)^2 + \frac{1}{(2\alpha')^2} (X'^i)^2 \right\} \tag{2}$$

where in both cases there is a sum over $i = 2, 3, \ldots D-1$. From all that it is clear that only X^i and P_τ^i $(i = 2, 3, \ldots D-1)$ remain as independent degrees of freedom in this gauge. Therefore it is enough to determine the transverse coordinates. As we are in an orthonormal gauge the transverse coordinates X^i satisfy

$$\ddot{X}^i(\sigma, \tau) - X''^i(\sigma, \tau) = 0 \qquad i = 2, 3, \ldots, D-1 \tag{3}$$

Let us consider the explicit form of these solutions.

<u>The Open String</u>

From (III - 3.3) we obtain

$$X^i(\sigma, \tau) = q^i + \alpha_0^i \tau + i \sum_{\substack{n=-\infty \\ n \neq 0}}^{+\infty} \frac{1}{n} \alpha_n^i e^{-in\tau} \cos n\sigma \tag{4}$$

and from (1) and (III - 2.1)

$$X_-' = \frac{1}{2\alpha' P_+} \dot{X}^i X'^i \tag{5}$$

Hence

$$X_-' = -\frac{i}{2\alpha' P_+} \sum_{k=-\infty}^{+\infty} \alpha_k^i e^{-ik\tau} \cos k\sigma \sum_{l=-\infty}^{+\infty} \alpha_l^i e^{-il\tau} \sin l\sigma =$$

$$= -\frac{i}{2\alpha' P_+} \sum_{n=-\infty}^{+\infty} e^{-in\tau} \sum_{m=-\infty}^{+\infty} \alpha_m^i \alpha_{n-m}^i \sin (n-m)\sigma \cos m\sigma =$$

$$= -\frac{i}{4\alpha' P_+} \sum_{n=-\infty}^{+\infty} e^{-in\tau} \left\{ \sin n\sigma \sum_{m=-\infty}^{+\infty} \alpha_{n-m}^i \alpha_m^i + \sum_{m=-\infty}^{+\infty} \alpha_{n-m}^i \alpha_m^i \sin (n-2m)\sigma \right\}$$

The last term is zero since the terms m and $m' = n - m$ cancel each other. Introducing

$$\alpha_{n,-} \equiv \frac{1}{4\alpha' P_+} \sum_{m=-\infty}^{+\infty} \alpha_{n-m}^i \alpha_m^i \tag{6}$$

we obtain

$$X_-' = -i \sum_{n=-\infty}^{+\infty} \alpha_{n-} e^{-in\tau} \sin n\sigma \qquad (1)$$

and hence the $X_-(\sigma,\tau)$, solution of the equations of motion, is

$$X_-(\sigma,\tau) = q_- + \hat{\alpha}_{0,-}\,\tau + i \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n} \alpha_{n,-} e^{-in\tau} \cos n\sigma \qquad (2)$$

We still must check that the constraint (6.2) is satisfied. We have

$$(2\alpha'\pi)^2 (P_\tau^i)^2 = \left\{ \sum_{k=-\infty}^{+\infty} \alpha_k^i e^{-ik\tau} \cos k\sigma \right\} \left\{ \sum_{\ell=-\infty}^{+\infty} \alpha_\ell^i e^{-i\ell\tau} \cos \ell\sigma \right\} =$$

$$= \sum_{n=-\infty}^{+\infty} e^{-in\tau} \sum_{m=-\infty}^{+\infty} \alpha_m^i \alpha_{n-m}^i \cos m\sigma \cos (n-m)\sigma =$$

$$= \frac{1}{2} \sum_{n=-\infty}^{+\infty} e^{-in\tau} \left\{ \cos n\sigma \sum_{m=-\infty}^{+\infty} \alpha_m^i \alpha_{n-m}^i + \right.$$

$$\left. + \sum_{m=-\infty}^{+\infty} \alpha_m^i \alpha_{n-m}^i \cos (n-2m)\sigma \right\} \qquad (3)$$

and similarly

$$(X'^i)^2 = -\sum_{n=-\infty}^{+\infty} e^{-in\tau} \sum_{m=-\infty}^{+\infty} \alpha_m^i \alpha_{n-m}^i \sin m\sigma \sin (n-m)\sigma$$

$$= +\frac{1}{2} \sum_{n=-\infty}^{+\infty} e^{-in\tau} \left\{ \cos n\sigma \sum_{m=-\infty}^{+\infty} \alpha_m^i \alpha_{n-m}^i - \right.$$

$$\left. - \sum_{m=-\infty}^{+\infty} \alpha_m^i \alpha_{n-m}^i \cos (n-2m)\sigma \right\} \qquad (4)$$

Hence

$$\frac{1}{2\pi P_+} \left\{ \pi^2 (P_\tau^i)^2 + \frac{1}{(2\alpha')^2} (X'^i)^2 \right\} =$$

$$= \frac{1}{2\pi P_+ (2\alpha')^2} \sum_{n=-\infty}^{+\infty} e^{-in\tau} \cos n\sigma \sum_{m=-\infty}^{+\infty} \alpha_m^i \alpha_{n-m}^i =$$

$$= \frac{1}{2\alpha'\pi} \sum_{n=-\infty}^{+\infty} \alpha_{n-} e^{-in\tau} \cos n\sigma \qquad (5)$$

and this coincides with $P_{\tau-}$ computed from (2) only if $\hat{\alpha}_{0-} = \alpha_{0-}$. Then all together we have

$$X^i(\sigma,\tau) = q^i + \alpha_0^i \tau + i \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{1}{m} \alpha_m^i e^{-im\tau} \cos m\sigma \qquad (1)$$

$$X_+(\sigma,\tau) = 2\alpha' p_+ \tau \qquad (2)$$

$$X_-(\sigma,\tau) = q_- + \alpha_{0-} \tau + i \sum_{\substack{m=-\infty \\ m\neq 0}}^{+\infty} \frac{1}{m} \alpha_{m,-} e^{-im\tau} \cos m\sigma \qquad (3)$$

$$\alpha_{m,-} = \frac{1}{4\alpha' p_+} \sum_{m=-\infty}^{+\infty} \alpha_m^i \alpha_{m-m}^i \,, \qquad m \in \mathbb{Z} \qquad (4)$$

where X_+ has been obtained from (5.6).

The closed string

From (III - 5.5) we obtain

$$X^i(\sigma,\tau) = q^i + 2\alpha_0^i \tau + \frac{i}{2} \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{1}{m} \left(\alpha_m^i e^{-2im(\tau-\sigma)} + \tilde{\alpha}_m^i e^{-2im(\tau+\sigma)} \right) \qquad (5)$$

and from (6.5)

$$X_-'(\sigma,\tau) = \frac{1}{2\alpha' p_+} \left\{ \sum_{k=-\infty}^{+\infty} \left[\alpha_k^i e^{-2ik(\tau-\sigma)} + \tilde{\alpha}_k^i e^{-2ik(\tau+\sigma)} \right] \right\} \cdot$$

$$\cdot \left\{ \sum_{\ell=-\infty}^{+\infty} \left[-\alpha_\ell^i e^{-2i\ell(\tau-\sigma)} + \tilde{\alpha}_\ell^i e^{-2i\ell(\tau+\sigma)} \right] \right\} =$$

$$= \frac{1}{2\alpha' p_+} \sum_{k=-\infty}^{+\infty} \sum_{\ell=-\infty}^{+\infty} \left\{ -\alpha_k^i \alpha_\ell^i e^{-2i(k+\ell)(\tau-\sigma)} + \tilde{\alpha}_k^i \tilde{\alpha}_\ell^i e^{-2i(k+\ell)(\tau+\sigma)} \right\} =$$

$$= \frac{1}{2\alpha' p_+} \sum_{m=-\infty}^{+\infty} \left\{ -e^{-2im(\tau-\sigma)} \sum_{m=-\infty}^{+\infty} \alpha_m^i \alpha_{m-m}^i + e^{-2im(\tau+\sigma)} \sum_{m=-\infty}^{+\infty} \tilde{\alpha}_m^i \tilde{\alpha}_{m-m}^i \right\} \qquad (6)$$

Introducing

$$\alpha_{m,-} \equiv \frac{1}{2\alpha' p_+} \sum_{m=-\infty}^{+\infty} \alpha_m^i \alpha_{m-m}^i \,, \qquad \tilde{\alpha}_{m,-} = \frac{1}{2\alpha' p_+} \sum_{m=-\infty}^{+\infty} \tilde{\alpha}_m^i \tilde{\alpha}_{m-m}^i \qquad (7)$$

we can write

$$X_-'(\sigma\tau) = -\sum_{m=-\infty}^{+\infty} \left\{ \alpha_{m,-} e^{-2im(\tau-\sigma)} - \tilde{\alpha}_{m,-} e^{-2im(\tau+\sigma)} \right\} \qquad (8)$$

and hence the solution of the equations of motion is

$$X_-(\sigma,\tau) = q_- + 2\hat{\alpha}_{0,-}\tau - (\alpha_{0,-} - \tilde{\alpha}_{0,-})\sigma +$$

$$+ \frac{i}{2} \sum_{\substack{m=-\infty \\ m\neq 0}}^{+\infty} \frac{1}{m} \left[\alpha_{m-} e^{-2im(\tau-\sigma)} + \tilde{\alpha}_{m-} e^{-2im(\tau+\sigma)} \right] \quad (1)$$

The linear term must cancel do to periodicity conditions and hence

$$\alpha_{0,-} = \tilde{\alpha}_{0,-} \quad (2)$$

Therefore

$$X_-(\sigma,\tau) = q_- + 2\hat{\alpha}_{0,-}\tau + \frac{i}{2} \sum_{\substack{m=-\infty \\ m\neq 0}}^{+\infty} \frac{1}{m} \left[\alpha_{m-} e^{-2im(\tau-\sigma)} + \tilde{\alpha}_{m-} e^{-2im(\tau+\sigma)} \right] \quad (3)$$

being q_- and $\hat{\alpha}_{0,-}$ integration constants. Let us now check (6.2)

$$(2\pi\alpha')^2 (P_\tau^i)^2 + (X'^i)^2 = \left\{ \sum_{k=-\infty}^{+\infty} \left[\alpha_k^i e^{-2ik(\tau-\sigma)} + \tilde{\alpha}_k^i e^{-2ik(\tau+\sigma)} \right] \right\}^2$$

$$+ \left\{ \sum_{k=-\infty}^{+\infty} \left[-\alpha_k^i e^{-2ik(\tau-\sigma)} + \alpha_k^i e^{-2ik(\sigma+\tau)} \right] \right\}^2 =$$

$$= \left\{ \sum_{k=-\infty}^{+\infty} \alpha_k^i e^{-2ik(\tau-\sigma)} \right\}^2 + \left\{ \sum_{k=-\infty}^{+\infty} \tilde{\alpha}_k^i e^{-2ik(\tau+\sigma)} \right\}^2 =$$

$$= 4\alpha' P_+ \sum_{m=-\infty}^{+\infty} \left[\alpha_{m-} e^{-2im(\tau-\sigma)} + \tilde{\alpha}_{m,-} e^{-2im(\tau+\sigma)} \right] \quad (4)$$

using this we can compute $P_{\tau-}$ using (6.2) and this coincides with the value computed from (3) only if

$$\hat{\alpha}_{0,-} = \alpha_{0,-} + \tilde{\alpha}_{0,-} \quad (5)$$

Then

$$X^i(\sigma,\tau) = q^i + 2\alpha_0^i \tau + \frac{i}{2} \sum_{\substack{m=-\infty \\ m\neq 0}}^{+\infty} \frac{1}{m} \left[\alpha_m^i e^{-2im(\tau-\sigma)} + \tilde{\alpha}_m^i e^{-2im(\tau+\sigma)} \right] \quad (6)$$

$$X_+(\sigma,\tau) = 2\alpha' P_+ \tau \quad (7)$$

$$X_-(\sigma,\tau) = q_- + 2\alpha_{0,-}\tau + \frac{i}{2} \sum_{\substack{m=-\infty \\ m\neq 0}}^{+\infty} \frac{1}{m} \left[\alpha_{m,-} e^{-2im(\tau-\sigma)} + \tilde{\alpha}_{m,-} e^{-2im(\tau+\sigma)} \right] \quad (8)$$

$$\alpha_{m-} = \frac{1}{2\alpha' P_+} \sum_{m=-\infty}^{+\infty} \alpha_m^i \alpha_{m-m}^i \,, \qquad \tilde{\alpha}_{m-} = \frac{1}{2\alpha' P_+} \sum_{m=-\infty}^{+\infty} \tilde{\alpha}_m^i \tilde{\alpha}_{m-m}^i \quad (9)$$

$$\alpha_0^i = \tilde{\alpha}_0^i \,, \qquad \alpha_{0-} = \tilde{\alpha}_{0-} \quad (10)$$

V - MASS, MOMENTUM AND ANGULAR MOMENTUM OF THE STRING

The Open String

Let us begin considering the covariant solution given in (III-3.3). Then the barycenter of the string is given by

$$\frac{1}{\pi}\int_0^\pi d\sigma\, X^\mu(\sigma,\tau) = q^\mu + \alpha_0^\mu \tau \tag{1}$$

and the total momentum is

$$p^\mu = \int_0^\pi d\sigma\, p^\mu_\tau(\sigma,\tau) = \frac{1}{2\alpha'\pi}\int_0^\pi d\sigma\, \dot X^\mu(\sigma,\tau) = \frac{1}{2\alpha'}\alpha_0^\mu \Rightarrow$$

$$p^\mu = \frac{1}{2\alpha'}\alpha_0^\mu \tag{2}$$

Furthermore the angular momentum is

$$M^{\mu\nu} = \int_0^\pi d\sigma\,[X^\mu p^\nu_\tau - X^\nu p^\mu_\tau] = \frac{1}{2\pi\alpha'}\int_0^\pi d\sigma\,[X^\mu \dot X^\nu - X^\nu \dot X^\mu] =$$

$$= \frac{1}{2\alpha'\pi}\int_0^\pi d\sigma\,\Big\{\Big[q^\mu + \alpha_0^\mu\tau + i\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\frac{1}{m}\alpha_m^\mu e^{-im\tau}\cos m\sigma\Big]\cdot\Big[\sum_{k=-\infty}^{+\infty}\alpha_k^\nu e^{-ik\sigma}\cos k\sigma\Big] - (\mu\leftrightarrow\nu)\Big\} \tag{3}$$

Since

$$\int_0^\pi d\sigma\,\cos m\sigma\cos k\sigma = \frac{1}{2}\pi\delta_{mk}\,, \quad m,k\geq 0 \tag{4}$$

we obtain

$$M^{\mu\nu} = q^\mu p^\nu - q^\mu p^\nu - \frac{i}{2\alpha'}\sum_{m=1}^{\infty}\frac{1}{m}(\alpha_{-m}^\mu\alpha_m^\nu - \alpha_{-m}^\nu\alpha_m^\mu) \tag{5}$$

Furthermore we would like to calculate the mass squared, M^2, of the string and its intrinsic angular momentum in terms of α_m^μ. From the constraint $L_0 = 0$ and (III-4.2)

$$-4\alpha' L_0 = \alpha_0^\mu\alpha_{0\mu} + 2\sum_{m=1}^{\infty}\alpha_{m\mu}^*\alpha_m^\mu \tag{6}$$

and using (2) as well as (6)

$$\alpha' M^2 = -\frac{1}{2\alpha'}\sum_{m=1}^{\infty}\alpha_{m\mu}^*\alpha_m^\mu \tag{7}$$

and it is not obvious that $M^2 \geq 0$.

In order to see this property let us calculate this gauge independent quantity in the light-cone gauge. We have

$$M^2 = p_\mu p^\mu = 2 p_+ p_- - (p^i)^2 \tag{1}$$

From (IV-10.8) and (IV-10.6) we get

$$2 p_+ p_- = \frac{4}{(2\alpha')^2} \sum_{m=-\infty}^{+\infty} \alpha_m^i \alpha_{-m}^i \quad , \quad (p^i)^2 = \frac{4}{(2\alpha')^2} \alpha_0^i \alpha_0^i \tag{2}$$

and hence

$$\alpha' M^2 = \frac{4}{(2\alpha')} \sum_{m=1}^{\infty} \alpha_m^{i*} \alpha_m^i \tag{3}$$

which clearly shows that $M^2 \geq 0$.

Let us now return to the covariant gauge and consider (4.5)

$$M^{\mu\nu} = \ell^{\mu\nu} + S^{\mu\nu} \tag{4}$$

$$\ell^{\mu\nu} = q^\mu p^\nu - q^\nu p^\mu \quad , \quad S^{\mu\nu} = -\frac{i}{2\alpha'} \sum_{m=1}^{\infty} (\alpha_{-m}^\mu \alpha_m^\nu - \alpha_{-m}^\nu \alpha_m^\mu)$$

The classical spin of the string can be computed using

$$J^2 = \frac{1}{2} \left[S^{\mu\nu} S_{\mu\nu} - \frac{2}{M^2} (p^\nu S_{\nu\rho}) (p_\mu S^{\mu\rho}) \right] \tag{5}$$

In order to evaluate this quantity let us consider an orthogonal gauge where X^0 is proportional to τ and therefore $\alpha_m^0 \equiv 0$ $(m \neq 0)$. In this gauge $S^{\mu 0} = S^{0\mu} = 0$. Now let us choose a frame of reference where the string is at rest, i.e. $p^i \equiv 0$. Then the last term in (5) vanishes and

$$J^2 = \frac{1}{2} S_{ke} S^{ke} = \tag{6}$$

$$= -\frac{1}{(2\alpha')^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn} \left\{ (\vec{\alpha}_m^* \cdot \vec{\alpha}_n^*)(\vec{\alpha}_m \cdot \vec{\alpha}_n) - (\vec{\alpha}_m^* \cdot \vec{\alpha}_n)(\vec{\alpha}_m \cdot \vec{\alpha}_n^*) \right\}$$

In the same gauge

$$\alpha'^2 M^4 = \frac{1}{(2\alpha')^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (\vec{\alpha}_m^* \cdot \vec{\alpha}_m)(\vec{\alpha}_n^* \cdot \vec{\alpha}_n) \tag{7}$$

Let us now compare these two expressions. Using the Schwartz inequality

$$(\vec{\alpha}_m^* \cdot \vec{\alpha}_n)(\vec{\alpha}_m \cdot \vec{\alpha}_n^*) \leq (\vec{\alpha}_m^* \cdot \vec{\alpha}_m)(\vec{\alpha}_n^* \cdot \vec{\alpha}_n) \tag{1}$$

we get

$$(2\alpha')^2 J^2 \leq - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn} (\vec{\alpha}_m^* \cdot \vec{\alpha}_n^*)(\vec{\alpha}_m \cdot \vec{\alpha}_n) +$$

$$+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn} (\vec{\alpha}_m^* \alpha_m)(\vec{\alpha}_n^* \alpha_n) \leq$$

$$= - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn} |(\vec{\alpha}_m \cdot \vec{\alpha}_n)|^2 + \Big[\sum_{n=1}^{\infty} (\vec{\alpha}_n^* \vec{\alpha}_n) \Big]^2 \qquad \Rightarrow$$

$$J^2 \leq \alpha'^2 M^4 \tag{2}$$

Finally we give an example that shows how the classical Regge trajectories arise and how the inequality (2) can be saturated. Let us consider

$$X^0 = 2\alpha' p^0 \tau \quad , \quad X^i = 0 \qquad i = 3, 4, \ldots D-1$$

$$X^1 = \frac{L}{2n} \cos n\tau \cos n\sigma = \frac{L}{4n} \cos n\sigma \, (e^{in\tau} + e^{-in\tau}) \tag{3}$$

$$X^2 = \frac{L}{2n} \sin n\tau \cos n\sigma = - \frac{iL}{4n} \cos n\sigma \, (e^{in\tau} - e^{-in\tau})$$

Comparing this with the general expression (III-3.3) we obtain

$$\alpha_0^\mu = (2\alpha' p^0, 0, 0, \ldots, 0) \tag{4}$$

$$\alpha_n^\mu = \frac{L}{4} (0, -i, 1, 0, \ldots, 0)$$

Then

$$\alpha_{n\mu}^* \alpha_n^\mu = - \frac{1}{8} L^2 \quad , \quad \alpha_{n\mu}^* \alpha_n^{\mu *} = \alpha_{n\mu} \alpha_n^\mu = 0 \tag{5}$$

and from (2.6) and (2.7)

$$\alpha'^2 M^4 = \frac{1}{(2\alpha')^2} \frac{L^4}{64} \quad , \quad J^2 = \frac{1}{(2\alpha')^2} \frac{L^4}{64} \frac{1}{n^2} \tag{6}$$

Therefore

$$|J| = \frac{\alpha'}{n} M^2 \tag{7}$$

which gives the classical Regge trajectories for the open string. Notice that for the first mode ($n=1$) the inequality (3.2) is saturated (leading Regge trajectory).

The Closed String

In the covariant gauge and using (III-5.6) we find for the barycenter coordinate of the string

$$\frac{1}{\pi}\int_0^\pi d\sigma\, X^\mu(\sigma,\tau) = q^\mu + 2\alpha_0^\mu \tau \tag{1}$$

and for the total momentum of the string

$$p^\mu = \frac{1}{2\pi\alpha'}\int_0^\pi d\sigma\, \dot{X}^\mu(\sigma,\tau) = \frac{1}{2\pi\alpha'}\,\pi\, 2\alpha_0^\mu \Rightarrow$$

$$p^\mu = \frac{1}{2\alpha'}\, 2\alpha_0^\mu \tag{2}$$

Similarly

$$M^{\mu\nu} = \frac{1}{2\pi\alpha'}\int_0^\pi d\sigma\, [X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu] =$$

$$= \frac{1}{2\pi\alpha'}\int_0^\pi d\sigma \Big\{ [q^\mu + 2\alpha_0^\mu \tau + \frac{i}{2}\sum_{\substack{m=-\infty \\ m\neq 0}}^{+\infty} \frac{1}{m}[\alpha_m^\mu e^{-2im(\tau-\sigma)} + \tilde{\alpha}_m^\mu e^{-2im(\tau+\sigma)}]] \cdot$$

$$\cdot \Big[\sum_{\ell=-\infty}^{+\infty} [\alpha_\ell^\nu e^{-2i\ell(\tau-\sigma)} + \tilde{\alpha}_\ell^\nu e^{-2i\ell(\tau+\sigma)}]\Big] - (\mu \leftrightarrow \nu)\Big\}$$

Since

$$\int_0^\pi d\sigma\, e^{i(m-n)\sigma} = \pi\,\delta_{mn} \tag{3}$$

we get

$$M^{\mu\nu} = q^\mu p^\nu - q^\nu p^\mu - \frac{i}{2\alpha'}\sum_{m=1}^{\infty}\frac{1}{m}[\alpha_{-m}^\mu \alpha_m^\nu - \alpha_{-m}^\nu \alpha_m^\mu + \tilde{\alpha}_{-m}^\mu \tilde{\alpha}_m^\nu - \tilde{\alpha}_{-m}^\nu \tilde{\alpha}_m^\mu] \tag{4}$$

In order to calculate M^2 let us take into account the constraints $L_0 = \tilde{L}_0 = 0$ given in (III-6.1); then

$$-2\alpha' L_0 = \alpha_0^\mu \alpha_{0\mu} + 2\sum_{m=1}^{\infty} \alpha_{m\mu}^* \alpha_m^\mu = 0$$

$$-2\alpha' \tilde{L}_0 = \tilde{\alpha}_0^\mu \tilde{\alpha}_{0\mu} + 2\sum_{m=1}^{\infty} \tilde{\alpha}_{m\mu}^* \alpha_m^\mu = 0 \tag{5}$$

and using (4.2)

$$\alpha' M^2 = -\frac{2}{\alpha'} \sum_{m=1}^{\infty} \alpha_{m\mu}^* \alpha_m^\mu = -\frac{2}{\alpha'} \sum_{m=1}^{\infty} \tilde{\alpha}_{m\mu}^* \tilde{\alpha}_m^\mu \tag{1}$$

As before in this formalism it is obvious that $M^2 \geq 0$. To see this let us work in the light-cone gauge. Then the solution is given in (IV-9.6). We have

$$2 P_+ P_- = \frac{2}{\alpha'} P_+ \alpha_{0-} = \frac{1}{\alpha'^2} \sum_{m=-\infty}^{+\infty} \alpha_m^i \alpha_{-m}^i = \frac{1}{\alpha'^2} \sum_{m=-\infty}^{+\infty} \tilde{\alpha}_m^i \tilde{\alpha}_{-m}^i \tag{2}$$

$$(P^i)^2 = \frac{1}{\alpha'^2} \alpha_0^i \alpha_0^i = \frac{1}{\alpha'^2} \tilde{\alpha}_0^i \tilde{\alpha}_0^i \tag{3}$$

and hence

$$\alpha' M^2 = \frac{2}{\alpha'} \sum_{m=1}^{\infty} \alpha_m^{i*} \alpha_m^i = \frac{2}{\alpha'} \sum_{m=1}^{\infty} \tilde{\alpha}_m^{i*} \tilde{\alpha}_m^i \tag{4}$$

and clearly $M^2 \geq 0$

We can compute as before

$$J^2 = -\frac{1}{(2\alpha')^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m\,n} \Big\{ (\vec{\alpha}_m^* \cdot \vec{\alpha}_n^*)(\vec{\alpha}_m \cdot \vec{\alpha}_n) - (\vec{\alpha}_m^* \vec{\alpha}_n)(\vec{\alpha}_m \cdot \vec{\alpha}_n^*)$$

$$+ (\vec{\tilde{\alpha}}_m^* \cdot \vec{\tilde{\alpha}}_n^*)(\vec{\tilde{\alpha}}_m \cdot \vec{\tilde{\alpha}}_n) - (\vec{\tilde{\alpha}}_m^* \cdot \vec{\tilde{\alpha}}_n)(\vec{\tilde{\alpha}}_m \cdot \vec{\tilde{\alpha}}_n^*) +$$

$$+ 2 (\vec{\alpha}_m^* \cdot \vec{\tilde{\alpha}}_n^*)(\vec{\alpha}_m \cdot \vec{\tilde{\alpha}}_n) - 2 (\vec{\alpha}_m^* \vec{\tilde{\alpha}}_n)(\vec{\alpha}_m \cdot \vec{\tilde{\alpha}}_n^*) \Big\} \tag{5}$$

$$\alpha'^2 M^4 = \frac{16}{(2\alpha')^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (\vec{\alpha}_m^* \vec{\alpha}_m)(\vec{\alpha}_n^* \vec{\alpha}_n) = \frac{16}{(2\alpha')^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (\vec{\tilde{\alpha}}_m^* \cdot \vec{\tilde{\alpha}}_m)(\vec{\tilde{\alpha}}_n^* \vec{\tilde{\alpha}}_n)$$

$$= \frac{16}{(2\alpha')^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (\vec{\alpha}_m^* \vec{\alpha}_m)(\vec{\tilde{\alpha}}_n^* \vec{\tilde{\alpha}}_n) \tag{6}$$

and proceeding as before

$$J^2 \leq (\alpha'/2)^2 M^4 \tag{7}$$

Finally let us consider an example

$$X^0 = 2\alpha' P^0 \tau \quad , \quad X^i = 0 \qquad i = 3, 4, \ldots, D-1$$

$$X^1 = \frac{L}{4\pi} \cos 2n\sigma \cos 2n\tau = \frac{L}{16\pi} \Big\{ e^{-2in(\tau-\sigma)} + e^{2in(\tau-\sigma)} + e^{-2in(\tau+\sigma)} + e^{2in(\tau+\sigma)} \Big\} \tag{8}$$

$$\delta^2 = \frac{L}{4m} \cos 2m\sigma \sin 2m\tau = -\frac{iL}{16m} \Big\{ -e^{-2im(\tau-\sigma)} + e^{+2im(\tau-\sigma)} - e^{-2im(\tau+\sigma)} + e^{+2im(\tau+\sigma)} \Big\}$$

Comparing with (III - 5.5)

$$\alpha_m^\mu = \tilde{\alpha}_m^\mu = -\frac{2iL}{16} (0, 1, i, 0, \dots 0) \tag{1}$$

and hence

$$(\vec{\alpha}_m \cdot \vec{\alpha}_m) = (\vec{\alpha}_m^* \cdot \vec{\alpha}_m^*) = 0 \quad , \quad (\vec{\alpha}_m^* \cdot \vec{\alpha}_m) = -\frac{L^2}{32} \tag{2}$$

Hence

$$\alpha'^2 M^4 = \frac{1}{(2\alpha')^2} \frac{L^4}{64} \quad , \quad J^2 = \frac{1}{(2\alpha')^2} \frac{L^4}{256} \frac{1}{m^2} \tag{3}$$

Therefore

$$|J| = \frac{\alpha'}{2m} M^2 \tag{4}$$

which gives the classical Regge trajectories for the closed string. As previously the leading Regge trajectory, which saturates the inequality (5.7), is obtained for $n = 1$.

VI - HAMILTONIAN FORMALISM : GENERAL IDEAS

In our study the parameter τ has been considered as an evolution parameter. It is therefore normal to consider

$$p^\mu_\tau \equiv - \frac{\partial L}{\partial \dot{X}_\mu} \tag{1}$$

as the momentum conjugate to $X^\mu(\sigma, \tau)$, where σ denotes the collection of points along the trajectory. However, if we try to express L as function of X^μ, X'^μ and p^μ_τ it turns out to be impossible because of the constraints (II - 6.4)

$$p_\tau \cdot X' = 0 \quad , \quad (2\pi\alpha')^2 p_\tau^2 + X'^2 = 0 \tag{2}$$

In order to introduce a Hamiltonian formalism we may proceed in two ways

1) To disregard the constraints and compute the Poisson brackets and then afterwards to impose the constraints on the dynamical system. This is the Dirac method [P.A.M. Dirac, Can J. Phys 2, 129 (1950), Proc. Roy. Soc. A246, 326 (1958)] of quantization and it is covariant, but leads, when quantized to an indefinite metric space

2) To reduce the number of degrees of freedom and eliminate all redundant variables. Then we have only to assume canonical Poisson brackets for the independent variables. This leads when quantized to a positive metric, but the procedure is not explicitly covariant.

Let us present the first method. We shall follow Dirac and Fadeev [L. D. Fadeev Theor. and Math. Phys. 1, 1 (1970)] in the treatment of Lagrangians with constraints. Let us consider a dynamical system described by an action functional

$$S = \int dt \; L(q_m, \dot{q}_m) \tag{1}$$

where $L(q_m, \dot{q}_m)$ is a Lagrangian depending on the generalized coordinates q_m $(m = 1, 2, \ldots N)$ and the corresponding velocities $\dot{q}_m$. For this Lagrangian we have the Euler - Lagrange equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_m}\right) - \frac{\partial L}{\partial q_m} = 0 \qquad m = 1, 2, \ldots N \tag{1}$$

The first step for constructing the Hamiltonian is the definition of the momentum variables

$$p_m \equiv \frac{\partial L}{\partial \dot{q}_m} \qquad m = 1, 2, \ldots, N \tag{2}$$

Then if the matrix

$$W_{mm} \equiv \frac{\partial p_m}{\partial \dot{q}_m} = \frac{\partial^2 L}{\partial \dot{q}_m \partial \dot{q}_m} \tag{3}$$

is non-singular, equation (2) can be solved for the velocities and this allows the introduction of the Legendre transform of L

$$H_0(p_m, q_m) = \sum_{m=1}^{N} p_m \dot{q}_m - L(q_m, \dot{q}_m) \tag{4}$$

Then from (4) and with help of (1) and (2) we obtain the Hamiltonian equations of motion

$$\dot{q}_m = \frac{\partial H_0}{\partial p_m}, \qquad \dot{p}_m = -\frac{\partial H_0}{\partial q_m} \qquad m = 1, 2, \ldots N \tag{5}$$

Introducing the canonical Poisson brackets, for two arbitrary functions f, g of phase space (q_m, p_m)

$$\{f, g\} = \sum_{m=1}^{N} \left(\frac{\partial f}{\partial q_m}\frac{\partial g}{\partial p_m} - \frac{\partial f}{\partial p_m}\frac{\partial g}{\partial q_m}\right) \tag{6}$$

equations (5) read

$$\dot{q}_m = \{q_m, H_0\}, \qquad \dot{p}_m = \{p_m, H_0\} \qquad m = 1, 2, \ldots, N \tag{7}$$

and we see that H_0 is the generator of time translations, i.e. the Hamiltonian.

However if the matrix (3) is singular, not all the velocities can be solved from (2) as functions of the p's and q's. This means that not all momenta defined in (2) are independent; more precisely, if the rank of W_{mm} is $R < N$, there will be $N - R \geq 1$ functional relations

$$\Phi_\alpha(q_m, p_m) = 0 \qquad \alpha = R+1, \ldots, N \tag{8}$$

Which are the so called primary constraints. From (2.7), N-R momenta can be expresed in terms of R independent momenta p_R $(k = 1, 2 \ldots R)$

$$p_\alpha = \psi_\alpha (q_m, p_r) \tag{4}$$

$m = 1, 2 \ldots N$, $r = 1, 2 \ldots R$, $\alpha = R+1, \ldots N$

Accordingly R velocities $\dot{q}_r$ can be solved from (2.2) and (4) in terms of the generalized coordinates q_m, the independent momenta p_r and the unsolved velocities $\dot{q}_\alpha$

$$\dot{q}_r = S_r (q_m, p_r, \dot{q}_\alpha) \tag{2}$$

Now we introduce the Legendre transform of L

$$H_0 (q_m, p_r, \dot{q}_\alpha) = \sum_{m=1}^{N} p_m \dot{q}_m - L (q_m, \dot{q}_m) \tag{3}$$

Computing the partial derivatives of H_0 and using (2.2)

$$\frac{\partial H_0}{\partial \dot{q}_\alpha} = 0 \tag{4a}$$

$$\frac{\partial H_0}{\partial q_m} = \sum_{\alpha=R+1}^{N} \frac{\partial \psi_\alpha}{\partial q_m} \dot{q}_\alpha - \frac{\partial L}{\partial q_m} \tag{4b}$$

$$\frac{\partial H_0}{\partial p_r} = \dot{q}_r + \sum_{\alpha=R+1}^{N} \frac{\partial \psi_\alpha}{\partial p_r} \dot{q}_\alpha \tag{4c}$$

From (4.a) we see that H_0 depends only on the generalized coordinates q_m and the independent momenta p_r. Furthermore (4b) and (4c) with the Heep of (2.1) give the Hamiltom equations of motion

$$\dot{p}_m = - \frac{\partial H_0}{\partial q_m} + \sum_{\alpha=R+1}^{N} \frac{\partial \psi_\alpha}{\partial q_m} \dot{q}_\alpha \qquad m = 1, 2 \ldots, N \tag{5a}$$

$$\dot{q}_r = \frac{\partial H_0}{\partial p_r} - \sum_{\alpha=R+1}^{N} \dot{q}_\alpha \frac{\partial \psi_\alpha}{\partial p_r} \qquad r = 1, 2 \ldots, R \tag{5b}$$

These equations give the time evolution of all momenta and of the coordinates $q_1 \ldots q_R$ in terms of the unsolved velocities $\dot{q}_{R+1} \ldots \dot{q}_N$ which remain, at this stage, as arbitrary functions of time.

At this stage we must analyze the evolution of the constraints (2.8), and require that they must be preserved through

the time evolution. This condition will give rise to new equations that determine some of the unsolved $\dot{q}_\alpha$ and/or provide new constraints. If new constraints appear, the process repeats itself. This recursive procedure will end up when no more constraints appear. All the constraints so obtained, using the equations of motion (3.5) are called secondary constraints. If at the end of the process some subset of the $\dot{q}_\alpha$ still remain undetermined, they will appear as arbitrary functions of time in the general solutions of the equations of motion. The undetermined velocities correspond then to the gauge invariances of the theory.

We emphasize that equations (3.5), as they stand, cannot be expressed through Poisson brackets, as we did before (2.7) because canonical Poisson brackets, involving dependent momenta p_α, are inconsistent. Then, in order to introduce canonical Poisson brackets we must enlarge phase space, from the constrained hypersurface U (2.8) to a full $2N$-dimensional space (q_n, p_n), and introduce the concept of Dirac's weak inequalities. Two functions $f(q_n, p_n)$ and $g(q_n, p_n)$ are weakly equivalent $f \approx g$, if they coincide on the constrained hypersurface U. In particular, introducing the functions

$$\chi_\alpha(p_n, q_n) \equiv p_\alpha - \psi_\alpha(q_n, p_r) \qquad \alpha = r+1, \dots N \tag{1}$$

the hypersurface U is defined by the weak equations

$$\chi_\alpha \approx 0 \tag{2}$$

Turning back to the equations of motion (3.5) we rewrite them as weak equations in terms of χ_α

$$\begin{aligned} \dot{q}_n &\approx \frac{\partial H_0}{\partial p_n} + \sum_{\alpha=r+1}^{N} \dot{q}_\alpha \frac{\partial \chi_\alpha}{\partial p_n} \\ \dot{p}_n &\approx -\frac{\partial H_0}{\partial q_n} - \sum_{\alpha=r+1}^{N} \dot{q}_\alpha \frac{\partial \chi_\alpha}{\partial q_n} \end{aligned} \tag{3}$$

where we have added equations for the unsolved velocities $\dot{q}_\alpha$, which are empty (they say simply $\dot{q}_\alpha = \dot{q}_\alpha$). Now equations (3) can be expressed in terms of Poisson brackets:

$$\dot{q}_m \approx \{ q_m , H_0 + \sum_{\alpha=R+1}^{N} \dot{q}_\alpha \chi_\alpha \} \tag{1}$$

$$\dot{p}_m \approx \{ p_m , H_0 + \sum_{\alpha=R+1}^{N} \dot{q}_\alpha \chi_\alpha \}$$

Notice that the operational idea behind weak identities is to compute first all Poisson brackets, as if all momenta were independent, and impose the constraints only at the end.

In general equations (1) are not the final answers to our problem because, as we said before, we must impose that the primary constraints will be preserved in time. This yields the consistency conditions

$$\{ \chi_\beta , H_0 + \sum_{\alpha=R+1}^{N} \dot{q}_\alpha \chi_\alpha \} \approx 0 \tag{2}$$

Then, if the primary constraints satisfy

$$\{ \chi_\alpha, \chi_\beta \} \approx 0 \quad , \quad \{ \chi_\alpha , H_0 \} \approx 0 \tag{3}$$

equation (2) is trivially satisfied. Such χ_α's satisfying (3) are called first-class primary constraints. In this case (1) give the final expression of the Hamiltonian equations of motion and all $\dot{q}_\alpha$ ($\alpha = R+1, \dots, N$) remain arbitrary functions of time, associated to the gauge invariances of the action. If (3) are not both satisfied we have to go through the procedure described above. We leave the discussion here because in the cases that we will be interested in (3) are satisfied. In this cases the time evolution is generated by the Hamiltonian

$$H = H_0 + \sum_{\alpha=R+1}^{N} v_\alpha \chi_\alpha \tag{4}$$

where v_α are indetermined quantities multiplying the constraints.

Example : Free Point-like Particle

Let us apply this formalism to the free point-like relativistic particle. From (I-1.1)

$$L(X, \dot{X}) \equiv - m \left[\dot{X}^\mu(\tau) \, \dot{X}_\mu(\tau) \right]^{1/2} \tag{5}$$

and the momenta are

$$p^\mu(\tau) \equiv -\frac{\partial L}{\partial \dot{X}_\mu} = m \frac{\dot{X}^\mu(\tau)}{[\dot{X}^\nu(\tau)\,\dot{X}_\nu(\tau)]^{1/2}} \tag{1}$$

which are constrained :

$$p_\mu(\tau)\, p^\mu(\tau) = m^2 \tag{2}$$

Then the naive Hamiltonian H_0 is

$$H_0 = -p^\mu \frac{\partial X_\mu}{\partial \tau} - L = 0 \tag{3}$$

As we have only one constraint (2) there are $(D-1)$ independent momenta p^i and the remaining one p^0 is given by

$$p^0 = [\vec{p}^2 + m^2]^{1/2} \tag{4}$$

Accordingly the Hamiltonian equations of motion (3.5) are

$$\dot{p}^\mu(\tau) = 0 \quad , \quad \dot{X}^i(\tau) = \frac{p^i(\tau)\,\dot{X}^0(\tau)}{p^0(\tau)} \tag{5}$$

and the velocity $\dot{X}^0(\tau)$ remains indetermined.

Notice that since $H_0 = 0$ and we have only one constraint equations (5.3) are trivially satisfied and therefore the Hamiltonian is

$$H = \nu\,(p^2 - m^2) \tag{6}$$

So that the equations of motion can be written in terms of the Poisson brackets

$$\dot{X}^\mu = \{X^\mu, H\} = -2\nu\, p^\mu \quad , \quad \dot{p}^\mu = \{p^\mu, H\} = 0 \tag{7}$$

where the first equation specified for $\mu = 0$ tells us $-2\nu = \dot{X}^0 / p^0$.

To quantize the theory we make the usual transition from Poisson brackets to commutators with the appropriate factor $i\hbar$. The constraint now becomes

$$\hat{\chi} = (\partial_\mu \partial^\mu + m^2) \tag{8}$$

To find the second - quantized field theory, we consider the

state described by a field $\phi(\xi^\mu, \tau)$ and we impose the constraint

$$\hat{\chi}\, \phi(\xi^\mu, \tau) = (\partial^\mu \partial_\mu + m^2)\, \phi(\xi^\mu, \tau) = 0 \qquad (1)$$

We also impose the Schrödinger equation

$$i \frac{\partial}{\partial \tau} \phi = H \phi \qquad (2)$$

The right-hand side of this equation vanishes and we find that ϕ is independent of τ. In the second-quantization the only equation is the Klein-Gordon equation

$$(\partial^\mu \partial_\mu + m^2)\, \phi(\xi^\mu) = 0 \qquad (3)$$

VII - STRING : COVARIANT FORMALISM AND QUANTIZATION

In the string case the canonical variables are $X^\mu(\sigma,\tau)$ and $P^\mu_\tau(\sigma,\tau)$ for which we will assume the canonical Poisson brackets

$$\{ X^\mu(\sigma,\tau) , X^\nu(\sigma',\tau) \} = \{ P^\mu_\tau(\sigma,\tau) , P^\nu_\tau(\sigma',\tau) \} = 0$$

$$\{ X^\mu(\sigma,\tau) , P^\nu_\tau(\sigma',\tau) \} = -\eta^{\mu\nu}\,\delta(\sigma-\sigma') \qquad \sigma,\sigma' \in [0,\pi] \qquad (1)$$

Let us remember that the momenta and angular momenta of the string are given by

$$P^\mu \equiv \int_0^\pi d\sigma\; P^\mu_\tau(\sigma,\tau) \qquad (2)$$

$$M^{\mu\nu} \equiv \int_0^\pi d\sigma\; [\, X^\mu(\sigma,\tau)\, P^\nu_\tau(\sigma,\tau) - X^\nu(\sigma,\tau)\, P^\mu_\tau(\sigma,\tau) \,]$$

It is an easy exercise, using (1), that these quantity satisfy the Poincaré algebra

$$\{ P^\mu , P^\nu \} = 0$$

$$\{ M^{\lambda\rho} , P^\mu \} = -\eta^{\lambda\mu} P^\rho + \eta^{\rho\mu} P^\lambda \qquad (3)$$

$$\{ M^{\lambda\rho} , M^{\mu\nu} \} = -\eta^{\lambda\mu} M^{\rho\nu} - \eta^{\rho\nu} M^{\lambda\mu} + \eta^{\rho\mu} M^{\lambda\nu} + \eta^{\lambda\nu} M^{\rho\mu}$$

From the Nambu-Goto Lagrangian it follows that we have the primary constraints (II-6.4)

$$P_\tau \cdot X' = 0 \qquad , \qquad (2\pi\alpha')^2 P_\tau^2 + X'^2 = 0 \qquad (4)$$

and the naive Hamiltonian is

$$H_0 = \int_0^\pi d\sigma\; [\, - P^\mu_\tau \dot{X}_\mu - L_{N.G} \,] = 0 \qquad (5)$$

as a consequence of (II-6.10a) and (II-2.6)

The Open String

For open strings we deal with the constraints in the follo-

wing way : We extend analitically the string from $[0, \pi]$ to $[-\pi, +\pi]$. Taking into account (III - 3.4) that

$$P^\mu_\tau(\sigma, \tau) = P^\mu_\tau(-\sigma, \tau) \quad , \quad X'^\mu(\sigma, \tau) = - X'^\mu(\sigma, \tau) \tag{1}$$

the two constraints (1.4) are unified in

$$[\, 2\alpha' \pi P_\tau(\sigma, \tau) + X'(\sigma, \tau) \,]^2 = 0 \tag{2}$$

Let us now introduce the constrained functionals

$$L_f \equiv \frac{1}{4\pi} \int_{-\pi}^{+\pi} d\sigma \; f(\sigma,\tau) \, [\, \pi (2\alpha')^{1/2} P_\tau(\sigma, \tau) + (2\alpha')^{-1/2} X'(\sigma, \tau) \,]^2 \tag{3}$$

where $f(\sigma, \tau)$ is an arbitrary function. Obviously (2) implies that $L_f = 0$. Notice that (2) can be substituted by $\{ L_f = 0 \}$ if $\{ f \}$ is some complete set. For instance (2) is equivalent to

$$L_m \equiv - \frac{1}{4\pi} \int_{-\pi}^{+\pi} d\sigma \; e^{i m (\sigma + \sigma)} \, [\, \pi (2\alpha')^{1/2} P_\tau(\sigma, \tau) + (2\alpha')^{-1/2} X'(\sigma, \tau) \,]^2 = 0 \tag{4}$$

[Let us begin checking that (4) coincide with the Virasoro gauges introduced in (III - 4.1) for the gauge used there. From (III - 4.7)

$$[\, \pi (2\alpha')^{1/2} P_\tau(\sigma, \tau) + (2\alpha')^{-1/2} X'(\sigma, \tau) \,]^2 = - 2 \sum_{m=-\infty}^{+\infty} L_m \, e^{-im(\sigma + \sigma)} \tag{5}$$

and the result follows.]

Before going on a word of caution : When you compute Poisson brackets after analytic continuation you must use instead of (1.6)

$$\{ X^\mu(\sigma, \tau) \, , \, P^\nu_\tau(\sigma', \tau) \} = - \eta^{\mu\nu} \, [\, \delta(\sigma - \sigma') + \delta(\sigma + \sigma') \,] \qquad \sigma, \sigma' \in [-\pi, +\pi] \tag{6}$$

We have now an infinite number of constraints $\{ L_m = 0 \, , \, m \in \mathbb{Z} \}$. Since $H_0 = 0$, it is clear that in order to establish that they are first class primary constraints we must see that $[\, L_m, L_{m'} \,] \approx 0$ (see (VI - 5.3)). Let us define

$$A^\mu(\sigma, \tau) \equiv \pi (2\alpha')^{1/2} P^\mu_\tau(\sigma, \tau) + (2\alpha')^{-1/2} X'^\mu(\sigma, \tau) \tag{7}$$

Then

$$\{L_f, L_g\} = \frac{1}{(4\pi)^2} \int_{-\pi}^{+\pi} d\sigma \int_{-\pi}^{\pi} d\sigma' \, f(\sigma\tau)\, g(\sigma'\tau) \, \{ A^2(\sigma,\tau) \, , \, A^2(\sigma',\tau) \} \tag{1}$$

But

$$\{ A^2(\sigma,\tau) \, , \, A^2(\sigma'\tau) \} = 4 \, A_\mu(\sigma\tau) \, A_\nu(\sigma'\tau) \, \{ A^\mu(\sigma\tau) \, , \, A^\nu(\sigma'\tau) \} =$$

$$= 4\pi A_\mu(\sigma\tau) \, A_\nu(\sigma'\tau) \, [\, \{ P^\mu_\tau(\sigma,\tau) \, , \, X'^\nu(\sigma'\tau) \} + \{ X'^\mu(\sigma,\tau) \, , \, P^\nu_\tau(\sigma'\tau) \} \,]$$

$$= + 4\pi \, A^\mu(\sigma\tau) \, A_\mu(\sigma'\tau) \left\{ \frac{\partial}{\partial\sigma'} [\delta(\sigma-\sigma') + \delta(\sigma+\sigma')] - \frac{\partial}{\partial\sigma} [\delta(\sigma-\sigma') + \delta(\sigma+\sigma')] \right\}$$

$$= 4\pi \, A^\mu(\sigma\tau) \, A_\mu(\sigma'\tau) \left\{ \frac{\partial}{\partial\sigma'} \delta(\sigma-\sigma') - \frac{\partial}{\partial\sigma} \delta(\sigma-\sigma') \right\} \tag{2}$$

Introducing (2) in (1) and integrating by parts:

$$\{L_f, L_g\} = \frac{1}{4\pi} \int_{-\pi}^{+\pi} d\sigma \int_{-\pi}^{+\pi} d\sigma' \, \delta(\sigma-\sigma') \left\{ - f(\sigma\tau) \frac{\partial g(\sigma'\tau)}{\partial\sigma'} A_\mu(\sigma\tau) A^\mu(\sigma'\tau) - \right.$$

$$- f(\sigma\tau) \, g(\sigma'\tau) \, A_\mu(\sigma\tau) \frac{\partial A^\mu(\sigma'\tau)}{\partial\sigma'} + \frac{\partial f(\sigma\tau)}{\partial\sigma} g(\sigma'\tau) \, A_\mu(\sigma\tau) \, A^\mu(\sigma'\tau)$$

$$\left. + f(\sigma\tau) \, g(\sigma'\tau) \frac{\partial A_\mu(\sigma\tau)}{\partial\sigma} A_\mu(\sigma'\tau) \right\} =$$

$$= \frac{1}{4\pi} \int_{-\pi}^{+\pi} d\sigma \left\{ - f(\sigma\tau) \frac{\partial g(\sigma\tau)}{\partial\tau} + \frac{\partial f(\sigma\tau)}{\partial\tau} g(\sigma\tau) \right\} A^2(\sigma\tau) \quad \Rightarrow$$

$$\{L_f, L_g\} = L_{f \circledast g} \tag{3}$$

$$f \circledast g = \frac{\partial f(\sigma\tau)}{\partial\sigma} g(\sigma\tau) - f(\sigma\tau) \frac{\partial g(\sigma,\tau)}{\partial\sigma}$$

In particular

$$\{ L_n, L_m \} = i \, (m-n) \, L_{n+m} \tag{4}$$

which is the so-called Virasoro algebra. We have therefore proved that $\{ L_n = 0 \, , \, n \in \mathbb{Z} \}$ are first-class primary constraints.

The effective Hamiltonian that generates the motion equations will be of the form $H = L_h$ for some appropiate choice of the function h. There is some arbitrariness for the choice of this function; indeed, any specification of a definite form of H defines a gauge for the system. To see this, let us recall that the gauge invariance manifest itself in the impossibility of inverting

$p^\mu_\tau = p^\mu_\tau (\dot X', X')$ to obtain $\dot X^\mu = \dot X^\mu (p_\tau, X')$. However, once a Hamiltonian is defined, we can express the velocities in terms of the coordinate and momenta through Hamilton equations $\dot X^\mu = \{ X^\mu, H \}$. A choice of H is then equivalent to a gauge specification.

In order to go on we would like to compute $\{ X^\mu(\sigma,\tau), L_h \}$ and $\{ p^\mu_\tau(\sigma,\tau), L_h \}$.

$$\{ X^\mu(\sigma,\tau), L_h \} = \frac{1}{4\pi} \int_{-\pi}^{+\pi} d\sigma' \, h(\sigma'\tau) \{ X^\mu(\sigma,\tau), A^2(\sigma'\tau) \} =$$

$$= \frac{1}{2\pi} \int_{-\pi}^{+\pi} d\sigma' \, h(\sigma'\tau) A_\nu(\sigma'\tau) \{ X^\mu(\sigma,\tau), A^\nu(\sigma'\tau) \} =$$

$$= -\frac{1}{2} (2\alpha')^{1/2} \{ h(\sigma\tau) A^\mu(\sigma\tau) + h(-\sigma\tau) A^\mu(-\sigma\tau) \}$$

$$\{ p^\mu_\tau(\sigma,\tau), L_h \} = \frac{1}{2\pi} \int_{-\pi}^{+\pi} d\sigma' \, h(\sigma'\tau) A_\nu(\sigma'\tau) \{ p^\mu_\tau(\sigma\tau), A^\nu(\sigma'\tau) \}$$

$$= -\frac{1}{2\pi} \frac{1}{(2\alpha')^{1/2}} \{ h'(\sigma\tau) A^\mu(\sigma\tau) + h(\sigma\tau) A'^\mu(\sigma\tau) + h'(-\sigma\tau) A^\mu(-\sigma\tau) + h(-\sigma\tau) A'^\mu(-\sigma\tau) \}$$

and hence

$$\{ X^\mu(\sigma,\tau), L_h \} = -\frac{1}{2} [h(\sigma\tau) + h(-\sigma\tau)] 2\pi\alpha' p^\mu_\tau(\sigma\tau) - \frac{1}{2} [h(\sigma\tau) - h(-\sigma\tau)] X'^\mu(\sigma\tau) \tag{1}$$

$$\{ p^\mu_\tau(\sigma,\tau), L_h \} = -\frac{1}{2} [h(\sigma\tau) - h(-\sigma\tau)] \frac{1}{2\pi\alpha'} [X'^\mu(\sigma\tau) - X'^\mu(-\sigma\tau)] - \frac{1}{2} [h'(\sigma\tau) + h'(-\sigma\tau)] p^\mu_\tau(\sigma\tau) \tag{2}$$

We see from (1) that the even part of $h(\sigma,\tau)$ transforms X^μ into p^μ_τ, and thus generates a dynamical evolution of the system. The odd part corresponds to a reparametrization of the points of the string at fixed τ. Then, if $h(\sigma\tau) = -h(-\sigma,\tau)$,

$$\delta X^\mu(\sigma,\tau) = \epsilon \{ X^\mu(\sigma,\tau), L_h \} = -\epsilon \, h(\sigma\tau) X'^\mu(\sigma\tau) \tag{3}$$

which is the change in parametrization induced by $\sigma \to \sigma - \epsilon h(\sigma\tau)$.

A very convenient gauge is the one where

$$H = L_0 \tag{4}$$

then from equations (4.1) and (4.2) ($h(\sigma\tau) \equiv -1$)

$$\dot{X}^\mu(\sigma\tau) = \{ X^\mu(\sigma,\tau) , H \} = 2\pi\alpha' P^\mu_\tau(\sigma,\tau)$$

$$\dot{P}^\mu_\tau(\sigma,\tau) = \{ P^\mu_\tau(\sigma,\tau) , H \} = \frac{1}{2\pi\alpha'} X''^\mu(\sigma,\tau) \qquad (1)$$

From these equations we recover immediately the equations of motion

$$\ddot{X}^\mu(\sigma,\tau) - X''^\mu(\sigma,\tau) = 0 \qquad (2)$$

and inserting (1) into (1.4) we obtain

$$\dot{X} \cdot X' = 0 \quad , \quad \dot{X}^2 + X'^2 = 0 \qquad (3)$$

and therefore the choice (4.4) corresponds to an orthonormal gauge.

Furthermore (III-3.3) implies

$$X^\mu(\sigma,\tau) = q^\mu + \alpha_0^\mu \tau + i \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{1}{m} \alpha_m^\mu e^{-2mi\tau} \cos m\sigma$$

$$P^\mu_\tau(\sigma,\tau) = \frac{1}{2\pi\alpha'} \sum_{m=-\infty}^{+\infty} \alpha_m^\mu e^{-2mi\tau} \cos m\sigma \qquad (4)$$

$$X'^\mu(\sigma,\tau) = -i \sum_{m=-\infty}^{+\infty} \alpha_m^\mu e^{-2mi\tau} \sin m\sigma$$

From here and using

$$\frac{1}{2\pi} \int_{-\pi}^{+\pi} d\sigma \cos m\sigma \cos k\sigma = \frac{1}{2} [\delta_{k,m} + \delta_{k,-m}] \qquad (5)$$

we obtain

$$\alpha_0^\mu = \alpha' \int_{-\pi}^{+\pi} d\sigma \, P^2_\mu(\sigma\tau)$$

$$q^\mu = \frac{1}{2\pi} \int_{-\pi}^{+\pi} d\sigma \, [X^\mu(\sigma,\tau) - 2\pi\alpha' \tau P^\mu_\tau(\sigma\tau)] \qquad (6)$$

$$\alpha_m^\mu = \frac{1}{2\pi} \int_{-\pi}^{+\pi} d\sigma \, [2\pi\alpha' P^\mu_\tau(\sigma,\tau) - im X^\mu(\sigma,\tau)] e^{2mi\tau} \cos m\sigma \qquad m \in \mathbb{Z}$$

Then it is easy to check that the Poisson brackets are

$$\{ \alpha_m^\mu , \alpha_n^\nu \} = 2im\alpha' \eta^{\mu\nu} \delta_{m,-n}$$

$$\{ q^\mu , \alpha_m^\nu \} = -2\alpha' \eta^{\mu\nu} \delta_{m,0} \qquad (7)$$

Let us check :

$$\{ \alpha_m^\mu , \alpha_n^\nu \} = \frac{1}{(2\pi)^2} \int_{-\pi}^{+\pi} d\sigma \int_{-\pi}^{+\pi} d\sigma' \, e^{i(m+n)\tau} \cos m\sigma \cos n\sigma' \cdot$$

$$\cdot \{ 2\pi\alpha' \Pi^\mu_\tau(\sigma,\tau) - i m X^\mu(\sigma,\tau) \, , \, 2\pi\alpha' \Pi^\nu_\tau(\sigma',\tau) - i n X^\nu(\sigma',\tau) \} =$$

$$= \frac{1}{(2\pi)^2} \int_{-\pi}^{+\pi} d\sigma \int_{-\pi}^{+\pi} d\sigma' \, e^{i(m+n)\tau} \cos m\sigma \cos n\sigma' \{ -i m 2\pi\alpha' \eta^{\mu\nu} [\delta(\sigma-\sigma') + \delta(\sigma+\sigma')]$$

$$+ i n 2\pi\alpha' \eta^{\mu\nu} [\delta(\sigma-\sigma') + \delta(\sigma+\sigma')] \}$$

$$= \frac{i\alpha'}{2\pi} \eta^{\mu\nu} \int_{-\pi}^{+\pi} d\sigma \{ -2m \cos m\sigma \cos n\sigma \, e^{i(m+n)\tau}$$

$$+ 2n \cos m\sigma \cos n\sigma \, e^{i(m+n)\tau} \}$$

$$= i\alpha' \eta^{\mu\nu} \frac{1}{2} \{ -2m \, e^{i(m+n)\tau} [\cancel{\delta_{m,n}} + \delta_{m,-n}] + 2n \, e^{i(m+n)\tau} [\cancel{\delta_{m,n}} + \delta_{m,-n}] \}$$

$$= 2im\alpha' \eta^{\mu\nu} \delta_{m,-n}$$

$$\{ q^\mu , \alpha_n^\nu \} = \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} d\sigma \int_{-\pi}^{\pi} d\sigma' \, e^{in\tau} \cos m\sigma \{ \dot{X}^\mu(\tau\sigma) - 2\pi\alpha' \Pi^\mu_\tau(\tau,\sigma) \, , \, \dot{X}^\nu(\tau,\sigma) + 2\pi\alpha' \Pi^\nu_\tau(\tau,\sigma') \}$$

$$= \frac{1}{(2\pi)^2} \int_{-\pi}^{+\pi} d\sigma \int_{-\pi}^{+\pi} d\sigma' \, e^{in\tau} \cos m\sigma \{ -2\pi\alpha' \eta^{\mu\nu} [\delta(\sigma-\sigma') + \delta(\sigma+\sigma')]$$

$$+ 2\pi\alpha' in \eta^{\mu\nu} [\delta(\sigma-\sigma') + \delta(\sigma+\sigma')] \} =$$

$$= \frac{1}{2\pi} \eta^{\mu\nu} \int_{-\pi}^{+\pi} d\sigma \, e^{in\tau} \cos m\sigma \{ -2\alpha' + 2in\alpha' \} = -\delta_{m0} 2\alpha' \eta^{\mu\nu} \qquad \text{Q.E.D.}$$

From (III.4-2)

$$L_m = -\frac{1}{4\alpha'} \sum_{n=-\infty}^{+\infty} \alpha_n^\mu \, \alpha_{m-n\,\mu} \qquad m \in \mathbb{Z} \tag{1}$$

and

$$H_0 = L_0 = -\frac{1}{4\alpha'} [(\alpha_0 \cdot \alpha_0) + 2 \sum_{n=1}^{\infty} (\alpha_n \cdot \alpha_{-n})] \tag{2}$$

The Closed String

For closed strings we rewrite the constraints (1.4) as

$$[2\alpha' n \Pi^\mu_\tau \pm X'^\mu]^2 = 0 \tag{3}$$

As before, we reduce (6.3) to a discrete infinite set of constraints

$$L_m = 0 \quad , \quad \tilde{L}_m = 0 \quad , \quad m \in \mathbb{Z} \tag{1}$$

where

$$L_m(+1) \equiv \tilde{L}_m \quad , \quad L_m(-1) \equiv L_m$$

$$L_m(\lambda) \equiv -\frac{1}{4\pi} \int_0^{2\pi} d\sigma \; e^{2im(\lambda\sigma + \tau)} A^\mu(\sigma,\tau;\lambda) \tag{2}$$

$$A^\mu(\sigma,\tau;\lambda) \equiv \pi (2\alpha')^{1/2} P^\mu_\tau(\sigma,\tau) + \lambda (2\alpha')^{-1/2} X'^\mu(\sigma,\tau)$$

Then

$$\{ L_m(\lambda) , L_{m'}(\lambda') \} = \frac{1}{(4\pi)^2} \int_0^{2\pi} d\sigma \int_0^{2\pi} d\sigma' \; e^{2im(\lambda\sigma+\tau)} e^{2im'(\lambda'\sigma'+\tau)} \{ A^2(\sigma\tau;\lambda) , A^2(\sigma'\tau;\lambda') \}$$

but

$$\{ A^2(\sigma\tau;\lambda) , A^2(\sigma'\tau;\lambda') \} = 4 A_\mu(\sigma\tau;\lambda) A_\nu(\sigma'\tau;\lambda') \{ A^\mu(\sigma\tau;\lambda) , A^\nu(\sigma'\tau;\lambda') \} =$$

$$= 4\pi A_\rho(\sigma\tau;\lambda) A^\mu(\sigma'\tau;\lambda') \left\{ \lambda' \frac{\partial\varepsilon}{\partial\sigma'} \delta(\sigma-\sigma') - \lambda \frac{\partial\varepsilon}{\partial\sigma} \delta(\sigma-\sigma') \right\}$$

Now integrating by parts

$$\{ L_m(\lambda) , L_{m'}(\lambda') \} = \frac{1}{4\pi} \int_0^{2\pi} d\sigma \; e^{2i(m+m')\tau} e^{2i(m\lambda\sigma + m'\lambda'\sigma')} \delta(\sigma-\sigma')$$

$$\cdot \left\{ -2im A_\rho(\sigma\tau;\lambda) A^\mu(\sigma\tau;\lambda') - \lambda' A_\mu(\sigma\tau;\lambda) \frac{\partial\varepsilon A^\mu(\sigma\tau;\lambda')}{\partial\sigma} \right.$$

$$\left. + 2im A_\mu(\sigma\tau;\lambda) A^\mu(\sigma\tau;\lambda') + \lambda \frac{\partial\varepsilon A_\mu(\sigma\tau;\lambda)}{\partial\sigma} A^\mu(\sigma\tau;\lambda') \right\} =$$

$$= \frac{1}{4\pi} \int_0^{2\pi} d\sigma \; e^{2i(m+m')\tau} e^{2i\lambda(m+\lambda' m')\sigma} \cdot \left\{ -2i(m-m') A_\mu(\sigma\tau;\lambda) A^\mu(\sigma\tau;\lambda') \right.$$

$$\left. + (\lambda-\lambda') \left[2\alpha'\pi^2 (P_\tau \cdot P'_\tau) - \frac{1}{2\alpha'} (X' \cdot X'') \right] \right\} \tag{3}$$

Then clearly

$$\{ L_m , L_{m'} \} = 2i(m-m') L_{m+m'} \tag{4}$$

$$\{ \tilde{L}_m , \tilde{L}_{m'} \} = 2i(m-m') \tilde{L}_{m+m'}$$

Furthermore

$$\{L_n, \tilde{L}_m\} = -\frac{1}{4\pi}\int_0^\pi d\sigma\, e^{2i(n+m)\tau}\, e^{-2i(n-m)\sigma}\Big\{-2i(n-m)\Big[2\alpha'\pi^2(P_\tau P_\tau)$$

$$-\frac{1}{2\alpha'}(X'.X')\Big] + 2\Big[2\pi^2\alpha'(P_\tau P_\tau') - \frac{1}{2\alpha'}(X'.X'')\Big]\Big\} =$$

$$= -\frac{1}{4\pi}\int_0^\pi d\sigma\, e^{2i(n+m)\tau}\frac{\partial}{\partial\sigma}\Big\{e^{-2i(n-m)\sigma}\Big[2\alpha'\pi^2(P_\tau P_\tau) - \frac{1}{2\alpha'}(X'.X')\Big]\Big\} = 0$$

and therefore the Virasoro algebra is

$$\{L_n, L_m\} = 2i(m-n)\, L_{n+m}$$

$$\{\tilde{L}_n, \tilde{L}_m\} = 2i(m-n)\, \tilde{L}_{n+m} \qquad (1)$$

$$\{L_n, \tilde{L}_m\} = 0$$

From this and (1.5) we have proved that (7.1) are first-class primary constraints.

A useful choice of the Hamiltonian is

$$H = L_0 + \tilde{L}_0 \qquad (2)$$

Notice

$$\{X^\mu(\sigma,\tau), H\} = -\frac{1}{4\pi}\int_0^\pi d\sigma'\,\Big\{X^\mu(\sigma,\tau),\ 4\pi^2\alpha' P_\tau^2(\sigma'\tau) + \frac{1}{\alpha'}X'^2(\sigma'\tau)\Big\} =$$

$$= -2\pi\alpha'\int_0^\pi d\sigma'\, P_{\tau\nu}(\sigma'\tau)\,\{X^\mu(\sigma,\tau), P_\tau^\nu(\sigma'\tau)\} = 2\pi\alpha' P_\tau^\mu(\sigma\tau)$$

$$\{P_\tau^\mu(\sigma,\tau), H\} = -\frac{1}{2\pi\alpha'}\int_0^\pi d\sigma'\, X'_\nu(\sigma'\tau)\,\{P_\tau^\mu(\sigma\tau), X'^\nu(\sigma'\tau)\} =$$

$$= -\frac{1}{2\pi\alpha'}\int_0^\pi d\sigma'\, X'^\mu(\sigma'\tau)\frac{\partial}{\partial\sigma'}\delta(\sigma-\sigma') = \frac{1}{2\pi\alpha'}X''^\mu(\sigma,\tau) \quad \Rightarrow$$

$$\dot{X}^\mu(\sigma,\tau) = \{X^\mu(\sigma,\tau), H\} = 2\pi\alpha' P_\tau^\mu(\sigma,\tau)$$

$$\dot{P}_\tau^\mu(\sigma,\tau) = \{P^\mu(\sigma,\tau), H\} = \frac{1}{2\pi\alpha'}X''^\mu(\sigma,\tau) \qquad (3)$$

and this implies, as before, that the choice (2) corresponds to an orthonormal gauge.

Therefore using (7.3) as well as (III - 6.2) we obtain that the L_m here introduced correspond to those of (III - 6.1). Furthermore

$$X^\mu(\sigma,\tau) = q^\mu + 2\alpha_0\tau + \frac{i}{2}\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\frac{1}{m}\left\{\alpha_m^\mu\, e^{-2im(\tau-\sigma)} + \tilde\alpha_m^\mu\, e^{-2im(\tau+\sigma)}\right\}$$

$$P_\tau^\mu(\sigma,\tau) = \frac{1}{2\pi\alpha'}\sum_{m=-\infty}^{+\infty}\left\{\alpha_m^\mu\, e^{-2im(\tau-\sigma)} + \tilde\alpha_m^\mu\, e^{-2im(\tau+\sigma)}\right\} \qquad (1)$$

$$X'^\mu(\sigma,\tau) = \sum_{m=-\infty}^{+\infty}\left\{-\alpha_m^\mu\, e^{-2im(\tau-\sigma)} + \tilde\alpha_m^\mu\, e^{-2im(\tau+\sigma)}\right\}$$

Then it is easy to check that

$$q^\mu = \int_0^\pi d\sigma\left\{\frac{1}{\pi}X^\mu(\sigma,\tau) - 2\alpha'\tau P_\tau^\mu(\sigma,\tau)\right\}$$

$$\alpha_m^\mu = \int_0^\pi d\sigma\left\{\alpha' P_\tau^\mu(\sigma,\tau) - \frac{im}{\pi}X^\mu(\sigma,\tau)\right\}e^{+2im(\tau-\sigma)} \qquad (2)$$

$$\tilde\alpha_m^\mu = \int_0^\pi d\sigma\left\{\alpha' P_\tau^\mu(\sigma,\tau) - \frac{im}{\pi}X^\mu(\sigma,\tau)\right\}e^{+2im(\tau+\sigma)}$$

Using now (1.4)

$$\{q^\mu, \alpha_m^\nu\} = \{q^\mu, \tilde\alpha_m^\nu\} = -\delta_{m0}\alpha'\eta^{\mu\nu}$$

$$\{\alpha_m^\mu, \alpha_n^\nu\} = \{\tilde\alpha_m^\mu, \tilde\alpha_n^\nu\} = 2im\alpha'\eta^{\mu\nu}\delta_{m,-n} \qquad (3)$$

$$\{\alpha_m^\mu, \tilde\alpha_n^\nu\} = 0$$

Finally from (III - 6.2)

$$L_m = -\frac{1}{2\alpha'}\sum_{n=-\infty}^{+\infty}\alpha_{m-n}^\mu\,\alpha_{n\mu}\,, \qquad \tilde L_m = -\frac{1}{2\alpha'}\sum_{n=-\infty}^{+\infty}\tilde\alpha_{m-n}^\mu\,\tilde\alpha_{n\mu} \qquad (4)$$

and

$$H = -\frac{1}{2\alpha'}\left\{(\alpha_0\cdot\alpha_0) + 2\sum_{m=1}^{\infty}\left[(\alpha_m\cdot\alpha_{-m}) + (\tilde\alpha_m\cdot\tilde\alpha_{-m})\right]\right\} \qquad (5)$$

Now we are ready to introduce second quantization in this Hamiltonian formalism for the open and closed string.

Covariant Quantization

In order to quantize the theory in the covariant formalism we do in the equal time canonical Poisson brackets the substitution

$$i\{A, B\} \longrightarrow [A, B] \tag{1}$$

and hence

$$[X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)] = [P^\mu_\tau(\sigma,\tau), P^\nu_\tau(\sigma',\tau)] = 0$$

$$[X^\mu(\sigma,\tau), P^\nu_\tau(\sigma',\tau)] = -i\eta^{\mu\nu}\delta(\sigma-\sigma') \qquad \sigma,\sigma' \in [0,\pi] \tag{2}$$

Using the definitions of the momentum and angular momentum of the string

$$P^\mu \equiv \int_0^\pi d\sigma\, P^\mu_\tau(\sigma,\tau)$$

$$M^{\mu\nu} \equiv \int_0^\pi d\sigma\, [X^\mu(\sigma,\tau) P^\nu_\tau(\sigma,\tau) - X^\nu(\sigma,\tau) P^\mu_\tau(\sigma,\tau)] \tag{3}$$

we get immediately

$$[P^\mu, P^\nu] = 0$$

$$[M^{\lambda\rho}, P^\mu] = -i(\eta^{\lambda\mu}P^\rho - \eta^{\rho\mu}P^\lambda) \tag{5}$$

$$[M^{\lambda\rho}, M^{\mu\nu}] = -i(\eta^{\lambda\mu}M^{\rho\nu} + \eta^{\rho\nu}M^{\lambda\mu} - \eta^{\rho\mu}M^{\lambda\nu} - \eta^{\lambda\nu}M^{\rho\mu})$$

i.e. the Poincare algebra is maintained in the D-dimensional Minkowski space

The Open String

The solutions of the equations of motion are now (III-3.3)

$$X^\mu(\sigma,\tau) = q^\mu + \alpha_0^\mu \tau + i\sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{1}{m}\alpha_m^\mu e^{-im\tau}\cos m\sigma \tag{4}$$

$$P^\mu_\tau(\sigma,\tau) = \frac{1}{2\pi\alpha'}\sum_{m=-\infty}^{+\infty}\alpha_m^\mu e^{-im\tau}\cos m\sigma \tag{6}$$

$$X'^\mu(\sigma,\tau) = -i\sum_{m=-\infty}^{+\infty}\alpha_m^\mu e^{-im\tau}\sin m\sigma \tag{7}$$

where q^μ and α_m^μ ($m \in \mathbb{Z}$) are operators. Since (8.6)

$$q^\mu = \frac{1}{2\pi} \int_{-\pi}^{+\pi} d\sigma \; [\, \dot{X}^\mu(\sigma,\tau) - 2\pi\alpha' P_\tau^\mu(\sigma,\tau) \,] \tag{1}$$

$$\alpha_m^\mu = \frac{1}{2\pi} \int_{-\pi}^{+\pi} d\sigma \; [\, 2\pi\alpha' P_\tau^\mu(\sigma,\tau) - i m X^\mu(\sigma,\tau) \,] e^{im\tau} \cos m\sigma \quad , \; m \in \mathbb{Z}$$

we obtain immediately

$$[\, \alpha_m^\mu \; , \; \alpha_m^\nu \,] = - 2m\alpha' \eta^{\mu\nu} \delta_{m,-m}$$

$$[\, q^\mu \; , \; \alpha_m^\nu \,] = - i 2\alpha' \eta^{\mu\nu} \delta_{m,0} \tag{2}$$

Remember that the baricenter coordinate and the momentum are

$$\frac{1}{\pi} \int_0^\pi d\sigma \; \dot{X}^\mu(\sigma,\tau) = q^\mu + \alpha_0^\mu \tau \tag{3}$$

$$P^\mu = \int_0^\pi d\sigma \; P_\tau^\mu(\sigma,\tau) = \frac{1}{2\alpha'} \alpha_0^\mu \tag{4}$$

The above relations suggest that the string is a system that contains the baricenter degrees of freedom (q^μ, $\alpha_0^\mu = 2\alpha' P^\mu$) plus an infinite set of quantum harmonic oscillators (α_m^μ, $m \neq 0$). In order to make this more evident let us introduce

$$a_m^\mu \equiv (2\alpha' m)^{-1/2} \alpha_m^\mu \quad , \quad a_m^{\mu\,+} \equiv (2\alpha' m)^{-1/2} \alpha_{-m}^\mu \quad , \; m > 0 \tag{5}$$

then we can write (2) as

$$[\, a_m^\mu \; , \; a_m^{\nu\,+} \,] = - \eta^{\mu\nu} \delta_{m,m}$$

$$[\, a_m^\mu \; , \; a_m^\nu \,] = [\, a_m^{\mu\,+} \; , \; a_m^{\nu\,+} \,] = 0 \qquad m, m > 0 \tag{6}$$

$$[\, q^\mu \; , \; P^\nu \,] = - i \eta^{\mu\nu}$$

$$[\, a_m^\mu \; , \; q^\nu \,] = [\, a_m^{\mu\,+} \; , \; q^\nu \,] = [\, a_m^\mu \; , \; P^\nu \,] = [\, a_m^{\mu\,+} \; , \; P^\nu \,] = 0$$

We have now to define the Hilbert space on which the oscillators operate. So we introduce a vector $|0;P\rangle$ corresponding to the ground state of the string with momentum P^μ:

$$P^\mu \, |0;P\rangle = P^\mu \, |0;P\rangle \tag{7}$$

being p^μ the eigenvalue of the operator $\hat{p}^\mu$ acting on this state. Furthermore, in order to specify the harmonic operators we require

$$a_m^\mu \, |0; p\rangle = 0 \qquad m > 0 \tag{1}$$

The excitation levels of the string are now defined as any vector of the type

$$\left\{ \prod_m \left(a_m^{\mu m +} \right)^{\lambda_m} \right\} |0; p\rangle \qquad \lambda_m = \text{integer} \tag{2}$$

The space so constructed has an indefinite metric due to the presence of $\eta^{\mu\nu}$ in (11.6).

At this point we must remember that we are dealing with a constrained system, and that the constraints

$$\left[2\pi\alpha' \, p_\tau(\sigma, \tau) + X'(\sigma, \tau) \right]^2 = 0 \qquad \sigma \in [-\pi, +\pi] \tag{3}$$

are inconsistent with canonical commutation relations. In the classical theory the constraints amount to relations among the α_m^μ, expressed by the vanishing of the Virasoro gauges

$$L_m = 0 \qquad , \quad m \in \mathbb{Z} \tag{4}$$

In quantum theory L_m are promoted to operators and we must study the new situation.

Following Fubini and Veneziano [S. Fubini and G. Veneziano Nuovo Cim. 67A, 29 (1970)] we define the so-called Fubini and Veneziano fields

$$Q^\mu(z) = q^\mu - 2i\alpha' p^\mu \ln z - i(2\alpha')^{1/2} \sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} \left[a_m^{\mu +} z^m - a_m^\mu z^{-m} \right]$$

$$P^\mu(z) = iz \frac{dQ^\mu(z)}{dz} = 2\alpha' p^\mu + (2\alpha')^{1/2} \sum_{m=1}^{\infty} \sqrt{m} \left[a_m^{\mu +} z^m + a_m^\mu z^{-m} \right] \tag{5}$$

Notice that

$$Q^\mu\left(e^{i\tau}\right) = \bar{X}^\mu(\sigma = 0, \tau) \tag{6}$$

$$P^\mu\left(e^{i(\tau+\sigma)}\right) = \dot{\bar{X}}^\mu(\sigma, \tau) + \bar{X}'^\mu(\sigma, \tau)$$

Now it is clear from (6.1) that for L_m, $m \neq 0$, no ordering problems appear, but for L_0 we have ordering problems. Now we define, taking into account (2.4),

$$L_m = -\frac{1}{4\pi}\frac{1}{2\alpha'}\int_{-\pi}^{+\pi} d\sigma\, e^{im(\tau+\sigma)} : P^2\left(e^{i(\tau+\sigma)}\right) : \qquad (4)$$

where : : denotes normal ordering. Defining $z \equiv e^{i(\tau+\sigma)}$ we can re-write this as

$$L_m = -\frac{1}{4\alpha'}\oint \frac{dz}{2i\pi z}\, z^m : P^2(z) : \qquad (2)$$

where the integration is performed in the positive sense around the origin. More generally we can define "generalized Virasoro gauges" through the formula

$$L_f \equiv \frac{1}{4\alpha'}\oint \frac{dz}{2i\pi z}\, f(z) : P^2(z) : \qquad (3)$$

where $f(z)$ is supposed sufficiently regular.

Let us now consider the algebraic properties of these operators. Let us start considering

$$P^\mu(x)\, P^\nu(y) = \left\{ 2\alpha' p^\mu + (2\alpha')^{1/2} \sum_{m=1}^{\infty} \sqrt{m}\, \left[a_m^\mu{}^+ x^m + a_m^\mu x^{-m} \right] \right\} \cdot$$

$$\cdot \left\{ 2\alpha' p^\nu + (2\alpha')^{1/2} \sum_{m=1}^{\infty} \sqrt{m}\, \left[a_m^\nu{}^+ y^m + a_m^\nu y^{-m} \right] \right\} =$$

$$= : P^\mu(x)\, P^\nu(y) : + 2\alpha' \sum_{m=1}^{\infty}\sum_{m=1}^{\infty} \sqrt{mm}\; x^{-m} y^m (-\eta^{\mu\nu}\delta_{mm})$$

$$= : P^\mu(x)\, P^\nu(y) : - 2\alpha' \eta^{\mu\nu} \sum_{m=1}^{\infty} m\,(y/x)^m \quad \Rightarrow$$

$$P^\mu(x)\, P^\nu(y) = : P^\mu(x)\, P^\nu(y) : - 2\alpha' \eta^{\mu\nu} \frac{xy}{(x-y)^2}, \quad |x| > |y| \qquad (5)$$

For $|x| \leq y$ the left hand side is not well defined (since the normal product is not singular at $x = y$) and we must be careful when writing $P^\mu(x)\, P^\nu(y)$ that the condition $|x| > |y|$ is satisfied. In this equation the last term in the right hand side plays the role of the "contraction" $\underline{P^\mu(x)\, P^\nu(y)}$ used in Wick theorem to express the product of operators in terms of their normal ordered product.

Notice furthermore that for $|x|>|y|$

$$:P^2(x):\,:P^2(y): \;=\; :P^2(x)\,P^2(y): \;+\; 4\,\underline{P_\mu(x)\,P_\nu(y)}\,:P^\mu(x)\,P^\nu(y):$$

$$+\; 2\,\underline{P^\nu(x)\,P^\nu(y)}\;\;\underline{P_\mu(x)\,P_\nu(y)} \quad \Rightarrow$$

$$:P^2(x):\,:P^2(y): \;=\; :P^2(x)\,P^2(y): \;-\; 4\,(2\alpha')\,\frac{xy}{(x-y)^2}\,:P(x)\,P(y):$$

$$+\; 2\,(2\alpha')^2\,D\,\frac{x^2y^2}{(x-y)^4}\,, \qquad |x|>|y| \tag{1}$$

The fact that the dimension D appears explicitly is the first indication of how crucially the quantum theory depends on D. Let us consider

$$[L_f, L_g] = \frac{1}{16\alpha'^2}\oint_{\Gamma_2}\frac{dy}{2i\pi y}\oint_{\Gamma_1}\frac{dx}{2i\pi x}\,f(x)\,g(y)\,:P^2(x)::P^2(y):$$

$$-\frac{1}{16\alpha'^2}\oint_{\Gamma_4}\frac{dy}{2i\pi y}\oint_{\Gamma_3}\frac{dx}{2i\pi x}\,f(x)\,g(y)\,:P^2(y)::P^2(x): \tag{2}$$

where Γ_1, Γ_2, Γ_3 and Γ_4 are arbitrary simple contours around the origin such that Γ_1 (Γ_4) encloses Γ_2 (Γ_3) in such a way that the products are well defined. We can choose $\Gamma_2 \equiv \Gamma_4$ and therefore we have the situation of the figure. Let us introduce in (2) the result (1). Then



$$[L_f, L_g] = \frac{1}{16\alpha'^2}\oint_{\Gamma_2}\frac{dy}{2i\pi y}\cdot\left\{\oint_{\Gamma_1}\frac{dx}{2i\pi x}\,f(x)\,g(y)\cdot\right.$$

$$\left[\,:P^2(x)\,P^2(y): \;-\; 8\alpha'\,\frac{xy}{(x-y)^2}\,:P(x)\,P(y): \;+\; 8\alpha'^2\,D\,\frac{x^2y^2}{(x-y)^4}\,\right] -$$

$$\left.-\oint_{\Gamma_3}\frac{dx}{2i\pi x}\,f(x)\,g(y)\left[\,:P^2(y)\,P^2(x): \;-\; 8\alpha'\,\frac{xy}{(x-y)^2}\,:P(x)\,P(y): \;+\; 8\alpha'^2\,D\,\frac{x^2y^2}{(x-y)^4}\,\right]\right\} \tag{3}$$

We will assume that $f(x)$ and $g(y)$ have no poles in the integration regions; then

$$[L_f, L_g] = \frac{1}{16\alpha'^2}\oint_{\Gamma_2}\frac{dy}{2i\pi y}\,g(y)\oint_{\Gamma_1-\Gamma_3}\frac{dx}{2i\pi x}\,f(x)\cdot$$

$$\cdot\left\{-8\alpha'\,\frac{xy}{(x-y)^2} \;+\; 8\alpha'^2\,D\,\frac{x^2y^2}{(x-y)^4}\right\} \tag{4}$$

The x integration is simply evaluated taking the residue at $x=y$

$$[L_f, L_g] = \oint_{r_2} \frac{dy}{2\pi i y} \frac{(-1)}{2\alpha'} y \left\{ f'(y)\, g(y) : P^2(y) : + f(y) g(y) : P(y) \cdot P'(y) : \right\}$$

$$+ \frac{D}{12} \oint_{r_2} \frac{dy}{2\pi i y} y^2 g(y) \left(\frac{d^3}{dy^3} y f(y) \right) =$$

$$= \oint_{r_2} \frac{dy}{2\pi i y} \left(-\frac{1}{2\alpha'} \right) y \frac{1}{2} \left[f'(y) g(y) - f(y) g'(y) \right] : P^2(y) :$$

$$- \frac{D}{24} \oint \frac{dy}{2\pi i} \left\{ \left(\frac{d^2}{dy^2} y f(y) \right) \left(\frac{d}{dy} y g(y) \right) - \left(\frac{d}{dy} y f(y) \right) \left(\frac{d^2}{dy^2} y g(y) \right) \right\} \Rightarrow$$

$$[L_f, L_g] = L_h + C_{fg}$$

$$h(y) \equiv y \left[g'(y) f(y) - g(y) f'(y) \right] \tag{1}$$

$$C_{fg} = -\frac{D}{24} \oint \frac{dy}{2\pi i} \left\{ (y f(y))'' (y g(y))' - (y f(y))' (y g(y))'' \right\}$$

In particular $f(x) = -x^m$, $g(x) = -x^m$ we obtain the Virasoro algebra [M. A. Virasoro Phys. Rev. D1, 2933 (1970)]

$$[L_m, L_m] = (m - m) L_{m+m} + \frac{D}{12} m (m^2 - 1) \delta_{m,-m} \tag{2}$$

where we have taken into account that

$$\oint \frac{dz}{2\pi i z} z^m = \delta_{m,0} \tag{3}$$

The difference between the operator algebra and the Poisson bracket algebra is thus essentially a c-number term which depends on the dimension of the space-time and plays an essential role in string theories.

Before going on we would like to obtain the explicit expression for L_m. From (13.2) and (12.5)

$$L_m = -\frac{1}{4\alpha'} \oint \frac{dz}{2\pi i z} z^m : P^2(z) :$$

$$= -\frac{1}{4\alpha'} \oint \frac{dz}{2\pi i z} z^m : \left\{ (2\alpha') p^\mu + (2\alpha')^{1/2} \sum_{k=1}^{\infty} \sqrt{k} \left[a_k^{\mu +} z^k + a_k^\mu z^{-k} \right] \right\} \cdot$$

$$\cdot \left\{ (2\alpha') p_\mu + (2\alpha')^{1/2} \sum_{e=1}^{\infty} \sqrt{e} \left[a_{\mu e}^{+} z^e + a_{e\mu} z^{-e} \right] \right\} =$$

$$= -\frac{1}{4\alpha'} \oint \frac{dz}{2i\pi z} \left\{ 4\alpha'^2 p^2 z^m + 2(2\alpha')^{3/2} \sum_{e=1}^{\infty} \sqrt{e} \left[(p.a_e^+) z^{m+e} + (p.a_e) z^{m-e} \right] \right.$$

$$\left. + 2\alpha' \sum_{e=1}^{\infty} \sum_{k=1}^{\infty} \sqrt{ek} \left[(a_k^+ a_e^+) z^{m+k+e} + 2(a_k^+ a_e) z^{m+k-e} + (a_k a_e) z^{m-k-e} \right] \right\}$$

Hence

$$L_0 = -\alpha' p^2 - \sum_{k=1}^{\infty} k (a_k^+ . a_k)$$

$$L_m = -(2\alpha')^{1/2} \sqrt{m} (p. a_m) - \sum_{k=1}^{\infty} \sqrt{k(m+k)} (a_k^+ . a_{m+k})$$

$$- \frac{1}{2} \sum_{k=1}^{m-1} \sqrt{k(m-k)} (a_k . a_{m-k}) \qquad m>0 \qquad (1)$$

$$L_{-m} = -(2\alpha')^{1/2} \sqrt{m} (p. a_m^+) - \sum_{k=1}^{\infty} \sqrt{k(m+k)} (a_{m+k}^+ . a_k)$$

$$- \frac{1}{2} \sum_{k=1}^{m-1} \sqrt{k(m-k)} (a_{m-k}^+ . a_k^+) = L_m^+ \qquad m>0$$

or equivalently using that $\alpha_0^\mu = 2\alpha' p^\mu$ and (11.5)

$$L_0 = -\frac{1}{2\alpha'} \left\{ \frac{1}{2} (\alpha_0 . \alpha_0) + \sum_{k=1}^{\infty} (\alpha_{-k} . \alpha_k) \right\}$$

$$L_m = -\frac{1}{2\alpha'} \left\{ \frac{1}{2} \sum_{k=1}^{m-1} (\alpha_k . \alpha_{m-k}) + \sum_{k=0}^{\infty} (\alpha_{-k} . \alpha_{m+k}) \right\} \quad , m>0 \qquad (2)$$

$$L_{-m} = -\frac{1}{2\alpha'} \left\{ \frac{1}{2} \sum_{k=1}^{m-1} (\alpha_{-m+k} . \alpha_{-k}) + \sum_{k=0}^{\infty} (\alpha_{-m-k} . \alpha_k) \right\} \quad , m>0$$

The commutation rules (15.2) can in principle be deduced from (1) or (2) working carefully the normal ordering.

The constraints $L_m = 0$ cannot be imposed as operator equations for they are inconsistent with canonical commutation relations. This situation is similar to the one which arises in the covariant quantization of the electromagnetic field with the Lorentz gauge condition $\partial_\mu A^\mu(x) = 0$. So following the Gupta-Bleuler procedure, a weaker condition would be to consider the constraints as equations determining the physical states :

$$L_m |\psi\rangle_{ph} = 0 \quad , \quad m \in \mathbb{Z} \qquad (3)$$

However these conditions are too strong because they are not

satisfied even for the ground state $|0;p\rangle$ if $m \leq 0$, as can be seen from (16.1). This forces us to still weaker conditions. Notice that

$$L_m |0;p\rangle = 0 \quad , m > 0$$

$$L_0 |0;p\rangle = -\alpha' p^2 |0;p\rangle \equiv \alpha(0) |0;p\rangle \quad \text{(mass shell condition)} \tag{1}$$

Then it is natural to assume that physical states satisfy

$$(L_m - \alpha(0)\,\delta_{m,0})\,|\psi\rangle = 0 \qquad m \geq 0$$

$$\langle\psi|\,(L_{-m} - \alpha(0)\,\delta_{m,0}) = 0 \qquad m \geq 0 \tag{2}$$

Because of the indefinite metric we are not sure that the solutions of these equations contain no negative norm states.

We now wish to show that the covariant quantization of the string is also ghost free. The Fock space of the states with which we are working is not positive definite and it is spanned by the states ($|0\rangle \equiv |0;p\rangle$)

$$|r\rangle \equiv \prod_{m=1}^{\infty} \prod_{\mu_m=0}^{D-1} [\,a^{+}_{m,\mu_m}]^{\lambda_{m,\mu_m}} |0\rangle \tag{3}$$

where $\{\lambda_{m,\mu_m}\}$ are all possible sets of non-negative integers. We define the "number" operator

$$R \equiv -\sum_{m=1}^{\infty} m\,(a^{+}_m \cdot a_m) \tag{4}$$

Notice

$$R\,|r\rangle = -\sum_{k=1}^{\infty} \prod_{m=1}^{\infty} \prod_{\mu_m=0}^{D-1} k\,(a^{\nu +}_k a_{k\nu})\,[\,a^{+}_{m,\mu_m}]^{\lambda_{m,\mu_m}} |0\rangle =$$

$$= \sum_{k=1}^{\infty} \prod_{m=1}^{\infty} \prod_{\mu_m=0}^{D-1} k\,a^{\nu +}_k\, \eta_{\nu,\mu_m}\, \lambda_{m,\mu_m}\, \delta_{k,m}\, [\,a^{+}_{m,\mu_m}]^{\lambda_{m,\mu_m} - \delta_{km}\delta_{\mu\mu_m}} |0\rangle \Rightarrow$$

$$R \equiv -\sum_{m=1}^{\infty} m\,(a^{+}_m \cdot a_m) \tag{5}$$

$$R\,|r\rangle = M\,|r\rangle \qquad M \equiv \sum_{m=1}^{\infty} \sum_{\mu_m=0}^{D-1} m\,\lambda_{m\mu_m}$$

The quantity $M = 0, 1, 2, \ldots$ is called the level number. We shall call R^M the space of states at the level M. Obviously this

space contains both positive and negative norm states. Notice that

$$L_0 = -\alpha' p^2 + R \tag{1}$$

and from (17.2) a physical state at the level M has a mass given by

$$p^2 = \frac{1}{\alpha'} [M - \alpha(0)] \qquad M = \text{level number} \tag{2}$$

Spurious states are those belonging to R^M of the form

$$|S\rangle \equiv \sum_{m=1}^{M} C_m L_{-m} |S_m\rangle \tag{3}$$

$|S_m\rangle$ being arbitrary. Notice that if $|\phi\rangle$ is a physical state as a consequence of (17.2)

$$\langle \phi | S \rangle = 0 \tag{4}$$

We shall first build a subspace of R^M, called the transverse subspace, which has obviously a positive definite norm. This will be our first step towards proving the no-ghost theorem. The transverse states are defined as follows: the momentum p^μ of the states we shall talk about is constrained by (2). We choose a frame

$$p^\mu \equiv \frac{1}{2\sqrt{\alpha' \alpha(0)}} (M, 2\alpha(0) - M, 0, 0, \ldots, 0) \qquad M = \text{level number} \tag{5}$$

we then pick up the light-like vector

$$k^\mu \equiv \frac{1}{2\sqrt{\alpha' \alpha(0)}} (1, -1, 0, 0, \ldots, 0) \tag{6}$$

Notice

$$(k \cdot p) = \frac{1}{2\alpha'} \tag{7}$$

Let us define the operators

$$K_m \equiv k \cdot \alpha_m = (2\alpha')^{1/2} \sqrt{m}\, (k \cdot a_m), \quad K_{-m} \equiv K_m^+ \quad , m > 0 \tag{8}$$

We now define the transverse states through the conditions

$$L_m |t\rangle = K_m |t\rangle = 0 \quad , \quad m \geq 1 \tag{9}$$

Let us see that the transverse states have positive semidefinite norm. Let us use light cone coordinates (IV-5.4)

$$\alpha_{m+} \equiv \frac{1}{\sqrt{2}} [\alpha_m^0 + \alpha_m^1] \quad , \quad \alpha_{m-} \equiv \frac{1}{\sqrt{2}} [\alpha_m^0 - \alpha_m^1] \tag{1}$$

then from (11.2)

$$[\alpha_{m+} , \alpha_{m+}] = [\alpha_{m-} , \alpha_{m-}] = 0$$

$$[\alpha_{m+} , \alpha_{m-}] = -2m\alpha' \delta_{m,-m} \tag{2}$$

$$K_m = [2\alpha' \alpha(0)]^{-1/2} \alpha_{m+} = \sqrt{\frac{m}{\alpha(0)}}\, a_{m+}$$

Since $K_m |t\rangle = 0$ we deduce that $|t\rangle$ does not contain any a_{m-}^+ creation operator and hence its norm is positive or zero. It is more difficult to prove that because of the L_m conditions, the transverse states have strictly positive norm and we need two lemmas to stablish this [P. Goddard and C. B. Thorn Phys. Lett. 40B, 235 (1972)]

Lemma 1 : Let us consider a transverse state $|t\rangle \in R^M$ such that $\langle t|t\rangle \neq 0$. For such a $|t\rangle$ the states

$$|\{\lambda,\mu\}, t\rangle \equiv L_{-1}^{\lambda_1} L_{-2}^{\lambda_2} \cdots L_{-m}^{\lambda_m} K_{-1}^{\mu_1} K_{-2}^{\mu_2} \cdots K_{-m}^{\mu_m} |t\rangle \tag{3}$$

where

$$\sum r\lambda_r + \sum s\mu_s \equiv N - M > 0 \tag{4}$$

form a linear independent span of some subspace of R^N. Further, this subspace containts no solution of

$$L_m |\phi\rangle = K_m |\phi\rangle = 0 \tag{5}$$

Proof :

(a) Suppose that we have a linear relation between the vectors (3)

$$\Big\{ \sum_{\{\lambda,\mu\}} C[\lambda,\mu] L_{-1}^{\lambda_1} \cdots L_{-m}^{\lambda_m} K_{-1}^{\mu_1} \cdots K_{-m}^{\mu_m} \Big\} |t\rangle = 0 \tag{6}$$

$|t\rangle$ being obtained by applying creation operators on the vacuum. Let us now consider (16.1) and (18.8). If we pick up, for instance, the terms with the a_i^+ oscillators in $L_{-1}^{\lambda_1} \cdots L_{-m}^{\lambda_m} K_{-1}^{\mu_1} \cdots K_{-m}^{\mu_m}$ we obtain a contribution of the form

$$(R.a_1^+)^{\lambda_1}\,(a_1^+.a_1^+)^{\lambda_2}\,(a_1^+a_2^+)^{\lambda_3}\cdots\,(a_1^+.a_{m-1}^+)^{\lambda_m}\,(R\,a_1^+)^{\mu_1} \qquad (4)$$

and these terms have to cancel with each other in order to satisfy (6). Consider the terms in this sum that maximize $\lambda_1 + 2\lambda_2 + \lambda_3 + \lambda_4 + \cdots + \lambda_m + \mu_1$. This maximal terms have to cancel with each other. But obviously they are independent unless all the terms which maximize the above quantity have the same $\lambda_1 \cdots \lambda_m \mu_1$. But if this is the case there is no cancellation possible, and we therefore conclude that $\lambda_1 = \lambda_2 = \cdots = \lambda_m = \mu_1 = 0$. It is then obvious that all $C[\lambda,\mu] = 0$.

(b) Let us prove that this subspace contains no solution of (19.5). We have (see (15.2) and use (18.8) and (11.2))

$$[L_m, L_m] = (m-m)\,L_{n+m} + \frac{D}{12}\,m\,(m^2-1)\,\delta_{m,-m}$$

$$[K_m, K_m] = 0 \qquad , m, m > 0 \qquad (2)$$

$$[L_m, K_m] = -m\,K_{m+m} \qquad , \quad K_0 \equiv 1$$

Let us first apply K_m to a state $|\{\lambda,\mu\}, t\rangle$. So we use $K_m L_{-m} = L_{-m} K_m + m\,K_{n-m}$. Commuting K_m through the $L_{-m}^{\lambda_m}$ we produce some further K_r with $r < m$. We move them to the right until those K_r which have $r > 0$ will annihilate on $|t\rangle$. Some K_r with $r < 0$ will have been created in the process. The L_{-m} may not be any more in the order of eq (19.6), but we can bring them back to this order using the first commutation relation of (2). And so we see that

$$K_m\,|\{\lambda,\mu\}, t\rangle = \sum_{\{\lambda',\mu'\}} C[\lambda'\mu']\,|\{\lambda'\mu'\}, t\rangle \qquad (3)$$

where

$$\mu' \equiv \sum s\mu_s' \;\geq\; \mu \equiv \sum s\mu_s \qquad (5)$$

If $n = 1$ the only commutator which does not produce K_r with $r < 0$ is $[K_1, L_{-1}] = 1$. For this commutator μ is not increased.

Let us now consider a superposition of $|\{\lambda,\mu\}, t\rangle$ vectors (with fixed $|t\rangle$) and ask that this superposition be annihilated by all K_m. The vectors in the superposition with maximal μ will

be mapped into vectors with greater value of μ, which them have to cancel by themselves. This is impossible because of the independence theorem already proved and hence we conclude that $\lambda_2 = \lambda_3 = \cdots = \lambda_m = 0$. Applying again K_2 on a term which contains only $L_{-1}^{\lambda_1}$ and K_{-m} operators, we see again that $\lambda_1 = 0$ is the only solution. So we conclude that

$$|\Phi\rangle = \sum_{\{\mu\}} C[\mu]\, K_{-1}^{\mu_1} \cdots K_{-m}^{\mu_m} |t\rangle \qquad (1)$$

Applying again now L_m to $|\Phi\rangle$ we see that, because the last relation of (20.2), we shall obtain a superposition of $|\{0, \mu\}, t\rangle$ vectors where μ' will be increased. Then the only solution is $\mu_1 = \cdots = \mu_m = 0$ which is forbidden by (19.4). This finishes the proof of the lemma.

In this first lemma, what we have done is the following: Because of the commutation relations

$$[R, L_{-m}] = m L_{-m} \quad , \quad [R, K_{-m}] = m K_{-m} \qquad (2)$$

L_{-m} and K_{-m} raise the eigenvalue of R by m units. So starting from a transverse state belonging to R^M we have constructed an independent set of vectors belonging to R^N: $N = \sum r\lambda_r + \sum s\mu_s + M > M$; these vectors are obviously orthogonal to all transverse vectors of R^N. Further, the subspace of these vectors does not contain any transverse vector itself.

In order to obtain a complete basis for R^N we still have to vary $|t\rangle$ and this is done in the following lemma.

<u>Lemma 2</u>. If $|t, M, \nu\rangle$ is an orthonormal basis for the subspace T^M of transverse states belonging to R^M, the states $|\{\lambda, \mu\}, t, M, \nu\rangle$ defined in equation (19.3) give a basis for the states with $R = \sum_r r\lambda_r + \sum s\lambda_s + M = N$, and, as N varies, for the whole Fock space. Further T^M is positive definite.

Proof:

$|t, N, \nu\rangle$ is a transversal state belonging to R^N. The subspace of the transverse states is called T^N. Let us call G^N the set of all $|\{\lambda, \mu\}, t, M, \nu\rangle$ belonging to R^N such that $\sum r\lambda_r + \sum s\mu_s > 0$. We construct T^N and G^N as follows:

We start from the vacuum $|0\rangle$, i.e. $N=0$. It satisfies $L_m|0\rangle = K_m|0\rangle = 0$ for $n \geq 1$. Hence $T^0 = \{|0\rangle\}$, $G^0 = \{|0\rangle\}$. From them we construct $G^1 = \{L_{-1}|0\rangle, K_{-1}|0\rangle\}$ and built its orthonormal complement. It is readely seen to be made of states annihilated by L_1 and K_1 and it is found to be

$$T^1 = \{ a^+_{1,i}|0\rangle \; ; \; i = 2, 3, \ldots D-1 \} \tag{1}$$

From T^0 and T^1 we build now

$$G^2 = \{ L_{-1} a^+_{1,i}|0\rangle, \; L_{-2}|0\rangle, \; K_{-1} a^+_{1i}|0\rangle, \; K_{-2}|0\rangle \} \tag{2}$$

and construct its orthogonal complement T^2 and so on. Let us make sure that this construction is indeed possible and that at each state T^N is made up of transverse vectors. We do it by recursion and assume that we have been able to construct $T^0, T^1, \ldots T^{N-1}$ and $G^0, G^1, \ldots G^{N-1}$. We now attempt to construct T^N and G^N.

G^N is constructed by raising all the states of $T^0, T^1, \ldots, T^{N-1}$ by means of L_{-m} and K_{-m} operators, as done before. All the states thus obtained are $|\{\lambda,\mu\}, t, M, \nu\rangle$ states. They are all independent. We have shown this in the first lemma for fixed $|t, M, \nu\rangle$. However, states with different $|t, M, \nu\rangle$ can never be dependent because is readely seen that

$$\langle\{\lambda'\mu'\}\, t' M' \nu' | \{\lambda\mu\}\, t M \nu\rangle = \delta_{MM'}\, \delta_{\nu\nu'}\, f(\lambda, \lambda', \mu, \mu', M, \nu) \tag{3}$$

This is due to the fact that L_{-m}, K_{-m} can be commuted to the left and K_m, L_m to the right until they annihilate and the only surviving contributions will come from c-numbers or to operators which are diagonal in $|M, \nu\rangle$ (in M is obvious, in ν because we have selected an orthonormal basis for $T^0, \ldots T^{N-1}$ by hypothesis). So the $|\{\lambda,\mu\}, t, M, \nu\rangle$ states form a linearly independent basis of G^N. We have used here the recursion hypothesis that T^M is made of transverse states.

We now construct T^N as the orthogonal complement of G^N and show that it is made up of transverse states. Since $|t, N, \nu\rangle$ belongs to T^N it has to be orthogonal to all vectors $|\{\lambda,\mu\}, t, M, \nu\rangle$, because

of their independence just proved. Hence

$$\langle t,N,v \mid L_{-1}^{\lambda_1} \cdots L_{-m}^{\lambda_m} K_{-1}^{\mu_1} \cdots K_{-m}^{\mu_m} \mid t,M,v' \rangle =$$

$$= \langle t,N,v \mid L_{-1} \left[L_{-1}^{\lambda_1 - 1} L_{-2}^{\lambda_2} \cdots L_{-m}^{\lambda_m} K_{-1}^{\mu_1} \cdots K_{-m}^{\mu_m} \right] \mid t M v' \rangle = 0 \qquad (1)$$

for all $\lambda_1 \cdots \lambda_m \mu_1 \cdots \mu_m M v$. The vectors

$$L_{-1}^{\lambda_1 - 1} L_{-2}^{\lambda_2} \cdots L_{-m}^{\lambda_m} K_{-1}^{\mu_1} \cdots K_{-m}^{\mu_m} \mid t M v' \rangle \qquad (2)$$

form a complete basis for R^{N-1} by hypothesis, and hence $\langle t,N,v \mid L_{-1} = 0$, identically. To prove that $\langle t,N,v \mid L_{-2} = 0$ we use the fact that by commuting L_{-m} with each other another independent basis of G^N is obtained by ordering the L_{-m} in the form $L_{-2}^{\lambda_2} L_{-1}^{\lambda_1} \cdots L_{-m}^{\lambda_m}$ (in fact any order will do) and because we have an independent basis for R^{N-2} we conclude that $\langle t,N,v \mid L_{-2} = 0$. Hence

$$\langle t,N,v \mid L_{-m} = 0 \qquad (3)$$

by repeated commutations of $L_{-1}, L_{-2}, \ldots$ and then we show easily that $\langle t,N,v \mid K_{-1}^{\mu_1} \cdots K_{-m}^{\mu_m} \mid t M v' \rangle = 0$ implies

$$\langle t,N,v \mid K_{-m} = 0 \qquad (4)$$

Hence the orthogonal complement of G^N, T^N, is made up of transverse states. We still have to ensure a point which has been used in the recursion hypothesis, namely we can construct an orthonormal basis for T^N. This will be ensured if we can show that if $|t\rangle$ is a zero norm state belonging to a semipositive Hilbert space, then it has to be orthogonal to all other states of this space, i.e. it has to be a null state. If there was such state in T^N, it would also belong to its orthonormal complement G^N, i.e. in G^N there would be a transverse state. But by Lemma 1 we know that this is impossible and hence T^N contains no zero norm states and hence we can apply the orthonormalization procedure to it, which closes the recursion hypothesis. Hence Lemma 2. In the wording of Lemma 2 the $|\{\lambda,\mu\},t,M,v\rangle$ states include the states $\lambda_i = \mu_i = 0$, i.e. transverse states.

Hence we have been able to construct, for each eigenvalue N of R, a positive definite subspace $T^N \subset R^N$ of transverse states, as well as a complete basis for $R^N = T^N \oplus G^N$. This is independent of D and the mass shell condition as well. Now we must consider the problem of the physical states which, at each level, span a subspace $P^N \subseteq R^N$.

Let us denote by $S^N(l_0) \subset R^N$ the spurious subspace such that if $|s, N\rangle \in S^N(l_0)$ then

$$L_0 |s, N\rangle = [\alpha(0) + \ell(0) - 1] |s, N\rangle \tag{1}$$

Therefore $S^N(1)$ is the on-shell spurious subspace, $S^N(0)$ is the spurious subspace one unit below mass shell, $S^N(-1)$ is two units below mass shell and so on.

Let $|p, N\rangle \in P^N$, it can be decomposed into

$$|p, N\rangle = |s, N\rangle + |k, N\rangle \tag{2}$$

where $|s, N\rangle \in S^N(1)$ and $|k, N\rangle \in R^N \ominus S^N(1) \equiv K^N(1)$. Notice that

$$(L_0 - \alpha(0)) |p, N\rangle = 0 \tag{3}$$

and

$$L_m |p, N\rangle = L_m |s, N\rangle + L_m |k, N\rangle = 0 \quad , \quad m \geq 1 \tag{4}$$

Before going on we need some new results. First we must prove that L_1 maps $S^N(1) \longrightarrow S^{N-1}(0)$ for any value of D if $\alpha(0) = 1$. The most general state belonging to $S^N(1)$ may be written

$$|s, N\rangle = L_{-1} |f, N-1\rangle + L_{-2} |f, N-2\rangle \tag{5}$$

where

$$L_0 |f, N\rangle = \alpha(0) |f, N\rangle \tag{6}$$

and therefore

$$L_0 |f, N-m\rangle = [\alpha(0) - m] |f, N-m\rangle \tag{7}$$

To check this assertion, observe that the Virasoro algebra may

be used to put all spurious states in this form, for example

$$L_{-3}\,|f, N-3\rangle = [L_{-1}, L_{-2}]\,|f, N-3\rangle =$$

$$= L_{-1}\{L_{-2}\,|f, N-3\rangle\} + L_{-2}\{-L_{-1}\,|f, N-3\rangle\} \qquad (1)$$

Now we act with L_1 on $|s, N\rangle$ to find

$$L_1\,|s, N\rangle = L_1 L_{-1}\,|f, N-1\rangle + L_1 L_{-2}\,|f, N-2\rangle =$$

$$= \{L_{-1}L_1 + 2L_0\}\,|f, N-1\rangle + \{L_{-2}L_1 + 3L_{-1}\}\,|f, N-1\rangle$$

$$= 2[\alpha(0) - 1]\,|f, N-1\rangle +$$

$$+ \{L_{-1}L_1 + L_{-2}L_1 + 3L_{-1}\}\,|f, N-1\rangle \qquad (2)$$

The second one is a state $|s, N-1\rangle$ and the first one disappears if $\alpha(0) = 1$. Then from this and (24.4), since $K(l_0)$ and $S(l_0)$ are independent physical

$$L_1\,|p, N\rangle = L_1\,|s, N\rangle = L_1\,|k, N\rangle = 0 \qquad (3)$$

We now wish to find a special operator

$$\tilde{L}_2 = \alpha L_1 L_1 + \beta L_2 \qquad (4)$$

that maps directly $S^N(1) \longrightarrow S^{N-2}(-1)$. We need some preliminary relations

$$[L_{-1}L_{-1}, L_1 L_1] = L_{-1}[L_{-1}, L_1]L_1 + L_1[L_{-1}, L_1]L_{-1} + [L_{-1}, L_1]L_{-1}L_1$$

$$+ L_1 L_{-1}[L_{-1}, L_1] = -2\{L_{-1}L_0L_1 + L_1L_0L_{-1} + L_0L_{-1}L_1 + L_1L_{-1}L_0\}$$

$$= -2\{2L_0L_{-1}L_1 - L_{-1}L_1 + L_1L_0L_{-1} + L_1L_{-1}L_0\} =$$

$$= -2\{2L_0L_1L_{-1} - 4L_0L_0 - L_1L_{-1} + 2L_0 + L_1L_0L_{-1} + L_1L_0L_{-1} - L_1L_{-1}\} =$$

$$= -2\{2L_0L_1L_{-1} - 4L_0L_0 - L_1L_{-1} + 2L_0 + 2L_0L_1L_{-1} + 2L_1L_{-1} - L_1L_{-1}\}$$

$$= 8\{L_0L_0 - \tfrac{1}{2}L_0 - \tfrac{1}{2}L_0L_1L_{-1}\}$$

$$[L_{-1}L_{-1}, L_2] = L_{-1}[L_{-1}, L_0] + [L_{-1}, L_2]L_{-1} = -3\{L_{-1}L_1 + L_1L_{-1}\} =$$

$$= 3\{L_1L_{-1} - 2L_0 + L_1L_{-1}\} = -6L_1L_{-1} + 6L_0$$

Hence

$$[\,L_{-1}L_{-1},\,L_1L_1\,] = 4\,(2L_0-1)\,L_0 - 8L_0L_1L_{-1}$$

$$[\,L_{-1}L_{-1},\,L_2\,] = [\,L_{-2},\,L_1L_1\,] = -6L_1L_{-1} + 6L_0 \qquad (4)$$

$$[\,L_1L_{-2},\,L_{-1}\,] = -2L_1 + 4L_1L_0$$

Therefore

$$[\,\tilde{L}_2,\,L_{-1}\,] = 4\alpha L_1L_0 + (3\beta - 2\alpha)\,L_1 \qquad (2)$$

$$[\,\tilde{L}_2,\,\tilde{L}_2^{+}\,] = \alpha^2\,[\,8L_0L_1L_{-1} - 4\,(2L_0-1)\,L_0\,] + 12\alpha\beta\,(L_1L_{-1} - L_0) + \beta^2\left(4L_0 + \tfrac{1}{2}D_0\right)$$

Let us rewrite the most general spurious state of $S^N(4)$ as

$$|S,N\rangle = L_{-1}\,|f,N-1\rangle + \tilde{L}_2^{+}\,|f,N-2\rangle \qquad (3)$$

Then we get

$$\tilde{L}_2\,|SN\rangle = \tilde{L}_2L_{-1}\,|f,N-1\rangle + \tilde{L}_2\tilde{L}_2^{+}\,|f,N-2\rangle =$$

$$= \{\,L_{-1}\tilde{L}_2 + 4\alpha L_1L_0 + (3\beta - 2\alpha)\,L_1\,\}\,|f,N-1\rangle + \{\,\tilde{L}_2^{+}L_2 + \alpha^2\,(\,8L_0L_1L_{-1}$$

$$- 4\,(2L_0-1)\,L_0\,] + 12\alpha\beta\,(L_1L_{-1} - L_0) + \beta^2\,(4L_0 + \tfrac{1}{2}D)\,\}\,|f,N-2\rangle \qquad (6)$$

Notice

$$L_1L_0\,|f,N-1\rangle = [\,\alpha(0)-1\,]\,L_1\,|f,N-1\rangle$$

$$L_0L_1L_{-1}\,|f,N-2\rangle = [\,L_{-1}L_1L_0 + 2L_0^2\,]\,|f,N-2\rangle = \qquad (5)$$

$$= \{\,L_{-1}L_1\,[\alpha(0)-2) + 2\,(\alpha(0)-2)^2\,\}\,|f,N-2\rangle$$

$$L_1L_{-1}\,|f,N-2\rangle = (L_{-1}L_1 + 2L_0)\,|f,N-2\rangle = (L_1L_1 + 2\,(\alpha(0)-2))\,|f,N-2\rangle$$

Hence

$$\tilde{L}_2\,|S,N\rangle = \{\,L_{-1}\tilde{L}_2 + [\,3\beta - 2\alpha + 4\,(\alpha(0)-1)\,\alpha\,]\,L_1\,\}\,|f,N-1\rangle$$

$$+ \{\,\tilde{L}_2^{+}L_2 + 4\,[-2\alpha^2 + 3\alpha\beta + 2\,(\alpha(0)-1)\,\alpha^2\,]\,L_{-1}L_1 +$$

$$+ [\,4\alpha^2 - 12\alpha\beta - 4\beta^2 + \tfrac{1}{2}D\beta^2 - 12\,(\alpha(0)-1)\,\alpha^2 + 12\,(\alpha(0)-1)\,\alpha\beta + 4\,(\alpha(0)-1)\,\beta^2$$

$$+ 8\,(\alpha(0)-1)^2\,\alpha^2\,]\,\}\,|f,N-2\rangle$$

In order to obtain a spurious state we must equalate to zero the coefficient of the term L_1. Then if $\alpha(0) = 1$ this is obtained with $\beta = 1$, $\alpha = 3/2$. Then

$$\tilde{L}_2 = L_2 + \frac{3}{2} L_1 L_1 \qquad (1)$$

and

$$\tilde{L}_2 \, |s, N\rangle = L_{-1} \tilde{L}_2 \, |f, N-1\rangle + \tilde{L}_2^{+} L_2 \, |f, N-2\rangle + \frac{1}{2}(D-26) \, |f, N-2\rangle \qquad (2)$$

Only for $D = 26$ we have $\tilde{L}_2 \, |s\, N\rangle = |s, N-2\rangle$ and we have proved that $\tilde{L}_2$ maps $S^N(1) \longrightarrow S^{N-2}(-1)$, as required. Note the peculiar way in which D enters through the anomaly term occurring in $[L_2, L_{-2}]$. If the anomaly term were absent, we could never obtain such a simple result. The result for $D = 26$ is that the two components, spurious and non-spurious, in our general physical state satisfy

$$L_m \, |s, N\rangle = L_m \, |k, N\rangle = 0 \qquad (3)$$

The fact that $|s, N\rangle$ is annihilated by the L_m implies that we have a state which is simultaneously spurious and physical; it is therefore a null state, that is

$$\langle s, N \,|\, s, N\rangle = 0 \qquad (4)$$

Concerning the state $|k, N\rangle$ we may write its most general form as

$$|k, N\rangle = \sum_{\{\mu\}} C[\mu] \, K_{-1}^{\mu_1} \cdots K_{-m}^{\mu_m} \, |t, N - \textstyle\sum m\mu_m\rangle \qquad (5)$$

Now acting with L_m is not difficult to show that $L_m \, |k, N\rangle = 0$ implies that $C[\mu] = 0$ $(\mu_m \neq 0)$ and therefore is a transversal state. Thus we have that for $\alpha(0) = 1$ and $D = 26$ any physical state can be written as

$$|p, N\rangle = |t, N\rangle + |\text{null state}\rangle \qquad (6)$$

The null state may be abandoned by redefinition since it does not alter the properties. Since $|t, N\rangle$ has been shown to have positive norm, this proves that all physical state have positive norm.

VII - A - APPENDIX

From (12.5)

$$Q^\mu(z) \equiv q^\mu - i(2\alpha')\, p^\mu \ln z - i(2\alpha')^{1/2} \sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} \left[a_m^{\mu +} z^m - a_m^\mu z^{-m} \right] \tag{1}$$

$$P^\mu(z) \equiv iz \frac{dQ^\mu(z)}{dz} = (2\alpha') p^\mu + (2\alpha')^{1/2} \sum_{m=1}^{\infty} \sqrt{m} \left[a_m^{\mu +} z^m + a_m^\mu z^{-m} \right]$$

Let us now introduce

$$Q^{(0)\mu}(z) \equiv q^\mu - i(2\alpha') p^\mu \ln z \tag{2}$$

$$Q^{(+)\mu}(z) \equiv -i(2\alpha')^{1/2} \sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} a_m^{\mu +} z^m \quad , \quad Q^{(-)\mu}(z) \equiv i(2\alpha')^{1/2} \sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} a_m^\mu z^{-m}$$

Then the only non-zero commutators are

$$\left[Q^{(0)\mu}(z) \, , \, Q^{(0)\nu}(z') \right] = \eta^{\mu\nu} (2\alpha') \ln(z/z') \tag{3}$$

$$\left[Q^{(+)\mu}(z) \, , \, Q^{(-)\nu}(z') \right] = \eta^{\mu\nu} (2\alpha') \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{z}{z'} \right)^m = -\eta^{\mu\nu} (2\alpha') \ln\left(1 - \frac{z}{z'}\right)$$

Hence

$$\left[Q^\mu(z) \, , \, Q^\nu(z') \right] = \eta^{\mu\nu} (2\alpha') \left\{ \ln \frac{z}{z'} + \sum_{\substack{m=-\infty \\ m \neq 0}}^{+\infty} \frac{1}{m} \left(\frac{z}{z'} \right)^m \right\} \tag{4}$$

If

$$z = e^{i\theta} \quad , \quad z' = e^{i\theta'} \tag{5}$$

we obtain

$$\left[Q^\mu(z) \, , \, Q^\nu(z') \right] = \eta^{\mu\nu} (2\alpha') \left\{ i(\theta - \theta') + \sum_{\substack{m=-\infty \\ m \neq 0}}^{+\infty} \frac{i}{m} e^{im(\theta - \theta')} \right\} \tag{6}$$

Let us now introduce

$$P^{(0)\mu}(z) = (2\alpha') p^\mu \tag{7}$$

$$P^{(+)\mu}(z) \equiv (2\alpha')^{1/2} \sum_{m=1}^{\infty} \sqrt{m}\, a_m^{\mu +} z^m \quad , \quad P^{(-)\mu}(z) \equiv (2\alpha')^{1/2} \sum_{m=1}^{\infty} \sqrt{m}\, a_m^\mu z^{-m}$$

Notice that

$$P^{(\alpha)\mu}(z) = i z \frac{d Q^{(\alpha)\mu}(z)}{dz} = \frac{d Q^{(\alpha)\mu}(z)}{d\theta} \qquad \alpha = 0, \pm \tag{1}$$

Then the only non-zero commutators between $Q^{(\alpha)\mu}(z)$ and $P^{(\beta)\nu}(z')$ are

$$[\, Q^{(0)\mu}(z) \,,\, P^{(0)\nu}(z') \,] = -i \eta^{\mu\nu} (2\alpha')$$

$$[\, Q^{(+)\mu}(z) \,,\, P^{(-)\nu}(z') \,] = -i \eta^{\mu\nu} (2\alpha') \sum_{m=1}^{\infty} e^{im(\theta-\theta')} \tag{2}$$

$$[\, Q^{(-)\mu}(z) \,,\, P^{(+)\nu}(z') \,] = -i \eta^{\mu\nu} (2\alpha') \sum_{m=1}^{\infty} e^{-im(\theta-\theta')}$$

and hence

$$[\, Q^{\mu}(z) \,,\, P^{\nu}(z') \,] = -i \eta^{\mu\nu} (2\alpha') \sum_{m=-\infty}^{+\infty} e^{im(\theta-\theta')} \tag{3}$$

Let us now remember that in the sense of distributions

$$\delta(\theta) = \frac{1}{2\pi} \sum_{m=-\infty}^{+\infty} e^{im\theta} \tag{4}$$

and therefore

$$[\, Q^{\mu}(z) \,,\, P^{\nu}(z') \,] = -2\pi i (2\alpha') \eta^{\mu\nu} \delta(\theta-\theta')$$

$$[\, P^{\mu}(z) \,,\, P^{\nu}(z') \,] = -2\pi i (2\alpha') \eta^{\mu\nu} \frac{\partial}{\partial\theta} \delta(\theta-\theta') \tag{5}$$

For any operator $A(z)$ we can write

$$A(z) = \sum_{m=-\infty}^{+\infty} \langle A(z) z^m \rangle z^{-m} \tag{6}$$

$$\langle O(z) \rangle \equiv \frac{1}{2\pi} \int_{-\pi}^{+\pi} d\theta \; O(e^{i\theta}) = \oint \frac{dz}{2\pi i z} O(z)$$

With this notation and (13.2)

$$L_m = \left\langle -\frac{1}{4\alpha'} z^m : P^2(z) : \right\rangle \tag{7}$$

Using this formalism we can now compute the commutators of L_m with $Q^{\mu}(z)$ and $P^{\mu}(z)$

$$[L_m, Q^\mu(z)] = \left\langle -\frac{1}{4\alpha'} z'^{m} [:P^2(z'):, Q^\mu(z)] \right\rangle =$$

$$= \left\langle -\frac{1}{2\alpha'} z'^{m} P_\nu(z') [P^\nu(z'), Q^\mu(z)] \right\rangle = \langle -2\pi i \delta(\theta-\theta') z^m P^\mu(z) \rangle =$$

$$= -i e^{+im\theta} P^\mu(z) = -i z^m P^\mu(z) \Rightarrow$$

$$[L_m, Q^\mu(z)] = z^m \left(z \frac{dQ^\mu(z)}{dz} \right) \tag{4}$$

$$[L_m, P^\mu(z)] = z^m \left(z \frac{d}{dz} + m \right) P^\mu(z)$$

Let us define the generalized projective spin S of an operator $O^S(z)$ by

$$[L_m, O^S(z)] = z^m \left(z \frac{d}{dz} + mS \right) O^S(z) \tag{2}$$

Then we see that $Q^\mu(z)$ is a generalized spin projective scalar ($S=0$) and $P^\mu(z)$ is a generalized projective vector ($S=1$).

In the future we will need frequently the operator

$$V(k;z) \equiv :\exp\left\{ -i \sum k_\mu \cdot Q^\mu(z) \right\}: \tag{3}$$

where k^μ is a vector with mass-dimensions. This can be written as

$$V(k;z) = \exp\left\{ -i \sum k_\mu Q^{(0)\mu}(z) \right\} \exp\left\{ -i \sum k_\mu Q^{(+)\mu}(z) \right\} \exp\left\{ -i \sum k_\mu Q^{(-)\mu}(z) \right\} =$$

$$= \lim_{z' \to z} \exp\left\{ -i \sum k_\mu Q^{(0)\mu}(z) \right\} \exp\left\{ -i \sum k_\mu Q^{(+)\mu}(z) \right\} \exp\left\{ -i \sum k_\mu Q^{(-)\mu}(z') \right\} \tag{5}$$

Let us now remember the Campbell-Hausdorff relation

$$e^A e^B = \exp\left\{ A + B + \frac{1}{2}[A,B] + \frac{1}{12}[[A,B],B] + \frac{1}{12}[[B,A],A] + \cdots \right\} \tag{5}$$

then using (1.3) we obtain immediately

$$V(k;z) = \lim_{z' \to z} \left(1 - \frac{z}{z'} \right)^{\alpha' k^2} \exp\left\{ -i \sum k_\mu Q^\mu(z) \right\} \tag{6}$$

Now we would like to calculate $[L_m, V(k;z)]$. Up to this end we consider

$$e^{\epsilon L_m} V(k;z)\, e^{-\epsilon L_m} = V(k,z) + \epsilon [L_m, V(k;z)] \qquad (1)$$

where terms of order $O(\epsilon^2)$ have been neglected. Let us now use (3.6)

$$e^{\epsilon L_m} V(k;z)\, e^{-\epsilon L_m} = \lim_{z'\to z} \left(1 - \frac{z}{z'}\right)^{\alpha' k^2} \exp\left\{ -i\, e^{\epsilon L_m} (k.Q(z))\, e^{-\epsilon L_m} \right\} =$$

$$= \lim_{z'\to z} \left(1 - \frac{z}{z'}\right)^{\alpha' k^2} \exp\left\{ -i\, k.Q(z) - i\, \epsilon [L_m, k Q(z)] \right\} =$$

$$= \lim_{z'\to z} \left(1 - \frac{z}{z'}\right)^{\alpha' k^2} \exp\left\{ -i\, k\, Q(z) - i\, \epsilon z^{m+1} \frac{d[k.Q(z)]}{dz} \right\} =$$

$$= \lim_{z'\to z} \left(1 - \frac{z}{z'}\right)^{\alpha' k^2} \exp\left\{ -i\, k.Q(z + \epsilon z^{m+1}) \right\} =$$

$$= \lim_{z'\to z} \left[\left(1 - \frac{z}{z'}\right)\left(1 - \frac{z+\epsilon z^{m+1}}{z'+\epsilon z'^{m+1}}\right)^{-1}\right]^{\alpha' k^2} \left(1 - \frac{z+\epsilon z^{m+1}}{z'+\epsilon z'^{m+1}}\right)^{\alpha' k^2} \exp\left\{ -i\, k.Q(z+\epsilon z^{m+1}) \right\}$$

Now let us consider

$$\lim_{z'\to z} \left\{ \frac{1 - \frac{z}{z'}}{1 - \frac{z+\epsilon z^{m+1}}{z'+\epsilon z'^{m+1}}} \right\}^{\alpha' k^2} = \lim_{z'\to z} \left\{ \frac{1}{1 - \epsilon \frac{z(z^m - z'^m)}{z'-z}} \right\}^{\alpha' k^2}$$

$$= \lim_{z'\to z} \left\{ 1 + \alpha' k^2 \epsilon \frac{z(z^m - z'^m)}{z'-z} \right\} = 1 - \epsilon\, \alpha' k^2 m z^m$$

Therefore

$$e^{\epsilon L_m} V(k;z)\, e^{-\epsilon L_m} = \left\{ 1 - \epsilon\, \alpha' k^2 m z^m \right\} V(k;\, z + \epsilon z^{m+1})$$

$$= V(k,z) - \epsilon\, \alpha' k^2 m z^m + \epsilon z^{m+1} \frac{dV(k;z)}{dz} \qquad (2)$$

The comparison of (1) and (2) gives the desired result

$$[L_m, V(k;z)] = z^m \left(z \frac{d}{dz} - m\, \alpha' k^2 \right) V(k;z) \qquad (3)$$

This result may be summarized by stating that $V(k;z)$ transforms as a generalized projective spin $S = -\alpha' k^2$

We will need still some further relations. Let us prove that for any $f(x)$ we can write

$$L_m \, f(L_0) = f(L_0 + m) \, L_m \tag{1}$$

Proof:

$[L_0, L_m] = -m L_m \Rightarrow \quad L_m L_0 = (L_0 + m) L_m$

$L_m L_0^2 = (L_0 + m) L_m L_0 = (L_0 + m)^2 L_m$

and in general

$$L_m L_0^k = (L_0 + m)^k L_m \tag{2}$$

and from here equation (1) follows.

Another interesting relation is

$$V(k; z) = z^{L_0} V(k; 1) z^{-L_0} \tag{3}$$

Which is clearly true for $z = 1$. This can be written as

$$V(k; z) = \exp\{L_0 \ln z\} V(k; 1) \exp\{-L_0 \ln z\} \tag{4}$$

and hence

$$\frac{dV(k; z)}{dz} = \frac{1}{z} L_0 V(k; z) - \frac{1}{z} V(k, z) L_0 = \frac{1}{z} [L_0, V(k, z)]$$

which coincides with (4.3).

Furthermore if $\alpha' k^2 = -1$ equation (4.3) can be written as

$$[L_m, z^m V(k, z)] = z^{m+1} \frac{d}{dz} [z^m V(k, z)] \quad , \quad \alpha' k^2 = -1 \tag{5}$$

This equation implies that for small λ

$$e^{\lambda L_m} [z^m V(kz)] e^{-\lambda L_m} = \left\{1 + \lambda z^{m+1} \frac{d}{dz}\right\} [z^m V(k, z)] + O(\lambda^2)$$

$$= z'^m V(k, z'^m) + O(\lambda^2) \tag{6}$$

where

$$z' = z + \lambda z^{m+1} + O(\lambda^2) \tag{7}$$

Equation (7) only gives z' to first order in λ. However, z' is uniquely extended to finite λ as a consequence of the group property by integrating the equation

$$\frac{dz'}{d\lambda} = (z')^{m+1} \tag{8}$$

The solution is

$$m = 0 \qquad z' = e^{\lambda} z \qquad , \qquad m \neq 0 \qquad z' = z\,(1 - m\lambda\, y^m)^{-1/m} \tag{4}$$

and hence

$$e^{\lambda L_m}\, [\, z^m\, V(k, z)\,]\, e^{-\lambda L_m} = z'^m\, V(k, z'^m) \qquad , \quad \alpha' k^2 = -1$$

$$z' = e^{\lambda} z \quad \text{if} \quad m = 0 \tag{2}$$

$$z' = z\,(1 - m\lambda z^m)^{-1/m} \quad \text{if} \quad m \neq 0$$

Another interesting property is the following one. Let us consider

$$V(k_1, z_1)\, V(k_2 z_2) = \exp\{-i k_1 Q^{(+)}(z_1)\} \exp\{-i k_1 Q^{(0)}(z_1)\} \exp\{-i k_1 Q^{(-)}(z_1)\}.$$

$$\exp\{-i k_2 Q^{(+)}(z_2)\} \exp\{-i k_2 Q^{(0)}(z_2)\} \exp\{-i k_2 Q^{(-)}(z_2)\} = \tag{3}$$

but

$$e^{-i k_1 Q^{(-)}(z_1)}\, e^{-i k_2 Q^{(+)}(z_2)} = e^{-i k_2 Q^{(+)}(z_2)}\, e^{-i k_1 Q^{(-)}(z_1)} \left(1 - \frac{z_2}{z_1}\right)^{-2\alpha'(k_1 k_2)} \tag{4}$$

$$e^{-i k_1 Q^{(0)}(z_1)}\, e^{-i k_2 Q^{(0)}(z_2)} = e^{-i k_2 Q^{(0)}(z_2)}\, e^{-i k_1 Q^{(0)}(z_1)} \left(\frac{z_1}{z_2}\right)^{-2\alpha'(k_1 k_2)}$$

Hence

$$V(k_1, z_1)\, V(k_2, z_2) = e^{-i k_2 Q^{(+)}(z_2)}\, e^{-i k_1 Q^{(+)}(z_1)}\, e^{-i k_1 Q^{(0)}(z_1)}\, e^{-i k_2 Q^{(0)}(z_2)}\, e^{-i k_2 Q^{(-)}(z_2)}.$$

$$\cdot e^{-i k_1 Q^{(-)}(z_1)} \left(1 - \frac{z_2}{z_1}\right)^{-2\alpha'(k_1 k_2)} =$$

$$= e^{-i k_2 Q^{(+)}(z_2)}\, e^{-i k_2 Q^{(0)}(z_2)}\, e^{-i k_1 Q^{(+)}(z_1)}\, e^{-i k_2 Q^{(-)}(z_2)}\, e^{-i k_1 Q^{(0)}(z_1)}\, e^{-i k_1 Q^{(-)}(z_1)}$$

$$\cdot \left(1 - \frac{z_2}{z_1}\right)^{-2\alpha'(k_1 k_2)} \left(\frac{z_1}{z_2}\right)^{-2\alpha'(k_1 k_2)} =$$

$$= V(k_2 z_2)\, V(k_1 z_1) \left(1 - \frac{z_2}{z_1}\right)^{-2\alpha'(k_1 k_2)} \left(\frac{z_1}{z_2}\right)^{-2\alpha'(k_1 k_2)} \left(1 - \frac{z_1}{z_2}\right)^{2\alpha'(k_1 k_2)} \Rightarrow$$

$$V(k_1, z_1)\, V(k_2 z_2) = V(k_2 z_2)\, V(k_1 z_1)\, (-1)^{2\alpha'(k_1 k_2)} \tag{5}$$

and a careful study of $z_1 \gtrless z_2$

$$V(k_1 z_1)\, V(k_2 z_2) = V(k_2 z_2)\, V(k_1 z_1)\, e^{-2\pi i \alpha'(k_1 k_2)\, \varepsilon(z_1 - z_2)} \tag{6}$$

VII-B- COVARIANT QUANTIZATION BASED ON BRS INVARIANCE

Let us consider the Lagrangian density for the bosonic string in D-dimensions. From (II-10.1)

$$L_0' = -\frac{1}{4\pi\alpha'}\ \bar{g}^{\alpha\beta}(\xi)\,\partial_\alpha X^\mu(\xi)\,\partial_\beta X_\mu(\xi)$$

$$(\xi^0, \xi') \equiv (\tau, \sigma) \tag{1}$$

$$\bar{g}^{\alpha\beta}(\xi) \equiv g^{\alpha\beta}(\xi)\ /\sqrt{-\det g(\xi)}$$

where the metric tensor has only two degrees of freedom since

$$\det \bar{g}(\xi) = -1 \tag{2}$$

For simplicity of calculation we regard $\bar{g}^{\alpha\beta}$ as possessing three modes and introducing a multiplier field E. We have

$$L_0 = -\frac{1}{4\pi\alpha'}\ \bar{g}^{\alpha\beta}(\xi)\,\partial_\alpha X^\mu(\xi)\,\partial_\beta X_\mu(\xi) + E(\xi)\,[\det \bar{g}(\xi) + 1] \tag{3}$$

Let us now consider an infinitesimal coordinate transformation

$$\xi^\alpha \longrightarrow \xi'^\alpha = \xi^\alpha + \epsilon^\alpha(\xi) \tag{4}$$

We can calculate the transformation laws of the terms appearing in (3) under the assumption that $X^\mu(\xi)$ is a scalar and $E(\xi)$ a scalar density.

$$\delta X^\mu(\xi) \equiv X'^\mu(\xi) - X^\mu(\xi) = X'^\mu(\xi' - \epsilon) - X^\mu = X'^\mu(\xi') - X^\mu(\xi) - \epsilon^\alpha\,\partial_\alpha X^\mu(\xi) \Rightarrow$$

$$\delta X^\mu(\xi) = -\epsilon^\alpha(\xi)\,\partial_\alpha X^\mu(\xi) \tag{5}$$

$$\delta g^{\alpha\beta}(\xi) \equiv g'^{\alpha\beta}(\xi) - g^{\alpha\beta}(\xi) = g'^{\alpha\beta}(\xi') - g^{\alpha\beta}(\xi) - \epsilon^\lambda(\xi)\,\partial_\lambda g^{\alpha\beta}(\xi) =$$

$$= \frac{\partial\xi'^\alpha}{\partial\xi^\rho}\,\frac{\partial\xi'^\beta}{\partial\xi^\sigma}\,g^{\rho\sigma}(\xi) - g^{\alpha\beta}(\xi) - \epsilon^\lambda(\xi)\,\partial_\lambda g^{\alpha\beta}(\xi) =$$

$$= [\,\delta^\alpha_\rho + \partial_\rho\epsilon^\alpha\,]\,[\,\delta^\beta_\sigma + \partial_\sigma\epsilon^\beta\,]\,g^{\rho\sigma} - g^{\alpha\beta} - \epsilon^\lambda\,\partial_\lambda g^{\alpha\beta} =$$

$$= \partial_\rho\epsilon^\alpha\,\delta^\beta_\sigma\,g^{\rho\sigma} + \partial_\sigma\epsilon^\beta\,\delta^\alpha_\rho\,g^{\rho\sigma} - \epsilon^\lambda\,\partial_\lambda g^{\alpha\beta} =$$

$$= \partial_\rho\epsilon^\alpha\,g^{\rho\beta} + \partial_\sigma\epsilon^\beta\,g^{\alpha\sigma} - \epsilon^\lambda\,\partial_\lambda g^{\alpha\beta}$$

$$\delta g^{\alpha\beta}(\xi) = g^{\lambda\beta}(\xi)\,\partial_\lambda\epsilon^\alpha(\xi) + g^{\alpha\lambda}(\xi)\,\partial_\lambda\epsilon^\beta(\xi) - \epsilon^\lambda(\xi)\,\partial_\lambda g^{\alpha\beta}(\xi) \tag{6}$$

$$\delta \det g = g^{11}\,\delta g^{22} + g^{22}\,\delta g^{11} - 2g^{12}\,\delta g^{12} =$$

$$= g^{11}\,[2g^{\rho 2}\partial_\rho\epsilon^2 + \epsilon^\rho\partial_\rho g^{22}] + g^{22}\,[2g^{\rho 1}\partial_\rho\epsilon^1 - \epsilon^\rho\partial_\rho g^{11}] - 2g^{12}\,[\,g^{\rho 2}\partial_\rho\epsilon^1 +$$

$$+ g^{1\rho}\partial_\rho\epsilon^2 - \epsilon^\rho\partial_\rho g^{12}] =$$

$$= 2\,[\,g^{1\alpha}\,g^{22} - g^{12}\,g^{\alpha 2}\,]\,\partial_\alpha \epsilon^1 + 2\,[\,g^{11}\,g^{\alpha 2} - g^{12}\,g^{1\alpha}\,]\,\partial_\alpha \epsilon^2$$

$$- g^{11}\,\epsilon^\alpha\,\partial_\alpha\, g^{22} - g^{22}\,\epsilon^\alpha\,\partial_\alpha\, g^{11} + 2\,g^{12}\,\epsilon^\alpha\,\partial_\alpha\, g^{12} \Rightarrow$$

$$\delta \det g\,(\xi) = 2 \det g\,(\xi)\,\partial_\alpha\,\epsilon^\alpha(\xi) - \epsilon^\alpha(\xi)\,\partial_\alpha\,[\det g\,(\xi)] \tag{1}$$

$$\delta\,\bar g^{\alpha\beta}(\xi) = \delta\,\{\sqrt{-\det g(\xi)}\;\; g^{\alpha\beta}(\xi)\,\} = \sqrt{-\det g\,(\xi)}\;\,\delta g^{\alpha\beta}(\xi) - \frac{1}{2}\,\frac{\delta[\det g\,(\xi)]}{\sqrt{-\det g(\xi)}}\; g^{\alpha\beta}(\xi) =$$

$$= \bar g^{\gamma\beta}\,\partial_\gamma\,\epsilon^\alpha + \bar g^{\alpha\gamma}\,\partial_\gamma\,\epsilon^\beta - \sqrt{-\det g}\;\,\epsilon^\gamma\,\partial_\gamma\, g^{\alpha\beta} - \bar g^{\alpha\beta}\,\partial_\gamma\,\epsilon^\gamma +$$

$$+ \frac{1}{2}\,\frac{1}{\sqrt{-\det g}}\; g^{\alpha\beta}\,\epsilon^\gamma\,\partial_\gamma\,[\det g\,] \Rightarrow$$

$$\delta\,\bar g^{\alpha\beta}(\xi) = [\,(\xi)\,\partial_\mu \bar g^{\alpha\beta}\,(\xi)\,\epsilon^\mu(\xi) - \partial_\gamma\,[\epsilon^\alpha(\xi)\,\bar g^{\gamma\beta}(\xi) + \epsilon^\beta(\xi)\,\bar g^{\alpha\gamma}(\xi)\,]\,\partial_\mu \epsilon^\gamma(\xi)\,]\,\delta\xi^\mu \tag{2}$$

$$\delta[\det \bar g\,(\xi)\,] = -\,\epsilon^\mu(\xi)\,\partial_\mu\,[\det \bar g\,(\xi)\,] \tag{3}$$

$$\delta E(\xi) = E'(\xi' - \epsilon) - E(\xi) = \left|\det\left(\frac{\partial \xi'^\alpha}{\partial \xi^\beta}\right)\right|^{-1} E(\xi) - E(\xi) - \epsilon^\mu\,\partial_\mu E \Rightarrow$$

$$\delta E(\xi) = -\,\partial_\mu\,[\,\epsilon^\mu(\xi)\,E(\xi)\,] \tag{5}$$

From all that

$$\delta\,\bar\xi^\mu(\xi) = -\,\epsilon^\alpha(\xi)\,\partial_\alpha\,\bar\xi^\mu(\xi)$$

$$\delta\,\bar g^{\alpha\beta}(\xi) = [\,(\xi)\,\partial_\mu \bar g^{\alpha\beta}\,(\xi)\,\epsilon^\mu(\xi)\,] - \partial_\gamma\,[\,\epsilon^\alpha(\xi)\,\bar g^{\gamma\beta}(\xi) + \epsilon^\beta(\xi)\,\bar g^{\alpha\gamma}(\xi)\,] \tag{5}$$

$$\delta\,[\det \bar g\,(\xi)\,] = -\,\epsilon^\mu(\xi)\,\partial_\mu\,[\det \bar g\,(\xi)\,]$$

$$\delta E(\xi) = -\,\partial_\mu\,[\,\epsilon^\mu(\xi)\,E(\xi)\,]$$

and similarly for $V_\alpha(\xi)$

$$\delta V_\alpha(\xi) = -\,V_\beta(\xi)\,\partial_\alpha\,\epsilon^\beta(\xi) - \epsilon^\beta(\xi)\,\partial_\beta\,V_\alpha(\xi) \tag{6}$$

Now we have

$$\delta\,[\,\bar g^{\alpha\beta}\,\partial_\alpha \bar\xi^\mu\,\partial_\beta \bar\xi^\nu\,] = [\,\partial_\lambda \bar g^{\alpha\beta}\,\epsilon^\lambda - \bar g^{\alpha\lambda}\,\partial_\lambda \epsilon^\beta - \bar g^{\beta\lambda}\,\partial_\lambda \epsilon^\alpha\,]\,\partial_\alpha \bar\xi^\mu\,\partial_\beta \bar\xi^\nu -$$

$$- \bar\xi^\mu\,\partial_\alpha\,[\,(\partial_\beta \bar g^{\lambda\gamma})\,\partial_\lambda \bar\xi^\nu\,\partial_\gamma \bar\xi^\beta + \bar g^{\lambda\gamma}\,\partial_\lambda \partial_\beta \bar\xi^\nu\,\partial_\gamma \bar\xi^\beta + \bar g^{\lambda\gamma}\,\partial_\lambda \bar\xi^\nu\,\partial_\beta \partial_\gamma \bar\xi^\beta\,] \tag{7}$$

$$= -\,[\,\bar\xi^\mu\,\partial_\alpha \partial_\beta \bar\xi^\nu\,\partial_\lambda \epsilon^\lambda + \bar\xi^\mu\,\partial_\alpha \partial_\beta \bar\xi^\lambda\,\partial_\lambda \epsilon^\nu + \bar\xi^\nu\,\partial_\alpha \partial_\beta \bar\xi^\lambda\,\partial_\lambda \epsilon^\mu\,]\,\bar g^{\alpha\beta} -$$

$$- \bar\xi^\mu\,\partial_\alpha \bar\xi^\nu\,\partial_\beta \bar\xi^\lambda\,\partial_\lambda\,\bar g^{\alpha\beta} - \bar\xi^\mu\,\partial_\alpha \bar\xi^\lambda\,\partial_\lambda \bar\xi^\nu\,\partial_\beta\,\bar g^{\alpha\beta} - \bar\xi^\nu\,\partial_\alpha \bar\xi^\lambda\,\partial_\lambda \bar\xi^\mu\,\partial_\beta\,\bar g^{\alpha\beta}$$

$$\delta [E (\det \bar{g} + 1)] = - \partial_\alpha [\epsilon^\alpha E] (\det \bar{g} + 1) - E \epsilon^\alpha \partial_\alpha (\det \bar{g}) = - \partial_\alpha [\epsilon^\alpha E (\det \bar{g} + 1)] \quad (1)$$

Hence using this two last results

$$\delta L_0 = - \partial_\alpha [\epsilon^\alpha (\xi) L_0 (\xi)] \quad (2)$$

This indicates the invariance of the action if $\epsilon' = 0$ at the boundary $\sigma = 0, \pi$.

The B.R.S transformation is defined by replacing the parameter $\epsilon^\alpha(\xi)$ with $\lambda C^\alpha(\xi)$ in eq. (2.5), where $C^\alpha(\xi)$ is the Faddeev-Popov ghost field and λ is an anticommuting c-number parameter.

$$\delta X^\mu (\xi) = - \lambda C^\alpha (\xi) \partial_\alpha X^\mu (\xi)$$

$$\delta \bar{g}^{\alpha\beta} (\xi) = \lambda \bar{g}^{\gamma\beta} (\xi) \partial_\gamma C^\alpha (\xi) + \lambda \bar{g}^{\alpha\gamma} (\xi) \partial_\gamma C^\beta (\xi) - \lambda \partial_\gamma [C^\gamma (\xi) \bar{g}^{\alpha\beta} (\xi)]$$

$$\delta [\det \bar{g} (\xi)] = - \lambda C^\alpha (\xi) \partial_\alpha [\det \bar{g} (\xi)] \quad (3)$$

$$\delta E (\xi) = - \lambda \partial_\alpha [C^\alpha (\xi) E (\xi)]$$

$$\delta [\partial_\alpha X^\mu (\xi)] = - \lambda \partial_\beta X^\mu (\xi) \partial_\alpha C^\beta (\xi) - \lambda C^\beta (\xi) \partial_\beta \partial_\alpha X^\mu (\xi)$$

We next determine $\delta C^\alpha(\xi)$ so that the BRS transformation has the nilpotency property; i.e. if we define $\delta \equiv \lambda \tilde{\delta}$, then $\tilde{\delta} [\tilde{\delta} X^\mu] = 0$.

$$\tilde{\delta} [\tilde{\delta} X^\mu (\xi)] = \tilde{\delta} [- C^\alpha (\xi) \partial_\alpha X^\mu (\xi)] = - [\tilde{\delta} C^\alpha (\xi)] \partial_\alpha X^\mu (\xi) + C^\alpha (\xi) [\tilde{\delta} \partial_\alpha X^\mu (\xi)]$$

$$= - \tilde{\delta} C^\alpha (\xi) \partial_\alpha X^\mu (\xi) - C^\alpha (\xi) \partial_\beta X^\mu (\xi) \partial_\alpha C^\beta (\xi) - C^\alpha (\xi) C^\beta (\xi) \partial_\beta \partial_\alpha X^\mu (\xi) = 0$$

$$[\tilde{\delta} C^\alpha (\xi) + C^\beta (\xi) \partial_\beta \epsilon^\alpha (\xi)] \partial_\alpha X^\mu (\xi) = 0 \quad (4)$$

and hence

$$\delta C^\alpha (\xi) = - \lambda C^\beta (\xi) \partial_\beta C^\alpha (\xi) \quad (5)$$

It is possible to check the nilpotency for the other fields, including the ghost field.

Finally, following T. Kugo and S. Uehara [Nucl. Phys. B197, 378-384 (1982)] we introduce Faddeev-Popov antighost fields $\bar{C}_\alpha(\xi)$ and so-called $B_\alpha(\xi)$-fields, the BRS transformation of which is defined as

$$\delta \bar{C}_\alpha(x) = i\lambda B_\alpha(x) \quad , \quad \delta B_\alpha(x) = 0 \tag{1}$$

As we will see below, the B-fields play the role of Lagrange multipliers for gauge fixing, the importance of which was first recognized by N. Nakanishi [Prog. Theor. Phys. 51, 1 (1972)] in Q.E.D. We should here note that the BRS transformation is defined, by eqs. (3.3), (3.5) and (1), prior to any particular choice of gauge-fixing conditions. This was made possible by virtue of the introduction of the B-fields.

The BRS transformation thus constructed satisfies nilpotency $\delta^2 = 0$. If we introduce the BRS charge Q_B by

$$\lambda \tilde{\delta}(*) = [\, i\lambda Q_B \,,\, * \,] \tag{2}$$

the nilpotency condition can be written

$$\{ Q_B, Q_B \} = 2Q_B^2 = 0 \tag{3}$$

We first recall that physical states are defined as BRS-invariant ones

$$Q_B |\, phys \rangle = 0 \tag{4}$$

so that any operators of the form $\tilde{\delta}(*)$ are null operators on the subspace of the physical states. So we can freely add such a null operator to the original Lagrangian without changing physics, and use it as a gauge-fixing and Faddeev-Popov ghost Lagrangian density.

Our choice of gauge in the following is of axial type. Lorentz covariance of the theory is manifest in this gauge because we impose the gauge condition only on $\bar{g}^{\alpha\beta}$ as

$$\bar{g}^{\alpha\beta}(x) = \begin{vmatrix} A_1(x) + A_2(x) & A_0(x) \\ A_0(x) & A_1(x) - A_2(x) \end{vmatrix} \tag{5}$$

Then we take as gauge condition

$$A_0(x) = A_1(x) \equiv 0 \tag{6}$$

Together with the equation of motion $\det \bar{g} = -1$, the above condition implies $\bar{g}^{\alpha\beta}(x) = \eta^{\alpha\beta}$ [$\eta^{\alpha\beta} = \mathrm{diag}(1, -1)$].

Then we can give the sum of the gauge-fixing term and the Faddeev-Popov ghost term concisely as

$$\lambda (L_{GF} + L_{FP}) = -i\lambda \tilde{\delta} \left(\bar{C}_0 A_0 + \bar{C}_1 A_1 \right) \tag{1}$$

The explicit form of these additional terms is

$$L_{GF} = \left[B_0(\xi) A_0(\xi) + B_1(\xi) A_1(\xi) \right]$$

$$L_{FP} = i \left\{ \bar{C}_0(\xi) \left[- C^\alpha(\xi) \partial_\alpha A_\beta(\xi) + \left(\partial_1 C^0(\xi) + \partial_0 C^1(\xi) \right) A_1(\xi) \right. \right. \tag{2}$$

$$\left. - \left(\partial_1 C^0(\xi) - \partial_0 C^1(\xi) \right) A_2(\xi) \right] + \bar{C}_1(\xi) \left[\left(\partial_1 C^0(\xi) + \partial_0 C^1(\xi) \right) A_0(\xi) \right.$$

$$\left. \left. - C^\alpha(\xi) \partial_\alpha A_1(\xi) + \left(\partial_0 C^0(\xi) - \partial_1 C^1(\xi) \right) A_2(\xi) \right] \right\}$$

where we have taken into account the transformation laws for δA_0 and δA_1 deduced from (3.3). Due to the nilpotency of the BRS transformation the Lagrangian density added (1) is manifestly BRS invariant. Therefore the variation of the action under BRS transformation is simply given by replacing the parameter $\epsilon^\alpha(\xi)$ by $\lambda C^\alpha(\xi)$ in eq (3.2)

$$\delta S = - \lambda \int d^2\xi \, \partial_\alpha \left[C^\alpha(\xi) L_0(\xi) \right] \tag{3}$$

where the BRS invariance of the action is assured if we impose the following boundary conditions on the ghost field

$$C^1(\xi) = 0 \qquad \text{at} \qquad \sigma = 0, \pi \tag{4}$$

The invariance of this boundary condition under the BRS transformation requires further (see (3.5))

$$\partial_0 C^1(\xi) = 0 \qquad \text{at} \qquad \sigma = 0, \pi \tag{5}$$

Then the total invariant Lagrangian density is

$$L(\xi) = - \frac{1}{4\pi\alpha'} \left\{ (A_1 + A_2) \partial_0 X^\mu \partial_0 X_\mu + (A_1 - A_2) \partial_1 X^\mu \partial_1 X_\mu + 2 A_0 \partial_0 X^\mu \partial_1 X_\mu \right\}$$

$$+ E \left\{ A_1^2 + A_2^2 - A_0^2 + 1 \right\} + B_0 A_0 + B_1 A_1$$

$$+ i \bar{C}_0 \left[- C^\alpha \partial_\alpha A_0 + (\partial_1 C^0 + \partial_0 C^1) A_1 - (\partial_1 C^0 - \partial_0 C^1) A_2 \right]$$

$$+ i \bar{C}_1 \left[(\partial_1 C^0 + \partial_0 C^1) A_0 - C^\alpha \partial_\alpha A_1 + (\partial_0 C^0 - \partial_1 C^1) A_2 \right] \tag{6}$$

This rather cumbersome form of the total Lagrangian density can be largely simplified by shifting the auxiliary fields

$$B_0 \longrightarrow B_0 + \frac{1}{4\pi\alpha'}\left\{\partial_0 \xi^\mu \partial_1 \xi_\mu + \partial_1 \xi^\mu \partial_0 \xi_\mu\right\} + E A_0 - i\partial_\alpha\left[\bar{c}_0 c^\alpha\right]$$
$$- i\bar{c}_1\left[\partial_1 c^0 + \partial_0 c^1\right]$$

$$B_1 \longrightarrow B_1 + \frac{1}{4\pi\alpha'}\left\{\partial_0 \xi^\mu \partial_0 \xi_\mu + \partial_1 \xi^\mu \partial_1 \xi_\mu\right\} + E A_1 - i\partial_\alpha\left[\bar{c}_1 c^\alpha\right] \tag{1}$$
$$- i\bar{c}_0\left[\partial_1 c^0 + \partial_0 c^1\right]$$

$$E \longrightarrow E + \frac{1}{1+A_2}\left\{-\frac{1}{4\pi\alpha'}\left[\partial_0 \xi^\mu \partial_0 \xi_\mu - \partial_1 \xi^\mu \partial_1 \xi_\mu\right]\right.$$
$$\left. - i\bar{c}_0\left[\partial_1 c^0 - \partial_0 c^1\right] + i\bar{c}_1\left[\partial_0 c^0 - \partial_1 c^1\right]\right\}$$

and we obtain

$$L(\xi) = \frac{1 + A_2 + A_1^2 - A_0^2}{1 + A_2}\left\{-\frac{1}{4\pi\alpha'}\eta^{\alpha\beta}\partial_\alpha \xi^\mu \partial_\beta \xi_\mu - i\bar{c}_\alpha \partial^\alpha{}_\beta c^\beta\right\}$$
$$+ E(1 - A_2^2) + B_0 A_0 + B_1 A_1 - i\partial_\alpha(A_0 \bar{c}_0 c^\alpha) - i\partial_\alpha(A_1 \bar{c}_1 c^\alpha) \tag{2}$$

where

$$\partial^\alpha{}_\beta = \begin{vmatrix} \partial_1 & -\partial_0 \\ -\partial_0 & \partial_1 \end{vmatrix} \tag{3}$$

The last two terms can be eliminated. Furthermore in our gauge $1 + A_2 + A_1^2 - A_0^2 = 1 + A_2$ since $A_1 = A_0 = 0$. Hence the original Lagrangian density can be substituted by

$$L(\xi) = -\frac{1}{4\pi\alpha'}\eta^{\alpha\beta}\partial_\alpha \xi^\mu(\xi)\partial_\beta \xi_\mu(\xi) - i\bar{c}_\alpha(\xi)\partial^\alpha{}_\beta c^\beta(\xi)$$
$$+ E(\xi)\left[1 - A_2^2(\xi)\right] + B_0(\xi) A_0(\xi) + B_1(\xi) A_1(\xi) \tag{4}$$

Obviously the BRS transformations of various variables are altered according to the above shift of auxiliary fields.

The canonical quantization of (4) is straightforward. We proceed as follows. First we give the mode expansion of each variable and then de-

termine the canonical commutation relations of each variable employing Dirac's method of quantization.

To carry out these steps, we apply the variational principle to the action

$$\delta S = \int d^2\xi\, \delta L = \int d^2\xi \left\{ \frac{1}{2\pi\alpha'} \partial_\alpha \left[\eta^{\alpha\beta} \partial_\beta X^\mu \right] \delta X_\mu - i\, \delta \bar{c}_\alpha \left[\partial^\alpha{}_\beta\, c^\beta \right] \right.$$

$$\left. + i \left[\partial^a{}_b\, \bar{c}_a \right] \delta c^b + \delta E \left[1 - A_2^2 \right] - 2E A_2 \delta A_2 + \delta B_0 A_0 + B_0 \delta A_0 + \delta B_1 A_1 + B_1 \delta A_1 \right\} +$$

$$+ \int d\tau \left\{ \frac{1}{2\pi\alpha'} \partial_1 X^\mu \delta X_\mu - i\, \bar{c}_0 \delta c^0 - i\, \bar{c}_1 \delta c^1 \right\}_{\sigma=0}^{\sigma=\pi} \qquad (1)$$

From (1) we get the equations of motion

$$(\partial_0^2 - \partial_1^2)\, X^\mu(\xi) = 0 \quad , \quad \partial^\alpha{}_\beta\, c^\beta(\xi) = 0 \quad , \quad \partial^\alpha{}_\beta\, \bar{c}_\alpha(\xi) = 0$$

$$A_0(\xi) = A_1(\xi) = A_2^2(\xi) - 1 = B_0(\xi) = B_1(\xi) = E(\xi) = 0 \qquad (2)$$

and the boundary conditions

$$\partial_1 X^\mu(\xi) = 0 \quad , \quad \bar{c}_0(\xi) = 0 \quad , \quad c^1(\xi) = 0 \qquad \text{at} \quad \sigma = 0, \pi \qquad (3)$$

Then we can derive the mode expansion of the relevant variables

$$X^\mu(\sigma,\tau) = q^\mu + \alpha_0^\mu \tau - i\sqrt{2\alpha'} \sum_{m=1}^{\infty} \frac{1}{\sqrt{m}} \left[a_m^\mu{}^+ e^{+im\tau} - a_m^\mu e^{-im\tau} \right] \cos m\sigma$$

$$c^0(\sigma,\tau) = \frac{1}{\sqrt{\pi}} c_0 + \frac{1}{\sqrt{\pi}} \sum_{m=1}^{\infty} \left[c_m^+ e^{im\tau} + c_m e^{-im\tau} \right] \cos m\sigma$$

$$c^1(\sigma,\tau) = + \frac{i}{\sqrt{\pi}} \sum_{m=1}^{\infty} \left[c_m^+ e^{im\tau} - c_m e^{-im\tau} \right] \sin m\sigma \qquad (4)$$

$$\bar{c}_0(\sigma,\tau) = \frac{i}{\sqrt{\pi}} \sum_{m=1}^{\infty} \left[\bar{c}_m^+ e^{im\tau} - c_m e^{-im\tau} \right] \sin m\sigma$$

$$\bar{c}_1(\sigma,\tau) = \frac{1}{\sqrt{\pi}} \bar{c}_0 + \frac{1}{\sqrt{\pi}} \sum_{m=1}^{\infty} \left[\bar{c}_m^+ e^{im\tau} + \bar{c}_m e^{-im\tau} \right] \cos m\sigma$$

Notice here that constant modes (zero-frequency modes) are present also for Faddeev-Popov ghost fields $c^0(\xi)$, $\bar{c}_1(\xi)$. Since Faddeev-Popov ghost fields $c^\alpha(\xi)$, $\bar{c}_\alpha(\xi)$ are hermitian, so the constant modes: $c_0 = c_0^+$, $\bar{c}_0 = \bar{c}_0^+$.

In order to determine the canonical commutation relations, we write down the conjugate momenta for each canonical variable

$$\Pi_{X_\mu} = - \frac{1}{2\pi\alpha'} \partial_0 X_\mu \quad , \quad \Pi_{c_0} = - i\, \bar{c}_1 \quad , \quad \Pi_{c_1} = -i\, \bar{c}_0 \qquad (5)$$

$$\Pi_{c^0} = \Pi_{\bar{c}^1} = \Pi_{A_0} = \Pi_{A_1} = \Pi_{A_2} = \Pi_{B_0} = \Pi_{B_1} = \Pi_E = 0 \qquad (6)$$

Equations (7.6) express the primary constraints. Applying Dirac's method of quantization, it can be easely seen that all constraints are of second class and that the redundant variables A_0, A_1, A_2, $\bar{C}_\alpha$, B_α and E can be eliminated from the theory. At this stage the Hamiltonian and the commutation relations are given by

$$\mathcal{H} = -\frac{1}{4\pi\alpha'} \partial_0 X^\mu \partial_0 X_\mu - \frac{1}{4\pi\alpha'} \partial_1 X^\mu \partial_1 X_\mu + i\bar{c}_0 \partial_1 c^0 + i\bar{c}_1 \partial_1 c^1 \tag{1}$$

$$\left[X^\mu(\tau,\sigma) , \dot{X}^\nu(\tau,\sigma') \right] = -i 2\pi\alpha' \eta^{\mu\nu} \delta(\sigma-\sigma')$$

$$\left\{ c^\alpha(\tau,\sigma) , \bar{c}_\beta(\tau,\sigma') \right\} = (1-\delta^\alpha_\beta)\,\delta(\sigma-\sigma') \tag{2}$$

Hence we get for the canonical commutation relations of mode variables

$$[q^\mu, p^\nu] = -i\eta^{\mu\nu} \quad , \quad [a^\mu_m, a^{\nu\,+}_m] = -\eta^{\mu\nu}\delta_{m,m}$$

$$\{\bar{c}_0, c_0\} = 1 \quad , \quad \{\bar{c}_m, c^+_m\} = \{c_m, \bar{c}^+_m\} = \delta_{mm} \tag{3}$$

We also obtain the mode expansion of the Hamiltonian

$$H = -\alpha' p^\mu p_\mu - \sum_{m=1}^{\infty} m\, a^{\mu\,+}_m a_{m\mu} + \sum_{m=1}^{\infty} m \left[\bar{c}^+_m c_m + c^+_m \bar{c}_m \right] \tag{4}$$

In order to define the quantum theory with indefinite metric, we must first construct the Fock space and then specify the physical subspace. As usual, the ground state of the Fock space $|0,p\rangle$ is defined by

$$p^\mu |0,p\rangle = p^\mu |0,p\rangle$$

$$a^\mu_m |0,p\rangle = c_m |0,p\rangle = \bar{c}_m |0,p\rangle = 0 \quad , \quad m \geq 1 \tag{5}$$

On the other hand, the constant modes of the ghost field require special consideration. For representation with respect to the constant mode, we take $|+\rangle$ as a state annihilated by c_0:

$$c_0 |+\rangle = 0 \tag{6}$$

Then

$$|-\rangle \equiv \bar{c}_0 |+\rangle \tag{7}$$

is such that $\bar{c}_0 |-\rangle = 0$. We use notations like $|\varphi\rangle_\pm$ indicating the

states $|\varphi\rangle \otimes |\pm\rangle$, where $|\varphi\rangle$ denotes the state other that the constant ghost modes. We see that the vacuum is doubly degenerate since the Hamiltonian (8.4) neither depends on c_0 nor on $\bar{c}_0$.

The ghost number charge Q_c is defined to be a generator of the scale transformations $c^\alpha \longrightarrow e^{\theta} c^\alpha$, $\bar{c}_\alpha \longrightarrow e^{-\theta}\bar{c}_\alpha$

$$Q_c \equiv \int_0^\pi d\sigma : c^\alpha \frac{\partial L}{\partial \dot{c}^\alpha} : = -i \int_0^\pi d\sigma : c^0 \bar{c}_1 + c^1 \bar{c}_0 : =$$

$$= -i \left\{ \frac{1}{2} (c_0 \bar{c}_0 - \bar{c}_0 c_0) + \sum_{m=1}^{\infty} [c_m^+ \bar{c}_m - \bar{c}_m^+ c_m] \right\} \qquad (1)$$

With this definition of Q_c, the degenerate vacua $|0,p\rangle_\pm$ have fractional ghost number $N_{FP} \equiv i Q_c$

$$N_{FP} |0;p\rangle_\pm = \pm \frac{1}{2} |0;p\rangle_\pm \qquad (2)$$

Correspondingly, any state in the Fock space has fractional ghost number, Notice that, in spite of the vacuum degeneracy N_{FP} is well defined and conserved.

This phenomenon of fractional ghost number causes one complication here. With the usual norm definition (remember $c_0^+ = c_0$, $\bar{c}_0^+ = \bar{c}_0$)

$$\pm\langle\varphi|\varphi\rangle_\pm = \langle\varphi|\varphi\rangle \langle\pm|\pm\rangle = \langle\varphi|\varphi\rangle \langle\pm| c_0 \bar{c}_0 + \bar{c}_0 c_0 |\pm\rangle = 0 \qquad (3)$$

The appropriate inner product can be defined by introducing the metric operator $\eta \equiv c_0 + \bar{c}_0$, then we define the product as

$$\pm\langle\varphi|\eta|\varphi\rangle_\pm = \langle\varphi|\varphi\rangle \langle\pm| c_0 + \bar{c}_0 |\mp\rangle = \langle\varphi|\varphi\rangle \langle\pm|\mp\rangle = 1 \qquad (4)$$

where we have assumed the normalization

$$\langle \pm | \mp \rangle = 1 \qquad (5)$$

We can get the BRS charge Q_B from the conserved Noether current J^α_{BRS}

$$Q_B \equiv \int_0^\pi d\sigma\, J^0_{BRS} = \int_0^\pi d\sigma \left\{ \frac{1}{4\pi\alpha'} c^0 (\partial_0 X^\mu \partial_0 X_\mu + \partial_1 X^\mu \partial_1 X_\mu) + \right.$$

$$\left. + \frac{1}{4\pi\alpha'} c^1 (\partial_0 X^\mu \partial_1 X_\mu + \partial_1 X^\mu \partial_0 X_\mu) + i [c^\alpha (\partial_\alpha c^0) \bar{c}_1 + c^\alpha (\partial_\alpha c^1) \bar{c}_0] \right\} \qquad (6)$$

In terms of the mode representation

$$Q_B = c_0 L + \bar{c}_0 M + \sqrt{2\alpha'}\,\tilde{Q}_B \tag{1}$$

$$L = \frac{1}{\sqrt{\pi}} \left\{ \alpha' p_\mu p^\mu + \frac{1}{2} \sum_{m=1}^{\infty} m \left(a_m^\mu a_{m\mu}^+ + a_m^{\mu +} a_{m\mu} \right) - \sum_{m=1}^{\infty} m \left(c_m^+ \bar{c}_m - c_m \bar{c}_m^+ \right) \right\} \tag{2}$$

$$M = \frac{1}{\sqrt{\pi}} \sum_{m=1}^{\infty} 2m\, c_m^+ c_m \tag{3}$$

$$\begin{aligned} \tilde{Q}_B = {} & \frac{1}{\sqrt{\pi}} \sum_{m=1}^{\infty} \sqrt{m}\, p^\mu \left(a_{m\mu}^+ c_m + c_m^+ a_{m\mu} \right) \\ & + \frac{1}{2\sqrt{2\pi\alpha'}} \sum_{m=1}^{\infty} \sum_{m=1}^{m-1} \sqrt{(m-m)\,m} \left(a_{m-m}^{\mu +} a_{m\mu}^+ c_m + c_m^+ a_{m-m}^\mu a_{m\mu} \right) \\ & + \frac{1}{\sqrt{2\pi\alpha'}} \sum_{m=1}^{\infty} \sum_{m=1}^{\infty} \sqrt{(m+m)\,m} \left(a_{m+m}^{\mu +} a_{m\mu} c_m + c_m^+ a_m^{\mu +} a_{m+m\mu} \right) \\ & + \frac{1}{\sqrt{2\pi\alpha'}} \sum_{m=1}^{\infty} \sum_{m=1}^{\infty} \left\{ m \left(\bar{c}_{m+m}^+ c_m c_m - c_m^+ c_m^+ \bar{c}_{m+m} \right) + (m+2m) \left(\bar{c}_m^+ c_m^+ c_{m+m} + c_{m+m}^+ c_m \bar{c}_m \right) \right\} \end{aligned} \tag{4}$$

The BRS charge formally defined above in fact contains a divergence. So we must find a suitable reordering of the mode operators in Q_B in order to make it well defined. The problem of ordering is relevant only to the $c_0 L$ term. Normal ordering of (2) is just $-H/\sqrt{\pi}$, so we redefine L as a well defined operator

$$L = -\frac{1}{\sqrt{\pi}} \left(H - \alpha(0) \right) \tag{5}$$

where $\alpha(0)$ is a suitable finite constant. Let us now calculate Q_B^2. We get immediately

$$\begin{aligned} Q_B^2 = {} & c_0^2 L^2 + \bar{c}_0^2 M^2 + 2\alpha' \tilde{Q}_B^2 + \bar{c}_0 c_0 \left[M, L \right] + LM + \sqrt{2\alpha'}\, c_0 \left[L, \tilde{Q}_B \right] \\ & + \sqrt{2\alpha'}\, \bar{c}_0 \left[M, \tilde{Q}_B \right] \end{aligned} \tag{6}$$

The first two terms disappear because $c_0^2 = \bar{c}_0^2 = 0$. Furthermore

$$\left[M, L \right] = -\frac{1}{\pi} \sum_{m=1}^{\infty} \sum_{m=1}^{\infty} 2m\,m \left[c_m^+ c_m,\ c_m^+ \bar{c}_m + \bar{c}_m^+ c_m \right] =$$

$$= -\frac{1}{\pi} \sum_{m=1}^{\infty} \sum_{m=1}^{\infty} 2m\,m \left\{ -c_m^+ \delta_{mm} c_m + c_m^+ \delta_{mm} c_m \right\} = 0 \tag{7}$$

$$\begin{aligned} LM = \frac{1}{\pi} \Bigg\{ & \alpha' p^\mu p_\mu \sum_{m=1}^{\infty} 2m\, c_m^+ c_m + \sum_{m=1}^{\infty} \sum_{m=1}^{\infty} m\, 2m\, a_m^{\mu +} a_{m\mu} c_m^+ c_m - \\ & - \sum_{m=1}^{\infty} \sum_{m=1}^{\infty} 2m\,m \left\{ \bar{c}_m^+ c_m c_m^+ c_m + c_m^+ \bar{c}_m c_m^+ c_m \right\} \Bigg\} \end{aligned} \tag{8}$$

A straightforward calculation gives

$$Q_B^2 = \frac{2}{\Gamma} \left\{ \left(\frac{D-26}{12} \right) \sum_{m=1}^{\infty} m^3 C_m^+ C_m - \left(\frac{D-26}{12} - \alpha(0) + 1 \right) \sum_{m=1}^{\infty} m C_m^+ C_m \right\}$$

Then the nilpotency of the BRS transformation holds if and only if

$$D = 26 \quad , \quad \alpha(0) = 1$$

Now it is possible to prove that the physical states form a definite positive Hilbert space.

VIII - TRANSVERSE GAUGE AND QUANTIZATION

We have obtained previously the following equations (IV-5.6)

$$X_+ = 2\alpha' p_+ \tau \quad , \quad p_{\tau +} = \frac{1}{\pi} p_+ \tag{1}$$

which define the transverse gauge. From (IV-6.2) we have

$$X_-' = \frac{\pi}{p_+} p_\tau^i X^{i\prime} \quad , \quad p_{\tau -} = \frac{1}{2\pi p_+} \left\{ \pi^2 (p_\tau^i)^2 + \frac{1}{(2\alpha')^2} (X'^i)^2 \right\} \tag{2}$$

equations which express the primary constraints. Remember that here $i = 2, 3, \dots D-1$. We have furthermore the equations of motion

$$\ddot{X}^i - X''^i = 0 \quad , \quad \dot{p}_\tau^i = \frac{1}{2\pi\alpha'} \dot{X}^i \tag{3}$$

The open string

In this case (IV-6.4)

$$X^i(\sigma, \tau) = q^i + \alpha_0^i \tau + i \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{1}{m} \alpha_m^i e^{-im\tau} \cos m\sigma \tag{4}$$

and from (IV-8.3) and (IV-8.4)

$$X_-(\sigma, \tau) = q_- + \alpha_{0-} \tau + i \sum_{\substack{m=-\infty \\ m\neq 0}}^{+\infty} \frac{1}{m} \alpha_{m-} e^{-im\tau} \cos m\sigma \tag{5}$$

$$\alpha_{m-} = \frac{1}{4\alpha' p_+} \sum_{m=-\infty}^{+\infty} \alpha_m^i \alpha_{m-m}^i \qquad m \in \mathbb{Z} \tag{6}$$

In this gauge the independant variables are $X^i(\sigma, \tau)$, $p_\tau^i(\sigma, \tau)$, p_+ and q_- and, as usual, we postulate among them the following Poisson brackets

$$\{ X^i(\sigma, \tau), X^j(\sigma', \tau) \} = \{ p_\tau^i(\sigma, \tau), p^j(\sigma', \tau) \} = 0$$

$$\{ X^i(\sigma, \tau), p_\tau^j(\sigma', \tau) \} = \delta^{ij} \delta(\sigma - \sigma') \quad , \quad \sigma, \sigma' \in [0, \pi] \tag{7}$$

$$\{ q_-, q_- \} = \{ p_+, p_+ \} = 0 \quad , \quad \{ q_-, p_+ \} = -1$$

$$\{ q_-, X'^i(\sigma, \tau) \} = \{ q_-, p_\tau^i(\sigma, \tau) \} = \{ p_+, X^{i\prime}(\sigma, \tau) \} = \{ p_+, p_\tau^i(\sigma, \tau) \} = 0$$

The Hamiltonian for this problem is

$$H = \pi\alpha' \int_0^{\pi} d\sigma \left\{ [P_\tau^i(\sigma,\tau)]^2 + \frac{1}{(2\pi\alpha')^2} [X'^i(\sigma,\tau)]^2 \right\} \tag{1}$$

Let us check that using this Hamiltonian we recover the equations of motion.

$$\dot{X}^i(\sigma,\tau) = \{ X^i(\sigma,\tau) , H \} = \pi\alpha' \int_0^{\pi} d\sigma' \{ X^i(\sigma,\tau) , [P_\tau^j(\sigma',\tau)]^2 \} =$$

$$= 2\pi\alpha' \int_0^{\pi} d\sigma' P_\tau^j(\sigma',\tau) \{ X^i(\sigma,\tau) , P_\tau^j(\sigma',\tau) \} = 2\pi\alpha' P_\tau^i(\sigma,\tau)$$

$$\dot{P}^i(\sigma,\tau) = \{ P^i(\sigma,\tau) , H \} = \frac{1}{4\pi\alpha'} \int_0^{\pi} d\sigma' \{ P^i(\sigma,\tau) , [X'^j(\sigma',\tau)]^2 \} =$$

$$= \frac{1}{2\pi\alpha'} \int_0^{\pi} d\sigma' X'^j(\sigma',\tau) \{ P^i(\sigma,\tau) , X'^j(\sigma',\tau) \} = -\frac{1}{2\pi\alpha'} \int_0^{\pi} d\sigma' X'^j(\sigma',\tau) \frac{\partial}{\partial\sigma'} \delta(\sigma-\sigma')$$

$$= \frac{1}{2\pi\alpha'} X''^i(\sigma,\tau)$$

Hence

$$\dot{X}^i(\sigma,\tau) = 2\pi\alpha' P_\tau^i(\sigma,\tau) \quad , \quad \dot{P}^i(\sigma,\tau) = \frac{1}{2\pi\alpha'} X''^i(\sigma,\tau) \tag{2}$$

$$\dot{P}_+ = \dot{q}_- = 0$$

which are the correct equations of motion, as can be seen from the results on page 1.

Notice that in terms of not independent variables H can also be written as

$$H = 2\alpha' P_+ P_- \tag{3}$$

as an immediate consequence of (1.2).

Furthermore from (1.4)

$$P_\tau^i = \frac{1}{2\pi\alpha'} \sum_{m=-\infty}^{+\infty} \alpha_m^i e^{-im\tau} \cos m\sigma \tag{4}$$

$$X'^i = -i \sum_{m=-\infty}^{+\infty} \alpha_m^i e^{-im\tau} \sin m\sigma$$

Furthermore from (1) and (IV-7.5) we get

$$H = \frac{1}{4\alpha'} \sum_{m=-\infty}^{+\infty} \alpha_m^i \, \alpha_{-m}^i \tag{1}$$

and since

$$\alpha_0^i = 2\alpha' \, p^i \tag{2}$$

this can also be written as

$$H = \alpha' (p^i)^2 + \frac{1}{2\alpha'} \sum_{m=1}^{\infty} \alpha_m^i \, \alpha_{-m}^i \tag{3}$$

Finally for the mass we get

$$M^2 = 2 p_+ p_- - (p^i)^2 = \frac{1}{2\alpha'^2} \sum_{m=1}^{\infty} \alpha_m^{i*} \, \alpha_m^i \tag{4}$$

Notice furthermore that from (2.4)

$$\alpha_m^i \, e^{-im\tau} + \alpha_m^{i*} \, e^{+im\tau} = 4\alpha' \int_0^\pi d\sigma \; p_\tau^i(\sigma,\tau) \cos m\sigma$$

$$\alpha_m^i \, e^{-im\tau} - \alpha_m^{i*} \, e^{+im\tau} = \frac{2i}{\pi} \int_0^\pi d\sigma \; x'^i(\sigma,\tau) \sin m\sigma \qquad m \in \mathbb{Z} \tag{5}$$

and hence

$$\alpha_m^i = \int_0^\pi d\sigma \; e^{im\tau} \left\{ 2\alpha' \, p_\tau^i(\sigma,\tau) \cos m\sigma + \frac{i\pi}{\pi} \, x'^i(\sigma,\tau) \sin m\sigma \right\} \qquad m \in \mathbb{Z} \tag{6}$$

$$\alpha_m^{i*} = \alpha_{-m}^i$$

From here and using (1, 7)

$$\{ \alpha_m^i , \alpha_n^j \} = \int_0^\pi d\sigma \int_0^\pi d\sigma' \, e^{i(m+n)\tau} \left\{ 2\alpha' \, p_\tau^i(\sigma\tau) \cos m\sigma + \frac{\pi}{i} x'^i(\sigma\tau) \sin m\sigma , \right.$$

$$\left. \frac{\pi}{i} x'^j(\sigma'\tau) \sin n\sigma' + 2\alpha' \, p_\tau^j(\sigma'\tau) \cos n\sigma' \right\}$$

$$= \frac{2\alpha' i}{\pi} \int_0^\pi d\sigma \int_0^\pi d\sigma' \, e^{i(m+n)\tau} \delta_{ij} \left\{ -\cos m\sigma \sin n\sigma' \frac{\partial}{\partial\sigma'} \delta(\sigma-\sigma') + \right.$$

$$\left. + \sin m\sigma \cos n\sigma' \frac{\partial}{\partial\sigma} \delta(\sigma-\sigma') \right\}$$

$$= \frac{2\alpha' i}{\pi} \delta_{ij} \int_0^\pi d\sigma \int_0^\pi d\sigma' \, e^{i(m+n)\tau} \delta(\sigma-\sigma') (m-n) \cos m\sigma \cos n\sigma$$

$$= \frac{2\alpha' i}{\pi} \delta_{ij} \int_0^\pi d\sigma \; (m-n) \, e^{i(m+n)\tau} \cos m\sigma \cos n\sigma = -2i\alpha' m \delta_{ij} \delta_{m,-n}$$

i.e.

$$\{\alpha_m^i\ ,\ \alpha_m^j\} = -2i\alpha' m\,\delta^{ij}\,\delta_{m,-m} \qquad \forall m \in \mathbb{Z} \tag{4}$$

Let us now quantize this system. We postulate the commutation relations

$$\begin{aligned}
&[X^i(\sigma,\tau)\ ,\ X^j(\sigma',\tau)] = [P^i_\tau(\sigma,\tau)\ ,\ P^j_\tau(\sigma',\tau)] = 0\\
&[X^i(\sigma,\tau)\ ,\ P^j_\tau(\sigma',\tau)] = i\,\delta^{ij}\,\delta(\sigma-\sigma') \qquad ,\quad \sigma,\sigma' \in [0,\pi]\\
&[q_-\ ,\ q_-] = [p_+\ ,\ p_+] = 0 \qquad ,\qquad [q_-\ ,\ p_+] = -i\\
&[q_-\ ,\ X^i(\sigma,\tau)] = [p_+\ ,\ P^i_\tau(\sigma,\tau)] = [q_-\ ,\ P^i_\tau(\sigma,\tau)] = [p_+\ ,\ X^i(\sigma,\tau)] = 0
\end{aligned} \tag{2}$$

and using (3.6)

$$[\alpha_m^i\ ,\ \alpha_m^j] = 2\alpha' m\,\delta^{ij}\,\delta_{m,-m} \qquad ,\ \forall m \in \mathbb{Z} \tag{3}$$

$$\alpha_m^i = \alpha_{-m}^{i\,+}$$

or equivalently

$$\begin{aligned}
&[\alpha_m^i\ ,\ \alpha_m^j] = [\alpha_m^{i\,+}\ ,\ \alpha_m^{j\,+}] = 0\\
&[\alpha_m^i\ ,\ \alpha_m^{j\,+}] = +2\alpha' m\,\delta^{ij}\,\delta_{mm}
\end{aligned} \qquad m, m \geq 0 \tag{4}$$

We can now introduce

$$\begin{aligned}
\alpha_m^i &\equiv \sqrt{2m\alpha'}\ a_m^i\\
\alpha_m^{i\,+} &\equiv \sqrt{2m\alpha'}\ a_m^{i\,+}
\end{aligned} \qquad m \geq 1 \tag{5}$$

and (4) reduce to

$$\begin{aligned}
&[a_m^i\ ,\ a_m^j] = [a_m^{i\,+}\ ,\ a_m^{j\,+}] = 0\\
&[a_m^i\ ,\ a_m^{j\,+}] = \delta^{ij}\,\delta_{mm}
\end{aligned} \qquad m, m \geq 1 \tag{6}$$

Now because of the disappearance of the time component the space of vectors

$$\prod_{m=1}^{\infty} \prod_{i=2}^{D-1} \left[a_m^{i\,+} \right]^{\lambda_{mi}} |0\rangle \,, \qquad \lambda_{mi} = \text{integer}$$

has positive definite metric. The states of the string are hence described by purely transverse oscillators.

Using (VII - 12.5) we define

$$\alpha_{m,-} \equiv + \frac{1}{4\pi P_+} \frac{1}{2\alpha'} \int_{-\pi}^{+\pi} d\sigma \, e^{im(\tau+\sigma)} : \left[\partial_+ X^i(\tau,\sigma) \right]^2 : \qquad \forall m \in \mathbb{Z} \tag{2}$$

$$\partial_+ X^i(z) = 2\alpha' p^i + (2\alpha')^{1/2} \sum_{m=1}^{\infty} \sqrt{m} \left[a_m^{i\,+} z^m + a_m^i z^{-m} \right] \tag{3}$$

Furthermore from (4.5) and $p^i \equiv \alpha_0^i / 2\alpha'$ we can write

$$\partial_+ X^i(z) = \sum_{m=+\infty}^{+\infty} \alpha_m z^{-m} \tag{4}$$

Let us remember that (2) can be written as

$$\alpha_{m,-} = + \frac{1}{4\alpha' P_+} \oint \frac{dz}{2\pi i} z^m : \left[\partial_+ X^i(z) \right]^2 : \tag{5}$$

and we get immediately

$$\alpha_{m,-} = \frac{1}{4\alpha' P_+} \sum_{m=-\infty}^{+\infty} : \alpha_m \alpha_{n-m} : \tag{6}$$

i.e. it coincides with (3.6) but for $m=0$ where normal ordering ambiguities can appear. For the Hamiltonian, where normal ordering ambiguities appear, we have

$$H = 2\alpha' P_+ P_- = P_+ \alpha_{0,-} - \alpha(0) \tag{7}$$

It is interesting to introduce

$$\hat{\alpha}_m^i \equiv \alpha_m^i \,, \qquad \hat{\alpha}_{m,-} \equiv P_+ \alpha_{m,-} \tag{8}$$

Now let us compute some needed commutators. Using (4.3) we have

$$\left[\hat{\alpha}_m^i \,, \hat{\alpha}_n^j \right] = 2\alpha' m \, \delta_{ij} \, \delta_{m,-n}$$

Now we have

$$[\hat\alpha^i_m, \hat\alpha^j_{m,-}] = \frac{1}{4\alpha'}\sum_{k=-\infty}^{+\infty} [\hat\alpha^i_m, :\hat\alpha^j_{m-k}\hat\alpha^j_k:] = \frac{1}{4\alpha'}\sum_{k=-\infty}^{+\infty} [\hat\alpha^i_k, \hat\alpha^j_{m-k}\hat\alpha^j_k]$$

$$= \frac{1}{4\alpha'}\sum_{k=-\infty}^{+\infty}\{\hat\alpha^j_{m-k}\, 2\alpha' m\, \delta_{-k,m}\,\delta_{ij} + 2\alpha' m\,\delta_{m,k-m}\,\hat\alpha^j_k\} =$$

$$= \frac{1}{2}\{m\,\delta_{ij}\,\hat\alpha^j_{m+m} + m\,\hat\alpha^j_{m+m}\,\delta_{ij}\} \Rightarrow$$

$$[\hat\alpha^i_m, \hat\alpha_{m,-}] = m\,\hat\alpha^i_{m+m} \tag{1}$$

The commutator $[\hat\alpha_{m-}, \hat\alpha_{m-}]$ can be computed directly on taking into account (5.5) and the results on page (VII-15) can also be obtained. Then we get, all together,

$$[\hat\alpha^i_m, \hat\alpha^j_m] = 2\alpha' m\,\delta_{ij}\,\delta_{m,-m}$$

$$[\hat\alpha^i_m, \hat\alpha_{m,-}] = m\,\hat\alpha^i_{m+m} \tag{2}$$

$$[\hat\alpha_{m,-}, \hat\alpha_{m,-}] = (m-m)\,\hat\alpha_{m+m,-} + \frac{(D-2)}{12}\,m(m^2-1)\,\delta_{m,-m}$$

Furthermore

$$\hat\alpha_{m-} = \frac{1}{4\alpha'}\sum_{m=-\infty}^{+\infty} :\hat\alpha_m\,\hat\alpha_{n-m}: \tag{3}$$

$$H = 2\alpha' P_+ P_- = \hat\alpha_{0,-} - \alpha(0) \tag{4}$$

It is possible to implement Lorentz transformations and translations in the transverse gauge provided that one can define generators of the Poincaré group satisfying the usual algebra of this commutators. The natural choices for P^μ and $M^{\mu\nu}$ are of course, as in the covariant quantization, the energy and angular momentum of the string given in (V-2.4). From (VII-1.3) you would expect

$$[M^i_-, M^j_-] = \frac{1}{2}[M^{i0} - M^{i1}, M^{j0} - M^{j1}] =$$

$$= -\frac{i}{2}\{M^{ij} + \delta_{ij}M^{01} + \delta_{ij}M^{10} - M^{ij}\} = 0 \tag{5}$$

Now we must compute this directly and see under which conditions this turns out to be true. It is possible to check, using similar

methods that no problem appear for the rest of the commutators. From (I-2.4)

$$M^i_- = \ell^i_- + i \frac{1}{P_+} E^i$$

$$\ell^i_- \equiv q^i P_- - q_- P^i \tag{1}$$

$$E^i \equiv \frac{1}{2\alpha'} \sum_{m=1}^{\infty} \frac{1}{m} \left[\hat{\alpha}_{-m,-} \hat{\alpha}^i_m - \hat{\alpha}^i_{-m} \hat{\alpha}_{m,-} \right]$$

Since

$$q^i = + i \frac{\partial}{\partial P^i} \quad , \quad q_- = - i \frac{\partial}{\partial P_+} \tag{2}$$

we get

$$[\ell^i_-, \ell^j_-] = [q^i P_-, q^j P_-] - [q^i P_-, q_- P^j] - [q_- P^i, q^j P_-] + [q_- P^i, q_- P^j] =$$

$$= - P_- [q^i, P^j] q_- - q_- [P^i, P^j] P_- = - i \delta^{ij} \{ P_- q_- - q_- P_- \} = 0 \Rightarrow$$

$$[\ell^i_-, \ell^j_-] = 0 \tag{3}$$

From (3)

$$\left[q_-, \frac{1}{P_+} \right] = i \frac{1}{(P_+)^2} \tag{4}$$

and since (3.2) $\hat{\alpha}^i_0 = 2\alpha' P^i$ and hence

$$[q^i, \hat{\alpha}^j_m] = 2\alpha' i \delta_{m0} \delta^{ij} \tag{5}$$

From these and (6.3)

$$[q^i, \hat{\alpha}_{m,-}] = \frac{1}{4\alpha'} \sum_{m=-\infty}^{+\infty} [q^i, \hat{\alpha}^j_{m-m} \hat{\alpha}^j_m] =$$

$$= \frac{1}{4\alpha'} \sum_{m=-\infty}^{+\infty} 2\alpha' i \delta^{ij} \{ \delta_{mm} \hat{\alpha}^j_m + \delta_{m0} \hat{\alpha}^j_{m-m} \} \Rightarrow$$

$$[q^i, \hat{\alpha}_{m,-}] = i \hat{\alpha}^i_m \tag{6}$$

Furthermore

$$[q^i, E^j] = \frac{1}{2\alpha'} \sum_{m=1}^{\infty} \frac{1}{m} [q^i, \hat{\alpha}_{-m,-} \hat{\alpha}^j_m - \hat{\alpha}^j_{-m} \hat{\alpha}_{m,-}] =$$

$$= \frac{1}{2\alpha'} i \sum_{m=1}^{\infty} \frac{1}{m} [\hat{\alpha}^i_{-m} \hat{\alpha}^j_m - \hat{\alpha}^j_{-m} \hat{\alpha}^i_m] \Rightarrow$$

$$[q^i, E^j] = \frac{i}{2\alpha'} \sum_{m=1}^{\infty} \frac{1}{m} [\hat{\alpha}^i_{-m} \hat{\alpha}^j_m - \hat{\alpha}^j_{-m} \hat{\alpha}^i_m] \equiv i E^{ij} \tag{1}$$

Therefore

$$[p^i_-, \frac{1}{p_+} E^j] = [q^i p_- - q_- p^i, \frac{1}{p_+} E^j] =$$

$$= q^i \frac{1}{p_+} [p_-, E^j] + q^i [p_-, \frac{1}{p_+}] E^j + \frac{1}{p_+} [q^i, E^j] p_- + [q^i, \frac{1}{p_-}] E^j p_-$$

$$- q_- \frac{1}{p_+} [p^i, E^j] - q_- [p^i, \frac{1}{p_+}] E^j - \frac{1}{p_+} [q_-, E^j] p^i - [q_-, \frac{1}{p_+}] E^j p_i =$$

$$= \frac{1}{p_+} i E^{ij} p_- - \frac{i}{(p_+)^2} E^j p^i \Rightarrow$$

$$[p^i_-, \frac{1}{p_+} E^j] = - \frac{i}{(p_+)^2} p^i E^j + \frac{1}{p_+} i E^{ij} p_- \tag{2}$$

and we get

$$[M^i_-, M^j_-] = i [p^i_-, \frac{1}{p_+} E^j] - i [p^j_-, \frac{1}{p_+} E^i] - [\frac{1}{p_+} E^i, \frac{1}{p_+} E^j] \tag{3}$$

which implies

$$[M^i_-, M^j_-] = - \frac{1}{(p_+)^2} [2 p_+ p_- E^{ij} + [E^i, E^j] + E^i p^j - E^j p^i] \tag{4}$$

Finally taking into account (6.4) we obtain

$$[M^i_-, M^j_-] = - \frac{1}{(p_+)^2} C^{ij}_- \tag{5}$$

$$C^{ij} \equiv \frac{1}{\alpha'} [\hat{\alpha}_{0,-} - \alpha(0)] E^{ij} + [E^i, E^j] + E^i p^j - E^j p^i$$

Let us now consider this expression which is superficially quartic in transverse oscillations. It is not difficult to convince oneself that the quartic terms cancel each other. Note that $C^{ij} = - C^{ji}$ and furthermore commutes with the "number" operator

$$N \equiv \sum_{m=1}^{\infty} \hat{\alpha}^i_{-m} \alpha^i_m \tag{6}$$

Hence

$$C^{ij} = \sum_{r=1}^{\infty} C_r [\hat{\alpha}^i_{-r} \hat{\alpha}^j_r - \hat{\alpha}^j_{-r} \hat{\alpha}^i_r] \tag{7}$$

Notice that the coefficients C_r can be evaluated in the following way

$$\langle 0 | \hat{\alpha}_m^k \, C^{ij} \, \hat{\alpha}_{-m}^e | 0 \rangle = \sum_{r=1}^{\infty} C_r \langle 0 | \hat{\alpha}_m^k \left[\hat{\alpha}_{-r}^i \hat{\alpha}_r^j - \hat{\alpha}_{-r}^j \alpha_r^i \right] \hat{\alpha}_{-m}^e | 0 \rangle =$$

$$= \sum_{r=1}^{\infty} C_r \, 2\alpha' r \, \delta_{rm} \langle 0 | \delta_{je} \hat{\alpha}_m^k \hat{\alpha}_{-r}^i - \delta_{ie} \hat{\alpha}_m^k \alpha_{-r}^j | 0 \rangle \Rightarrow$$

$$\langle 0 | \hat{\alpha}_m^k \, C^{ij} \, \hat{\alpha}_{-m}^e | 0 \rangle = (2\alpha')^2 \, C_m \, m^2 \left[\delta_{ki} \delta_{je} - \delta_{kj} \delta_{ie} \right] \tag{1}$$

Let us now compute directly this quantity using (8.6). We have

$$\langle 0 | \hat{\alpha}_m^k \, E^i \, p^j \, \hat{\alpha}_{-m}^e | 0 \rangle = \frac{1}{2\alpha'} \sum_{r=1}^{\infty} \frac{1}{r} \langle 0 | \hat{\alpha}_m^k \left[\hat{\alpha}_{-r,-} \hat{\alpha}_r^i - \hat{\alpha}_{-r}^i \hat{\alpha}_{r,-} \right] p^j \hat{\alpha}_{-m}^e | 0 \rangle \tag{2}$$

since $2\alpha' p^j = \hat{\alpha}_0^j$, $[p^j, \hat{\alpha}_0^j] = 0$ and $|0\rangle$ is an eigenvector of p^j with eigenvalue p^j

$$\langle 0 | \hat{\alpha}_m^k \, E^i \, \hat{\alpha}_{-m}^e | 0 \rangle = \frac{1}{2\alpha'} p^j \sum_{r=1}^{\infty} \frac{1}{r} \langle 0 | \hat{\alpha}_m^k \left[\hat{\alpha}_{-r,-} \hat{\alpha}_r^i - \hat{\alpha}_{-r}^i \hat{\alpha}_{r,-} \right] \hat{\alpha}_{-m}^e | 0 \rangle =$$

$$= p^j \sum_{r=1}^{\infty} \langle 0 | \delta_{rm} \delta_{ie} \hat{\alpha}_m^k \hat{\alpha}_{-r,-} - \delta_{mr} \delta_{ki} \hat{\alpha}_{r-} \hat{\alpha}_{-,m}^e | 0 \rangle =$$

$$= p^j \, m \left(\delta_{ie} \hat{\alpha}_0^k - \delta_{ki} \hat{\alpha}_0^e \right) \Rightarrow$$

$$\langle 0 | \hat{\alpha}_m^k \left(E^i p^j - E^j p^i \right) \hat{\alpha}_{-m}^e | 0 \rangle =$$

$$= (2\alpha') \, m \left[\delta_{ie} p^j p^k - \delta_{ik} p^j p^e - \delta_{je} p^i p^k + \delta_{jk} p^i p^e \right] \tag{3}$$

Let us now consider

$$\frac{1}{\alpha'} \langle 0 | \hat{\alpha}_m^k \left(\hat{\alpha}_{0,-} - \alpha(0) \right) E^{ij} \hat{\alpha}_{-m}^e | 0 \rangle =$$

$$= \frac{2}{(2\alpha')^2} \sum_{r=1}^{\infty} \frac{1}{r} \langle 0 | \hat{\alpha}_m^k \left[\hat{\alpha}_{0,-} - \alpha(0) \right] \left[\hat{\alpha}_{-r}^i \hat{\alpha}_r^j - \hat{\alpha}_{-r}^j \hat{\alpha}_r^i \right] \hat{\alpha}_{-m}^e | 0 \rangle \tag{4}$$

but

$$\frac{2}{(2\alpha')^2} \sum_{r=1}^{\infty} \frac{1}{r} \langle 0 | \hat{\alpha}_m^k \left[\hat{\alpha}_{0,-} - \alpha(0) \right] \hat{\alpha}_{-r}^i \hat{\alpha}_r^j \hat{\alpha}_{-m}^e | 0 \rangle =$$

$$= \frac{2}{(2\alpha')} \delta_{je} \langle 0 | \hat{\alpha}_m^k \left[\hat{\alpha}_{0,-} - \alpha(0) \right] \hat{\alpha}_{-m}^i | 0 \rangle =$$

$$= \frac{2}{(2\alpha')} \delta_{je} \left\{ - \alpha(0) \, (2\alpha') \, m \, \delta_{ik} + \langle 0 | \hat{\alpha}_m^k \hat{\alpha}_{-m}^i \hat{\alpha}_{0-} | 0 \rangle + m \langle 0 | \hat{\alpha}_m^k \hat{\alpha}_{-m}^i | 0 \rangle \right\}$$

$$= \frac{2}{(2\alpha')} \delta_{je} \{ - \alpha(0)\,(2\alpha')\, m\, \delta_{ik} + 2m\alpha' \delta_{ki} \langle 0 | \hat{\alpha}_{0,-} | 0 \rangle + 2\alpha' m^2 \delta_{ki} \} =$$

$$= 2\, \delta_{je}\, \delta_{ik}\, m \{ - \alpha(0) + m + 2\alpha' p_+ p_- + \alpha(0) \} \qquad (4)$$

Where (6.4) has been used. Therefore

$$\frac{2}{(2\alpha')} \langle 0 | \hat{\alpha}_m^k \left(\hat{\alpha}_{0,-} - \alpha(0) \right) E^{ij} \hat{\alpha}_{-m}^e | 0 \rangle =$$

$$= 2m \left[m + 2\alpha' p_+ p_- \right] \left[\delta_{ki}\, \delta_{je} - \delta_{kj}\, \delta_{ie} \right] \qquad (2)$$

Finally

$$\langle 0 | \hat{\alpha}_m^k\, E^{ij}\, E^{ij}\, \hat{\alpha}_{-m}^e | 0 \rangle =$$

$$= \frac{1}{(2\alpha')^2} \sum_{r=1}^{\infty} \sum_{t=1}^{\infty} \frac{1}{rt} \langle 0 | \hat{\alpha}_m^k \left[\hat{\alpha}_{-r,-} \hat{\alpha}_r^i - \hat{\alpha}_{-r}^i \hat{\alpha}_{r,-} \right] \left[\hat{\alpha}_{-t,-} \hat{\alpha}_t^j - \hat{\alpha}_{-t}^j \hat{\alpha}_{t,-} \right] \hat{\alpha}_{-m}^e | 0 \rangle$$

$$= \frac{1}{(2\alpha')^2} \sum_{r=1}^{\infty} \sum_{t=1}^{\infty} \frac{1}{rt} \langle 0 | \left[\hat{\alpha}_m^k \hat{\alpha}_{-r,-} \hat{\alpha}_r^i - 2\alpha' m\, \delta_{mr}\, \delta_{ki} \hat{\alpha}_{m-} \right] \left[2\alpha' m\, \delta_{tm}\, \delta_{je}\, \hat{\alpha}_{-m,-} - \hat{\alpha}_{-t}^i \hat{\alpha}_{t,-} \hat{\alpha}_{-m}^e \right] | 0 \rangle =$$

$$= \frac{1}{(2\alpha')^2} \{ - \sum_{r=1}^{\infty} \sum_{t=1}^{\infty} \frac{1}{rt} \langle 0 | \hat{\alpha}_m^k \hat{\alpha}_{-r,-} \hat{\alpha}_r^i \hat{\alpha}_{-t}^j \hat{\alpha}_{t,-} \hat{\alpha}_{-m}^e | 0 \rangle +$$

$$+ (2\alpha')\, \delta_{ki} \sum_{r=1}^{\infty} \frac{1}{r} \langle 0 | \hat{\alpha}_{m,-} \hat{\alpha}_{-r}^j \hat{\alpha}_{r,-} \hat{\alpha}_{-m}^e | 0 \rangle + (2\alpha')\, \delta_{je} \sum_{r=1}^{\infty} \frac{1}{r} \langle 0 | \hat{\alpha}_m^k \hat{\alpha}_{-r,-} \hat{\alpha}_r^i \hat{\alpha}_{-m,-} | 0 \rangle$$

$$+ (2\alpha')^2\, \delta_{ki}\, \delta_{je} \langle 0 | \hat{\alpha}_{m,-} \hat{\alpha}_{-m,-} | 0 \rangle \} \qquad (3)$$

Then

$$\langle 0 | \hat{\alpha}_{m,-} \hat{\alpha}_{-m,-} | 0 \rangle = 2m \langle 0 | \hat{\alpha}_{0-} | 0 \rangle + \frac{1}{12} (D-2)\, m\, (m^2-1) =$$

$$= 2m \left[(2\alpha') p_+ p_- + \alpha(0) \right] + \frac{D-2}{12} m (m^2-1) \qquad (4)$$

$$\sum_{r=1}^{\infty} \frac{1}{r} \langle 0 | \hat{\alpha}_m^k \hat{\alpha}_{-r,-} \hat{\alpha}_r^i \hat{\alpha}_{-m,-} | 0 \rangle = \sum_{r=1}^{\infty} m \langle 0 | \hat{\alpha}_{m-r}^k \hat{\alpha}_{r-m}^i | 0 \rangle =$$

$$= \sum_{r=1}^{m} m \langle 0 | \hat{\alpha}_{m-r}^k \hat{\alpha}_{r-m}^i | 0 \rangle = m \langle 0 | \hat{\alpha}_0^k \hat{\alpha}_0^i | 0 \rangle + \sum_{r=1}^{m-1} m \langle 0 | \hat{\alpha}_{m-r}^k \hat{\alpha}_{r-m}^i | 0 \rangle =$$

$$= (2\alpha')^2 m\, p^k p^i + m \sum_{r=1}^{m-1} (2\alpha')\, (m-r)\, \delta_{ki} =$$

$$= (2\alpha')^2 m\, p^k p^i + (2\alpha') \frac{1}{2} m^2 (m-1)\, \delta_{ki} \qquad (5)$$

$$\sum_{r=1}^{\infty} \frac{1}{r} \langle 0 | \hat{\alpha}_{m,-} \hat{\alpha}^{j}_{-r} \hat{\alpha}_{r,-} \hat{\alpha}^{e}_{-m} | 0 \rangle = m \sum_{r=1}^{m-1} \langle 0 | \hat{\alpha}^{i}_{m-r} \hat{\alpha}^{e}_{r-m} | 0 \rangle =$$

$$= (2\alpha')^2 m \, p^j p^e + (2\alpha') \frac{1}{2} m^2 (m-1) \delta^{je} \qquad (1)$$

$$\sum_{r=1}^{\infty} \sum_{t=1}^{\infty} \frac{1}{rt} \langle 0 | \hat{\alpha}^{k}_{m} \hat{\alpha}_{-r,-} \hat{\alpha}^{i}_{r} \hat{\alpha}^{j}_{-t} \hat{\alpha}_{t,-} \hat{\alpha}^{e}_{-m} | 0 \rangle = - m^2 \sum_{r=1}^{m-1} \sum_{t=1}^{m-1} \frac{1}{rt} \langle 0 | \hat{\alpha}^{k}_{m-r} \hat{\alpha}^{i}_{r} \hat{\alpha}^{j}_{-t} \hat{\alpha}^{e}_{-m+t} | 0 \rangle =$$

$$= - m^2 \left\{ \frac{1}{m^2} \langle 0 | \hat{\alpha}^{k}_{0} \hat{\alpha}^{i}_{m} \hat{\alpha}^{j}_{-m} \hat{\alpha}^{e}_{0} | 0 \rangle + \frac{1}{m} \sum_{r=1}^{m-1} \frac{1}{r} \langle 0 | \hat{\alpha}^{k}_{m-r} \hat{\alpha}^{i}_{r} \hat{\alpha}^{j}_{-m} \hat{\alpha}^{e}_{0} | 0 \rangle + \right.$$

$$\left. + \frac{1}{m} \sum_{r=1}^{m-1} \frac{1}{r} \langle 0 | \hat{\alpha}_{0} \hat{\alpha}^{i}_{m} \hat{\alpha}^{j}_{-r} \hat{\alpha}^{e}_{-m+r} | 0 \rangle + \sum_{r=1}^{m-1} \sum_{t=1}^{m-1} \frac{1}{rt} \langle 0 | \hat{\alpha}^{k}_{m-r} \hat{\alpha}^{i}_{r} \hat{\alpha}^{j}_{-t} \hat{\alpha}^{e}_{-m+t} | 0 \rangle \right\} =$$

$$= - m^2 \left\{ (2\alpha')^3 p^e p^k \frac{1}{m} \delta^{ij} + \sum_{r=1}^{m-1} \sum_{t=1}^{m-1} \frac{1}{rt} \langle 0 | \hat{\alpha}^{k}_{m-r} \hat{\alpha}^{j}_{-t} \hat{\alpha}^{i}_{r} \hat{\alpha}^{e}_{-m+t} | 0 \rangle \right.$$

$$\left. + (2\alpha') \delta^{ij} \sum_{r=1}^{m-1} \frac{1}{r} \langle 0 | \hat{\alpha}^{k}_{m-r} \hat{\alpha}^{e}_{-m+r} | 0 \rangle \right\} =$$

$$= - m^2 \left\{ (2\alpha')^3 \frac{1}{m} p^k p^e \delta^{ij} + (2\alpha')^2 \delta^{ij} \delta^{ke} \sum_{r=1}^{m-1} \frac{1}{r} (m-r) + (2\alpha') \delta^{ij} \sum_{t=1}^{m-1} \frac{1}{t} \langle 0 | \hat{\alpha}^{k}_{e} \hat{\alpha}^{j}_{-t} | 0 \rangle \right\} =$$

$$= - m^2 \left\{ (2\alpha')^3 \frac{1}{m} p^k p^e \delta^{ij} + (2\alpha')^2 \delta^{ij} \delta^{ke} \sum_{r=1}^{m-1} \frac{1}{r} (m-r) + (2\alpha')^2 \delta^{ij} \delta^{kj} (m-1) \right\} \qquad (2)$$

Hence

$$\langle 0 | \hat{\alpha}^{k}_{m} [E^i, E^j] \hat{\alpha}^{e}_{-m} | 0 \rangle = (\delta^{kj} \delta^{ie} - \delta^{ki} \delta^{je}) \left[2m \left(2\alpha' p_+ p_- + a(0) \right) + \frac{D-2}{12} m (m^2-1) \right]$$

$$+ (2\alpha') m \left[\delta^{je} p^i p^k - \delta^{ie} p^j p^k \right] + \frac{1}{2} m^2 (m-1) \left[\delta^{je} \delta^{ki} - \delta^{ie} \delta^{kj} \right]$$

$$+ (2\alpha') m \left[\delta^{ki} p^j p^e - \delta^{kj} p^i p^e \right] + \frac{1}{2} m^2 (m-1) \left[\delta^{je} \delta^{ki} - \delta^{ie} \delta^{kj} \right]$$

$$- m^2 (m-1) \left[\delta^{ie} \delta^{kj} - \delta^{je} \delta^{ki} \right] \qquad (2)$$

Now using all that we get

$$\langle 0 | \hat{\alpha}^{k}_{m} C^{ij} \hat{\alpha}^{e}_{-m} | 0 \rangle = \left[\delta^{ik} \delta^{je} - \delta^{jk} \delta^{ie} \right] m \cdot$$

$$\cdot \left\{ 2m^2 - 2a(0) - \frac{D-2}{12} (m^2-1) \right\} \qquad (3)$$

In order that this cancels for any value of m we must impose the conditions

$$D = 26 \quad , \quad \alpha(0) = 1 \tag{1}$$

Therefore for arbitrary values of D and $\alpha(0)$, the theory is non-covariant. From (5.7) and $\alpha(0) = 1$ we get that the mass of the ground state is $M_0^2 = -1/\alpha'$, ie. the existence of a tachyon.

The closed string

In this case (IV-8.5)

$$X^i(\sigma,\tau) = q^i + 2\alpha_0^i \tau + \frac{i}{2} \sum_{\substack{m=-\infty \\ m\neq 0}}^{+\infty} \frac{1}{m} \left\{ \alpha_m^i e^{-2im(\tau-\sigma)} + \tilde{\alpha}_m^i e^{-2im(\tau+\sigma)} \right\} \tag{2}$$

and from (IV-9.6)

$$X_-(\sigma,\tau) = q_- + 2\alpha_{0-}\tau + \frac{i}{2} \sum_{\substack{m=-\infty \\ m\neq 0}}^{+\infty} \frac{1}{m} \left\{ \alpha_{m,-} e^{-2im(\tau-\sigma)} + \tilde{\alpha}_{m,-} e^{-2im(\tau+\sigma)} \right\} \tag{3}$$

$$\alpha_{m-} \equiv \frac{1}{2\alpha' p_+} \sum_{m=-\infty}^{+\infty} \alpha_m^i \alpha_{m-m}^i \quad , \quad \tilde{\alpha}_{m-} \equiv \frac{1}{2\alpha' p_+} \sum_{m=-\infty}^{+\infty} \tilde{\alpha}_m^i \tilde{\alpha}_{m-m}^i \tag{4}$$

$$\alpha_0^i = \tilde{\alpha}_0^i \quad , \quad \alpha_{0-} = \tilde{\alpha}_{0-} \tag{5}$$

As for the open string the independent variables are $X^i(\sigma,\tau)$, $p^i(\sigma,\tau)$ q_- and p_+ and they have the Poisson brackets (1.7) and the Hamiltonian is given as before (2.1). It is easy to check that equations (2.2) and (2.3) are still true. Furthermore from (2)

$$p_\tau^i = \frac{1}{2\pi\alpha'} \sum_{m=-\infty}^{+\infty} \left[\alpha_m^i e^{-2im(\tau-\sigma)} + \tilde{\alpha}_m^i e^{-2im(\tau+\sigma)} \right] \tag{6}$$

$$X'^i = -\sum_{m=-\infty}^{+\infty} \left[\alpha_m^i e^{-2im(\tau-\sigma)} - \tilde{\alpha}_m^i e^{-2im(\tau+\sigma)} \right]$$

From (9.4) and the form of H

$$H = \frac{1}{2\alpha'} \sum_{m=-\infty}^{+\infty} \left[\alpha_m^i \alpha_{-m}^i + \tilde{\alpha}_m^i \tilde{\alpha}_{-m}^i \right] \tag{7}$$

Since

$$2\alpha' p_+^i = \alpha_0^i + \tilde{\alpha}_0^i = 2\alpha_0^i \tag{8}$$

We can write

$$H = \alpha' (p^i)^2 + \frac{1}{\alpha'} \sum_{m=1}^{\infty} [\alpha_m^i \alpha_{-m}^i + \tilde{\alpha}_m^i \tilde{\alpha}_{-m}^i] \tag{1}$$

Finally for the mass we get

$$M^2 = 2 p_+ p_- - (p^i)^2 = \frac{1}{\alpha'^2} \sum_{m=1}^{\infty} [\alpha_m^i \alpha_{-m}^i + \tilde{\alpha}_m^i \tilde{\alpha}_{-m}^i] \tag{2}$$

or equivalently

$$M^2 = \frac{1}{\alpha'^2} \sum_{m=1}^{\infty} [\alpha_m^i \alpha_m^{i*} + \tilde{\alpha}_m^i \tilde{\alpha}_m^{i*}] \tag{3}$$

Notice furthermore that from (VII-9.2)

$$\alpha_m^i = \int_0^{\pi} d\sigma \left\{ \alpha' p_\tau^i(\sigma,\tau) - \frac{im}{\pi} x^i(\sigma\tau) \right\} e^{2im(\tau-\sigma)}$$

$$\tilde{\alpha}_m^i = \int_0^{\pi} d\sigma \left\{ \alpha' p_\tau^i(\sigma,\tau) - \frac{im}{\pi} x^i(\sigma\tau) \right\} e^{2im(\tau+\sigma)} \tag{4}$$

Now we can calculate the Poisson-Brackets for this quantities. Let us introduce

$$\alpha_m^i(-1) \equiv \alpha_m^i \qquad \alpha_m^i(+1) = \tilde{\alpha}_m^i \tag{5}$$

Then

$$\{\alpha_m^i(\lambda), \alpha_m^j(\lambda')\} = \int_0^{\pi} d\sigma \int_0^{\pi} d\sigma' \, e^{2im(\tau+\lambda\sigma)} e^{2im(\tau+\lambda'\sigma')}$$

$$\cdot \left\{ \alpha' p_\tau^i(\sigma,\tau) - \frac{\pi}{im} x^{i\prime}(\sigma\tau) \;,\; \alpha' p_\tau^j(\sigma',\tau) - \frac{\pi}{im'} x^{j\prime}(\sigma'\tau) \right\}$$

$$= \int_0^{\pi} d\sigma \oint_0^{\pi} d\sigma' \, e^{2i(m+m')\tau} e^{2i(\lambda m\sigma + \lambda' m'\sigma')} \left\{ \frac{\pi}{im} \alpha' \delta_{ij} \delta(\sigma-\sigma') - \frac{\pi}{im'} \alpha' \delta_{ij} \delta(\sigma-\sigma') \right\}$$

$$= \frac{i\alpha'}{\pi} (m-m') \delta_{ij} \int_0^{\pi} d\sigma \, e^{2i(m+m')\tau} e^{2i\lambda(m+\lambda\lambda' m')\sigma} \Rightarrow$$

$$\{\alpha_m^i(\lambda), \alpha_{m'}^j(\lambda')\} = -i\alpha' m (1+\lambda\lambda') \delta_{ij} \delta_{m,-\lambda\lambda' m'} \tag{6}$$

and therefore

$$\{ \alpha_m^i , \alpha_m^j \} = \{ \tilde{\alpha}_m^i , \tilde{\alpha}_m^j \} = - 2i m \alpha' \delta_{m,-m} \delta^{ij}$$

$$\{ \alpha_m^i , \tilde{\alpha}_m^j \} = 0 \qquad (1)$$

Let us now proceed to quantize the system. We postulate the equal-time commutation relations (4.2) and using (13.4) we obtain the commutation rules

$$[\alpha_m^i , \alpha_m^j] = [\tilde{\alpha}_m^i , \tilde{\alpha}_m^j] = 2\alpha' m \delta_{m,-m} \delta^{ij}$$

$$[\alpha_m^i , \tilde{\alpha}_m^j] = 0 \qquad (2)$$

$$\alpha_m^i = \alpha_{-m}^{i\,+} \quad , \quad \tilde{\alpha}_m^i = \tilde{\alpha}_{-m}^{i\,+}$$

or equivalently

$$[\alpha_m^i , \alpha_m^{j+}] = [\alpha_m^{i+} , \alpha_m^j] = [\tilde{\alpha}_m^i , \tilde{\alpha}_m^{j+}] = [\tilde{\alpha}_m^{i+} , \tilde{\alpha}_m^j] = 0$$

$$[\alpha_m^i , \tilde{\alpha}_m^j] = [\alpha_m^i , \tilde{\alpha}_m^{j+}] = [\alpha_m^{i+} , \tilde{\alpha}_m^j] = [\alpha_m^{i+} , \tilde{\alpha}_m^{j+}] = 0 \qquad m, n \geq 0 \qquad (3)$$

$$[\alpha_m^i , \alpha_m^{j+}] = [\tilde{\alpha}_m^i , \tilde{\alpha}_m^{j+}] = 2\alpha' m \delta_{mn} \delta^{ij}$$

We can now introduce

$$\alpha_m^i = \sqrt{2\alpha' m}\; a_m^i \quad , \quad \tilde{\alpha}_m^i = \sqrt{2\alpha' m}\; \tilde{a}_m^i \quad , \quad m \geq 1 \qquad (4)$$

and (3) reduce to

$$[a_m^i , a_m^j] = [a_m^{i+} , a_m^{j+}] = [\tilde{a}_m^i , \tilde{a}_m^j] = [\tilde{a}_m^{i+} , \tilde{a}_m^{j+}] = 0$$

$$[a_m^i , \tilde{a}_m^j] = [a_m^i , \tilde{a}_m^{j+}] = [a_m^{i+} , \tilde{a}_m^j] = [a_m^{i+} , \tilde{a}_m^{j+}] = 0 \qquad m, n \geq 1 \qquad (5)$$

$$[a_m^i , a_n^{j+}] = [\tilde{a}_m^i , \tilde{a}_n^{j+}] = \delta_{mn} \delta^{ij}$$

Now because of the disappearance of the time component the space of vectors

$$\prod_{m=1}^{\infty} \prod_{i=2}^{D-1} \prod_{j=2}^{D-1} [a_m^{i+}]^{\lambda_{mi}} [\tilde{a}_m^{j+}]^{\tilde{\lambda}_{mj}} |0\rangle \qquad (6)$$

has positive definite metric. The states of the string are hence described by purely transverse oscillators.

As before we introduce

$$\alpha_{m,-} = \frac{1}{2\alpha' p_+} \sum_{m=-\infty}^{+\infty} : \alpha^i_{n-m}\, \alpha^i_m : \,, \qquad \tilde{\alpha}_{m,-} = \frac{1}{2\alpha' p_+} \sum_{m=-\infty}^{+\infty} : \tilde{\alpha}^i_{n-m}\, \tilde{\alpha}^i_m : \tag{1}$$

which coincide with (12.4) for $n \neq 0$, but for $n=0$ there are in (12.4) normal ordering ambiguities. For the Hamiltonian, where normal ordering ambiguities appear, we have

$$H = 2\alpha' p_+ p_- = p_+ (\alpha_{0,-} + \tilde{\alpha}_{0,-}) - \alpha(0) \tag{2}$$

It is interesting to introduce

$$\hat{\alpha}^i_m \equiv \alpha^i_m \,, \qquad \hat{\tilde{\alpha}}^i_m \equiv \tilde{\alpha}^i_m \tag{3}$$

$$\hat{\alpha}_{m,-} \equiv p_+ \alpha_{m,-} \,, \qquad \hat{\tilde{\alpha}}_{m,-} \equiv p_+ \tilde{\alpha}_{m,-}$$

Then we have

$$[\, \hat{\alpha}^i_m(\lambda) \,,\, \hat{\alpha}^j_n(\lambda') \,] = (1 + \lambda\lambda')\, m\alpha' \delta^{ij}\, \delta_{m,-n} \tag{4}$$

where we have used the notation introduced in (13.5). Notice

$$\hat{\alpha}_{m,-}(\lambda) = \frac{1}{2\alpha'} \sum_{m=-\infty}^{+\infty} : \hat{\alpha}^i_{m-n}(\lambda)\, \hat{\alpha}^i_n(\lambda) : \tag{5}$$

As before we need some commutators

$$[\, \hat{\alpha}^i_m(\lambda) \,,\, \hat{\alpha}_{n,-}(\lambda') \,] = \frac{1}{2\alpha'} \sum_{k=-\infty}^{\infty} [\, \hat{\alpha}^i_m(\lambda) \,,\, : \hat{\alpha}^j_{n-k}(\lambda')\, \hat{\alpha}^j_k(\lambda') : \,] =$$

$$= \frac{1}{2\alpha'} \sum_{k=-\infty}^{+\infty} [\, \hat{\alpha}^i_m(\lambda) \,,\, \hat{\alpha}^j_{n-k}(\lambda')\, \hat{\alpha}^j_k(\lambda') \,] =$$

$$= \frac{1}{2\alpha'} \sum_{k=-\infty}^{+\infty} \delta^{ij} \{\, \hat{\alpha}^j_{n-k}(\lambda')\, (1+\lambda\lambda')\, m\alpha'\, \delta_{m,-k} + (1+\lambda\lambda')\, m\alpha'\, \delta_{m,n-m}\, \hat{\alpha}^j_k(\lambda') \,\}$$

$$= \frac{1}{2}(1+\lambda\lambda') \{\, m\, \hat{\alpha}^i_{n+m}(\lambda') + m\, \hat{\alpha}^i_{n+m}(\lambda') \,\} \Rightarrow$$

$$[\, \hat{\alpha}^i_m(\lambda) \,,\, \hat{\alpha}_{n,-}(\lambda') \,] = (1+\lambda\lambda')\, m\, \hat{\alpha}^i_{n+m}(\lambda) \tag{6}$$

Furthermore

$$[\, \hat{\alpha}_{m,-}(\lambda) \,,\, \hat{\alpha}_{n,-}(\lambda') \,] = \frac{1}{(2\alpha')} \sum_{k=+\infty}^{+\infty} [\, : \hat{\alpha}^i_{m-k}(\lambda)\, \hat{\alpha}^i_k(\lambda) : \,,\, \hat{\alpha}_{n-}(\lambda') \,]$$

$$= \frac{1}{2\alpha'} \sum_{k=-\infty}^{+\infty} [\hat{\alpha}^i_{m-k}(\lambda)\hat{\alpha}^i_k(\lambda) , \hat{\alpha}_{m-}(\lambda')] =$$

$$= \frac{1}{2\alpha'}(1+\lambda\lambda') \sum_{k=-\infty}^{+\infty} \{ k\hat{\alpha}^i_{m-k}(\lambda)\hat{\alpha}^i_{m+k}(\lambda) + (m-k)\hat{\alpha}^i_{m+m-k}(\lambda)\hat{\alpha}^i_k(\lambda) \} \quad (1)$$

But

$$\hat{\alpha}^i_\ell(\lambda)\hat{\alpha}^i_k(\lambda) = :\hat{\alpha}^i_\ell(\lambda)\hat{\alpha}^i_k(\lambda): + 2\alpha'(D-2)\,\ell\,\Theta(\ell)\,\delta_{\ell,-k} \quad (2)$$

Then

$$[\hat{\alpha}_{m,+}(\lambda), \hat{\alpha}_{m,-}(\lambda')] = \frac{1+\lambda\lambda'}{2\alpha'} \sum_{k=-\infty}^{+\infty} \{ k :\hat{\alpha}^i_{m-k}(\lambda)\hat{\alpha}^i_{m+k}(\lambda): + (m-k):\hat{\alpha}^i_{m+m-k}(\lambda)\hat{\alpha}^i_k(\lambda):$$

$$+ 2\alpha'(D-2)\,k(m-k)\,\Theta(m-k)\,\delta_{m,-m} - 2\alpha'(D-2)(m-k)\,k\,\Theta(-k)\,\delta_{m,-m}\}$$

$$= \frac{1+\lambda\lambda'}{2\alpha'} \Bigg\{ \sum_{k=-\infty}^{+\infty} \{ (k-m):\hat{\alpha}^i_{m+m-k}(\lambda)\hat{\alpha}^i_k(\lambda): + (m-k):\hat{\alpha}^i_{m+m-k}(\lambda)\hat{\alpha}^i_k(\lambda): +$$

$$+ \delta_{m,-m}\,2\alpha'(D-2) \sum_{k=1}^{m-1} k(m-k) \Bigg\} \quad (3)$$

But

$$\sum_{k=1}^{m-1} km = \frac{1}{2}m^2(m-1) \quad , \quad \sum_{k=1}^{m-1} k^2 = \frac{1}{6}(m-1)m(2m-1) \quad (4)$$

Therefore all together

$$[\hat{\alpha}^i_m(\lambda), \hat{\alpha}^j_m(\lambda')] = (1+\lambda\lambda')\,m\,\alpha'\,\delta_{ij}\,\delta_{m,-m}$$

$$[\hat{\alpha}^i_m(\lambda), \hat{\alpha}_{m,-}(\lambda')] = (1+\lambda\lambda')\,m\,\hat{\alpha}^i_{m+m}(\lambda) \quad (5)$$

$$[\hat{\alpha}_{m,-}(\lambda), \hat{\alpha}_{m,-}(\lambda')] = (1+\lambda\lambda') \Big\{ (m-m)\hat{\alpha}_{m+m,-}(\lambda) + \frac{D-2}{6} m(m^2-1)\delta_{m,-m} \Big\}$$

and

$$\hat{\alpha}_{m,-}(\lambda) = \frac{1}{2\alpha'} \sum_{m=-\infty}^{+\infty} :\hat{\alpha}_m(\lambda)\hat{\alpha}_{-m}(\lambda): \quad (6)$$

$$H = 2\alpha' P_+ P_- = \hat{\alpha}_{0,-}(-1) + \hat{\alpha}_{0,-}(+1) - \alpha(0) \quad (7)$$

VIII -A- COMPACTIFICATION

In order to illustrate the compactification phenomena let us consider the $D = 26$ closed bosonic string. We will work in the light-cone gauge. Let us now consider this string with one coordinate compactified to a circle, say $0 \leq X^{25} \leq 2\pi R$. The equations of motion for the transverse modes are

$$\ddot{X}^i(\sigma,\tau) - X''^i(\sigma,\tau) = 0 \tag{1}$$

The coordinates X^α ($\alpha = 2, 3, \ldots, 24$) all have the usual mode expansion

$$X^\alpha(\sigma,\tau) = q^\alpha + 2\tau\,\alpha_0^\alpha + \frac{i}{2}\sum_{\substack{m=-\infty \\ m\neq 0}}^{+\infty}\frac{1}{m}\left\{\alpha_m^\alpha\, e^{-2im(\tau-\sigma)} + \tilde{\alpha}_m^\alpha\, e^{-2im(\tau+\sigma)}\right\} \tag{2}$$

$$\alpha_0^\alpha \equiv \alpha' p^\alpha$$

However the expansion $X^{25}(\sigma,\tau)$ is modified. First the momentum, p^{25}, in the compact dimension is quantized

$$p^{25} \equiv \frac{S}{R} \qquad , \quad S = \text{integer} \tag{3}$$

In addition string theories give rise to a new phenomenon which has no counterpart in field theories of point particles, namely the existence of solitons which are consequences of the multiply connected structure of configuration space. The function $X^{25}(\sigma,\tau)$ maps the circle $0 \leq \sigma \leq \pi$ into the circle $0 \leq X^{25} \leq 2\pi R$, and such maps fall into homotopy classes characterized by a winding number L that counts how many times X^{25} maps around the circle. Thus

$$X^{25}(\sigma,\tau) = q^{25} + 2\alpha' p^{25}\tau + 2LR\sigma + \frac{i}{2}\sum_{\substack{m=-\infty \\ m\neq 0}}^{+\infty}\frac{1}{m}\left\{\alpha_m^{25}\, e^{-2im(\tau-\sigma)} + \tilde{\alpha}_m^{25}\, e^{-2im(\tau+\sigma)}\right\} \tag{4}$$

String configurations with non-vanishing, integer L are solitons. They must be included in the spectrum of interacting string theories, since a string with $L=0$ can split to form two strings with $L=+1$ and $L=-1$. Such solitons will always arise whenever a closed string propagates on a nonsimply connected manifold.

Let us remember that the Hamiltonian that reproduces the equation of motion is

$$H = 2\alpha' P_+ P_- = \frac{1}{4\pi\alpha'} \int_0^\pi d\sigma \left\{ [\dot{X}^i(\sigma\tau)]^2 + [X'^i(\sigma\tau)]^2 \right\} \tag{1}$$

Hence

$$H = \alpha' p^{i\,2} + \frac{1}{\alpha'} L^2 R^2 + \frac{1}{\alpha'} \sum_{m=1}^{\infty} [\alpha_{-m}^i \alpha_m^i + \tilde{\alpha}_{-m}^i \tilde{\alpha}_m^i] - 4 \tag{2}$$

where the two has appear due to normal ordering. Let us now introduce the number operators

$$N \equiv \frac{1}{2\alpha'} \sum_{m=1}^{\infty} \alpha_{-m}^i \alpha_m^i \tag{3}$$

$$\tilde{N} \equiv \frac{1}{2\alpha'} \sum_{m=1}^{\infty} \tilde{\alpha}_{-m}^i \tilde{\alpha}_m^i$$

and hence

$$H = \alpha' (p^a)^2 + \alpha' \frac{S^2}{R^2} + \frac{1}{\alpha'} L^2 R^2 + 2N + 2\tilde{N} - 4 \tag{4}$$

Now we define the $(\text{mass})^2$ operator for the effective 25-dimensional theory as

$$M^2 \equiv 2 P_+ P_- - (p^a)^2 \Rightarrow$$

$$\alpha' M^2 = 2N + 2\tilde{N} + \alpha' \frac{S^2}{R^2} + \frac{1}{\alpha'} L^2 R^2 - 4 \tag{5}$$

Let us return to the normal space time, then

$$[N - \tilde{N}, X^i(\sigma\tau)] = \frac{1}{2\alpha'} \sum_{k=1}^{m} [\alpha_{-m}^k \alpha_m^k - \tilde{\alpha}_{-m}^k \tilde{\alpha}_m^k, q^i + 2\tau\alpha_0^i +$$

$$+ \frac{i}{2} \sum_{\substack{m=-\infty \\ m\neq 0}}^{+\infty} \frac{1}{m} [\alpha_m^i e^{-2im(\tau-\sigma)} + \tilde{\alpha}_m^i e^{-2im(\tau+\sigma)}]]$$

$$= \frac{i}{2} \frac{1}{2\alpha'} \sum_{m=1}^{\infty} \sum_{\substack{m=-\infty \\ m\neq 0}}^{+\infty} \frac{1}{m} \left\{ \alpha_{-m}^k [\alpha_m^k, \alpha_m^i] e^{-2im(\tau-\sigma)} + [\alpha_{-m}^k, \alpha_m^i] \alpha_m^k e^{-2im(\tau-\sigma)} \right.$$

$$\left. - \tilde{\alpha}_{-m}^k [\tilde{\alpha}_m^k, \tilde{\alpha}_m^i] e^{-2im(\tau+\sigma)} - [\tilde{\alpha}_{-m}^k, \tilde{\alpha}_{m\,i}] \tilde{\alpha}_m^k e^{-2im(\tau-\sigma)} \right\}$$

$$= \frac{i}{2} \sum_{n=1}^{\infty} \left\{ - \alpha_{-n}^{i} e^{+2in(\tau-\sigma)} - \alpha_{rn}^{i} e^{-2in(\tau-\sigma)} + \tilde{\alpha}_{-n}^{i} e^{+2in(\tau+\sigma)} + \tilde{\alpha}_{n}^{i} e^{-2in(\tau+\sigma)} \right\}$$

$$= \frac{i}{2} \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \left\{ - \alpha_{n}^{i} e^{-2in(\tau-\sigma)} + \tilde{\alpha}_{n}^{i} e^{-2in(\tau+\sigma)} \right\} \Rightarrow$$

$$[N - \tilde{N}, X^{i}(\sigma,\tau)] = \frac{i}{2} \frac{\partial X^{i}(\sigma,\tau)}{\partial \sigma} \qquad (1)$$

and therefore

$$e^{-2i\Delta(N-\tilde{N})} X^{i}(\sigma,\tau) e^{+2i\Delta(N-\tilde{N})} = X^{i}(\sigma+\Delta,\tau) \qquad (2)$$

But the operator $U(\Delta) = \exp[2i\Delta(N-\tilde{N})]$, which shifts σ by an amount Δ, is equal to the identity operator on physical states, thus ensuring the absence of distinguished point on the string and therefore $N = \tilde{N}$. Let us return to the compactified case with winding number L. Then equation (1) is true for $i = 2, \ldots 24$. and in order that is true also for $i = 25$ we must write

$$[N - \tilde{N} - LR p^{25}, X^{i}(\sigma,\tau)] = \frac{i}{2} \frac{\partial X^{i}(\sigma,\tau)}{\partial \sigma} \qquad (3)$$

and therefore

$$N = \tilde{N} + LS \qquad (4)$$

and hence

$$M^2 = 4N - 2LS + \alpha' \frac{S^2}{R^2} + \frac{1}{\alpha'} L^2 R^2 - 4 \qquad (5)$$

Let us now examine the massless spectrum of the effective 25-dimensional theory. This corresponds to the states

$$\tilde{\alpha}_{-1}^{i} \alpha_{-1}^{j} |0\rangle \quad , \quad 2 \leq i, j \leq 24 \qquad (6)$$

($N = \tilde{N} = 1$) which contains a scalar particle (the dilaton), a second order antisymmetric tensor and a second order traceless symmetric tensor (the graviton). Furthermore there are also massless

$$\tilde{\alpha}_{-1}^{i} \alpha_{-1}^{25} |0\rangle \quad , \quad \alpha_{-1}^{i} \tilde{\alpha}_{-1}^{25} |0\rangle \qquad (7)$$

that corresponds to the gauge bosons expected from the usual Kaluza-Klein compactification on a circle with isometry $U(1)$.

However for the particular choice of the radius of the circle

$$R \equiv \sqrt{\alpha'} \tag{1}$$

the solitons give origem to four more massless vectors

$$\alpha_{-1}^{i} \, | M = L = \pm 1 \rangle \quad , \quad \tilde{\alpha}_{-1}^{i} \, | M = -L = \pm 1 \rangle \tag{2}$$

and therefore we have a total of six massless vectors, which can be classified according to the adjoint representation of $SU(2) \otimes SU(2)$, with the diagonal generators having eigenvalues $\frac{1}{2}(M+L)$ and $\frac{1}{2}(M-L)$. This construction is a physical realization of the work of Frenkel and Kac on representations of Kac-Moody Lie algebras. Their work suggests that this construction can be generalized to produce massless vector mesons in the adjoint representation of any simple laced Lie group G, i.e. a Lie group whose root vectors all have the same length. This restricts G to $SU(N)$, $SO(2N)$, E_6, E_7, E_8 or direct products of these.

IX - THE INTERACTING STRING

We shall choose the following normalization of the fully connected part of the S-matrix

$$\langle k_1 \dots k_j \,|S|\, k_{j+1} \dots k_N \rangle \equiv$$

$$\equiv -i\,(2\pi)^D\,\delta^{(D)}\Big(\sum_{i=1}^{j} k_i + \sum_{i=j+1}^{N} k_i\Big)\,\langle k_1 \dots k_j \,|T|\, k_{j+1} \dots k_N \rangle \qquad (1)$$

To compute the invariant scattering amplitude we shall define the propagator. In the usual field theory the propagator for the scalar particle is given by

$$\frac{i}{(2\pi)^D}\,\frac{1}{p^2 - M^2 + i\epsilon} \qquad (2)$$

For the open string and in covariant quantization we have from (IV-1.7) and (VII-18.1)

$$\alpha' M^2 = -\alpha(0) - \sum_{m=1}^{\infty} m\,(a_m^+ \cdot a_m)$$

$$L_0 = -\alpha' p^2 - \sum_{m=1}^{\infty} m\,(a_m^+ \cdot a_m) \qquad (3)$$

and hence

$$\alpha'\,(p^2 - M^2) = -\,[L_0 - \alpha(0)] = -\,(L_0 - 1) \qquad (4)$$

Therefore we define the string propagator as

$$D(p^2) = -\,\frac{i\alpha'}{(2\pi)^D}\,\frac{1}{L_0 - 1 + i\epsilon} \qquad (5)$$

We see that whenever p^2 equals one of the eigenvalues of the M^2 operator, the propagator blows up and the term $i\epsilon$ is needed. The propagator can be represented in two different ways:



(a) (b) (6)

In Fig (6a) we remember that this is a string propagator and we represent the two boundaries of that string. The string itself spans through its evolution the blanch spoe in between. Diagrams

built in accordance with this rule are usually called dual diagrams. The orientation of the lines is added in order to be able to compute the factors arising from internal symmetries, as we shall see below. In Fig (1.6b) the string picture has been forgotten and we simply represent the propagator by a line. One obtains then a set of diagrams called Feynman like diagrams (F.L.D.)

Notice that the propagator can also be written as

$$D(p^2) = -\frac{i\alpha'}{(2\pi)^D} \int_0^1 du\, u^{L_0 - 2} \tag{1}$$

Now we need to introduce a vertex for a three string interaction. We will define it as

$$\mathcal{V}(k_i) \equiv \frac{i\,g}{|\alpha'|^\beta} (2\pi)^D V(k_i; 1) \tag{2}$$

$$V(k_i; z) \equiv : \exp\left\{ -i \sum k_i^\mu Q_\mu(z) \right\}$$

where k_i^μ is the momentum of the incoming scalar. The coupling constant g is a dimensionless quantity and $\beta \equiv (6-D)/4$. Two possible representations of the vertex are shown in next figure: a) in terms



(3)

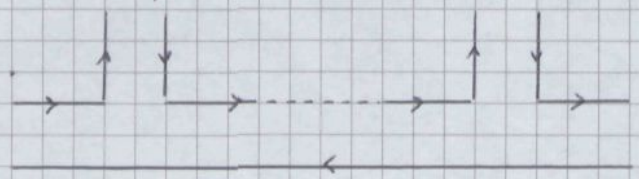
(a) (b)

of quark lines and b) where only the absorption of a scalar is indicated in the F.L.D. In the following we shall work with the dimensionless quantities $1/(L_0 - 1)$ and $V(k_i)$ and leave all normalization factor aside.

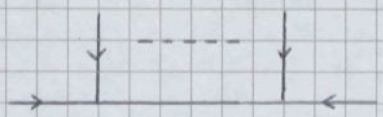
We first shall concentrate on tree-graphs, i.e. those which do not contain any loops, and more particularly "multiperipheral" trees

$$B_N \equiv \langle 0, -k_1 | V(k_2, 1) \frac{1}{L_0 - 1} V(k_3, 1) \frac{1}{L_0 - 1} \cdots \frac{1}{L_0 - 1} V(k_{N-1}, 1) | 0, k_N \rangle \tag{4}$$

where $|0, k_N\rangle$ is an incoming ground state scalar on the right and $\langle 0, -k_1|$ is an incoming ground state scalar on the left. So that such a graph is represented in the next figures:



(a)



(b)

By using the formalism now developed we can write the N-point function B_N in a form which exhibits explicitly both cyclic symmetry and factorizability. Using (2.1)

$$B_N = \int_0^1 \frac{dz_3}{z_3} \int_0^1 \frac{dz_4}{z_4} \cdots \int_0^1 \frac{dz_{N-1}}{z_{N-1}} \langle 0, -k_1 | V(k_2, 1) z_3^{L_0 - 1} V(k_3, 1) z_4^{L_0 - 1} \cdots z_{N-1}^{L_0 - 1} V(k_{N-1}, 1) | 0, k_N \rangle \quad (1)$$

Now using (VII-A-5.3)

$$z^{L_0} V(k, 1) z^{-L_0} = V(k, z) \quad (2)$$

and

$$(L_0 - 1) | 0, k_N \rangle = 0 \quad (3)$$

we get

$$B_N = \int_0^1 \frac{dz_3}{z_3} \int_0^1 \frac{dz_4}{z_4} \cdots \int_0^1 \frac{dz_{N-1}}{z_{N-1}} \langle 0, -k_1 | V(k_2, 1) V(k_3, z_3) V(k_4, z_3 z_4) \cdots$$

$$\cdots V(k_{N-2}, z_3 \cdots z_{N-2}) V(k_{N-1}, z_3 \cdots z_{N-1}) | 0, k_N \rangle \quad (4)$$

The change of variables

$$y_3 = z_3 \,, \quad y_4 = z_3 z_4 \,, \quad \cdots \quad y_{N-1} = z_3 z_4 \cdots z_{N-1} \quad (5)$$

leads to an integration region described by

$$1 \geq y_3 \geq y_4 \geq \cdots \geq y_{N-1} \geq 0 \quad (6)$$

and the Jacobian of the transformation is

$$\frac{\partial (y_3 \cdots y_{N-1})}{\partial (z_3 \cdots z_{N-1})} = \begin{vmatrix} 1 & z_4 & z_4 z_5 & \cdots & z_4 z_5 \cdots z_{N-2} & z_4 z_5 \cdots z_{N-1} \\ 0 & z_3 & z_3 z_5 & \cdots & z_3 z_5 \cdots z_{N-2} & z_3 z_5 \cdots z_{N-1} \\ 0 & 0 & z_3 z_4 & \cdots & z_3 z_4 z_6 \cdots z_{N-2} & z_3 z_4 z_6 \cdots z_{N-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & z_3 \cdots z_{N-3} & z_3 \cdots z_{N-3} z_{N-1} \\ 0 & 0 & 0 & \cdots & 0 & z_3 \cdots z_{N-2} \end{vmatrix} \Rightarrow$$

$$\frac{\partial(y_3, y_4, \dots, y_{N-1})}{\partial(z_3, z_4, \dots, z_{N-1})} = \frac{y_3\ y_4 \dots y_{N-1}}{z_3\ z_4 \dots z_{N-1}} \tag{1}$$

Hence

$$B_N = \int_0^1 \frac{dy_3}{y_3} \cdots \int_0^1 \frac{dy_{N-1}}{y_{N-1}}\ \theta(1-y_3)\,\theta(y_3-y_4) \cdots \theta(y_{N-2}-y_{N-1})\,\theta(y_{N-1})$$

$$\langle 0, -k_1 |\ V(k_2, 1)\ V(k_3, y_3)\ V(k_4, y_4) \cdots V(k_{N-1}, y_{N-1})\ |0, k_N\rangle \tag{2}$$

As it is desirable to describe all the external lines in as a symmetrical a way as possible, we should try to make $V(k_1, y_1)$ and $V(k_N, y_N)$ appear in the formula. Notice

$$\lim_{y_N \to 0} \frac{1}{y_N}\ V(k_N, y_N)\ |0,0\rangle =$$

$$= \lim_{y_N \to 0} \frac{1}{y_N} \exp\left(-i\,k_N \cdot Q^{(+)}(y_N)\right) \exp\left(-i\,k_N \cdot Q^{(0)}(y_N)\right) \exp\left(-i\,k_N \cdot Q^{(-)}(y_N)\right) |0,0\rangle =$$

$$= \lim_{y_N \to 0} \frac{1}{y_N} \exp\left\{-i\,(k_N \cdot q) - 2\alpha'\,(k_N \cdot p)\ln y_N\right\} |0,0\rangle$$

$$= \lim_{y_N \to 0} \frac{1}{y_N} \exp\left\{-\alpha' k_N^2 \ln y_N\right\} \exp\left\{-i\,(k_N \cdot q)\right\} \exp\left\{-2\alpha'\,(k_N \cdot p)\ln y_N\right\} |0,0\rangle$$

If we assume that this particles are such that

$$\alpha' k_i^2 = -1 \tag{3}$$

we get

$$\lim_{y_N \to 0} \frac{1}{y_N}\ V(k_N, y_N)\ |0,0\rangle = \lim_{y_N \to 0} \exp\left[-i\,(k_N \cdot q)\right] \exp\left[-2\alpha' \ln y_N \cdot (k_N \cdot p)\right] |0,0\rangle$$

$$= \lim_{y_N \to 0} \exp\left\{-i\,(k_N \cdot q)\right\} |0,0\rangle = \exp\left\{-i\,(k_N \cdot q)\right\} |0,0\rangle = |0, k_N\rangle \tag{4}$$

Similarly

$$\lim_{y_1 \to \infty} \langle 0, 0|\ y_1\ V(k_1, y_1) =$$

$$= \lim_{y_1 \to \infty} y_1\ \langle 0,0|\ \exp\left\{-i\,k_1 \cdot Q^{(+)}(y_1)\right\} \exp\left\{-i\,k_1 \cdot Q^{(0)}(y_1)\right\} \exp\left\{-i\,k_1 \cdot Q^{(-)}(y_1)\right\} =$$

$$= \lim_{y_1 \to \infty} y_1\ \langle 0,0|\ \exp\left\{-i\,(k_1 \cdot q) - 2\alpha'\,(k_1 \cdot p)\ln y_1\right\} =$$

$$\lim_{y_1 \to \infty} y_1 \langle 0,0| \exp\{\alpha' k_1^2 \ln y_1\} \exp\{-2\alpha'(k_1 \cdot p) \ln y_1\} \exp\{-i(k_1 \cdot q)\} =$$

$$= \lim_{y_1 \to \infty} \langle 0,0| \exp\{-i(k \cdot q)\} = \langle 0, -k_1 \qquad (1)$$

i.e.

$$\lim_{y_N \to 0} \frac{1}{y_N} V(k_N, y_N)|0,0\rangle = e^{-i(k_N \cdot q)}|0,0\rangle \equiv |0, k_N\rangle \qquad (2)$$

$$\lim_{y_1 \to \infty} y_1 \langle 0,0| V(k_1, y_1) = \langle 0,0| e^{-i(k_1 \cdot q)} \equiv \langle 0, -k_1|$$

Using these expressions we get

$$B_N = \lim_{\substack{y_1 \to \infty,\ y_2 = 1 \\ y_N \to 0}} \int \left\{ \prod_{i=3}^{N-1} \frac{dy_i}{y_i} \right\} \left\{ \prod_{j=2}^{N-1} \theta(y_j - y_{j+1}) \right\} \frac{y_1}{y_N}$$

$$\cdot \langle 0,0| V(k_1, y_1)\, V(k_2, y_2)\, V(k_3, y_3) \cdots V(k_{N-1}, y_{N-1})\, V(k_N, y_N)|0,0\rangle \qquad (3)$$

where $y_i \in [0,1]$ $\quad i = 3, \ldots N-1$.

Let us now try to evaluate this expected value. We will need

$$\exp\{-i k_i \cdot Q^{(0)}(y_i)\} = y_i^{-\alpha' k_i^2} \exp\{-i k_i \cdot q\} \exp\{-(2\alpha')(p \cdot k_i) \ln y_i\} \qquad (4)$$

$$\exp\{-i k_i \cdot Q^{(-)}(y_i)\} \cdot \exp\{-i k_{i+1} \cdot Q^{(+)}(y_{i+1})\} =$$

$$= \left(1 - \frac{y_{i+1}}{y_i}\right)^{-2\alpha'(k_i \cdot k_{i+1})} \exp\{-i k_{i+1} \cdot Q^{(+)}(y_{i+1})\} \exp\{-i k_i \cdot Q^{(-)}(y_i)\} \qquad (5)$$

which are easily obtained using the Campbell-Hausdorff identity (VII-A-3.5) and the commutation relations (VII-A-1.3). Now let us compute

$$A \equiv \langle 0,0| V(k_1, y_1)\, V(k_2, y_2)\, V(k_3, y_3)|0,0\rangle \quad , \quad \alpha' k_i^2 = -1 \quad , \quad k_1 + k_2 + k_3 = 0 \qquad (6)$$

We have using (4) and (5) repeatedly

$$A = \langle 0,0| V(k_1, y_1)\, V(k_2, y_2) \exp\{-i k_3 \cdot Q^{(+)}(y_3)\} \exp\{-i k_3 \cdot Q^{(0)}(y_3)\} \exp\{-i k_3 \cdot Q^{(-)}(y_3)\}|0,0\rangle$$

$$= \langle 0,0| V(k_1, y_1)\, V(k_2, y_2) \exp\{-i k_3 \cdot Q^{(+)}(y_3)\} \exp\{-i k_3 \cdot Q^{(0)}(y_3)\}|0,0\rangle =$$

$$= y_3 \langle 0,0| V(k_1,y_1)\, V(k_2,y_2)\, \exp\{-i k_3 Q^{(+)}(y_3)\}\, |0,k_3\rangle =$$

$$= y_3 \langle 0,0| V(k_1,y_1) \exp\{-i k_2 Q^{(+)}(y_2)\} \exp\{-i k_2 Q^{(0)}(y_2)\} \exp\{-i k_2 Q^{(-)}(y_2)\} \exp\{-i k_3 Q^{(+)}(y_3)\} |0,k_3\rangle$$

$$= y_3 \left(1-\frac{y_3}{y_2}\right)^{-2\alpha'(k_2\cdot k_3)} \langle 0,0| V(k_1,y_1) \exp\{-i k_2 Q^{(+)}(y_2)\} \exp\{-i k_2 Q^{(0)}(y_2)\} \exp\{-i k_3 Q^{(+)}(y_3)\} |0,k_3\rangle =$$

$$= y_3 \left(1-\frac{y_3}{y_2}\right)^{-2\alpha'(k_2\cdot k_3)} \langle 0,0| V(k_1,y_1) \exp\{-i k_2 Q^{(+)}(y_2)\} \exp\{-i k_3 Q^{(+)}(y_3)\} \exp\{-i k_2 Q^{(0)}(y_2)\} |0,k_3\rangle$$

$$= y_2 y_3 \left(1-\frac{y_3}{y_2}\right)^{-2\alpha'(k_2\cdot k_3)} y_2^{-2\alpha'(k_2\cdot k_3)} \langle 0,0| V(k_1,y_1) \exp\{-i k_2 Q^{(+)}(y_2)\} \exp\{-i k_3 Q^{(+)}(y_3)\} |0,k_2+k_3\rangle$$

$$= y_2 y_3 (y_2-y_3)^{-2\alpha'(k_2\cdot k_3)} \langle 0,0| \exp\{-i k_1 Q^{(+)}(y_1)\} \exp\{-i k_1 Q^{(0)}(y_1)\} \exp\{-i k_1 Q^{(-)}(y_1)\} \cdot$$

$$\cdot \exp\{-i k_2 Q^{(+)}(y_2)\} \exp\{-i k_3 Q^{(+)}(y_3)\} |0,k_2+k_3\rangle$$

$$= y_2 y_3 (y_2-y_3)^{-2\alpha'(k_2\cdot k_3)} \left(1-\frac{y_2}{y_1}\right)^{-2\alpha'(k_1\cdot k_2)} \langle 0,0| \exp\{-i k_1 Q^{(0)}(y_1)\} \exp\{-i k_2 Q^{(+)}(y_2)\} \cdot$$

$$\cdot \exp\{-i k_1 Q^{(-)}(y_1)\} \exp\{-i k_3 Q^{(+)}(y_3)\} |0,k_2+k_3\rangle$$

$$= y_2 y_3 (y_2-y_3)^{-2\alpha'(k_2\cdot k_3)} \left(1-\frac{y_2}{y_1}\right)^{-2\alpha'(k_1\cdot k_2)} \left(1-\frac{y_3}{y_1}\right)^{-2\alpha'(k_1\cdot k_3)} \cdot$$

$$\cdot \langle 0,0| \exp\{-i k_1 Q^{(0)}(y_1)\} \exp\{-i k_2 Q^{(+)}(y_2)\} \cdot \exp\{-i k_3 Q^{(+)}(y_3)\} |0,k_2+k_3\rangle =$$

$$= y_2 y_3 (y_2-y_3)^{-2\alpha'(k_2\cdot k_3)} \left(1-\frac{y_2}{y_1}\right)^{-2\alpha'(k_1\cdot k_2)} \left(1-\frac{y_3}{y_1}\right)^{-2\alpha'(k_1\cdot k_3)} \langle 0,0| \exp\{-i k_1 Q^{(0)}(y_1)\} |0,k_2+k_3\rangle$$

$$= y_1 y_2 y_3 (y_2-y_3)^{-2\alpha'(k_2\cdot k_3)} \left(1-\frac{y_2}{y_1}\right)^{-2\alpha'(k_1\cdot k_2)} \left(1-\frac{y_3}{y_1}\right)^{-2\alpha'(k_1\cdot k_3)} y_1^{-2\alpha' k_1(k_2+k_3)} \langle 0,0|\, 0,\, k_1+k_2+k_3\rangle$$

Hence

$$\langle 0,0|\, V(k_1,y_1)\, V(k_2,y_2)\, V(k_3,y_3)\, |0,0\rangle = y_1 y_2 y_3 (y_1-y_2)^{-2\alpha' k_1\cdot k_2} (y_1-y_3)^{-2\alpha' k_1\cdot k_3} (y_2-y_3)^{-2\alpha' k_2\cdot k_3} \qquad (1)$$

and in general

$$\langle 0,0|\, \frac{V(k_1,y_1)}{y_1} \cdots \frac{V(k_N,y_N)}{y_N} |0,0\rangle = \prod_{i<j=1}^{N} (y_i-y_j)^{-2\alpha'(k_i\cdot k_j)} \qquad (2)$$

From this we obtain

$$B_N = \lim_{\substack{y_1\to\infty,\ y_2=1\\ y_N\to 0}} y_1^2\, y_2 \int_0^1 \left\{ \prod_{i=3}^{N-1} dy_i \right\} \left\{ \prod_{j=2}^{N-1} \theta(y_j - y_{j+1}) \right\} \cdot$$

$$\cdot \left\{ \prod_{1=i<j}^{N} (y_i - y_j)^{-2\alpha'(k_i k_j)} \right\} \tag{1}$$

Let us now try to evaluate B_4.

$$B_4 = \lim_{\substack{y_1\to\infty,\ y_2=1\\ y_4\to 0}} y_1^2 \int_0^1 dy_3\ (y_1-y_2)^{-2\alpha'(k_1 k_2)} (y_1-y_3)^{-2\alpha'(k_1 k_3)} (y_1-y_4)^{-2\alpha'(k_1 k_4)}$$

$$\cdot (y_2-y_3)^{-2\alpha'(k_2 k_3)} (y_2-y_4)^{-2\alpha'(k_2 k_4)} (y_3-y_4)^{-2\alpha'(k_3 k_4)}$$

$$= \lim_{\substack{y_1\to\infty,\ y_2=1\\ y_4\to 0}} y_1^2\ y_1^{-2\alpha' k_1 (k_2+k_3+k_4)} \int_0^1 dy_3\ (1-y_3)^{-2\alpha'(k_2 k_3)}\ y_3^{-2\alpha'(k_3 k_4)}$$

$$= \int_0^1 dy\ (1-y)^{-2\alpha'(k_2 k_3)}\ y^{-2\alpha'(k_3 k_4)} \tag{2}$$

Remember that

$$\int_0^1 dt\ t^{x-1} (1-t)^{y-1} = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)} \tag{3}$$

Let us now apply this formula to (3)

$$x = 1 - 2\alpha'(k_3 k_4) = 1 - \alpha'\left[s - k_3^2 - k_4^2\right] = -1 - \alpha' s \equiv -\alpha(s)$$

$$y = 1 - 2\alpha'(k_2 k_3) = 1 - \alpha'\left[t - k_2^2 - k_3^2\right] = -1 - \alpha' t = -\alpha(t) \tag{4}$$

where we used

$$s \equiv (k_1+k_2)^2 = (k_3+k_4)^2 \quad , \quad t = (k_1+k_4)^2 = (k_2+k_3)^2 \tag{5}$$

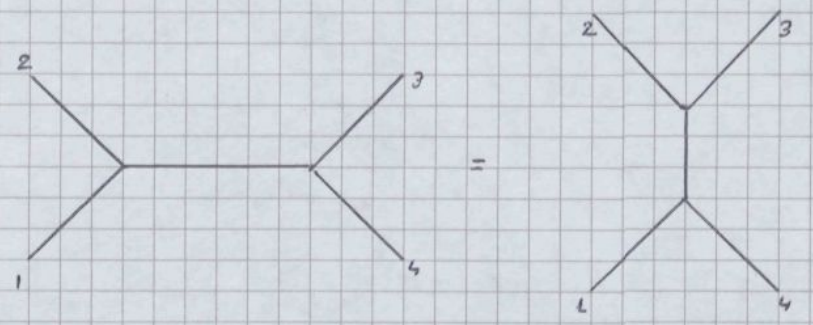
$$\alpha(x) = \alpha(0) + \alpha' x = 1 + \alpha' x$$

Hence

$$B_4 = \frac{\Gamma[-\alpha(s)]\ \Gamma[-\alpha(t)]}{\Gamma[-\alpha(s)-\alpha(t)]} \tag{6}$$

which is the Veneziano formula. This four-point function is symmetric

under the exchange of s and t. This was not imposed a priori but it is a consequence of our choice of vertices. Our FLD's hence have properties which are very much unlike those of ordinary Feynman graphs.



(1)

In field theory both diagrams have to be added and contribute to the S-matrix. In dual models, only one of them must be included because in contains these two diagrams in a "hidden" way.

Let us consider some general properties of B_N. From (VII-15.2) it is clear that the operators L_1, L_0, L_{-1} form an SU(1,1) subalgebra of the Virasoro algebra:

$$[L_1, L_0] = L_1 \quad , \quad [L_{-1}, L_0] = -L_{-1} \quad , \quad [L_{+1}, L_{-1}] = 2L_0 \tag{2}$$

Let us now remember (IH-A-6.2) which implies

$$e^{\lambda L_m} \left\{ \frac{V(k,z)}{z} \right\} e^{-\lambda L_m} = \left(\frac{z'}{z} \right)^{m+1} \frac{V(k,z')}{z'} \tag{3}$$

and in particular

$$e^{\lambda L_{-1}} \left\{ \frac{V(k,z)}{z} \right\} e^{-\lambda L_{-1}} = \left\{ \frac{V(k,z')}{z'} \right\} \quad , \quad z' = z + \lambda$$

$$e^{\lambda L_0} \left\{ \frac{V(k,z)}{z} \right\} e^{-\lambda L_0} = \left(e^{\lambda/2} \right)^2 \left\{ \frac{V(k,z')}{z'} \right\} \quad , \quad z' = \frac{e^{\lambda/2} z}{e^{-\lambda/2}} \tag{4}$$

$$e^{\lambda L_1} \left\{ \frac{V(k,z)}{z} \right\} e^{-\lambda L_1} = (1+\lambda z')^2 \left\{ \frac{V(k,z')}{z'} \right\} \quad , \quad z' = \frac{z}{-\lambda z + 1}$$

and therefore in all this cases we have

$$e^{\lambda L_m} \left\{ \frac{V(k,z)}{z} \right\} e^{-\lambda L_m} = (a - cz')^2 \left\{ \frac{V(k,z')}{z'} \right\} \tag{5}$$

$$z' = \frac{az + b}{cz + d} \quad , \quad ad - bc = 1$$

From all that we see that if

$$\Lambda(\xi) \equiv \exp\left\{ i \sum_{m=-1}^{+1} \xi_m L_m \right\} \tag{1}$$

then

$$\Lambda(\xi) \left\{ \frac{V(k,z)}{z} \right\} \Lambda^{-1}(\xi) = (a - cz')^2 \left\{ \frac{V(k,z')}{z'} \right\} \tag{2}$$

$$z' = \frac{az + b}{cz + d} \quad , \quad ad - bc = 1$$

and therefore $\Lambda(\xi)$ induces Möbius transformations. Let us now consider

$$A(z,k) \equiv \left\{ \prod_{i=1}^{N} \frac{|z_i - z_{i+1}|}{z_i} \right\} \langle 0,0 | V(k,z_1) \dots V(k_N z_N) | 0,0 \rangle \quad , \quad z_{N+1} \equiv z_1 \tag{3}$$

Then

$$\Lambda(\xi) A(z,k) \Lambda^{-1}(\xi) = \left\{ \prod_{i=1}^{N} \frac{|z_i - z_{i+1}|}{|z_i' - z_{i+1}'|} (a - cz_i')^2 \right\} A(z',k) \tag{4}$$

but

$$\frac{|z_i - z_{i+1}|}{|z_i' - z_{i+1}'|} (a - cz_i')^2 = \frac{cz_{i+1} + d}{cz_i + d} \tag{5}$$

and therefore

$$\Lambda(\xi) A(z,k) \Lambda^{-1}(\xi) = A(z',k) \tag{6}$$

i.e. the quantity (3) is invariant under a common Möbius transformation of all the z variables.

In order to express B_N in a completely Möbius-invariant form we need an invariant integration measure. We choose any three of the variables z to be unintegrated and set equal to definite fixed values. Then formally we introduce the measure

$$d\mu_N(z) \equiv \frac{(z_a - z_b)(z_a - z_c)(z_b - z_c)}{dz_a\, dz_b\, dz_c} \prod_{i=1}^{N+1} \theta(z_i - z_{i+1}) \prod_{j=1}^{N} \frac{dz_j}{|z_j - z_{j+1}|} \tag{7}$$

since

$$z_i' - z_{i+1}' = \frac{z_i - z_{i+1}}{(cz_i + d)(cz_{i+1} + d)} \quad , \quad dz_i' = \frac{dz_i}{(cz_i + d)^2} \tag{8}$$

it is easy to probe that $\mu_N(z)$ is Möbius invariant. Then using (9.3) and (9.7) we can write

$$B_N = \int d\mu_N(z) \left\{ \prod_{i=1}^{N} \frac{|z_i - z_{i+1}|}{z_i} \right\} \langle 0,0| V(k_1,z_1) \cdots V(k_N z_N) |0,0\rangle \qquad (1)$$

This form of B_N reduces to equation (5.3) when we let $z_1 \to \infty$, $z_2 \to 1$, $z_N \to 0$, which can be implemented by a Möbius transformation.

Substituting in (1) the identity (6.2) and using (9.7) we obtain

$$B_N = \frac{(z_a - z_b)(z_a - z_c)(z_b - z_c)}{dz_c\, dz_b\, dz_c} \int dz_1\, dz_2 \cdots dz_N\ \theta(z_1 - z_2) \cdots \theta(z_{N-1} - z_N) \cdot \prod_{1=i<j}^{N} (z_i - z_j)^{-2\alpha'(k_i \cdot k_j)} \qquad (2)$$

Which is the so called Koba - Nielsen form [Z. Koba and H.B. Nielsen Nucl. Phys. B12, 517 (1969)]. Furthermore let us introduce

$$z_i = e^{i\theta_i} \quad \Rightarrow \quad z_i - z_j = 2i\, e^{i(\theta_i + \theta_j)/2} \sin\frac{\theta_i - \theta_j}{2} \qquad (3)$$

then we can write

$$B_N = \frac{\sin[(\theta_a - \theta_b)/2]\, \sin[(\theta_a - \theta_c)/2]\, \sin[(\theta_b - \theta_c)/2]}{d\theta_a\, d\theta_b\, d\theta_c} \cdot \int_{\substack{\text{cyclic} \\ \text{ordering}}} d\theta_1 \cdots d\theta_N \prod_{1=i<j}^{N} \left\{ \sin\frac{\theta_i - \theta_j}{2} \right\}^{-2\alpha'(k_i \cdot k_j)} \qquad (4)$$

and in this form cyclic symmetry is manifest.

The cyclic symmetry of B_N can also be understood in terms of the algebraic properties of equation (1). Consider the Möbius transformation defined by the conditions $z'_{N-1} = z_N$, $z'_N = z_1$, and $z'_1 = z_2$. This transformations brings B_N to the form

$$B_N = \int d\mu_N(z') \left\{ \prod_{i=1}^{N} \frac{|z'_i - z'_{i+1}|}{z'_i} \right\} \langle 0,0| V(k_1, z_2)\, V(k_2 z_3) \cdots V(k_{N-1} z_N)\, V(k_N z_1) |0,0\rangle \qquad (5)$$

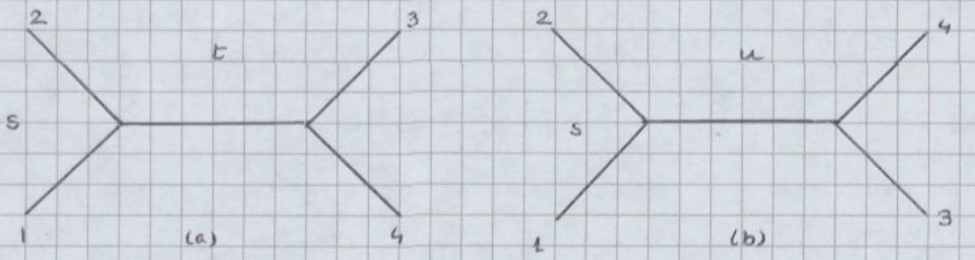
Cyclic symmetry now follows if we can push the factor $V(k_N, z_1)$ to the left past the other vertex operator since then we are back to the original formula with each momentum vector replaced by the next adjacent one.

When (VII-A-6.6) is applied

$$V(k_1 z_2)\, V(k_2 z_3) \cdots V(k_N z_1) = V(k_N z_1)\, V(k_1 z_2) \cdots V(k_{N-1} z_N) \exp\left\{ -2\pi i \alpha' k_N \sum_{i=1}^{N-1} k_i \right\} \tag{1}$$

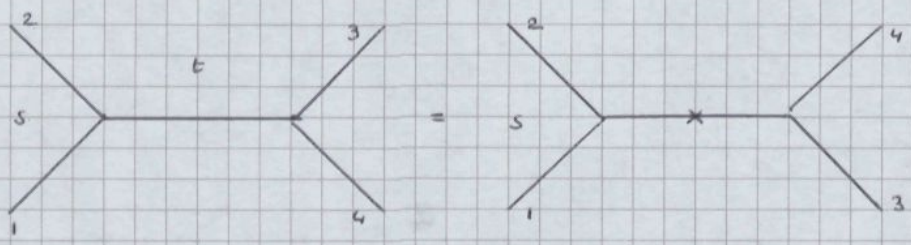
But since the sum of all momenta is zero and $\alpha' k_N^2 = -1$ the last factor drops out, and the cyclic invariance is proved.

A useful concept in planar models is the "twist". It plays an essential role in the construction of the higher order diagrams required for unitarization. In a conventional field theory the two diagrams shown



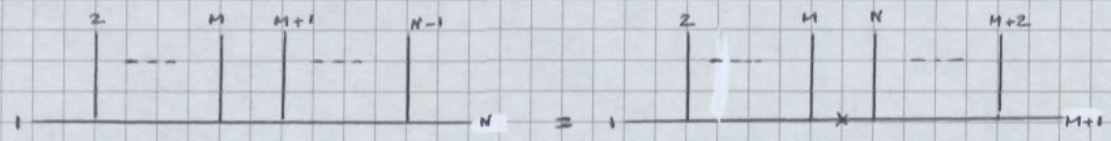
(2)

in figure (2) are not distinguished. However, in a dual model a distinction is clearly called for since fig (2a) has poles in s and t channels, while fig (2.b) has poles in the s and u channels. Therefore we indicate a twisted line in the way shown in Fig (3). This notation also applies



(3)

to N-particle amplitudes. In the case of multiperipheral trees, for example, one has the relation shown in fig (4). It turns out to be useful



(4)

to construct an operator Ω to implement the twisting of a line. Specifically we desire

$$\Omega\, V(k_1, 1) \otimes V(k_2, 1) \otimes \cdots \otimes V(k_{N-1}, 1) \,|0, k_N\rangle =$$

$$= V(k_N, 1) \otimes V(k_{N-1}, 1) \otimes \cdots \otimes V(k_2, 1) \,|0, k_1\rangle \tag{5}$$

Let us see that the appropiate operator is

$$\Omega \equiv (-1)^R \, e^{-L_{-1}} \quad , \quad R \equiv L_0 + \alpha' p^2 \tag{1}$$

In analogy with (5.3) we can write

$$V(k_1, 1) \, D \, V(k_2, 1) \, D \cdots D \, V(k_{N-1}, 1) \, |0, k_N\rangle$$

$$= \lim_{\substack{y_1 \to 1 \\ y_N \to 0}} \int dy_2 \cdots dy_{N-1} \prod_{i=1}^{N-1} \theta(y_i - y_{i+1}) \, \frac{V(k, y_1)}{y_1} \, \frac{V(k_2 y_2)}{y_2} \cdots \frac{V(k_N y_N)}{y_N} \, |0, 0\rangle \tag{2}$$

From (VII-A-6.2)

$$e^{-L_{-1}} \, \frac{V(k y)}{y} \, e^{+L_{-1}} = \frac{V(k, y-1)}{y-1} \tag{3}$$

and hence

$$\Omega \, \frac{V(k, y_1)}{y_1} \, \frac{V(k_2 y_2)}{y_2} \cdots \frac{V(k_N y_N)}{y_N} \, |0, 0\rangle =$$

$$= (-1)^R \, \frac{V(k_1, y_1 - 1)}{y_1 - 1} \, \frac{V(k_2, y_2 - 1)}{y_2 - 1} \cdots \frac{V(k_N, y_N - 1)}{y_N - 1} \, |0, 0\rangle \tag{5}$$

We can notice that $(-1)^R = (-1)^{\alpha' p^2} (-1)^{L_0}$ and since from (VII-A-6.2)

$$e^{+i\pi L_0} \, \frac{V(k, y-1)}{y-1} \, e^{-i\pi L_0} = - \frac{V(k, 1-y)}{1-y} \tag{5}$$

Hence

$$\Omega \, \frac{V(k_1, y_1)}{y_1} \cdots \frac{V(k_N y_N)}{y_N} \, |0, 0\rangle = (-1)^{\alpha' p^2} (-1)^N \, \frac{V(k_1, 1-y_1)}{1-y_1} \cdots \frac{V(k_N, 1-y_N)}{1-y_N} \, |0, 0\rangle \tag{6}$$

Now let us use the interchange rule (VII-A-6.6). We need

$$V(k_i, 1-y_i) \, V(k_j, 1-y_j) = V(k_j, 1-y_j) \, V(k_i, 1-y_i) \, e^{2\pi i \alpha' (k_i, k_j)} \qquad i < j \tag{7}$$

Hence

$$\Omega \, \frac{V(k_1, y_1)}{y_1} \cdots \frac{V(k_N y_N)}{y_N} \, |0, 0\rangle =$$

$$= (-1)^{\alpha' p^2} (-1)^N \, e^{2\pi i \alpha' \sum_{i<j}^{N} (k_i, k_j)} \, \frac{V(k_N, 1-y_N)}{1-y_N} \cdots \frac{V(k_1, 1-y_1)}{1-y_1} \, |0, 0\rangle$$

$$= (-1)^{\alpha' p^2} \, e^{\pi i \alpha' p^2} \, \frac{V(k_N, 1-y_N)}{1-y_N} \cdots \frac{V(k_1, 1-y_1)}{1-y_1} \, |0, 0\rangle \tag{8}$$

Hence

$$\Omega \; \frac{V(k_1, y_1)}{y_1} \; \cdots\cdots \; \frac{V(k_N, y_N)}{y_N} \; |0,0\rangle = \frac{V(k_N, 1-y_N)}{1-y_N} \; \cdots\cdots \; \frac{V(k_1, 1-y_1)}{1-y_1} \; |0,0\rangle \tag{1}$$

Now using (12.2) and introducing the new variables $y_i' \equiv 1 - y_i$ we obtain (11.5). A second application of Ω returns the tree to its original form. Thus

$$\Omega^2 = 1 \tag{2}$$

Up to here we have deduced some important properties of the multiperipheral diagrams. One essential ingredient in the determination of the vertex is that it should satisfy Ward identities guaranteeing the absence of ghost. Remember that the distinction between ghost and non-ghost resonances holds only on-the-mass shell. Let us now write our amplitude B_N given in (3.1) as

$$B_N = \langle \psi_1 | \int_0^1 dx \; x^{L_0 - 2} | \psi_2 \rangle \tag{3}$$

$$|\psi_2\rangle \equiv V(k_{i+1}, 1) \; D \; V(k_{i+2}, 1) \; D \cdots D \; V(k_{N-1}, 1) \; |0, k_N\rangle$$

$$\langle \psi_1 | \equiv \langle -k_1, 0 | \; V(k_2, 1) \; D \cdots\cdots \; D \; V(k_i, 1)$$

where we have distinguished the internal line connecting the vertices with incoming particles with momentum k_i and k_{i+1}. Now we put this distinguished internal line on the mass shell corresponding to the $\tilde{N}$-th level of the dual spectrum. Since the mass shell condition is $L_0 - 1 = 0$, where

$$L_0 = -\alpha' p^2 + R = -\alpha' p^2 - \sum_{m=1}^{\infty} m (a_m^+ \cdot a_m) \equiv L_0(p^2) \tag{4}$$

then if we have $k \equiv k_1 + k_2 + \cdots + k_i = -(k_{i+1} + \cdots + k_N)$ we get

$$2\alpha' k^2 = 2(\tilde{N} - 1)$$

Now let us define

$$\mathcal{M}_{\tilde{N}}(m) \equiv \oint \frac{dz}{2i\pi z} \; z^{L_0(k^2) - 1 + m} \tag{5}$$

Let us notice $\mathcal{M}_{\tilde{N}}(0) = 1$ when acting on this level and $\mathcal{M}_{\tilde{N}}(0) = 0$ otherwise.

Then $\mathcal{U}_{\tilde{N}}(0)$ is the projection operator acting from the full Fock space onto this level. Replacement of the propagator in (13.3) by $\mathcal{U}_{\tilde{N}}(0)$ gives the expression for the complete residue, $R_{\tilde{N}}$, at this $\tilde{N}$-th level

$$R_{\tilde{N}} = \langle \Psi_1 | \mathcal{U}_{\tilde{N}}(0) | \Psi_2 \rangle \tag{2}$$

and we must prove that $R_{\tilde{N}} \geq 0$. Let us notice that since $\mathcal{U}_{\tilde{N}}^2(0) = \mathcal{U}_{\tilde{N}}(0)$ this can be rewritten as

$$R_{\tilde{N}} = \langle \tilde{\Psi}_1 | \tilde{\Psi}_2 \rangle \quad , \quad |\tilde{\Psi}_i\rangle = \mathcal{U}_{\tilde{N}}(0) |\Psi_i\rangle \qquad i = 1, 2 \tag{3}$$

The absence of ghosts will follow if we can prove that $|\tilde{\Psi}_i\rangle$ are physical states, i.e.

$$L_m |\tilde{\Psi}_i\rangle = 0 \quad (m > 0) \quad , \quad (L_0 - 1) |\tilde{\Psi}_i\rangle = 0 \qquad i = 1, 2 \tag{4}$$

The second identity is obvious since $(L_0 - 1)\mathcal{U}_{\tilde{N}}(0) = 0$. So we have to require that

$$L_m \mathcal{U}_{\tilde{N}}(0) |\Psi_i\rangle = 0 \qquad m > 0 \tag{5}$$

From (VII-A-5.1) we have

$$L_m \mathcal{U}_{\tilde{N}}(0) = \mathcal{U}_{\tilde{N}}(m) L_m \tag{6}$$

and furthermore

$$\mathcal{U}_{\tilde{N}}(m)(L_0 + m - 1) = 0 \tag{7}$$

From this we are led to require that $|\Psi_i\rangle$ vectors satisfy the Virasoro identities

$$W_m |\Psi_i\rangle \equiv (L_0 - L_m + m - 1) |\Psi_i\rangle = 0 \quad , \quad m \geq 1 \tag{8}$$

Now we need the commutator of W_m with $V(k_i, 1)$. From (VII-A-4.1)

$$[W_m, V(k_i, 1)] = [L_0 - L_m, V(k_i, 1)] = m\alpha' k_i^2 V(k_i, 1) \tag{9}$$

and hence

$$\left[W_m, V(k_i, 1) \frac{1}{L_0 - 1} \right] = [W_m, V(k_i, 1)] \frac{1}{L_0 - 1} + V(k_i, 1) \left[W_m, \frac{1}{L_0 - 1} \right] \tag{10}$$

Let us compute the second commutator

$$\left[W_m , \frac{1}{L_0 - 1} \right] = - \left[L_m , \frac{1}{L_0 - 1} \right] = - L_m \frac{1}{L_0 - 1} + \frac{1}{L_0 - 1} L_m =$$

$$= - \frac{1}{L_0 + m - 1} L_m + \frac{1}{L_0 - 1} L_m = + \frac{m}{L_0 - 1} - \frac{m}{L_0 - 1} - \frac{1}{L_0 + m - 1} L_m + \frac{1}{L_0 - 1} L_m \Rightarrow$$

$$\left[W_m , \frac{1}{L_0 - 1} \right] = \frac{m}{L_0 - 1} - \frac{m}{(L_0 - 1)(L_0 + m - 1)} W_m \qquad (1)$$

Therefore

$$\left[W_m , V(k_0, 1) \frac{1}{L_0 - 1} \right] = m (\alpha' k_i^2 + 1) V(k_i, 1) \frac{1}{L_0 - 1} - \frac{m}{(L_0 - 1)(L_0 + m - 1)} W_m \qquad (2)$$

and since we have assumed that all absorbed scalars are ground state tachyons on the mass shell $\alpha' k_i^2 = -1$ and

$$\left[W_m , V(k_i, 1) \frac{1}{L_0 - 1} \right] = - \frac{m}{(L_0 - 1)(L_0 + m - 1)} W_m \qquad (3)$$

Using this relation we can convince oneself easily that the condition (14.8) is equivalent to $W_m |0, k_N\rangle = 0$, but this is clearly true. Therefore we have been able to prove the absence of ghosts in multiperipheral diagrams.

Before going on we would like to construct a projection operator which project any state of the Fock space into the space of transversal states. Up to this end let us consider the space R^M and chose a frame such that p^M is given as in (VII-18.8). Let us consider the vector k^M defined in (VII-18.5), then

$$k \cdot P(z) = 1 + \sum_{m=1}^{\infty} \left[K_{-m} z^m + K_m z^{-m} \right] \qquad (5)$$

where the K_m are defined in (VII-18.8). Notice that

$$K_m = \oint \frac{dz}{2\pi i z} z^m \left[k \cdot P(z) \right] \qquad (6)$$

Let us furthermore define the operators

$$D_m = \oint \frac{dz}{2\pi i z}\, z^m \, \frac{1}{k.P(z)} \tag{1}$$

Equivalent expressions are

$$K_m = \langle\, z^m\, [k.P(z)] \,\rangle \quad , \qquad D_m = \langle\, z^m \, \frac{1}{[k.P(z)]} \,\rangle \tag{2}$$

Let us remember (VII-20.2)

$$\begin{aligned}
&[L_m, L_m] = (m-m)\, L_{m+m} + \frac{D}{12}\, m\,(m^2-1)\, \delta_{m,-m} \\
&[K_m, K_m] = 0 \\
&[L_m, K_m] = -m\, K_{m+m} \qquad\qquad K_1 = 0
\end{aligned} \tag{3}$$

and furthermore

$$\begin{aligned}
&[K_m, D_m] = 0 \\
&[D_m, D_m] = 0 \\
&[L_m, D_m] = -(2m+m)\, D_{m+m}
\end{aligned} \tag{4}$$

The first two are immediate consequences of (2). The last one is obtained immediately if we take into account that, as can be proved easily from (VII-A-3.1), $1/k.P(z)$ has generalized projective spin $S=-1$, then

$$[L_m, D_m] = \oint \frac{dz}{2\pi i z}\, z^m\, [L_m, \frac{1}{k.P(z)}] = \oint \frac{dz}{2\pi i z}\, z^{m+m} \left\{ z\frac{d}{dz} - m \right\} \frac{1}{k.P(z)} =$$

$$= -(2m+m) \oint \frac{dz}{2\pi i z}\, z^{m+m}\, \frac{1}{k.P(z)} = -(2m+m)\, D_{m+m} \qquad \text{Q.E.D.}$$

Now let us introduce

$$E \equiv (D_0 - 1)\,[L_0 - \alpha(0)] + \sum_{m=1}^{\infty} (D_{-m} L_m + L_{-m} D_m) \tag{5}$$

Let us now consider the commutators of E with K_m, D_m and L_m

$$[K_m, E] = [K_m, D_0 L_0] - \alpha(0)\,[K_m, D_0] - i\,[K_m, L_0]$$

$$+ \sum_{m=1}^{m} [K_m, D_{-m} L_m] + \sum_{m=1}^{\infty} [K_m, L_{-m} D_m] =$$

$$= - D_0 [L_0, K_m] + [L_0, K_m] + \sum_{m=1}^{\infty} \{ - D_{-m} [L_m, K_m] - [K_{-m}, K_m] D_m \}$$

$$= + m D_0 K_m - m K_m + \sum_{m=1}^{\infty} \{ m D_{-m} K_{m+m} + m K_{m-m} D_m \}$$

$$= m \sum_{m=-\infty}^{+\infty} K_{m-m} D_m - m K_m \qquad (1)$$

But using (15.5)

$$\sum_{m=-\infty}^{+\infty} K_{m-m} D_m = \oint \frac{dz}{2\pi i z} \sum_{m=-\infty}^{+\infty} z^m K_{m-m} \frac{1}{h \cdot \ell(z)} =$$

$$= \oint \frac{dz}{2\pi i z} z^m \sum_{m=-\infty}^{+\infty} z^{-(m-m)} K_{m-m} \frac{1}{h \cdot \ell(z)} = \oint \frac{dz}{2\pi i z} z^m = \delta_{m0} \Rightarrow$$

$$\sum_{m=-\infty}^{+\infty} K_{m-m} D_m = \delta_{m0} \qquad (2)$$

Hence

$$[K_m, E] = - m K_m \qquad (3)$$

Furthermore

$$[D_m, E] = [D_m, D_0 L_0] + \alpha(0) [D_m, D_0] - [D_m, L_0]$$

$$+ \sum_{m=1}^{\infty} [D_m, D_{-m} L_m] + \sum_{m=1}^{\infty} [D_m, L_{-m} D_m] =$$

$$= - D_0 [L_0, D_m] + [L_0, D_m] + \sum_{m=1}^{\infty} \{ - D_{-m} [L_m, D_m] - [L_{-m}, D_m] D_m \}$$

$$= m D_0 D_m - m D_m + \sum_{m=1}^{\infty} \{ (2m + m) D_{-m} D_{m+m} + (-2m + m) D_{m-m} D_m \}$$

$$= - m D_m + \sum_{m=-\infty}^{+\infty} (m + 2m) D_{m+m} D_{-m} \qquad (4)$$

But

$$\sum_{m=-\infty}^{+\infty} (m + 2m) D_{m+m} D_{-m} = 0 \qquad (5)$$

since the term with m cancels with the term $-(m+m)$. Therefore

$$[D_m, E] = - m D_m \qquad (6)$$

Finally

$$[L_m, E] = [L_m, D_0 L_0] - \alpha(0)[L_m, D_0] - [L_m, L_0]$$

$$+ \sum_{m=1}^{m} \{ [L_m, D_{-m} L_m] + [L_m, L_{-m} D_m] \} =$$

$$= m D_0 L_m - 2m D_m L_0 + 2m \alpha(0) D_m - m L_m + \sum_{m=1}^{m} \{ (m-m) D_{-m} L_{m+m}$$

$$- (2m-m) D_{m-m} L_m + \frac{D}{12} m (m^2-1) D_m \delta_{m,-m} - (2m+m) L_{-m} D_{m+m} +$$

$$+ (m+m) L_{m-m} D_m + \frac{D}{12} m (m^2-1) D_m \delta_{m,m} \} =$$

$$= -m L_m + 2m \alpha(0) D_m + \frac{D}{12} m (m^2-1) D_m + m D_0 L_m - 2m D_m L_0 +$$

$$+ \sum_{m=1}^{m} \{ (m-m) D_{-m} L_{m+m} - (2m-m) D_{m-m} L_m \}$$

$$+ \sum_{m=1}^{m} \{ (m+m) L_{m-m} D_m - (2m+m) L_{-m} D_{m+m} \}$$

$$= -m L_m + 2m \alpha(0) D_m + \frac{D}{12} m (m^2-1) D_m + m D_0 L_m - 2m D_m L_0$$

$$- m D_0 L_m + 2m D_m L_0 - \sum_{m=1}^{m} (m+m) D_m L_{m-m} + \sum_{m=1}^{m} (m+m) L_{m-m} D_m$$

$$= -m L_m + 2m \alpha(0) D_m + \frac{D}{12} m (m^2-1) D_m - \sum_{m=1}^{m} (m+m)(2m-m) D_m \qquad (1)$$

but

$$\sum_{m=1}^{m} (m+m)(2m-m) = \sum_{m=1}^{m} [2m^2 + mm - m^2] = 2m^2 \cdot m + m \frac{m(m+1)}{2} - \frac{m}{6} (m+1)(2m+1)$$

$$= \frac{1}{6} m (13 m^2 - 1) \qquad (2)$$

Hence

$$[L_m, E] = -m L_m + \frac{m}{12} \{ (D-26)(m^2-1) + 24(\alpha(0)-1) \} D_m \qquad (3)$$

Hence we have obtained

$$[K_m, E] = -m K_m$$

$$[D_m, E] = -m D_m \qquad (4)$$

$$[L_m, E] = -m L_m + \frac{m}{12} \{ (D-26)(m^2-1) + 24[\alpha(0)-1] \} D_m$$

Now let us remember the results obtained in VII. Let us prove that on a transverse state belonging to T^N we have

$$E\,|t, N, v\rangle = 0 \tag{1}$$

Notice from (15.5) and (16.1)

$$D_m = \oint \frac{dz}{2\pi i z}\, z^m \left\{ 1 + \sum_{m=1}^{\infty} \left[K_{-m} z^m + K_m z^{-m} \right] \right\}^{-1} =$$

$$D_m = \oint \frac{dz}{2\pi i z}\, z^m \sum_{k=0}^{\infty} (-1)^k \left\{ \sum_{m=1}^{\infty} \left[K_{-m} z^m + K_m z^{-m} \right] \right\}^k \tag{2}$$

Notice that $(D_0 - 1)$ as well as D_m with $m \geq 1$ contains operators K_m with $m \geq 1$. Then (1) is an immediate consequence of (VII-18.9). Hence we have indeed found a subspace $T^N \subset R^N$ on which $E = 0$. In order to find the other eigenvectors of E we apply it on a basis of R^N:

$$E\, L_{-1}^{\lambda_1} \cdots L_{-m}^{\lambda_m} K_{-1}^{\mu_1} \cdots K_{-m}^{\mu_m} |t, N, v\rangle \tag{3}$$

Let us assume $D = 26$ and $\alpha(0) = 1$, then we have

$$E\, L_{-m} = L_{-m}\,(E - m) \quad , \quad E\, K_{-m} = K_{-m}\,(E - m) \tag{4}$$

and hence

$$E\, L_{-m}^{\lambda} = L_{-m}^{\lambda}\,(E - \lambda m) \quad , \quad E\, K_{-m}^{\lambda} = K_{-m}^{\lambda}\,(E - \lambda m) \tag{6}$$

and therefore

$$E\, L_{-1}^{\lambda_1} \cdots L_{-m}^{\lambda_m} K_{-1}^{\mu_1} \cdots K_{-m}^{\mu_m} |t, N, v\rangle =$$

$$= -\left(\sum_{r=1}^{n} r\lambda_r + \sum_{r=1}^{m} r\mu_r \right) L_{-1}^{\lambda_1} \cdots L_{-m}^{\lambda_m} K_{-1}^{\mu_1} \cdots K_{-m}^{\mu_m} |t, N, v\rangle \tag{7}$$

Therefore for $\alpha(0) = 1$ and $D = 26$ we have been able to prove that:

1) The eigenvectors of E span the whole Fock space
2) The eigenvalues of E are negative integers
3) The eigenvalue zero corresponds to, and only to, the transverse subspace.

It is now easy to construct a projection operator onto the trans-

verse subspace. Define the operator

$$\mathcal{J} \equiv \oint \frac{dz}{2\pi i z} z^{E} \tag{1}$$

It is clear that $\mathcal{J}$ is a Hermitian projection operator unto the transverse subspace of each level N. Furthermore from (18.4) and assuming $\alpha(0) = 1$, $D = 26$ we have

$$X_m E = (E - m) X_m \qquad X \equiv K, D, L \tag{2}$$

Hence

$$X_m E^k = (E - m)^k X_m \tag{3}$$

and therefore

$$X_m \mathcal{J} = \oint \frac{dz}{2\pi i z} z^{E-m} X_m \tag{4}$$

but $E \leq 0$ and hence for $m \geq 1$ the integral on the right hand side vanishes. Then

$$X_m \mathcal{J} = 0 = \mathcal{J} X_{-m} \quad , \quad m \geq 1 \quad , \quad X = K, D, L \tag{5}$$

From (17.2) with $m = 0$ we have

$$1 - D_0 = \sum_{m=1}^{\infty} [\, B_{-m} K_m + K_{-m} D_m \,] \tag{6}$$

and hence

$$(D_0 - 1)\, \mathcal{J} = \mathcal{J}\, (D_0 - 1) = 0 \tag{7}$$

Therefore

$$E\, \mathcal{J} = \mathcal{J} E = 0 \tag{8}$$

and

$$\mathcal{J}^2 = \mathcal{J} \tag{9}$$

So for $\alpha(0) = 1$ and $D = 26$, $\mathcal{J}$ is a projection operator, correctly normed on the subspace T^N

We now wish to consider the commutation relations of the projection operator $\mathcal{J}$ with a vertex which we shall take to absorb a tachyon with

momentum k_i and to join to internal lines with momentum q_1 and $q_2 = q_1 + k_i$. Now let us suppose that the internal legs are on the mass shell, (VII-18.7) implies $2\alpha'(q_2 \cdot k) = 2\alpha'(q_1 \cdot k) = 1$ and therefore we have $2\alpha'(k \cdot k_i) = 0$. This is kinematically possible but, of course, means we are working in a special frame of reference. In this frame it is clear from (VII-18.6) and (VII-A-2.3) that

$$[\, k\,\ell(z),\ k\,\ell(z')\,] = 0 \tag{1}$$

and therefore

$$[\, D_m,\ V(k_i, 1)\,] = 0 \tag{2}$$

Furthermore from (VII-A-4.3)

$$[\, L_m,\ V(k_i, 1)\,] = [\, L_0,\ V(k_i, 1)\,] + m\, V(k_i, 1) \tag{3}$$

Then

$$\mathcal{H}_{N_1}(0)\,[\, E,\ V(k_i, 0)\,]\,\mathcal{H}_{N_2}(0) =$$

$$= \mathcal{H}_{N_1}(0)\,[\,(D_0 - 1)(L_0 - 1) + \sum_{m=1}^{\infty}(D_{-m} L_m + L_{-m} D_m),\ V(k_i, 0)\,]\,\mathcal{H}_{N_2}(0) =$$

$$= \sum_{m=1}^{\infty} \mathcal{H}_{N_1}(0)\,\{\, D_{-m}\,[\, L_m,\ V(k_i, 1)\,] + [\, L_{-m},\ V(k_i, 1)\,]\, D_m \,\}\,\mathcal{H}_{N_2}(0) =$$

$$= \sum_{m=1}^{\infty} \mathcal{H}_{N_1}(0)\,\{\, D_{-m}\,[\, L_0,\ V(k_i, 1)\,] + [\, L_0,\ V(k_i, 1)\,]\, D_m$$

$$+ m\, V(k_1, 1)\, D_{-m} - m\, V(k_1, 1)\, D_m \,\}\,\mathcal{H}_{N_2}(0)$$

$$= \sum_{m=1}^{\infty} \mathcal{H}_{N_1}(0)\,\{\, D_{-m}\, L_0\, V(k_i, 1) - V(k_i, 1)\, L_0\, D_m$$

$$+ m\, V(k_1, 1)\, D_{-m} - m\, V(k_1, 1)\, D_m \,\}\,\mathcal{H}_{N_1}(0) = 0$$

where we have used (16.5) and the fact that $L_0 = 1$. Therefore

$$\mathcal{H}_{N_1}(0)\,[\, E,\ V(k_i, 0)\,]\,\mathcal{H}_{N_2}(0) = 0 \tag{4}$$

and hence

$$\mathcal{H}_{N_1}(0)\,[\, \vec{J},\ V(k_i, 1)\,]\,\mathcal{H}_{N_2}(0) = 0 \tag{5}$$

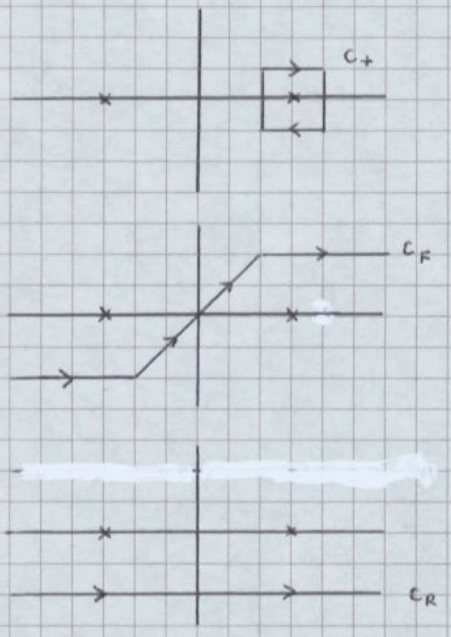
Thus, in the special frame already mentioned, the projection operator and the vertex commute, provided the two legs are on the mass shell.

X - LOOP DIAGRAMS

In complete analogy with the Yang - Mills theory it is not possible to define the loops in the dual model simply by using the propagadors and rentices which we have used so far. The reason is that the Feynman rules have been derived in a particular gauge and that the disappearance of unphysical states from tree diagrams which is due to gauge invariance does not guarantee that these states do not appear in the loops if we use the same set of Feynman rules. In fact to ensure unitarity one has to modify the Feynman rules by introducing the so called Faddeev - Popov ghosts. An alternative derivation of the rules found by Faddeev and Popov is offered by Feynman's tree theorem [R.P. Feynman "Magic Without Magic" Ed. J. Klauder, W.H. Freeman, San Francisco (1972)]. Let us sketch very briefly Feynman's argument for a single - loop diagram containing scalar particles. Let us consider the invariant functions

$$\Delta_i(x; m^2) \equiv \int_{C_i} \frac{d^4q}{(2\pi)^4} \frac{e^{-iq\cdot x}}{q^2 - m^2} \qquad (1)$$

where C_i is some integration contour in the q^0 - complex plane. Some contours of interest are



$\Delta_+(x; m^2)$ (Green's function)

$\Delta_F(x; m^2)$ (Feynman's function) (2)

$\Delta_R(x; m^2)$ (Retarded function)

where the poles $q^0 = \pm E(\vec{q}) \equiv \pm\sqrt{\vec{q}^2 + m^2}$ are represented by x. Then equivalent expressions are

$$\Delta_{C_i}(x; m^2) = \int \frac{d^4q}{(2\pi)^4} \Delta_i(q; m^2)\, e^{-iq\cdot x} \qquad (3)$$

where

$$\Delta_+(q;m^2) = -2\pi i\,\delta(q^2-m^2)\,\theta(q^0) \tag{1}$$

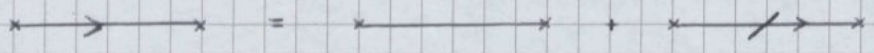
$$\Delta_F(q;m^2) = \frac{1}{q^2-m^2+i\epsilon}, \qquad \Delta_R(q;m^2) = \frac{1}{q^2-m^2-i\epsilon q^0}, \quad \epsilon \downarrow 0$$

Then we get immediately from (1) or (1.2)

$$\frac{1}{q^2-m^2-i\epsilon q^0} = \frac{1}{q^2-m^2+i\epsilon} + 2\pi i\,\delta(q^2-m^2)\,\theta(q^0) \tag{2}$$

$$\Delta_R(x;m^2) = \Delta_F(x;m^2) - \Delta_+(x;m^2) \tag{3}$$

We will use to represent the last equation the graphical notation

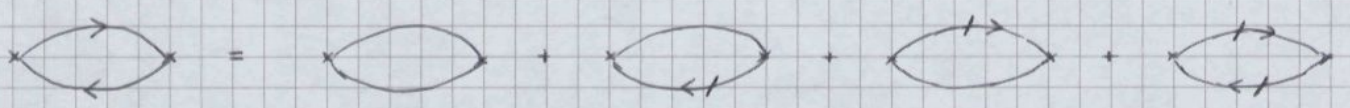


(4)

We shall now insert this identity into the integral

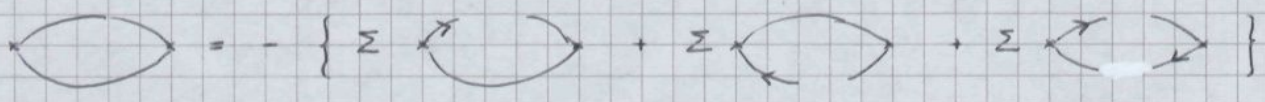
$$\int d^4x\ \Delta_R(x)\ \Delta_R(-x) \tag{5}$$

and so obtain the diagrammatic equation



(6)

(We forget about problems of ultraviolet divergences, which disappear when we consider single loops with three or more internal lines; the self-energy graph is the simplest graph to consider and illustrates all the main points). Now by the retardedness property the integral in (5) vanishes, since one or other of the two factors in the integrand is always zero. Notice that on the right hand side of Fig (6) one term is the Feynman integral (with Feynman propagators) and all the other terms have at least one line with a Δ_+ Green function so that the corresponding line is on the mass shell. Thus we can rewrite Fig(6) as in Fig (7),

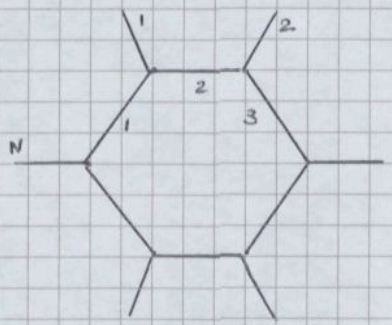


(7)

where we have cut open the lines on the mass shell and replaced the integration over the positive sheet of the mass-shell hyperboloid by a

summation sign indicating a phase space integration (which is the same thing). In other words, the Feynman loop (on the left) is expressed in terms of sums over trees with external lines on the mass shell (on the right). This is Feynman's tree theorem for ϕ^3 theory in the case of the self-energy single loop. Feynman's proposal is to use this identity to define loops in the Yang-Mills theories. Now the entering and leaving particles on the "cut" lines are not only specified by their momenta and quantum numbers but also by their polarization tensors. Since we want to include in the unitarity sum only those states which appear in the tree diagrams, we have to include only the transverse states. The result is identical to the one obtained by Faddeev and Popov.

The generalization to the single loop with N external lines can



(1)

be written, in momentum space as,

$$\int \frac{d^4q}{(2\pi)^4} \prod_{i=1}^{N} \frac{g_i}{q_i^2 - m_i^2 + i\epsilon} = - \sum_{\substack{\text{ways of} \\ \text{cutting}}} \int \frac{d^4q}{(2\pi)^4} \prod_{i=1}^{N} g_i \, \Delta^{(0)}(q_i, m_i^2) \qquad (2)$$

where

$$\Delta^{(0)}(q_i, m^2) = \begin{cases} \dfrac{1}{q_i^2 - m_i^2 + i\epsilon} & \text{if line } i \text{ is not cut} \\ 2\pi i \, \delta(q_i^2 - m_i^2) \, \theta(q_i^0) & \text{if line } i \text{ is cut} \end{cases} \qquad (3)$$

q_i is the momentum of the i-th internal line, measured in the direction of the loop momentum q, g_i is the coupling constant associated with the emission of the i-th external particles, and the sum over ways of cutting refers to $(2^N - 1)$ terms corresponding to the possibility of each internal line being cut (put on mass shell) or not, but excludes the possibility of no lines being cut (since this is the left hand side).

The importance of the tree theorem is revealed when we consider theories such as that of Yang-Mills or the dual models, in which the tree

graph expressions are known for external states on the mass-shell and physical and in which the physical states satisfy a bootstrap property: the totality of physical states completely factorizes the residues of the poles in the tree graphs amplitudes.

In such theories the propagator commonly used to construct the tree graphs is not unitary (or not projected), in that it propagates a larger spectrum that the physical spectrum. The deficiency between the two spectra can be gauge transformed away in tree graphs by the use of certain Ward identities arising from the symmetries of the coupling. In loops this deficiency cannot be gauge transformed away and so the use of non-unitary (or not-projected) propagators is incorrect and leads to discontinuity formulae with spurious intermediate states in addition to physical ones. Feynman's proposal is to use the equation in Figure (2.7), (and its generalization to N internal lines) as the definition of the appropiate single-loop graph since it will satisfy the desired analyticity properties, provided when one joins legs of trees one sums only over physical states - or equivalently - uses a physical state projection operator.

Let us remember that in dual models the equivalent to the Feynman propagator is (IX-1.5)

$$\frac{1}{p^2 - m^2 + i\epsilon} \longrightarrow -\alpha' \frac{1}{L_0 - 1 + i\epsilon} \tag{1}$$

where

$$L_0 = -\alpha' p^2 + R \tag{2}$$

Let us remember furthermore that for $D = 26$ and $\alpha(0) = 1$ the projector on transversal states is the Brink-Olive projection operator (IX-20.1)

$$\mathcal{J} = \oint \frac{dz}{2\pi i z} z^E \tag{3}$$

Furthermore the projection operator onto the level N is given by

$$\oint \frac{dz}{2\pi i z} z^{R-N} \tag{4}$$

In dual theories Ward identities are valid graph by graph and so the equation of Fig (2.7) and its generalizations are valid as they stand as candidates for loop amplitudes, with only the appropriate re-interpretation of the vertices and propagators,

$$g_i \longrightarrow g\, V(k_i, \epsilon) \tag{1}$$

and instead of (3.3) we now have

$$\Delta(q_i) = \begin{cases} -\alpha' \int_0^1 \frac{dx_i}{x_i}\, x_i^{L_0 - 1} & \text{, if line } i \text{ is uncut} \\ \sum_{N_i=0}^{\infty} \left[2\pi i \alpha' \, \theta(q_i^0)\, \delta\left[2\alpha' q_i^2 - 2(N_i - 1) \right] \right] \left\{ \oint \frac{dx_i}{2\pi i x_i}\, x_i^{R - N_i} \right\} \mathcal{P} & \text{, if line } i \text{ is cut} \end{cases} \tag{2}$$

In the last expression the sum is over the various levels, i.e. the various mass shells, indicated by the function in the first square bracket, the second bracket, $\{\ \}$ projects from the full Fock space onto the subspace contributing to the mass shell labelled by N_i and $\mathcal{P}$ projects onto the physical subspace at that level. The summation over all states leaving and entering a cut loop is a trace over all oscillator modes. Thus, instead of (3.2) we have that our planar-loop amplitude is

$$-g^R \sum_{\substack{\text{ways of} \\ \text{cutting}}} \int \frac{d^{26} q}{(2\pi)^{26}}\, \mathrm{Tr}\left[\Delta(q_1)\, V(k_1, \epsilon_1)\, \Delta(q_2) \cdots \Delta(q_N)\, V(k_N, \epsilon_N) \right] \tag{3}$$

Now we shall evaluate a typical term of the above sum. We take as typical contribution to (3) the term in which just internal line 1 (joining internal lines 1 and N, see figure (3.1)) is on the mass shell and so bears the projection operator

$$\oint \frac{dy}{2\pi i y}\, y^E \oint \frac{dx_1}{2\pi i x_1}\, x_1^{R - N_1} \tag{5}$$

where its momentum q_1 satisfies the mass shell condition

$$2\alpha' q_1^2 = m^2(N_1) = 2(N_1 - 1) \tag{5}$$

All the other lines bear the conventional propagators

$$-\alpha' \int_0^1 \frac{dx_i}{x_i}\, x_i^{L_0 - 1} \tag{6}$$

Thus if we define

$$x_1^{L_0-1} V(k_1,t)\, x_2^{L_0-1} V(k_2,t) \cdots x_N^{L_0-1} V(k_N,t) \equiv P(x_1,\dots x_N) \equiv P(x) \qquad (1)$$

and we take into account that in the level N_E $L_0 - 1 \equiv R - N_1$ as it is deduced immediately, we need to evaluate

$$\oint \frac{dy}{2\pi i y} \oint \frac{dx_1}{2\pi i x_1} \left\{ \prod_{i=2}^{N} \int_0^1 \frac{dx_i}{x_i} \right\} \mathrm{Tr}\left[y^E P(x) \right] \qquad (2)$$

since it is proportional to the loop - momentum integrand of the term we want.

Now let us illustrate the method we shall use to evaluate the trace in (2). in the trivial case

$$f(y) = \mathrm{Tr}\left[y^{a^+ a} \right] \qquad (3)$$

$$a a^+ - a^+ a = 1$$

We differentiate with respect to y

$$y \frac{d f(y)}{dy} = \mathrm{Tr}\left[y^{a^+ a}\, a^+ a \right] \qquad (4)$$

Notice

$$\mathrm{Tr}\left[y^{a^+ a}\, a^+ a \right] = \mathrm{Tr}\left[a\, y^{a^+ a}\, a^+ \right] = \mathrm{Tr}\left[y^{a^+ a + 1}\, a a^+ \right] =$$

$$= y\, \mathrm{Tr}\left[y^{a^+ a} (1 + a^+ a) \right] \qquad (5)$$

where we used

$$a (a^+ a)^m = (1 + a^+ a)^m a \qquad (6)$$

Hence

$$y \frac{d f(y)}{dy} = y f(y) + y^2 \frac{d f(y)}{dy} \Rightarrow$$

$$(1 - y) \frac{d f(y)}{dy} = f(y) \qquad (7)$$

and hence

$$f(y) = \frac{\mathrm{const}}{1 - y} \qquad (8)$$

We apply now this idea to the more complicated trace under the integration (6.2) and again we differentiate with respect to y

$$y \frac{\partial}{\partial y} \mathrm{Tr}\,[\,y^E\, \mathfrak{L}(x)\,] = \mathrm{Tr}\,[\,y^E\, E\, \mathfrak{L}(x)\,] =$$

$$= \mathrm{Tr}\,\{\, y^E\,(D_0 - 1)(L_0 - 1)\,\mathfrak{L}(x)\,\} + \sum_{m=1}^{\infty} \mathrm{Tr}\,\{\, y^E\,(D_{-m} L_m + L_{-m} D_m)\,\mathfrak{L}(x)\,\} =$$

$$= x_1 \frac{\partial}{\partial x_1} \mathrm{Tr}\,\{\, y^E\,(D_0 - 1)\,\mathfrak{L}(x)\,\} + \sum_{m=1}^{\infty} \mathrm{Tr}\,\{\, y^E\,(D_{-m} L_m + L_{-m} D_m)\,\mathfrak{L}(x)\,\} \qquad (1)$$

where the explicit form of $\mathfrak{L}(x)$ has been taken into account in order to eliminate the factor $(L_0 - 1)$ in favour of a differentiation. We shall now imitate the procedure in our simple example by considering the term with $D_{-m} L_m$ and obtaining an equation for it by moving the L_m to the right around the trace (or loop). From the results (VII-A-5.4) and (VII-A-5.5)

$$(L_m - L_0 + 1)\, x^{L_0 - 1}\, V(k,1) = \{\, x^{L_0 + m - 1} L_m - x^{L_0 - 1}\,(L_0 - 1)\,]\, V(k,1) =$$

$$= \{\, x^{L_0 + m - 1}\,(L_m - L_0 + 1) - (1 - x^m)\, x^{L_0 - 1}\,(L_0 - 1)\,\}\, V(k,1) =$$

$$= \{\, x^{L_0 + m - 1}\,[\, V(k,1)\,(L_m - L_0 + 1) + m\, V(k;1)\,] - (1 - x^m)\, x^{(L_0 - 1)}\,(L_0 - 1)\, V(k,1) =$$

$$= x^{L_0 + m - 1}\, V(k,1)\,(L_m - L_0 + 1) + \{\, m\, x^m - (1 - x^m)(L_0 - 1)\,\}\, x^{(L_0 - 1)}\, V(k,1) \Rightarrow$$

$$(L_m - L_0 + 1)\, x^{L_0 - 1}\, V(k;1) = x^{L_0 + m - 1}\, V(k,1)\,(L_m - L_0 + 1)$$

$$- x \frac{d}{dx} \left\{ (1 - x^m)\, x^{L_0 - 1}\, V(k,1) \right\} \qquad (2)$$

Let us remember furthermore (IX-16.4) and (IX-18.4)

$$[\, L_m, D_m\,] = -(2m + m)\, D_{m+m}\,, \qquad [\, L_m, E\,] = -m\, L_m \qquad (3)$$

From the last equation $L_m E = (E - m) L_m$ and this implies

$$L_m\, f(E) = f(E - m)\, L_m \qquad (4)$$

Now let us consider, for instance, the trace

$$T_2 \equiv Tr\left[y^E D_{-m} L_m \rho(x_1,x_2)\right] = Tr\left[y^E D_{-m} L_m x_1^{L_0-1} V(k_1,1)\, x_2^{L_0-1} V(k_2,1)\right] =$$

$$= Tr\left[y^E D_{-m} (L_m - L_0 + 1)\, \rho(x_1,x_2)\right] + Tr\left[y^E D_{-m} (L_0 - 1)\, \rho(x_1,x_2)\right]$$

$$= Tr\left[y^E D_{-m} (L_m - L_0 + 1)\, \rho(x_1,x_2)\right] + x_1 \frac{\partial}{\partial x_1} Tr\left[y^E D_{-m}\, \rho(x_1,x_2)\right]$$

$$= x_1^m\, Tr\left[y^E D_{-m}\, x_1^{L_0-1} V(k_1,1)\,(L_m - L_0 + 1)\, x_2^{L_0-1} V(k_2,1)\right] + x_1 \frac{\partial}{\partial x_1} x_1^m\, Tr\left[y^E D_{-m}\, \rho(x_1,x_2)\right]$$

$$= x_1^m x_2^m\, Tr\left[y^E D_{-m}\, \rho(x_1,x_2)\,(L_m - L_0 + 1)\right] + \left\{ x_1 \frac{\partial}{\partial x_1} x_1^m - x_2 \frac{\partial}{\partial x_2} x_1^m (1-x_2^m)\right\} Tr\left[y^E D_{-m}\, \rho(x_1,x_2)\right]$$

but

$$(L_m - L_0 + 1)\, y^E = \frac{1}{y^m}\, y^E L_m - y^E (L_0 - 1) \qquad (1)$$

Hence

$$T_2 = \frac{x_1^m x_2^m}{y^m}\, Tr\left[y^E L_m D_{-m}\, \rho(x_1,x_2)\right] - x_1^m x_2^m\, Tr\left[y^E (L_0-1) D_{-m}\, \rho(x_1,x_2)\right]$$

$$+ \left\{ x_1 \frac{\partial}{\partial x_1} x_1^m - x_2 \frac{\partial}{\partial x_2} x_1^m (1-x_2^m)\right\} Tr\left[y^E D_{-m}\, \rho(x_1,x_2)\right] =$$

$$= \frac{x_1^m x_2^m}{y^m}\, T_2 - m \frac{x_1^m x_2^m}{y^m}\, Tr\left[y^E D_0\, \rho(x_1,x_2)\right] - x_1^m x_2^m\, Tr\left[y^E D_{-m} (L_0 - 1)\, \rho(x_1,x_2)\right]$$

$$+ m\, x_1^m x_2^m\, Tr\left[y^E D_{-m}\, \rho(x_1,x_2)\right] + \left(x_1 \frac{\partial}{\partial x_1} x_1^m - x_2 \frac{\partial}{\partial x_2} x_1^m (1-x_2^m)\right) Tr\left[y^E D_{-m}\, \rho(x_1,x_2)\right] \Rightarrow$$

$$\left(1 - \frac{x_1^m x_2^m}{y^m}\right) T_2 = - m \frac{x_1^m x_2^m}{y^m}\, Tr\left[y^E D_0\, \rho(x_1,x_2)\right] +$$

$$+ \left\{ - x_1^{m+1} x_2^m \frac{\partial}{\partial x_1} - m\, x_1^m x_2^m + x_1 \frac{\partial}{\partial x_1} x_1^m - x_2 \frac{\partial}{\partial x_2} x_1^m (1-x_2^m)\right\} Tr\left[y^E D_{-m}\, \rho(x_1,x_2)\right] \Rightarrow$$

$$\left(1 - \frac{x_1^m x_2^m}{y^m}\right) Tr\left[y^E D_{-m} L_m\, \rho(x_1,x_2)\right] = - m \frac{x_1^m x_2^m}{y^m}\, Tr\left[y^E D_0\, \rho(x_1,x_2)\right]$$

$$+ \left\{ x_1 \frac{\partial}{\partial x_1} - x_2 \frac{\partial}{\partial x_2}\right\} x_1^m (1-x_2)^m\, Tr\left[y^E D_{-m}\, \rho(x_1,x_2)\right] \qquad (2)$$

Let us now introduce

$$x_{1k} \equiv x_1 \cdot x_2 \cdots x_k \tag{1}$$

$$\mathcal{D}_i \equiv x_1 \frac{\partial}{\partial x_1} - x_i \frac{\partial}{\partial x_i}$$

then we can prove in general

$$\left\{ 1 - \left(\frac{x_{1N}}{y} \right)^m \right\} \mathrm{Tr}\left[y^E D_{-m} L_m P(x) \right] = -m \left(\frac{x_{1N}}{y} \right)^m \mathrm{Tr}\left[y^E D_0 P(x) \right]$$

$$+ \sum_{i=2}^{N} \mathcal{D}_i \left\{ x_{1i-1}^m (1 - x_i^m) \, \mathrm{Tr}\left[y^E D_{-m} P(x) \right] \right\} \tag{2}$$

Notice that $\mathcal{D}_i$ commute with any function of x_1 and x_i which depends on the variables only via the product $x_1 x_i$. This allows us to write

$$\mathrm{Tr}\left[y^E D_{-m} L_m P(x) \right] = \frac{-m (x_{1N}/y)^m}{1 - (x_{1N}/y)^m} \mathrm{Tr}\left[y^E D_0 P(x) \right]$$

$$+ \sum_{i=2}^{N} \mathcal{D}_i \left\{ \frac{x_{1i-1}^m (1 - x_i^m)}{1 - (x_{1N}/y)^m} \mathrm{Tr}\left[y^E D_{-m} P(x) \right] \right\} \tag{3}$$

and similarly working the L_{-m} to the left we find

$$\mathrm{Tr}\left[y^E L_{-m} D_m P(x) \right] = \frac{-m (x_{1N}/y)^m}{1 - (x_{1N}/y)^m} \mathrm{Tr}\left[y^E D_0 P(x) \right]$$

$$+ \sum_{i=2}^{N} \mathcal{D}_i \left\{ \frac{x_{1i+1}^m (1 - x_i^m)}{y^m \left(1 - (x_{1N}/y)^m \right)} \mathrm{Tr}\left[y^E D_m P(x) \right] \right\} \tag{4}$$

Then we can write

$$y \frac{\partial}{\partial y} \mathrm{Tr}\left\{ y^E P(x) \right\} = x_1 \frac{\partial}{\partial x_1} \mathrm{Tr}\left\{ y^E (D_0 - 1) P(x) \right\} +$$

$$+ \mathrm{Tr}\left\{ y^E D_0 P(x) \right\} \sum_{m=1}^{\infty} \frac{-2m (x_{1N}/y)^m}{1 - (x_{1N}/y)^m} +$$

$$+ \sum_{i=2}^{N} \mathcal{D}_i \sum_{m=1}^{\infty} \left\{ \frac{x_{1i-1}^m (1 - x_i^m)}{1 - (x_{1N}/y)^m} \mathrm{Tr}\left[y^E D_{-m} P(x) \right] + \right.$$

$$\left. + \frac{x_{1i+1}^m (1 - x_i^m)}{y^m (1 - (x_{1N}/y)^m)} \mathrm{Tr}\left[y^E D_m P(x) \right] \right\} \tag{5}$$

Let us now introduce the so called partition function

$$f(x) \equiv \prod_{m=1}^{\infty} \frac{1}{(1-x^m)} \tag{1}$$

Some needed properties of it are

$$y \frac{\partial}{\partial y} \ln f^2(x_{iN}/y) = -\sum_{m=1}^{\infty} \frac{2m\,(x_{iN}/y)^m}{1-(x_{iN}/y)^m} \tag{2}$$

$$\mathcal{D}_i f(x_{iN}/y) = f(x_{iN}/y)\,\mathcal{D}_i \tag{3}$$

$$x_1 \frac{\partial}{\partial x_1} f^K(x_{iN}/y) = -y \frac{\partial}{\partial y} f^K(x_{iN}/y) \tag{4}$$

Using (2) we can write (9.5) as

$$y \frac{\partial}{\partial y} \mathrm{Tr}\{y^E \mathcal{L}(x)\} = x_1 \frac{\partial}{\partial x_1} \mathrm{Tr}\{y^E (D_0 - 1)\mathcal{L}(x)\} +$$

$$+ \mathrm{Tr}\{y^E D_0 \mathcal{L}(x)\}\, y \frac{\partial}{\partial y}\left[\ln f^2(x_{iN}/y)\right] + \sum_{i=2}^{N} \mathcal{D}_i A_i(x,y) \tag{5}$$

where

$$A_i(x,y) \equiv \sum_{m=1}^{\infty} \frac{(1-x_i^m)}{1-(x_{iN}/y)^m}\left\{x_{i-1}^m \mathrm{Tr}\left[y^E D_{-m}\mathcal{L}(x)\right] + \frac{x_{i+1}^m}{y^m}\mathrm{Tr}\left[y^E D_m \mathcal{L}(x)\right]\right\} \tag{6}$$

Now

$$y \frac{\partial}{\partial y}\left\{f^{-2}(x_{iN}/y)\,\mathrm{Tr}\left[y^E \mathcal{L}(x)\right]\right\} = -2 f^{-3}(x_{iN}/y)\left\{y \frac{\partial}{\partial y} f(x_{iN}/y)\right\}\mathrm{Tr}\left[y^E \mathcal{L}(x)\right]$$

$$+ f^{-2}(x_{iN}/y)\, x_1 \frac{\partial}{\partial x_1}\mathrm{Tr}\left[y^E (D_0 - 1)\mathcal{L}(x)\right] + \mathrm{Tr}\left[y^E D_0 \mathcal{L}(x)\right] f^{-2}(x_{iN}/y)\, x_1 \frac{\partial}{\partial x_1}\left[\ln f^2(x_{iN}/y)\right]$$

$$+ \sum_{i=2}^{N} \mathcal{D}_i\left[f^{-2}(x_{iN}/y)\, A_i(x,y)\right] =$$

$$= -2 f^{-3}(x_{iN}/y)\left\{x_1 \frac{\partial}{\partial x_1} f(x_{iN}/y)\right\}\mathrm{Tr}\left[y^E (D_0 - 1)\mathcal{L}(x)\right] +$$

$$+ 2 f^{-3}(x_{iN}/y)\left\{x_1 \frac{\partial}{\partial x_1} f(x_{iN}/y)\right\}\mathrm{Tr}\left[y^E D_0 \mathcal{L}(x)\right] + f^{-2}(x_{iN}/y)\, x_1 \frac{\partial}{\partial x_1}\mathrm{Tr}\left[y^E (D_0 - 1)\mathcal{L}(x)\right]$$

$$- \mathrm{Tr}\left[y^E D_0 \mathcal{L}(x)\right] f^{-2}(x_{iN}/y)\left\{x_1 \frac{\partial}{\partial x_1}\left[\ln f^2(x_{iN}/y)\right]\right\} + \sum_{i=2}^{N} \mathcal{D}_i\left[f^{-2}(x_{iN}/y)\, A_i(x,y)\right] \Rightarrow$$

$$y \frac{\partial}{\partial y}\left\{f^{-2}\left(\frac{x_{iN}}{y}\right)\mathrm{Tr}\left[y^E \mathcal{L}(x)\right]\right\} = x_1 \frac{\partial}{\partial x_1}\left\{f^{-2}\left(\frac{x_{iN}}{y}\right)\mathrm{Tr}\left[y^E (D_0 - 1)\mathcal{L}(x)\right]\right\}$$

$$+ \sum_{i=2}^{N} \mathcal{D}_i\left\{f^{-2}\left(\frac{x_{iN}}{y}\right) A_i(x,y)\right\} \tag{7}$$

Let us divide this by y and integrate from 1 to y

$$f^{-2}\left(\frac{x_{1N}}{y}\right) \mathrm{Tr}\left[y^E \rho(x)\right] - f^{-2}(x_{1N}) \mathrm{Tr}\left[\rho(x)\right] =$$

$$- x_1 \frac{\partial}{\partial x_1} \int_1^y \frac{dy'}{y'} f^{-2}\left(\frac{x_{1N}}{y'}\right) \mathrm{Tr}\left[y'^E (D_0 - 1) \rho(x)\right] +$$

$$+ \sum_{i=2}^{N} \mathcal{D}_i \int_1^y \frac{dy'}{y'} A_i(x, y') f^{-2}\left(\frac{x_{1N}}{y'}\right) \qquad (1)$$

Now let us multiply by $f^2(x_{1N}/y)$ and perform the x-integrations. In the shorthand

$$\int \frac{dx}{x} \equiv \oint \frac{dx_1}{2\pi i x_1} \prod_{i=2}^{N} \int_0^1 \frac{dx_i}{x_i} \qquad (2)$$

we have

$$\int \frac{dx}{x} \mathrm{Tr}\left[y^E \rho(x)\right] = \int \frac{dx}{x} \frac{f^2(x_{1N}/y)}{f^2(x_{1N})} \mathrm{Tr}\left[\rho(x)\right] +$$

$$+ \int \frac{dx}{x} f^2(x_{1N}/y) x_1 \frac{\partial}{\partial x_1} \int_1^y \frac{dy'}{y'} f^{-2}(x_{1N}/y') \mathrm{Tr}\left[y'^E (D_0 - 1) \rho(x)\right] +$$

$$+ \int \frac{dx}{x} \sum_{i=1}^{N} f^2(x_{1N}/y) \mathcal{D}_i \int_0^1 \frac{dy'}{y'} A_i(x, y') f^{-2}(x_{1N}/y') \qquad (3)$$

The last term gives a zero contribution since A_i vanishes when $x_i = 0$ or $x_i = 1$ and since the x_1 integration is closed. (Remember (10.3)). Using parts integration

$$\int \frac{dx}{x} \mathrm{Tr}\left[y^E \rho(x)\right] = \int \frac{dx}{x} \frac{f^2(x_{1N}/y)}{f^2(x_{1N})} \mathrm{Tr}\left[\rho(x)\right]$$

$$- \int \frac{dx}{x} \left\{x_1 \frac{\partial}{\partial x_1} f^2\left(\frac{x_{1N}}{y}\right)\right\} \int_1^y \frac{dy'}{y'} f^{-2}(x_{1N}/y') \mathrm{Tr}\left[y'^E (D_0 - 1) \rho(x)\right] \qquad (4)$$

We can now perform the y-integration. The last term is then seen to be zero because we can use (10.4) and perform a partial integration in y to arrive at

$$+ \int \frac{dx}{x} \oint \frac{dy}{2\pi i y} \mathrm{Tr}\left\{y^E (D_0 - 1) \rho(x)\right\} \qquad (5)$$

and this is zero because $\mathcal{J}(D_0 - 1) = 0$. (IX - 20.7). Then

$$\oint \frac{dy}{2\pi i y} \int \frac{dx}{x} \, \mathrm{Tr}\,[\, y^E P(x)\,] = \int \frac{dx}{x} \frac{1}{f^2(x_{1N})} \, \mathrm{Tr}\,[\, P(x)\,] \oint \frac{dy}{2\pi i y} \, f^2(x_{1N}/y) \qquad (1)$$

The last integral can be easely performed by distorting the contour to infinite and the result is 1. Then

$$\oint \frac{dy}{2\pi i y} \int \frac{dx}{x} \, \mathrm{Tr}\,[\, y^E P(x)\,] = \int \frac{dx}{x} \frac{1}{f^2(x_{1N})} \, \mathrm{Tr}\,[\, P(x)\,] \qquad (2)$$

Remember that for constructing E we need to choose some vector k^μ and therefore we introduced a priviliged Lorentz frame. Notice that in (2) all terms which appear to break Lorentz covariance have finally disappeared by some rather subtle integrations by parts.

Up to here we have dealt with a single term in the Feynman tree expansion equation (5.3); the term with just line 1 cut and so on mass shell. This term is enough to make the pattern clear for as we shall now see, all the other terms can be deduced from it. We shall rewrite (5.3) in a slightly different way:

$$- g^N \sum_{\substack{\text{ways of} \\ \text{cutting}}} \int \frac{d^{26} q}{(2\pi)^{26}} \prod_{i=1}^{N} E(q_i) \int_{C_i} \frac{dx_i}{x_i} \, \mathrm{Tr}\,\{\, x_1^{L_0-1} J_1 \, V(k_1, 1) \, . \, x_2^{L_0-1} J_2 \, V(k_2, 1) \cdots$$
$$\cdots x_N^{L_0-1} J_N \, V(k_N, 1) \,\} \qquad (3)$$

where

$$\left.\begin{aligned} E(q_i) &= \sum_{N_i=0}^{\infty} \alpha' \, \Theta(q_i^0) \, \delta\,[2\alpha' q_i^2 - 2(N_i - 1)] \\ J_i &= \oint \frac{dy_i}{2\pi i y_i} \, y_i^E \\ C_i &\text{ is a contour encircling the origin} \end{aligned}\right\} \text{ If line } i \text{ is not cut} \qquad (4)$$

$$\left.\begin{aligned} E(q_i) &= -\alpha' \\ J_i &= 1 \\ C_i &\text{ runs from } 0 \text{ to } 1 \end{aligned}\right\} \text{ if line } i \text{ is cut}$$

Let us now consider a term with several lines cut. From (IX-18.4) we see that J commutes with x^{L_0-1}. Furthermore from (IX-21.5) the J commute with the vertices and therefore all but one of the J's can be replaced by 1. But we have seen that if there was just

one ∂, say on line 1

$$\prod_{i=1}^{N}\left\{\int_{C_i}\frac{dx_i}{x_i}\right\}\mathrm{Tr}\left\{x_1^{L_0-1}\,\partial V(k_1;1)\,x_2^{L_0-1}V(k_2;1)\cdots x_N^{L_0-1}V(k_N;1)\right\}$$

$$=\prod_{i=1}^{N}\left\{\int_{C_i}\frac{dx_i}{x_i}\,x_i^{-\alpha_i}\right\}\prod_{m=1}^{\infty}(1-x_{1\cdots N}^{m})^{2}\,\mathrm{Tr}\left[x_1^{R}V(k_1,1)\cdots x_N^{R}V(k_N,1)\right]\qquad(1)$$

where we have taken into account that $L_0-1=-\alpha' p^2-1+R$ and $\alpha_i\equiv 1+\alpha' q_i^2$ is the trajectory function associated with the i-th internal line. Notice that the argument of the previous section is unaffected if some of the contours C_i enclose the origin instead of running from 0 to 1. Also notice that the only way the integral on the right-hand side of (1) distinguishes which particular lines are cut is via the assignment of contours C_i. In particular the integrand is quite independent of the choice of which lines are cut and it is this which will enable us to collect together different terms of the sum (12.3)

Since the operators R has a positive spectrum the expression

$$\prod_{m=1}^{\infty}(1-x_{1\cdots N}^{m})^{2}\,\mathrm{Tr}\left[x_1^{R}V(k_1,1)\cdots x_N^{R}V(k_N,1)\right]\qquad(2)$$

is analytic in the x_i at the origin and so possesses a multiple Taylor expansion in this variables

$$\sum_{m_i=0}^{\infty}\prod_{i=1}^{N}(x_i^{m_i})\;T_{m_1 m_2\cdots m_N}\qquad(3)$$

Now

$$-\alpha'\int_0^1\frac{dx_i}{x_i}\,x_i^{m_i-\alpha_i}=-\frac{\alpha'}{m_i-\alpha_i}=\frac{1}{2\alpha' q_i^2-2(m_i-1)}\qquad(4)$$

$$\oint\frac{dx_i}{2\pi i\,x_i}\,x_i^{m_i-\alpha_i}=\delta_{m_i,\alpha_i}\qquad(5)$$

Therefore

$$-g^{N}\sum_{\substack{\text{ways of}\\ \text{cutting}}}\int\frac{d^{26}q}{(2\pi)^{26}}\sum_{m_i=0}^{\infty}\prod_{i=1}^{N}\Delta_i^{(0)}(q_i;m_i)\,T_{m_1 m_2\cdots m_N}\qquad(6)$$

where $\Delta_i^{(0)}(q_i;m_i^2)$ is given in (4-3) with $m_i^2=2(m_i-1)/2\alpha'$. If we interchange the summation over m_i with the remaining sums and integrations we con-

use the Feynman tree theorem for ϕ^3 theory (3.2) to rewrite (13.6) as

$$\int \frac{d^{26}q}{(2\pi)^{26}} \sum_{m_i=0}^{\infty} \prod_{i=1}^{N} \frac{2\alpha'}{2\alpha' q_i^2 - 2(m_i - 1) + i\epsilon} \; T_{m_1, m_2 \cdots m_N} \tag{1}$$

So working both using (X.-13.4) we get that for a planar diagram with N vertices the quantity to consider is

$$F_N \equiv \int \frac{d^{26}q}{(2\pi)^{26}} \prod_{i=1}^{N} \left\{ -\alpha' \int_0^1 \frac{dx_i}{x_i} \right\} \prod_{m=1}^{\infty} (1 - x_{iN}^m)^2 \; Tr \left\{ x_1^{L_0 - 1} V(k_1, 1) \cdots x_N^{L_0 - 1} V(k_N, 1) \right\} \tag{2}$$

where we remember that $x_{iN} = x_i \cdots x_N$. The effect of inserting the projection operators is to produce the factor $\prod_{m=1}^{\infty} (1 - x_{iN}^m)^2$. This vector does not depend on the vector k^M used to define E and this ensures that (2) is Lorentz invariant. Since all the x_i play the same role and due to the cyclic properties of the trace we can write our final result as

$$F_N \equiv \int \frac{d^{26}q}{(2\pi)^{26}} \prod_{i=1}^{N} \left\{ -\alpha' \int \frac{dx_i}{x_i} \right\} \prod_{m=1}^{\infty} (1 - x_{iN}^m)^2 \; Tr \left\{ V(k_N, 1) \, x_N^{L_0 - 1} \cdots V(k_1, 1) \, x_1^{L_0 - 1} \right\} \tag{3}$$

In order to compute this trace we will use the method of Amati-Bouchiat-Gervais. We will write

$$L_0 = -\alpha' p^2 + \sum_{m=1}^{\infty} r_m \quad , \quad r_m \equiv -m \, (a_m^+ \cdot a_m)$$

$$V(k, 1) \equiv V^{(0)}(k) \prod_{m=1}^{\infty} V^{(m)}(k) \quad , \quad V^{(0)}(k) \equiv e^{-ikq} \tag{4}$$

$$V^{(m)}(k) \equiv \exp\left\{ -(2\alpha')^{1/2} \frac{1}{\sqrt{m}} (k \cdot a_m^+) \right\} \exp\left\{ +(2\alpha')^{1/2} \frac{1}{\sqrt{m}} (k \cdot a_m) \right\}$$

Then the trace appearing in (3) can be rewritten as

$$T_N \equiv T^{(0)} \prod_{i=1}^{\infty} T^{(m)}$$

$$T^{(0)} = Tr \left\{ V^{(0)}(k_N) \, x_N^{-\alpha' p^2 - 1} \, V^{(0)}(k_{N-1}) \, x_{N-1}^{-\alpha' p^2 - 1} \cdots V^{(0)}(k_1) \, x_1^{-\alpha' p^2 - 1} \right\} \tag{5}$$

$$T^{(m)} = Tr \left\{ V^{(m)}(k_N) \, x_N^{r_m} \, V^{(m)}(k_{N-1}) \, x_{N-1}^{r_m} \cdots V^{(m)}(k_1) \, x_1^{r_m} \right\}$$

due to the commutation relations (VII-11-6). Let us now introduce

$$\alpha_s = 1 - \alpha' (q + k_1 + \cdots + k_s)^2 \quad , \quad \alpha_0 \equiv 1 - \alpha' q^2 \tag{6}$$

Thus we get immediately

$$\mathrm{Tr}\left\{ V^{(0)}(k_N)\, x_N^{-\alpha' k^2 - 1} \cdots V^{(0)}(k_1)\, x_1^{-\alpha' k^2 - 1} \right\} = \prod_{i=1}^{N} x_i^{-\alpha_i - 1} \tag{1}$$

In order to treat in the calculations of $T^{(m)}$ all the space dimensions in a similar way we do a Wick rotation

$$b_m^i \equiv a_m^i \quad , \quad b_m^{i+} \equiv a_m^{i+} \quad , \quad \hat{k}^i \equiv k^i \quad , \quad i = 1, 2, \ldots D-1$$

$$b_m^D \equiv -i a_m^0 \quad , \quad b_m^{D+} \equiv -i a_m^{0+} \quad , \quad \hat{k}^D \equiv -i k^0 \tag{2}$$

Then

$$r_m = m \sum_{\mu=1}^{D} b_m^{\mu +} b_m^{\mu} \equiv m \left(b_m^+ . b_m \right)$$

$$V^{(m)}(k) = \exp\left\{ (2\alpha')^{1/2} \frac{1}{\sqrt{m}} (\hat{k} . b_m^+) \right\} \exp\left\{ - (2\alpha')^{1/2} \frac{1}{\sqrt{m}} (\hat{k} . b_m) \right\} \tag{3}$$

$$\left[b_m^{\mu} , b_m^{\nu +} \right] = \delta_{\mu\nu}$$

Due to these commutation relations $T^{(m)}$ can be computed independently for each value of μ and then multiply the so obtained results. Let us compute one of them. We will drop the index μ. It is useful to introduce coherent states defined as

$$|z\rangle \equiv e^{z b^+} |0\rangle \tag{4}$$

$z \equiv$ complex number

Clearly

$$e^{(z_2 b^+)} |z_1\rangle = |z_1 + z_2\rangle \tag{5}$$

Since

$$\left[b , (z b^+)^m \right] = m z (z b^+)^{m-1} \qquad m = 0, 1, 2, \ldots \tag{6}$$

we obtain

$$b\, e^{(z b^+)} = e^{(z b^+)} (b + z) \tag{7}$$

and hence

$$b\,|z\rangle = z\,|z\rangle \tag{1}$$

Furthermore from (15.10)

$$(z_1^* b)^m\, e^{z_2 b^+} = e^{z_2 b^+} \left[z_1^* b + z_1^* z_2 \right]^m \tag{2}$$

and hence

$$\langle z_1 | z_2 \rangle = e^{z_1^* z_2} \tag{3}$$

Also we can prove that

$$(b^+ b)^m\, (z b^+) = (z b^+) \left[(b^+ b) + 1 \right]^m \tag{4}$$

and hence

$$e^{\lambda (b^+ b)}\, (z b^+) = (z b^+)\, e^{\lambda \left[(b^+ b) + 1 \right]} \tag{5}$$

Therefore

$$e^{\lambda (b^+ b)}\, (z b^+)^m = (z b^+)^m\, e^{\lambda \left[(b^+ b) + m \right]} \tag{6}$$

and then we obtain

$$e^{\lambda (b^+ b)}\, |z\rangle = |e^{\lambda} z\rangle \tag{7}$$

For coherent states the unit operator can be written as

$$I = \frac{1}{\pi} \int d^2 z\; e^{-(z^* z)}\, |z\rangle\langle z| \quad , \quad d^2 z \equiv dx\, dy \quad , \quad z = x + iy \tag{8}$$

The integration extended to all complex plane, $x, y \in (-\infty, +\infty)$. Let us check this

$$I\,|z'\rangle = \frac{1}{\pi} \int d^2 z\; e^{-(z^* z)}\, e^{(z^* z')}\, e^{(z b^+)}\, |0\rangle =$$

$$= \frac{1}{\pi} \int dx\, dy\, \exp \left\{ -x^2 - y^2 + x' x + i y' x - i x' y + y' y + x b^+ + i y b^+ \right\} |0\rangle$$

$$= \frac{1}{\pi} \int_{-\infty}^{+\infty} dx\, \exp \left\{ -x^2 + (x' + i y' + b^+) x \right\} \frac{i}{\pi} \int_{-\infty}^{+\infty} dy\, \exp \left\{ -y^2 + (y' - i x' + i b^+) y \right\} |0\rangle$$

$$= \exp \left\{ \frac{1}{4} (x' + i y' + b^+)^2 \right\} \exp \left\{ \frac{1}{4} (-i x' + y' + i b^+)^2 \right\} |0\rangle =$$

$$= \exp (x' + i y')\, b^+\, |0\rangle = |z'\rangle \qquad \text{Q.E.D.}$$

Now we need to calculate

$$t^{(m)} = Tr \{ O_m(x_N \hat{k}_N) \cdots O_m(x_1 \hat{k}_1) \}$$

$$O_m(x, \hat{k}) \equiv \exp\{ \hat{q} b_m^+ \} \exp\{ -\hat{q} b_m \} u^{b_m^+ b_m} \qquad (1)$$

$$\hat{q} \equiv (2\alpha')^{1/2} \frac{1}{\sqrt{m}} \hat{k} \quad , \quad u \equiv x^m$$

Using (16.8)

$$t^{(m)} = \frac{1}{\pi} \int d^2 z \, \langle z | O_m(x_N \hat{k}_N) \cdots O_m(x_1 \hat{k}_1) | z \rangle \, e^{-z^* z} \qquad (2)$$

Let us now consider the effect on $O_m(x, \hat{k})$ on $|z\rangle$

$$O_m(x, \hat{k}) |z\rangle = \exp\{ \hat{q} b_m^+ \} \exp\{ -\hat{q} b_m \} u^{b_m^+ b_m} |z\rangle =$$

$$= \exp\{ \hat{q} b_m^+ \} \exp\{ -\hat{q} b_m \} |uz\rangle = \exp\{ -uz\hat{q} \} |uz + \hat{q}\rangle \Rightarrow$$

$$O_m(x, \hat{k}) |z\rangle = e^{-u\hat{q}z} |uz + \hat{q}\rangle \qquad (3)$$

Let us now define

$$z_e = u_e z_{e-1} + \hat{q}_e \quad , \quad z_0 \equiv z \qquad (4)$$

$$\beta_e = u_e \hat{q}_e z_{e-1} + \beta_{e-1} \quad , \quad \beta_0 = 0$$

Then

$$O_m(x_e \hat{k}_e) \cdots O_m(x_1 \hat{k}_1) |z\rangle = e^{-\beta_e} |z_e\rangle \qquad (5)$$

Relations (4) can be solved immediately with the result

$$z_e = z \left\{ \prod_{i=1}^{e} u_i \right\} + \sum_{i=1}^{e-1} \hat{q}_i (u_{i+1} \cdots u_e) + \hat{q}_e$$

$$\beta_e = z \sum_{k=1}^{e} \hat{q}_k \left(\prod_{i=1}^{k} u_i \right) + \sum_{1 \le k < j}^{e} \hat{q}_k \hat{q}_j (u_{k+1} u_{k+2} \cdots u_j) \qquad (6)$$

Using all that we obtain

$$t^{(m)} = \frac{1}{\pi} \int d^2z \; e^{-|z|^2} \langle z | z_N \rangle \; e^{-\beta_N} =$$

$$= \frac{1}{\pi} \int d^2z \; e^{-|z|^2} \; e^{z^* z_N} \; e^{-\beta_N} =$$

$$= \frac{1}{\pi} \int d^2z \; \exp \left\{ - A |z|^2 + Bz + C z^* + D \right\} \qquad (1)$$

where

$$A \equiv 1 - \prod_{i=1}^{N} u_i \equiv 1 - w \quad , \quad B \equiv - \sum_{\ell=1}^{N} \hat{q}_\ell \left(\prod_{i=1}^{\ell} u_i \right)$$

$$C \equiv \sum_{\ell=1}^{N-1} \hat{q}_\ell \left(\prod_{i=\ell+1}^{N} u_i \right) + \hat{q}_N \quad , \quad D \equiv - \sum_{\ell=2}^{N} \sum_{i=1}^{\ell-1} \hat{q}_i \hat{q}_\ell \, (u_{i+1} \cdots u_\ell) \qquad (2)$$

But

$$t^{(m)} = \frac{1}{\pi} \int_0^1 dx \int_0^1 dy \; \exp \left\{ - A x^2 - A y^2 + B x + i B y + C x - i C y + D \right\} =$$

$$= e^D \; \frac{1}{\sqrt{\pi}} \int_0^1 dx \; e^{-Ax^2 + (B+C)x} \; \frac{1}{\sqrt{\pi}} \int_0^1 dy \; e^{-Ay^2 + i(B-C)y} =$$

$$= e^D \; \frac{1}{\sqrt{A}} \; e^{\frac{1}{4} A^{-1} (B+C)^2} \; \frac{1}{\sqrt{A}} \; e^{-\frac{1}{4} A^{-1} (B-C)^2} = \frac{1}{A} \, e^D \, e^{BC/A}$$

$$= \frac{1}{1-w} \exp \left\{ - \frac{1}{1-w} \left\{ - D (1-w) - BC \right\} \right\} \qquad (3)$$

A careful inspection of all that allows us to write

$$t^{(m)} = \frac{1}{1-w} \exp \left\{ - \frac{1}{1-w} \sum_{i=1}^{N} \sum_{j=1}^{N} C_{ij} \, \hat{q}_i \, \hat{q}_j \right\}$$

$$C_{ij} = u_1 \cdots u_i \qquad j = N \qquad (4)$$

$$C_{ij} = (u_1 \cdots u_i)(u_{j+1} \cdots u_N) \qquad i \leq j \leq N-1$$

$$C_{ij} = (u_{j+1} \cdots u_i) \qquad i > j$$

Let us now perform the products of traces for $\mu = 1, 2, \ldots D$, then going back to Minkowski coordinates

$$T^{(m)} = \frac{1}{(1-w)^D} \exp\left\{ \frac{1}{1-w} \sum_{i=1}^{N} \sum_{j=1}^{N} (2\alpha') \frac{1}{m} C_{ij}^{\cdot} (k_i \cdot k_j) \right\} \qquad (4)$$

or equivalently

$$T^{(m)} = \frac{1}{(1-x_{1N})^D} \exp\left\{ \frac{1}{1-x_{1N}^m} \frac{2\alpha'}{m} \sum_{i=1}^{N} \sum_{j=1}^{N} C_{ij}^{m} (k_i \cdot k_j) \right\}$$

$$C_{ij} = x_1 \cdots x_i \qquad \text{if} \quad j = N \qquad (2)$$

$$C_{ij} = (x_1 \cdots x_i)(x_{j+1} \cdots x_N) \qquad \text{if} \quad i \le j \le N-1$$

$$C_{ij} = x_{j+1} \cdots x_i \qquad \text{if} \quad i > j$$

Notice that another equivalent way of writing C_{ij} is

$$C_{ij} = x_{j+1}\, x_{j+2} \cdots x_i \qquad \text{(in cyclic order)} \qquad (3)$$

and then

$$C_{ii} = x_{1N} \quad , \quad C_{ij} = x_{1N} / C_{ji} \quad \text{if} \quad i \neq j \qquad (4)$$

and therefore we can write

$$T^{(m)} = \frac{1}{(1-x_{1N}^m)^D} \prod_{1 \le i < j \le N} \exp\left\{ 2\alpha' (k_i \cdot k_j) \frac{C_{ji}^m + (x_{1N}/C_{ji})^m}{m(1-x_{1N}^m)} \right\} \cdot$$

$$\cdot \prod_{i=1}^{N} \exp\left\{ 2\alpha' k_i^2 \frac{x_{1N}^m}{m(1-x_{1N}^m)} \right\} \qquad (5)$$

Since $\sum k_i = 0 \Rightarrow \sum_{i=1}^{N} k_i^2 = -2 \sum_{1 \le i < j \le N} (k_i \cdot k_j)$ we get

$$T^{(m)} = \frac{1}{(1-x_{1N}^m)^D} \prod_{1 \le i < j \le N} \exp\left\{ 2\alpha' (k_i \cdot k_j) \frac{C_{ji}^m + (x_{1N}/C_{ji})^m - 2x_{1N}^m}{m(1-x_{1N}^m)} \right\} \qquad (6)$$

Hence

$$\mathcal{T} \equiv \prod_{m=1}^{\infty} T^{(m)} = \left[f(w) \right]^{+D} \prod_{m=1}^{\infty} \prod_{1 \le i < j \le N} \exp\left\{ 2\alpha' (k_i \cdot k_j) \frac{C_{ji}^m + (w/C_{ji})^m - 2w^m}{m(1-w^m)} \right\}$$

$w = x_1 \cdots x_N$, $C_{ji} = x_{i+1} \cdots x_j$ (in cyclic order) (7)

$$f(x) = \prod_{m=1}^{\infty} \frac{1}{(1-x^m)}$$

Furthermore

$$\prod_{m=1}^{\infty} \exp\left\{ 2\alpha'(k_i\cdot k_j)\ \frac{c_{ji}^m + (w/c_{ji})^m - 2w^m}{m\,(1-w^m)} \right\} =$$

$$= \exp\left\{ 2\alpha'(k_i\cdot k_j) \sum_{m=1}^{\infty} \frac{c_{ji}^m + (w/c_{ji})^m - 2w^m}{m\,(1-w^m)} \right\} =$$

$$= \exp\left\{ 2\alpha'(k_i\cdot k_j) \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{n}\left[(c_{ji}\,w^{m-1})^n + (w^m/c_{ji})^n - 2(w^m)^n \right] \right\} =$$

$$= \exp\left\{ 2\alpha'(k_i\cdot k_j) \sum_{m=1}^{\infty} \left\{ -\ln\left[1 - c_{ji}\,w^{m-1}\right] - \ln\left[1 - w^m/c_{ji}\right] + 2\ln\left[1-w^m\right] \right\} \right\}$$

$$= \exp\left\{ -2\alpha'(k_i\cdot k_j) \sum_{m=1}^{\infty} \ln \frac{(1 - c_{ji}\,w^{m-1})(1 - w^m/c_{ji})}{(1-w^m)^2} \right\}$$

$$= \prod_{m=1}^{\infty} \left\{ \frac{(1 - w^{m-1}c_{ji})(1 - w^m/c_{ji})}{(1-w^m)^2} \right\}^{-2\alpha'(k_i\cdot k_j)} \qquad (1)$$

and therefore we can write

$$\mathcal{C} \equiv \prod_{m=1}^{\infty} T^{(m)} = \left[f(w) \right]^{+D} \prod_{1\le i<j\le N} \prod_{m=1}^{\infty} \left\{ \frac{(1 - w^{m-1}c_{ji})(1 - w^m/c_{ji})}{(1-w^m)^2} \right\}^{-2\alpha'(k_i\cdot k_j)}$$

$$w = x_1 x_2 \cdots x_N \quad , \quad c_{ji} = x_{i+1}\, x_{i+2} \cdots x_j \quad \text{(cyclically)} \qquad (2)$$

$$f(w) = \prod_{m=1}^{\infty} (1-w^m)^{-1}$$

In order to write this in a more compact way let us introduce the Jacobi Θ functions

$$\Theta_1\left(\frac{\ln x}{2\pi i} \,\middle|\, \frac{\ln w}{2\pi i} \right) \equiv i\, x^{-1/2} f(w)\, w^{1/8} \prod_{m=0}^{\infty} (1 - x w^m)(1 - w^{m+1}/x) \qquad (3)$$

Let us denote

$$\Theta_1'\left(\frac{\ln x}{2\pi i} \,\middle|\, \frac{\ln w}{2\pi i} \right) \equiv \frac{d}{d\left(\frac{\ln x}{2\pi i}\right)} \Theta_1\left(\frac{\ln x}{2\pi i} \,\middle|\, \frac{\ln w}{2\pi i} \right) \qquad (4)$$

$$= 2\pi i \times \frac{d}{dx}\, \Theta_1\left(\frac{\ln x}{2\pi i} \,\middle|\, \frac{\ln w}{2\pi i} \right)$$

Then

$$\theta_1'\left(0 \,\middle|\, \frac{\ln w}{2\pi i}\right) = 2\pi i x\; i f(w)\, w^{1/8} \prod_{m=0}^{\infty} (1-xw^m)\left(1-\frac{w^{m+1}}{x}\right) \cdot$$

$$\cdot \left\{ -\frac{1}{2} x^{-3/2} + x^{-1/2} \sum_{k=0}^{\infty} \frac{w^{k+1}/x^2 - w^k}{(1-xw^k)\left(1-\frac{w^{k+1}}{x}\right)} \right\}\Bigg|_{x=1} =$$

$$= -2\pi f(w)\, w^{1/8} (1-x)(1-w) \prod_{m=1}^{\infty} (1-w^m)(1-w^{m+1}) \cdot$$

$$\cdot \left\{ -\frac{1}{2} + \frac{w-1}{(1-x)(1-w)} + \sum_{k=1}^{\infty} \left\{ \frac{1}{1-w^k} - \frac{1}{1-w^{k+1}} \right\} \right\}\Bigg|_{x=1}$$

$$= -2\pi f(w)\, w^{1/8} (1-x) \prod_{m=1}^{\infty} (1-w^m)^2 \left\{ -\frac{1}{2} - \frac{1}{1-x} - \frac{1}{1-w} \right\}\Bigg|_{x=1}$$

$$\theta_1'\left(0 \,\middle|\, \frac{\ln w}{2\pi i}\right) = 2\pi f(w)\, w^{1/8} \prod_{m=1}^{\infty} (1-w^m)^2 \tag{1}$$

and therefore

$$Z = [f(w)]^{+D} \prod_{1 \le i < j \le N} \left\{ -2\pi i\, C_{ji}^{1/2} \frac{\theta_1\left(\frac{\ln C_{ji}}{2\pi i} \,\middle|\, \frac{\ln w}{2\pi i}\right)}{\theta_1'\left(0 \,\middle|\, \frac{\ln w}{2\pi i}\right)} \right\}^{-2\alpha'(k_i \cdot k_j)} \tag{2}$$

Notice furthermore that

$$\prod_{1 \le i < j \le N} C_{ji}^{-\alpha'(k_i \cdot k_j)} = \frac{1}{x_2 x_3 \cdots x_N} = \frac{x_1}{w} \tag{3}$$

and therefore

$$Z = [f(w)]^{+D} \frac{x_1}{w} \prod_{1 \le i < j \le N} \left\{ -2\pi i \frac{\theta_1\left(\frac{\ln C_{ji}}{2\pi i} \,\middle|\, \frac{\ln w}{2\pi i}\right)}{\theta_1'\left(0 \,\middle|\, \frac{\ln w}{2\pi i}\right)} \right\}^{-2\alpha'(k_i \cdot k_j)} \tag{4}$$

which is the desired result

We must perform still the integration on q.

$$S_N \equiv \int d^D q \prod_{i=1}^{N} x_i^{-1-\alpha'(q+k_1+\cdots+k_{i-1})^2} \tag{5}$$

To see how to calculate this type of integrals let us consider some particular values of N. Let us begin doing a Wick rotation

$$S_N = i \int d^D \hat{q} \prod_{i=1}^{N} x_i^{-1 + \alpha' (\hat{q} + \hat{k}_{1,i-1})^2} \qquad \hat{k}_{1,i-1} \equiv \hat{k}_1 + \hat{k}_2 + \dots + \hat{k}_{i-1} \tag{1}$$

Then

$$S_N = \frac{i}{W} \int d^D \hat{q} \exp \left\{ \alpha' \hat{q}^2 \ln w + 2\alpha' \hat{q} \cdot \sum_{i=2}^{N} \hat{k}_{1,i-1} \ln x_i + \alpha' \sum_{i=2}^{N} \hat{k}^2_{1,i-1} \ln x_i \right\}$$

$$= \frac{i}{W} \exp \left\{ \alpha' \sum_{k=2}^{N} \hat{k}^2_{1,i-1} \ln x_i \right\} \int d^D \hat{q} \exp \left\{ \alpha' \ln w \left[\hat{q} + \frac{\sum_{i=2}^{N} \hat{k}_{1,i-1} \ln x_i}{\ln w} \right]^2 \right\}$$

$$\left\{ \cdot \exp - \alpha' \frac{\left[\sum_{i=2}^{N} \hat{k}_{1,i-1} \ln x_i \right]^2}{\ln w} \right\} =$$

$$= \frac{i}{W} \exp \left\{ \alpha' \sum_{k=2}^{N} \hat{k}^2_{1,i-1} \ln x_i \right\} \exp \left\{ - \alpha' \frac{1}{\ln w} \left[\sum_{i=2}^{N} \hat{k}_{1,i-1} \ln x_i \right]^2 \right\} \int d^D \hat{q} \left\{ - [1 - \alpha' \ln w] \hat{q}^2 \right\}$$

$$= \frac{i}{W} \left(\frac{\pi}{-\alpha' \ln w} \right)^{D/2} \exp \left\{ - \alpha' \sum_{k=2}^{N} k^2_{1,i-1} \ln x_i \right\} \exp \left\{ \frac{\alpha'}{\ln w} \left[\sum_{i=2}^{N} k_{1,i-1} \ln x_i \right]^2 \right\} \tag{1}$$

If $N = 2$

$$S_2 = \frac{i}{W} \left(\frac{\pi}{-\alpha' \ln w} \right)^{D/2} \exp \left\{ \ln x_2 \right\} \exp \left\{ \frac{\alpha'}{\ln w} k_1^2 \ln^2 x_2 \right\} \Rightarrow$$

$$S_2 = i \left(\frac{\pi}{-\alpha' \ln w} \right)^{D/2} \frac{1}{x_1} \sum_{1 \le i < j \le 2} \exp \left\{ - \alpha' (k_i \cdot k_j) \frac{[\ln C_{ji}]^2}{\ln w} \right\} \tag{2}$$

If $N = 3$

$$S_3 = \frac{i}{W} \left(\frac{\pi}{-\alpha' \ln w} \right)^{D/2} \exp \left\{ \ln (x_2 x_3) \right\} \exp \left\{ \frac{\alpha'}{\ln w} [k_1 \ln x_2 x_3 + k_2 \ln x_3]^2 \right\} \Rightarrow$$

$$S_3 = i \left(\frac{\pi}{-\alpha' \ln w} \right)^{D/2} \frac{1}{x_1} \sum_{1 \le i < j \le 3} \exp \left\{ - \alpha' (k_i \cdot k_j) \frac{[\ln C_{ji}]^2}{\ln w} \right\} \tag{3}$$

where in both cases the momentum conservation laws and the fact that $-\alpha' k_i^2 = 1$ has been used.

In general we have

$$S_N = i \left(\frac{\pi}{-\alpha' \ln w} \right)^{D/2} \frac{1}{x_1} \sum_{1 \le i < j \le N} \exp \left\{ - \alpha' (k_i \cdot k_j) \frac{[\ln C_{ji}]^2}{\ln w} \right\} \tag{4}$$

And writing all this together

$$F_N = i\, g^N (-\alpha')^N \int_0^1 dx_1 \cdots dx_N \left(\frac{\Gamma}{-\alpha' \ln w} \right)^{D/2} \frac{[f(w)]^{D-2}}{w^2} \cdot$$

$$\cdot \prod_{1 \le i < j \le N} \left\{ -2\pi i \exp\left\{ \frac{[\ln c_{ji}]^2}{2 \ln w} \right\} \frac{\theta_1\left(\frac{\ln c_{ji}}{2\pi i} \,\middle|\, \frac{\ln w}{2\pi i} \right)}{\theta_1'\left(0 \,\middle|\, \frac{\ln w}{2\pi i} \right)} \right\}^{-2\alpha' (k_i \cdot k_j)} \tag{1}$$

Introducing

$$\psi(x) \equiv \psi(x, w) \equiv -2\pi i \exp\left\{ \frac{[\ln x]^2}{2 \ln w} \right\} \frac{\theta_1\left(\frac{\ln c_{ji}}{2\pi i} \,\middle|\, \frac{\ln w}{2\pi i} \right)}{\theta_1'\left(0 \,\middle|\, \frac{\ln w}{2\pi i} \right)} \tag{2}$$

we can write for the planar amplitude

$$F_N = i\, g^N (-\alpha')^N \int_0^1 dx_1 \cdots dx_N \left(\frac{\Gamma}{-\alpha' \ln w} \right)^{D/2} \frac{[f(w)]^{D-2}}{w^2} \cdot$$

$$\cdot \prod_{1 \le i < j \le N} |\psi(c_{ji}, w)]^{-2\alpha' (k_j \cdot k_i)} \tag{3}$$

XI- SUPERSTRING THEORY

The study of dual resonance models began in 1968 with the discovery [G. Veneziano Nuovo Cim 57A, 190 (1968)] of an explicit four-particle amplitude that combines the narrow-resonance approximation with Regge behaviour and crossing symmetry. In 1971 a second dual model, originally called the "dual pion model" (DPM), containing a spectrum of fermions [P. Ramond Phys. Rev. D3, 2415 (1971)], was developed [A. Neveu y J.H. Schwarz Nucl. Phys B31, 86 (1971), Phys. Rev. D4, 1109 (1971)]. In an important paper [F. Gliozzi, J. Scherk and D.I. Olive Phys Lett 65B, 282 (1976), Nucl. Phys. B122, 253 (1977)] the authors showed that the Neveau-Ramond-Schwarz (NRS) model can be restricted in such a way that the spectrum of states is supersymmetric. However supersymmetry was in that model rather obscure although the model exhibited good properties as the absence of tachyons. That was the starting point of the work of M.B. Green and J.H. Schwarz [Nucl. Phys. B181, 502 (1981); B198, 252 (1982); B198, 441 (1982); Phys. Lett. 109B, 444 (1982)] who reformulated the theory in a manifestly supersymmetric manner in $D = 10$. The original superstring theory was formulated in the light cone gauge, since there was no known a covariant action. Further work has shown that such an action exist. It is nevertheless more convinient to start with the light cone approach.

The superstring is described, in the light cone gauge, by the transverse coordinates $X^i(\sigma,\tau)$ $(i = 2,\dots D = 10)$ and the fermionic degrees of freedom $S^{Aa}(\sigma,\tau)$. Both indices A and a are spinor indices in $D = 2$ and $D = 10$ dimensions, respectively. We use the notation $(\gamma^\mu)^{ab}$ for the space-time Dirac matrices which satisfy the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{1}$$

and their inducible representation has dimension $2^{D/2} = 2^5 = 32$. The Dirac matrices for the two-dimensional world-sheet are denoted by $(\rho^\alpha)^{AB}$ and they satisfy the Clifford-algebra

$$\{\rho^\alpha, \rho^\beta\} = 2\eta^{\alpha\beta} \qquad \eta^{\alpha\beta} = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2}$$

A possible representation of this matrices is

$$\rho^2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad , \quad \rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{1}$$

$$(\rho^\alpha)^* = -\rho^\alpha \quad , \quad \rho^0 \rho^{\alpha T} \rho^0 = -\rho^\alpha$$

and therefore this is the Majorana representation.

Supersymmetry requires the number of bosonic and fermionic degrees of freedom to be equal. There are 8 bosonic degrees of freedom corresponding to the 8 transverse coordinates $X^i(\sigma,\tau)$. On the other hand there are $2 \times 32 = 64$ complex fermionic coordinates $S^{Aa}(\sigma,\tau)$. This means that one must impose certain restrictions on these variables if one wants to obtain 8 fermionic degrees of freedom. These conditions are

i) <u>Majorana spinors</u>. They must satisfy a reality condition

$$[S^{Aa}(\sigma,\tau)]^* = S^{Aa}(\sigma,\tau) \tag{2}$$

which allows us to use for the Dirac matrices the Majorana representation

$$(\gamma^\mu)^* = -\gamma^\mu \quad , \quad (\rho^\alpha)^* = -\rho^\alpha \tag{3}$$

ii) <u>Weyl spinors</u>. We require that $S^{Aa}(\sigma,\tau)$ are Weyl spinors, i.e. they satisfy the chirality relation

$$(\gamma^{11})^{ab} S^{Ab} = \eta_A S^{Aa} \tag{4}$$

where $\eta_A = \pm 1$ is the chirality of the spinor S^A and γ^{11} is the chiral matrix defined as

$$\gamma^{11} \equiv \gamma^0 \gamma^1 \cdots \gamma^9 \tag{5}$$

It is easy to check that

$$(\gamma^{11})^2 = I \quad , \quad (\gamma^{11})^+ = \gamma^{11} \quad , \quad (\gamma^{11})^* = \gamma^{11} \tag{6}$$

It is convenient to introduce the Weyl projection operator h_A defined as

$$h_A \equiv \frac{1}{2}(1 - \eta_A \gamma^{11}) \quad , \quad h_A^2 = h_A \quad , \quad h_A = h_A^* = h_A^+ \tag{7}$$

so that

$$h_A S^A = 0 \qquad \text{for} \quad A = 1, 2 \tag{8}$$

We observe that S^1 and S^2 can have both the same or opposite chirality. This will be important when we classify the different kinds of supersymmetric theories.

Let us remark that the Majorana and Weyl conditions can be simultaneously imposed on a spinor in $D = 10$, so that they become Majorana - Weyl spinors. This is indeed possible since γ^{11} is a real matrix (2.6), so that if we take complex conjugates in equation (2.5) we find no contradiction. Notice for D dimensions in order to have

$$[\gamma^{D+1}]^2 = I \quad , \quad (\gamma^{D+1})^\dagger = \gamma^{D+1} \tag{1}$$

we must define

$$\gamma^{D+1} \equiv - [i]^{(D-1)(D+2)/2} \gamma^0 \gamma^1 \cdots \gamma^{D-1} \tag{2}$$

and then in the Majorana representation

$$(\gamma^{D+1})^* = (-1)^{D + \frac{1}{2}(D-1)(D+2)} \gamma^{D+1} \tag{3}$$

and in order to be able to define Majorana and Weyl spinors you must have $D + (D-1)(D+2)/2 =$ even and this fails for $D = 4$.

A Dirac spinor ψ in $D = 10$ has $2^{D/2} = 2^5 = 32$ complex components. If it is Majorana it has $2^{D/2} = 32$ real components and if Majorana - Weyl it has $2^{D/2}/2 = 2^{(D-2)/2} = 16$ real components. Thus the Lorentz group $SO(9,1)$ has two spinor representations of 16 dimensions, corresponding to opposite chiralities ($\eta_A = \pm 1$).

$$SO(D-1=9,1) = \begin{cases} \text{Vector representation} & D = 10 \\ \text{Spinor representations} & \begin{cases} 2^{(D-2)/2} = 16 \\ \underset{\sim}{2^{(D-2)/2}} = \underset{\sim}{16} \end{cases} \end{cases} \tag{4}$$

We need for our superstrings spinors $S^{Aa}(\sigma, \tau)$ two more conditions.

iii) Light - cone gauge condition. In the light - cone gauge the dynamical degrees of freedom are identified with the transverse variables which number $(D-2)$. Remember that for any vector V^M we have

Light-cone components $V_\pm = \frac{1}{\sqrt{2}} (V^0 \pm V^1)$

Transversal components $V^2 \ldots V^{D-1}$ (1)

Furthermore in the light-cone gauge we have for the bosonic string (IV-5.1)

$$X_+ = 2\alpha' P_+ \tau \tag{2}$$

This means that the Lorentz group $SO(9,1)$ is broken according to

$$SO(9,1) \xrightarrow[\text{fixing light-cone gauge}]{} SO(1,1) \otimes SO(8) \tag{3}$$

Here $SO(1,1)$ is the two dimensional Lorentz group which acts on the $(\bar{X}^0, \bar{X}^1)$ plane. Its action is very simple in terms of the light cone variables. Under a boost of $SO(1,1)$ we have

$$V'^0 = \frac{V^0 + v V^1}{\sqrt{1-v^2}} \quad , \quad V'^1 = \frac{v V^0 + V^1}{\sqrt{1-v^2}} \quad \Rightarrow \tag{4}$$

$$V'_+ = \sqrt{\frac{1+v}{1-v}}\, V_+ \quad , \quad V'_- = \sqrt{\frac{1-v}{1+v}}\, V_- \quad \Rightarrow \quad V'_+ V'_- = V_+ V_-$$

The breaking of the Lorentz group given in (3) means that the vectors and spinors in $D=10$ are also broken in the light-cone gauge. Thus a vector V^μ decomposes according to

$$V^\mu \longrightarrow V_+ \, , \, V_- \, , \, V^i \tag{5}$$

and using representation theory we can write eq (5) as

$$\underset{SO(9,1)}{\underset{\sim}{10}} = \underset{SO(1,1) \otimes SO(8)}{(\underset{\sim}{1}, \underset{\sim}{1}) \oplus (-\underset{\sim}{1}, \underset{\sim}{1}) \oplus (\underset{\sim}{0}, \underset{\sim}{8})} \tag{6}$$

We are now interested in the decomposition of the spinor representation of $SO(9,1)$. The Dirac matrices γ^μ can be decomposed according to

$$\gamma^\mu \longrightarrow \gamma_+ \equiv \frac{1}{\sqrt{2}} (\gamma^0 + \gamma^1) \quad , \quad \gamma_- \equiv \frac{1}{\sqrt{2}} (\gamma^0 - \gamma^1) \quad , \quad \gamma^i \tag{7}$$

Notice

$$(\gamma_+)^2 = (\gamma_-)^2 = 0 \quad , \qquad (\gamma_\pm)^+ = \gamma_\mp$$

$$\gamma_+ \gamma_- + \gamma_- \gamma_+ = 2 \tag{1}$$

$$\gamma_\mp \gamma^i + \gamma^i \gamma_\pm = 0 \qquad i = 2, \ldots D-1$$

These properties imply that

$$P_+ \equiv \frac{1}{2} \gamma_- \gamma_+ \quad , \qquad P_- \equiv \frac{1}{2} \gamma_+ \gamma_- \tag{2}$$

satisfy

$$[P_\pm]^+ = P_\pm \quad , \qquad [P_\pm]^2 = P_\pm \quad , \quad P_\pm P_\mp = 0 \quad , \quad P_+ + P_- = I \tag{3}$$

and therefore $P_\pm$ are orthogonal projection operators. In fact they yield the decomposition of a spinor ψ in $D=10$ into two $D=8$ spinors ψ_+ and ψ_-, as follows

$$\psi = \psi_+ + \psi_- \quad , \qquad P_\pm \psi \equiv \psi_\mp \tag{4}$$

and hence

$$\gamma_- \psi_+ = 0 \qquad \psi_+ \psi_- = 0 \tag{5}$$

Let us assume that ψ has definite chirality in $D=10$, i.e.

$$\gamma^{11} \psi = \eta \psi \tag{6}$$

For $D=8$ let us introduce

$$\hat{\gamma}^9 \equiv \gamma^2 \gamma^3 \ldots \gamma^8 \gamma^9 \tag{7}$$

which satisfies the properties

$$(\hat{\gamma}^9)^* = \hat{\gamma}^9 \quad , \quad (\hat{\gamma}^9)^+ = \hat{\gamma}^9 \quad , \quad [\hat{\gamma}^9]^2 = I \tag{8}$$

Notice that

$$\gamma^0 \gamma^1 \gamma^{11} = \gamma^{11} \gamma^0 \gamma^1 = \hat{\gamma}^9 \tag{9}$$

and furthermore

$$(P_+ - P_-)\, \gamma^{11} = \gamma^{11} (P_+ - P_-) = \hat{\gamma}^9$$

Then it is easy to check that

$$\hat{\gamma}^9 \psi_\pm = \pm \eta \psi_\pm \tag{1}$$

Proof:

$$\hat{\gamma}^9 \psi_\pm = \gamma'' \gamma^0 \gamma^1 P_\pm \psi = \frac{1}{2} \gamma'' \gamma^0 \gamma^1 (1 \mp \gamma^1 \gamma^0) \psi = \frac{1}{2} \gamma^0 \gamma^1 \psi'' \psi \pm \frac{1}{2} \gamma'' \psi =$$

$$= \frac{1}{2} \eta (\gamma^0 \gamma^1 \pm 1) \psi = \pm \frac{1}{2} \eta (1 \pm \gamma^0 \gamma^1) \psi = \pm \eta P_\pm \psi = \pm \eta \psi_\pm \quad \text{Q.E.D.}$$

Equation (1) simply means, in group theoretical language, that the 16 dimensional spinor representation of $SO(9,1)$ breaks into two 8 dimensional spinor representations of different chirality, i.e.

$$\underset{SO(9,1)}{\underset{\sim}{16}} \longrightarrow \underset{SO(8)}{\underset{\sim}{8}} + \underset{SO(8)}{\underset{\sim}{8}} \tag{2}$$

Let us now see how the spinor ψ_+ and ψ_- transform under the action of $SO(1,1)$. Let us consider the boost (4.5):

$$\Lambda^0{}_0 = \frac{1}{\sqrt{1-v^2}}, \quad \Lambda^0{}_1 = \frac{v}{\sqrt{1-v^2}}, \quad \Lambda^1{}_0 = \frac{v}{\sqrt{1-v^2}}, \quad \Lambda^1{}_1 = \frac{1}{\sqrt{1-v^2}}, \quad \Lambda^i{}_j = \delta^i{}_j \tag{3}$$

Under this transformation

$$\psi(x) \longrightarrow \psi'(x') = S(\Lambda)\, \psi(x) \tag{4}$$

where

$$S^{-1} \gamma^0 S = \frac{\gamma^0 + v \gamma^1}{\sqrt{1-v^2}}, \quad S^{-1} \gamma^1 S = \frac{v \gamma^0 + \gamma^1}{\sqrt{1-v^2}}, \quad S^{-1} \gamma^i S = \gamma^i \tag{5}$$

or equivalently

$$S^{-1}(\Lambda)\, \gamma_+ S(\Lambda) = e^{\alpha} \gamma_+, \quad S^{-1}(\Lambda)\, \gamma_- S(\Lambda) = e^{-\alpha} \gamma_-, \quad S^{-1}(\Lambda)\, \gamma^i S(\Lambda) = \gamma^i$$

$$e^{\alpha} \equiv \left(\frac{1+v}{1-v} \right)^{1/2} \tag{6}$$

and hence

$$S(\Lambda) = e^{\alpha/2} - \frac{1}{2} (e^{+\alpha/2} - e^{-\alpha/2})\, \gamma_+ \gamma_-$$

$$S^{-1}(\Lambda) = e^{-\alpha/2} + \frac{1}{2} (e^{\alpha/2} - e^{-\alpha/2})\, \gamma_+ \gamma_- \tag{7}$$

Since $P_\pm$ are invariant under these transformations we get

$$\psi_+(x) \longrightarrow \psi_+'(x') = \left(\frac{1+v}{1-v}\right)^{1/4} \psi_+(x)$$

$$\psi_-(x) \longrightarrow \psi_-'(x') = \left(\frac{1-v}{1+v}\right)^{1/4} \psi_-(x) \qquad (4)$$

We shall impose on S^A the following light-cone condition

$$(\gamma_+)^{ab} (S^A)^b = 0 \qquad A = 1,2 \qquad (2)$$

which reduces again the number of degrees of freedom by a half. and therefore we have that the S^A, for each value of A, has 8 real components. We need finally another factor one half in order to achieve supersymmetry.

iv) Equations of motion. This extra factor is obtained as usual from the equations of motion satisfied by the spinors S^A

$$\rho^\alpha \partial_\alpha S^a(\tau,\sigma) = 0 \qquad a = 1,2,\dots 32 \qquad (3)$$

This is a comun feature of any supersymmetric theory which is on-shell, or in other words when one does not use auxiliary fields for the closure of the supersymmetry algebra.

We have therefore

$$\{S^{Aa}\} = \underbrace{(2 \times 32 \times 2)}_{\text{real components}} \times \left(\frac{1}{2}\right)_{\text{Majoran}} \times \left(\frac{1}{2}\right)_{\text{chirality}} \times \left(\frac{1}{2}\right)_{\text{light cone}} \times$$

$$\times \left(\frac{1}{2}\right)_{\text{equat. motion}} = 8 \qquad (4)$$

which is the desired result.

We are now in conditions to build up a supersymmetric light-cone action for the superstring. The bosonic piece is very simple and is obtained from the Nambu-Goto action fixing the gauge:

$$\left.\begin{aligned} S_{NG} &= -\frac{1}{2\pi\alpha'} \int d\tau \int_0^\pi d\sigma \left[(\dot{X}\cdot X')^2 - \dot{X}^2 X'^2 \right]^{1/2} \\ (\dot{X}\cdot X') &= 0 \quad , \quad \dot{X}^2 + X'^2 = 0 \quad , \quad \dot{X}^2 \geq 0 \end{aligned}\right\} \Rightarrow$$

$$S_{NG}^{l.c.} = -\frac{1}{2\pi\alpha'} \int d\tau \int d\sigma\ \dot{X}^2 = -\frac{1}{4\pi\alpha'} \int d\tau \int d\sigma \left[\dot{X}^2 - X'^2\right]$$

$$S^{l.c.}_{boso} = \int d\tau \int_0^{\pi} d\sigma \left\{ -\frac{1}{4\pi\alpha'} \eta^{\alpha\beta} \partial_\alpha X^i(\sigma,\tau)\, \partial_\beta X^i(\sigma,\tau) \right\} \tag{1}$$

$$(\xi^0, \xi^1) \equiv (\tau, \sigma) \qquad , \qquad \eta^{\alpha\beta} = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}$$

Now we must add to this piece of the action the one that reproduces the spinor equation of motion (7.3). Let us remember some known fact: If $\psi(x)$ is a spinor we define $\bar{\psi}(x)$ as

$$\bar{\psi}(x) \equiv \psi(x)^+ \gamma^0 \tag{2}$$

Let us now introduce a matrix $\mathcal{C}$ such that

$$\mathcal{C}\, \gamma^\mu\, \mathcal{C}^{-1} = -(\gamma^\mu)^T \tag{3}$$

where T means transposed. Then an alternative definition of the Majorana spinors is

$$\psi(x) = [\, \mathcal{C}\, \bar{\psi}^T(x) \,] \tag{4}$$

For any D we have in the Majorana representation

$$\gamma^{0+} = \gamma^0 \quad , \quad \gamma^{i+} = -\gamma^i \quad , \quad \gamma^{\mu *} = -\gamma^\mu \quad \Rightarrow$$

$$\gamma^{0T} = -\gamma^0 \quad , \quad \gamma^{iT} = \gamma^i \quad , \quad \gamma^0 \gamma^{\mu T} \gamma^0 = -\gamma^\mu \tag{5}$$

and therefore

$$\mathcal{C} = -\gamma^0 \quad \Rightarrow \quad \mathcal{C}^2 = I, \quad \mathcal{C} = \mathcal{C}^{-1} = \mathcal{C}^\dagger = -\mathcal{C}^T = \mathcal{C}^* \tag{6}$$

Then it follows that for a generic matrix γ^A

$$\bar{\psi}_1(x)\, \gamma^A\, \psi_2(x) = -\psi_1^T(x)\, \mathcal{C}\, \gamma^A\, \psi_2(x) = +\psi_2^T(x)\, (\mathcal{C}\gamma^A)^T\, \psi_1(x) \tag{7}$$

and this implies that for Majorana fields

$$\psi^T(x)\, (\mathcal{C}\gamma^A)\, \psi(x) = 0 \qquad \text{if} \qquad \mathcal{C}\gamma^A = (\mathcal{C}\gamma^A)^T \tag{8}$$

For instance

$$(C\gamma^\mu)^T = \gamma^{\mu T} \mathcal{C}^T = -\gamma^{\mu T} \mathcal{C} = -\mathcal{C}\, \mathcal{C}^{-1} \gamma^{\mu T} \mathcal{C} = +\mathcal{C}\gamma^\mu \tag{9}$$

Going back to the superstring we note that if we want to construct a bilinear we associate to each spinor $S^{A\alpha}$,

$$\bar{\bar{S}}^{A\alpha} = S^{+B\beta} (\gamma^0)^{\beta\alpha} (\rho^0)^{BA} \tag{1}$$

where the matrices ρ^α have been given in (12.1). Then a natural candidate for the fermionic action density is given by

$$\bar{\bar{S}} \rho^\alpha \partial_\alpha S \equiv S^{+Aa} (\gamma^0)^{ab} (\rho^0 \rho^\alpha)^{AB} \partial_\alpha S^{Bb} \tag{2}$$

but this vanishes identically due to condition (7.2). Indeed

$$\gamma_+ S = 0 \Rightarrow S^+ \gamma_- = 0 \Rightarrow S^+ \gamma^0 \gamma^0 \gamma_- = \bar{S} \gamma^0 \gamma_- = S^+ \gamma_+ \gamma^0 = 0 \Rightarrow$$

$$\bar{S} \gamma_+ = 0 \Rightarrow$$

$$\gamma_+ S = 0 \,, \quad S^+ \gamma_- = 0 \,, \quad \bar{S} \gamma_+ = 0 \tag{3}$$

So that

$$\bar{S} S = \frac{1}{2} \bar{S} (\gamma_+ \gamma_- + \gamma_- \gamma_+) S = 0 \Rightarrow \bar{S} S = 0 \tag{4}$$

The unique choice that is left is

$$\bar{\bar{S}} \gamma_- \rho^\alpha \partial_\alpha S = S^{Aa} (\gamma^0 \gamma_-)^{ab} (\rho^0 \rho^\alpha)^{AB} \partial_\alpha S^{Bb} \tag{5}$$

Note

$$(\gamma^0 \gamma_-)^* = \gamma^0 \gamma_- \qquad (\rho^0 \rho^\alpha)^* = \rho^0 \rho^\alpha$$
$$(\gamma^0 \gamma_-)^T = \gamma^0 \gamma_- \qquad (\rho^0 \rho^\alpha)^T = \rho^0 \rho^\alpha \tag{6}$$

From all that we choose the action for superstring in the light-cone gauge that of Green-Schwarz

$$S^{l.c.} = \int d\tau \int_0^\pi d\sigma \left\{ -\frac{1}{4\pi\alpha'} \partial_\alpha X^i(\sigma,\tau) \partial^\alpha X^i(\sigma\tau) - \right.$$
$$\left. - \frac{i}{4\pi} \bar{\bar{S}}(\sigma,\tau) \gamma_- \rho^\alpha \partial_\alpha S(\sigma,\tau) \right\} \tag{7}$$

This action is invariant under the transverse rotations of $SO(8)$. This action is invariant under the supersymmetry transformations

$$\delta X^i = + (P_+)^{-1/2} \bar{\bar{\epsilon}} \gamma^i S$$

$$\delta S = + \frac{i}{2\alpha'} (P_+)^{-1/2} \gamma_+ \gamma_\mu \rho^\alpha \partial_\alpha X^\mu \epsilon \tag{1}$$

The normalization factor $(P_+)^{-1/2}$ is included for later convenience. The parameters $\epsilon^{A\alpha}$ are anticommuting Majorana-Weyl spinors and they are independent of σ and τ.

Taking into account that

$$\bar{\bar{S}}_1 \gamma_- \rho^\alpha S_2 = - \bar{\bar{S}}_2 \gamma_- \rho^\alpha S_1 \tag{2}$$

we obtain that the variation of the action under (1) is

$$\delta S^{e.c} = \delta_1 S^{e.c} + \delta S_2^{e.c}$$

$$\delta_1 S^{e.c} = \int d\tau \int_0^\pi d\sigma \left\{ + \frac{1}{2\pi\alpha'} [\partial_\alpha \partial^\alpha X^i] \delta X^i - \frac{i}{2\pi} \bar{\bar{S}} \gamma_- \rho^\alpha [\partial_\alpha \delta S] \right\}$$

$$\delta_2 S^{e.c.} = - \int d\tau \int_0^\pi d\sigma \, \partial_\alpha \left\{ \frac{1}{2\pi\alpha'} [\partial^\alpha X^i] \delta X^i - \frac{i}{4\pi} \bar{\bar{S}} \gamma_- \rho^\alpha \delta S \right\} \tag{3}$$

The second term is a surface term and it will be considered later. The first one is

$$+ \frac{1}{2\pi\alpha'} [\partial_\alpha \partial^\alpha X^i] \delta X^i - \frac{i}{2\pi} \bar{\bar{S}} \gamma_- \rho^\alpha [\partial_\alpha \delta S] =$$

$$= + \frac{(P_+)^{-1/2}}{2\pi\alpha'} [\partial_\alpha \partial^\alpha X^i] \bar{\bar{\epsilon}} \gamma^i S + \frac{(P_+)^{-1/2}}{4\pi\alpha'} [\partial_\alpha \partial_\beta X^\mu] \bar{\bar{S}} \gamma_- \rho^\alpha \gamma_+ \gamma_\mu \rho^\beta \epsilon$$

$$= + \frac{(P_+)^{-1/2}}{2\pi\alpha'} \left\{ [\partial_\alpha \partial^\alpha \bar{X}^i] \bar{\bar{\epsilon}} \gamma^i S + \frac{1}{2} [\partial_\alpha \partial^\alpha X^\mu] \bar{\bar{S}} \gamma_- \gamma_+ \gamma_\mu \epsilon \right\} =$$

$$= + \frac{(P_+)^{-1/2}}{2\pi\alpha'} \left\{ [\partial_\alpha \partial^\alpha X^i] \bar{\bar{\epsilon}} \gamma^i S + \frac{1}{2} [\partial_\alpha \partial^\alpha X^\mu] \bar{\bar{\epsilon}} \gamma_\mu \gamma_+ \gamma_- S \right\} \overset{(4.3)}{=}$$

$$= + \frac{(P_+)^{-1/2}}{2\pi\alpha'} \left\{ [\partial_\alpha \partial^\alpha X^i] \bar{\bar{\epsilon}} \gamma^i S + S \gamma_\mu \bar{\bar{\epsilon}} [\partial_\alpha \partial^\alpha X^\mu] \right\} =$$

$$= + \frac{(P_+)^{-1/2}}{2\pi\alpha'} \left\{ [\partial_\alpha \partial^\alpha X^i] \bar{\bar{\epsilon}} \gamma^i S - S \gamma_0 \bar{\bar{\epsilon}} [\partial_\alpha \partial^\alpha X^0] \right\} =$$

$$= + \frac{(P_+)^{-1/2}}{2\pi\alpha'} \frac{1}{2} \left\{ [\partial_\alpha \partial^\alpha X^0] \bar{\bar{\epsilon}} (\cancel{\gamma_+} + \gamma_-) S - [\partial_\alpha \partial^\alpha X^+] \bar{\bar{\epsilon}} (\cancel{\gamma_+} - \gamma_-) S \right\}$$

$$= + \frac{(P_+)^{-1/2}}{2\pi\alpha'} [\partial_\alpha \partial^\alpha X_+] \bar{\bar{\epsilon}} \gamma_- S \overset{(4.2)}{=} 0 \tag{4}$$

Then up to a divergence the Lagrangian density is invariant under the supersymmetry transformations (10.4).

As for any supersymmetric theory we now investigate the algebra generated by these supersymmetry transformations. We have

$$X^i \xrightarrow{\epsilon_1} X^i_{(1)} = X^i + (P_+)^{-1/2}\,\bar{\bar{\epsilon}}_1\,\gamma^i S \tag{1}$$

$$S \longrightarrow S_{(1)} = S + i\,(2\alpha')^{-1}\,(P_+)^{-1/2}\,\gamma_+\gamma_\mu\,\rho^\alpha\,\partial_\alpha X^\mu\,\epsilon_1$$

and

$$X^i_{(1)} \xrightarrow{\epsilon_2} X^i_{(21)} = X^i_{(1)} + (P_+)^{-1/2}\,\bar{\bar{\epsilon}}_2\,\gamma^i S_{(1)} \tag{2}$$

$$S_{(1)} \longrightarrow S_{(21)} = S_{(1)} + i\,(2\alpha')^{-1}\,(P_+)^{-1/2}\,\gamma_+\gamma_\mu\,\rho^\alpha\,\partial_\alpha X^\mu_{(1)}\,\epsilon_2$$

Then

$$X^i_{(21)} = X^i + i\,(2\alpha' P_+)^{-1}\,\bar{\bar{\epsilon}}_2\,\gamma^i\gamma_+\gamma_\mu\,\rho^\alpha\,\partial_\alpha X^\mu\,\epsilon_1 + (P_+)^{-1/2}(\bar{\bar{\epsilon}}_1 + \bar{\bar{\epsilon}}_2)\,\gamma^i S$$

$$S_{(21)} = S - i\,(2\alpha' P_+)^{-1}\,\gamma_+\gamma^i\rho^\alpha\,\partial_\alpha\,[\bar{\bar{\epsilon}}_1\,\gamma^i S]\,\epsilon_2 + i\,(2\alpha')^{-1}(P_+)^{-1/2}\gamma_+\gamma^\mu\rho^\alpha\partial_\alpha X^\mu(\epsilon_1+\epsilon_2) \tag{3}$$

We can now calculate

$$[\delta(\epsilon_1),\,\delta(\epsilon_2)]\,X^i \equiv X^i_{(12)} - X^i_{(21)} = \{+i\,(2\alpha' P_+)^{-1}\,\bar{\bar{\epsilon}}_2\,\gamma^i\gamma_+\gamma_\mu\,\rho^\alpha\,\partial_\alpha X^\mu\,\epsilon_1\} - \{1\leftrightarrow 2\} \tag{4}$$

but

$$\gamma_+\gamma_\mu\,\partial_\alpha X^\mu = \gamma_+\,[\gamma_+\,\partial_\alpha X_- + \gamma_-\,\partial_\alpha X_+] - \gamma_+\,\gamma^i\,\partial_\alpha X^i =$$

$$= 2\alpha' P_+\,\delta_{\alpha 0}\,\gamma_+\gamma_- - \gamma_+\,\gamma^i\,\partial_\alpha X^i \tag{5}$$

Hence

$$[\delta(\epsilon_1),\,\delta(\epsilon_2)]\,X^i = \{+i\,\bar{\bar{\epsilon}}_2\,\gamma^i\gamma_+\gamma_-\,\rho^0\,\epsilon_1 - i\,(2\alpha' P_+)^{-1}\,\bar{\bar{\epsilon}}_2\,\gamma^i\gamma_+\gamma^j\,\rho^\alpha\,\partial_\alpha X^j\,\epsilon_1\} - \{1\leftrightarrow 2\} \tag{6}$$

but

$$+i\,[\bar{\bar{\epsilon}}_2\,\gamma^i\gamma_+\gamma_-\,\rho^0\,\epsilon_1 - \bar{\bar{\epsilon}}_1\,\gamma^i\gamma_+\gamma_-\,\rho^0\,\epsilon_2] = +i\,[-\bar{\bar{\epsilon}}_1\,\gamma^i\gamma_-\gamma_+\rho^0\epsilon_2 - \bar{\bar{\epsilon}}_1\,\gamma^i\gamma_+\gamma_-\rho^0\epsilon_2] =$$

$$= -2i\,\bar{\bar{\epsilon}}_1\,\gamma^i\,\rho^\alpha\,\epsilon_2 \tag{7}$$

Similarly

$$-i\,(2\alpha' P_+)^{-1}\,\{\bar{\bar{\epsilon}}_2\,\gamma^i\gamma_+\gamma^j\,\rho^\alpha\,\partial_\alpha X^j\,\epsilon_1 - \bar{\bar{\epsilon}}_1\,\gamma^i\gamma_+\gamma^j\,\rho^\alpha\,\partial_\alpha X^j\,\epsilon_2\} =$$

$$= +i\,(2\alpha' P_+)^{-1}\,\partial_\alpha X^j \left\{ \bar{\bar{\epsilon}}_1 \left[\gamma^j \gamma_+ \gamma^i + \gamma^i \gamma_+ \gamma^j \right] \rho^\alpha \epsilon_2 \right\} = -2i\,(2\alpha' P_+)^{-1}\,\partial_\alpha X^i\, \bar{\bar{\epsilon}}_1 \gamma_+ \rho^\alpha \epsilon_2 \qquad (1)$$

We get therefore

$$[\delta(\epsilon_1), \delta(\epsilon_2)]\, X^i = \xi^\alpha \partial_\alpha X^i + a^i$$

$$\xi^\alpha \equiv -2i\,(2\alpha' P_+)^{-1}\, \bar{\bar{\epsilon}}_1 \gamma_+ \rho^\alpha \epsilon_2 \qquad (2)$$

$$a^i \equiv -2i\, \bar{\bar{\epsilon}}_1 \gamma^i \rho^0 \epsilon_2$$

Let us now consider

$$[\delta(\epsilon_1), \delta(\epsilon_2)]\, S \equiv S_{(12)} - S_{(21)} = \left\{ -i\,(2\alpha' P_+)^{-1}\, \gamma_+ \gamma^i \rho^\alpha \epsilon_1 \left(\bar{\bar{\epsilon}}_2 \gamma^i \partial_\alpha S \right) \right\} - \left\{ 1 \rightleftharpoons 2 \right\} \qquad (3)$$

We need now the Fierz transformation for $D = 10$. Let us introduce the set of 32^2 matrices $\Gamma^{(x)}$ with analogous properties as the usual 16 matrices for $D = 4$. Then

$$\left[\bar{\lambda}_1\, O\, \lambda_2 \right] \left[\bar{\lambda}_3\, \tilde{O}\, \lambda_4 \right] = -\frac{1}{32} \sum_x \left[\bar{\lambda}_1\, \Gamma^{(x)} \lambda_4 \right] \left[\bar{\lambda}_3\, \tilde{O}\, \Gamma^{(x)} O \lambda_2 \right] \qquad (4)$$

then

$$[\delta(\epsilon_1), \delta(\epsilon_2)]\, S^A = -i\,(2\alpha' P_+)^{-1}\, (\rho^\alpha)^{AB} \left\{ \gamma_+ \gamma^i \epsilon_1^B \left[\bar{\bar{\epsilon}}_2{}^C \gamma^i \partial_\alpha S^C \right] - \gamma_+ \gamma^i \epsilon_2^B \left[\bar{\bar{\epsilon}}_1{}^C \gamma^i \partial_\alpha S^C \right] \right\}$$

$$= +\frac{i}{32}\,(2\alpha' P_+)^{-1} \sum_x (\rho^\alpha)^{AB} \left\{ \cdot \Gamma^{(x)} \partial_\alpha S^C \left[\bar{\bar{\epsilon}}_2{}^C \gamma^i \Gamma^{(x)} \gamma_+ \gamma^i \epsilon_1^B \right] - \right.$$

$$\left. - \Gamma^{(x)} \partial_\alpha S^C \left[\bar{\bar{\epsilon}}_1{}^C \gamma^i \Gamma^{(x)} \gamma_+ \gamma^i \epsilon_2^B \right] \right\} =$$

$$= +\frac{i}{32}\,(2\alpha' P_+)^{-1} \sum_x (\rho^\alpha)^{AB}\, \Gamma^{(x)} \partial_\alpha S^C \left\{ \left[\bar{\bar{\epsilon}}_2{}^C \gamma^i \Gamma^{(x)} \gamma_+ \gamma^i \epsilon_1^B \right] - \right.$$

$$\left. - \left[\bar{\bar{\epsilon}}_1{}^C \gamma^i \Gamma^{(x)} \gamma_+ \gamma^i \epsilon_2^B \right] \right\} =$$

$$= +\frac{i}{32}\,(2\alpha' P_+)^{-1} \sum_x (\rho^\alpha)^{AB} (\rho^0)^{DC}\, \Gamma^{(x)} \partial_\alpha S^C \left\{ \left[\bar{\epsilon}_2^D \gamma^i \Gamma^{(x)} \gamma_+ \gamma^i \epsilon_1^B \right] - \right.$$

$$\left. - \left[\bar{\epsilon}_1^B \gamma^i \Gamma^{(x)} \gamma_+ \gamma^i \epsilon_2^B \right] \right\} =$$

$$= -\frac{i}{32}\,(2\alpha' P_+)^{-1} \sum_x (\rho^\alpha)^{AB} (\rho^0)^{DC}\, \Gamma^{(x)} \partial_\alpha S^C \left\{ \left[\bar{\epsilon}_1^D \gamma^i \Gamma^{(x)} \gamma_+ \gamma^i \epsilon_2^B \right] - \right.$$

$$\left. - \left[\bar{\epsilon}_2^D \gamma^i \Gamma^{(x)} \gamma_+ \gamma^i \epsilon_1^B \right] \right\} \qquad (1)$$

Let us now introduce

$$\tilde{\Gamma}^{(X)} \equiv \gamma^i \Gamma^{(X)} \gamma_+ \gamma^i \tag{1}$$

and then we can write

$$[\delta(\epsilon_1), \delta(\epsilon_2)] S^A = -\frac{i}{32} (2\alpha' p_+)^{-1} \sum_X (\rho^\alpha)^{AB} (\rho^0)^{DC} \Gamma^{(X)} \partial_\alpha S^C \cdot$$

$$\cdot \left\{ [\bar{\epsilon}_1^D \tilde{\Gamma}^{(X)} \epsilon_2^B] - [\bar{\epsilon}_1^B \gamma^0 \tilde{\Gamma}^{(X)T} \gamma^0 \epsilon_2^D] \right\} \tag{2}$$

Let us now divide the sum on X in two terms corresponding to

$$\gamma^0 \tilde{\Gamma}^{(Y)T} \gamma^0 = \tilde{\Gamma}^{(Y)}$$
$$\gamma^0 \tilde{\Gamma}^{(Z)T} \gamma^0 = -\tilde{\Gamma}^{(Z)} \tag{3}$$

Then

$$[\delta(\epsilon_1), \delta(\epsilon_2)] S^A = -\frac{i}{32} (2\alpha' p_+)^{-1} \cdot$$

$$\cdot \left\{ \sum_Y (\rho^\alpha)^{AB} (\rho^0)^{DC} \Gamma^{(Y)} \partial_\alpha S^C [\bar{\epsilon}_1^D \tilde{\Gamma}^{(X)} \epsilon_2^B - \bar{\epsilon}_1^B \tilde{\Gamma}^{(Y)} \epsilon_2^D] + \right.$$

$$\left. + \sum_Z (\rho^\alpha)^{AB} (\rho^0)^{DC} \Gamma^{(Z)} \partial_\alpha S^C [\bar{\epsilon}_1^D \tilde{\Gamma}^{(Z)} \epsilon_2^B + \bar{\epsilon}_1^B \tilde{\Gamma}^{(Z)} \epsilon_2^D] \right\} \tag{4}$$

Notice that a base for the 2×2 matrices is

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \rho^3 \equiv \rho^0 \rho^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5}$$

or equivalently

$$\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{7}$$

$$\rho_0 \rho^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \rho_1 \rho^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \rho^3 \rho^0 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

where

$$(\rho^0)^T = -\rho^0, \quad (\rho_\alpha \rho^0)^T = (\rho_\alpha \rho^0), \quad (\rho^3 \rho^0)^T = (\rho^3 \rho^0) \tag{8}$$

Then

$$\bar{\epsilon}_1^D \tilde{\Gamma}^{(y)} \epsilon_2^B - \bar{\epsilon}_1^B \tilde{\Gamma}^{(y)} \epsilon_2^D = (\rho^0)^{DB} V \tag{1}$$

$$\bar{\epsilon}_1^D \tilde{\Gamma}^{(z)} \epsilon_2^B + \bar{\epsilon}_1^B \tilde{\Gamma}^{(z)} \epsilon_2^D = (\rho_\alpha \rho^0)^{DB} W^\alpha + (\rho^3 \rho^0)^{DB} W$$

V, W^α and W can be computed easely and we get

$$\bar{\epsilon}_1^D \tilde{\Gamma}^{(y)} \epsilon_2^B - \bar{\epsilon}_1^B \tilde{\Gamma}^{(y)} \epsilon_2^D = -(\rho^0)^{DB} [\bar{\bar{\epsilon}}_1 \tilde{\Gamma}^{(y)} \epsilon_2] \tag{2}$$

$$\bar{\epsilon}_1^D \tilde{\Gamma}^{(z)} \epsilon_2^B + \bar{\epsilon}_1^B \tilde{\Gamma}^{(z)} \epsilon_2^D = (\rho_\alpha \rho^0)^{DB} [\bar{\bar{\epsilon}}_1 \rho^\alpha \tilde{\Gamma}^{(z)} \epsilon_2] + (\rho^3 \rho^0)^{DB} [\bar{\bar{\epsilon}}_1 \rho^3 \tilde{\Gamma}^{(z)} \epsilon_2]$$

Introducing this into (13.4) we get

$$[\delta(\epsilon_1), \delta(\epsilon_2)] S = -\frac{i}{16} (2\alpha' p_+)^{-1} \cdot \Big\{ \sum_y \Gamma^{(y)} \rho^\alpha \partial_\alpha S [\bar{\bar{\epsilon}}_1 \tilde{\Gamma}^{(y)} \epsilon_2] +$$

$$+ \sum_z (\rho^\alpha \rho^\beta) \Gamma^{(z)} \partial_\alpha S [\bar{\bar{\epsilon}}_1 \rho_\beta \tilde{\Gamma}^{(z)} \epsilon_2] + \sum_z \rho^3 \rho^\alpha \Gamma^{(z)} \partial_\alpha S [\bar{\bar{\epsilon}}_1 \rho^3 \tilde{\Gamma}^{(z)} \epsilon_2] \Big\} \tag{3}$$

Taking into account that $\rho^\alpha \rho^\beta = 2\eta^{\alpha\beta} - \rho^\beta \rho^\alpha$ and using the equations of motion (7.3)

$$[\delta(\epsilon_1), \delta(\epsilon_2)] S = -\frac{i}{16} (2\alpha' p_+)^{-1} \sum_z \Gamma^{(z)} \partial_\alpha S [\bar{\bar{\epsilon}}_1 \gamma^i \Gamma^{(z)} \gamma_+ \gamma^i \rho^\alpha \epsilon_2] \tag{4}$$

One must notice that $\rho^0 \rho^\alpha$ are not only symmetric but also diagonal matrices. This implies that all bilinear in (4) involve fermions with the same chirality, i.e. ϵ_1 and ϵ_2 have the same index A.

One possible way to construct the $\Gamma^{(x)}$ matrices is constructing the matrices

$$\gamma_{(m)}^{\mu_1 \cdots \mu_m} = \frac{1}{m!} \sum_P (-1)^{\epsilon_P} \gamma^{\mu_{P1}} \cdots \gamma^{\mu_{Pm}} \qquad m = 0, 1, \cdots, D = 10 \tag{5}$$

where the sum is over all permutations and ϵ_P is the parity of the permutation P. Notice that for a given m we have $\binom{D}{m}$ matrices of this type and all together we have

$$\sum_{m=0}^{D} \binom{D}{m} = 2^D = 2^{10} = 32^2 \tag{6}$$

Notice furthermore there up to some sign.

$$\{ \gamma^{11} \gamma_{(m)}^{\mu_1 \cdots \mu_m} \} \equiv \{ \gamma_{(D-m)}^{\mu_1 \cdots \mu_{D-m}} \} \tag{7}$$

Notice furthermore

$$\gamma_{(m)\,\mu_1 \cdots \mu_m}\, \gamma_{(m)}^{\mu_m \cdots \mu_1} = I \qquad \text{(no summation on } \mu\text{)} \tag{1}$$

Then we can write (14.4) as

$$[\,\delta(\epsilon_1),\,\delta(\epsilon_2)\,]\,S = \frac{i}{16}\,(2\alpha' p_+)^{-1} \sum_{m=0}^{10} \frac{1}{m!}\,\gamma_{(m)\,\mu_m \cdots \mu_1}\,\partial_\alpha S \cdot$$

$$\cdot\,[\,\bar{\epsilon}_1\,\gamma^i\,\gamma_{(m)}^{\mu_1 \cdots \mu_m}\,\gamma_+\,\gamma^i\,\rho^\alpha\,\epsilon_2\,] \tag{2}$$

where we must take into account only those terms such that (13.3)

$$\gamma^0\,[\,\gamma^i\,\gamma_{(m)}^{\mu_1 \cdots \mu_m}\,\gamma_+\,\gamma^i\,]^T\,\gamma^0 = -\,\gamma^i\,\gamma_{(m)}^{\mu_1 \cdots \mu_m}\,\gamma_+\,\gamma^i \tag{3}$$

Furthermore if λ is a Majorana - Weyl spinor with definite chirality η then

$$\gamma^{11}\lambda = \eta\lambda \quad \Rightarrow \quad \bar{\lambda}\,\gamma^{11} = -\,\eta\,\bar{\lambda} \quad \Rightarrow$$

$$\bar{\lambda}\,\gamma^i\,\gamma_{(m)}^{\mu_1 \cdots \mu_m}\,\gamma_+\,\gamma^i\,\lambda = -\,\bar{\lambda}\,\gamma^{11}\,\gamma^i\,\gamma_{(m)}^{\mu_1 \cdots \mu_m}\,\gamma_+\,\gamma^i\,\gamma^{11}\,\lambda = (-1)^m\,\bar{\lambda}\,\gamma^i\,\gamma_{(m)}^{\mu_1 \cdots \mu_m}\,\gamma_+\,\gamma^i\,\lambda \tag{4}$$

Due to the fact that ϵ_1 and ϵ_2 in (2) have the same chirality, as we have commented after equation (14.4), we can restrict (2) to even values of m and we can write

$$[\,\delta(\epsilon_1),\,\delta(\epsilon_2)\,]S = -\frac{i}{16}\,(2\alpha' p_+)^{-1}\,\Big\{\,\partial_\alpha S\,[\,\bar{\epsilon}_1\,\gamma^i\gamma_+\,\gamma^i\,\rho^\alpha\,\epsilon_2\,] +$$

$$+ \sum_{0 \le \mu < \nu \le 9} \gamma_\mu \gamma^\mu\,\partial_\alpha S\,[\,\bar{\epsilon}_1\,\gamma^i\,\gamma^\mu\,\gamma^\nu\,\gamma_+\,\gamma^i\,\rho^\alpha\,\epsilon_2\,] +$$

$$+ \sum_{0 \le \mu < \nu < \lambda < \sigma \le 9} \gamma_\sigma \gamma_\lambda \gamma_\nu \gamma_\mu\,\partial_\alpha S\,[\,\bar{\epsilon}_1\,\gamma^i\,\gamma^\mu\,\gamma^\nu\,\gamma^\lambda\,\gamma^\sigma\,\gamma_+\,\gamma^i\,\rho^\alpha\,\epsilon_2\,] +$$

$$+ \sum_{0 \le \mu < \nu < \lambda < \sigma \le 9} \gamma^{11}\,\gamma_\sigma \gamma_\lambda \gamma_\nu \gamma_\mu\,\partial_\alpha S\,[\,\bar{\epsilon}_1\,\gamma^i\,\gamma^\mu\,\gamma^\nu\,\gamma^\lambda\,\gamma^\sigma\,\gamma^{11}\,\gamma_+\,\gamma^i\,\rho^\alpha\,\epsilon_2\,] +$$

$$+ \sum_{0 \le \mu < \nu \le 9} \gamma^{11}\,\gamma_\nu \gamma_\mu\,\partial_\alpha S\,[\,\bar{\epsilon}_1\,\gamma^i\,\gamma^\mu\,\gamma^\nu\,\gamma^{11}\,\gamma_+\,\gamma^i\,\rho^\alpha\,\epsilon_2\,]$$

$$+ \gamma^{11}\,\partial_\alpha S\,[\,\bar{\epsilon}_1\,\gamma^i\,\gamma^\mu\,\gamma_+\,\gamma^i\,\rho^\alpha\,\epsilon_2\,]\,\Big\} \tag{5}$$

where the condition (3) must still be imposed. If we impose this condition the expression (5) reduces to

$$[\delta(\epsilon_1), \delta(\epsilon_2)]\, S = -\frac{i}{16}\,(2\alpha' p_+)^{-1} \Big\{ \partial_\alpha S\, [\bar{\bar{\epsilon}}_1 \gamma^i \gamma_+ \gamma^i \rho^\alpha \epsilon_2] +$$

$$+ \gamma_1 \gamma_0\, \partial_\alpha S\, [\bar{\bar{\epsilon}}_1 \gamma^i \gamma^0 \gamma^1 \gamma_+ \gamma^i \rho^\alpha \epsilon_2] +$$

$$+ \sum_{2 \le j \le k < \ell < m \le 9} \gamma_m \gamma_\ell \gamma_k \gamma_j\, \partial_\alpha S\, [\bar{\bar{\epsilon}}_1 \gamma^i \gamma^j \gamma^k \gamma^\ell \gamma^m \gamma_+ \gamma^i \rho^\alpha \epsilon_2]$$

$$+ \sum_{2 \le j < k < \ell < m \le 9} \gamma'' \gamma_m \gamma_\ell \gamma_k \gamma_j\, \partial_\alpha S\, [\bar{\bar{\epsilon}}_1 \gamma^i \gamma^j \gamma^k \gamma^\ell \gamma^m \gamma'' \gamma_+ \gamma^i \rho^\alpha \epsilon_2]$$

$$+ \gamma'' \gamma_1 \gamma_0\, \partial_\alpha S\, [\bar{\bar{\epsilon}}_1 \gamma^i \gamma^0 \gamma^1 \gamma'' \gamma_+ \gamma^i \rho^\alpha \epsilon_2]$$

$$+ \gamma''\, \partial_\alpha S\, [\bar{\bar{\epsilon}}_1 \gamma^i \gamma'' \gamma_+ \gamma^i \rho^\alpha \epsilon_2] \Big\} \qquad (1)$$

Notice that

$$\gamma^i (\gamma^j \gamma^k \gamma^\ell \gamma^m) \gamma^i = 0 \qquad 2 \le j < k < \ell < m \le 9 \qquad (2)$$

and therefore the terms 3th and 4th do not contribute. Notice furthermore

$$\gamma^i \gamma_+ \gamma^i = +8\, \gamma_+ \quad , \qquad \gamma_1 \gamma_0 = \gamma^0 \gamma^1 = \frac{1}{2}(\gamma_- \gamma_+ - \gamma_+ \gamma_-) = \gamma_- \gamma_+ - 1 \qquad (3)$$

Then

$$\partial_\alpha S\, [\bar{\bar{\epsilon}}_1 \gamma^i \gamma_+ \gamma^i \rho^\alpha \epsilon_2] = 8\, \partial_\alpha S\, [\bar{\bar{\epsilon}}_1 \gamma_+ \rho^\alpha \epsilon_2]$$

$$\gamma_1 \gamma_0\, \partial_\alpha S\, [\bar{\bar{\epsilon}}_1 \gamma^i \gamma^0 \gamma^k \gamma_+ \gamma^i \rho^\alpha \epsilon_2] = \partial_\alpha S\, [\bar{\bar{\epsilon}}_1 \gamma^i \gamma_+ \gamma^i \rho^\alpha \epsilon_2] = 8\, \partial_\alpha S\, [\bar{\bar{\epsilon}}_1 \gamma_+ \rho^\alpha \epsilon_2]$$

$$\gamma'' \gamma_1 \gamma_0\, \partial_\alpha S\, [\bar{\bar{\epsilon}}_1 \gamma^i \gamma^0 \gamma^1 \gamma'' \gamma_+ \gamma^i \rho^\alpha \epsilon_2] = \gamma''\, \partial_\alpha S\, [\bar{\bar{\epsilon}}_1 \gamma^i \gamma'' \gamma_+ \gamma^i \rho^\alpha \epsilon_2] = \qquad (5)$$

$$= -8\, \gamma''\, \partial_\alpha S\, [\bar{\bar{\epsilon}}_1 \gamma'' \gamma_+ \rho^\alpha \epsilon_2]$$

$$\gamma''\, \partial_\alpha S\, [\bar{\bar{\epsilon}}_1 \gamma^i \gamma'' \gamma_+ \gamma^i \rho^\alpha \epsilon_2] = -8 \gamma''\, \partial_\alpha S\, [\bar{\bar{\epsilon}}_1 \gamma'' \gamma_+ \rho^\alpha \epsilon_2]$$

Hence

$$[\delta(\epsilon_1), \delta(\epsilon_2)]\, S = -\frac{i}{\pi}\,(2\alpha' p_+)^{-1} \Big\{ \partial_\alpha S\, [\bar{\bar{\epsilon}}_2 \gamma_+ \rho^\alpha \epsilon_2] - \gamma''\, \partial_\alpha S\, [\bar{\bar{\epsilon}}_2 \gamma'' \gamma_+ \rho^\alpha \epsilon_2] \Big\} \qquad (5)$$

Let us now remember that since

$$\gamma'' S^A = \eta_A S^A \qquad A = 1, 2 \qquad (6)$$

in order to maintain this relation under supersymmetry transformations also

$$\gamma'' \epsilon^1_i = \eta_2 \epsilon^1_i \quad , \quad \gamma'' \epsilon^2_i = \eta_1 \epsilon^2_i \qquad i = 1,2 \tag{1}$$

Notice furthermore

$$\partial_\alpha S^A \left[\bar{\epsilon}_1 \gamma_+ \rho^\alpha \epsilon_2 \right] = \partial_\alpha S^A \left\{ \left[\bar{\epsilon}^1_1 \gamma_+ \epsilon^1_2 \right] + (-1)^\alpha \left[\bar{\epsilon}^2_1 \gamma_+ \epsilon^2_2 \right] \right\} \tag{2}$$

and

$$-\gamma'' \partial_\alpha S^A \left[\bar{\epsilon}_1 \gamma'' \gamma_+ \rho^\alpha \epsilon_2 \right] = \eta_A \partial_\alpha S^A \left\{ \eta_2 \left[\bar{\epsilon}^1_1 \gamma_+ \epsilon^1_2 \right] + (-1)^A \eta_1 \left[\bar{\epsilon}^2_1 \gamma_+ \epsilon^2_2 \right] \right\} \tag{3}$$

If $\eta_2 = \eta_1$, then (3) coincides with (2). This is also true if $\eta_2 = -\eta_1$, if we take into account the equations of motion

$$\partial_0 S^1 = -\partial_1 S^1 \quad , \quad \partial_0 S^2 = +\partial_1 S^2 \tag{4}$$

From all that

$$\left[\delta(\epsilon_1) , \delta(\epsilon_2) \right] S = -2i \left(2\alpha' p_+ \right)^{-1} \partial_\alpha S \left[\bar{\epsilon}_1 \gamma_+ \rho^\alpha \epsilon_2 \right] \tag{5}$$

Therefore taking into account (12.2) we have

$$\left[\delta(\epsilon_1), \delta(\epsilon_2) \right] X^i = \xi^\alpha \partial_\alpha X^i + a^i$$

$$\left[\delta(\epsilon_1) , \delta(\epsilon_2) \right] S = \xi^\alpha \partial_\alpha S \tag{6}$$

$$\xi^\alpha \equiv -2i \left(2\alpha' p_+ \right)^{-1} \left[\bar{\epsilon}_1 \gamma_+ \rho^\alpha \epsilon_2 \right] \quad , \quad a^i \equiv -2i \left[\bar{\epsilon}_1 \gamma^i \rho^0 \epsilon_2 \right]$$

We observe from this result that ξ^α describes an infinitesimal translation of the coordinates σ and τ, while a^i describes a translation of the X^i variables, both of which correspond to symmetries of the superstring action. ϵ^{Aa} may be regarded as parametrizing either two 10-dimensional supersymmetries or 16 two-dimensional supersymmetries

Noether's theorem implies that since the action is invariant under supersymmetry transformation must exists a conserved current. For supersymmetric transformation if we use the equations of motion we get from (10.3)

$$\delta_{SUSY} S^{c.c} = -\int d\tau \int_0^\pi d\sigma \partial_a \left\{ \frac{1}{2\pi\alpha'} \left[\partial^a X^i \right] \delta X^i - \frac{i}{4\pi} \bar{S} \gamma_- \rho^a \delta S \right\} \tag{7}$$

and arbitrary transformation can be written as

$$\delta S^{e.c} = -\int d\tau \int_0^{\pi} d\sigma \left\{ \frac{1}{2\pi\alpha'} \partial_\alpha X^i \partial^\alpha \delta X^i + \frac{i}{4\pi} \delta \bar{S} \gamma_- \rho^\alpha \partial_\alpha S + \frac{i}{4\pi} \bar{S} \gamma_- \rho^\alpha \partial_\alpha \delta S \right\} \quad (1)$$

and using the equations of motion

$$\delta S^{e.c} = -\int d\tau \int_0^{\pi} d\sigma \; \partial_\alpha \left\{ \frac{1}{2\pi\alpha'} [\partial^\alpha X^i] \delta X^i + \frac{i}{4\pi} \bar{S} \gamma_- \rho^\alpha \delta S \right\} \quad (2)$$

The comparison of (2) and (17.7) gives

$$\partial_\alpha \left\{ \bar{S} \gamma_- \rho^\alpha \delta S \right\} = 0 \quad (3)$$

and hence using (10.1)

$$\partial_\alpha \left\{ \bar{S} \gamma_- \gamma_+ \gamma_\mu \rho^\alpha \rho^\beta \partial_\beta X^\mu \epsilon \right\} = 0 \quad \Rightarrow \quad \partial_\alpha \left\{ \bar{\epsilon} \partial_\beta X^\mu \rho^\beta \rho^\alpha \gamma_\mu \gamma_+ \gamma_- S \right\} = 0$$

$$\Rightarrow \partial_\alpha \left\{ \bar{\epsilon} \partial_\beta X^\mu \rho^\beta \rho^\alpha \gamma_\mu S \right\} = 0 \Rightarrow$$

$$\partial_\alpha \left\{ \gamma_\mu [\rho^\beta \partial_\beta X^\mu] \rho^\alpha S \right\} = 0 \quad (4)$$

Then we define the supersymmetric charges as

$$Q^{Aa}(\tau) \equiv \frac{i}{2\pi\alpha' \sqrt{2}_+} \int_0^{\pi} d\sigma \left\{ \gamma_\mu [\rho^\beta \partial_\beta X^\mu] \rho^0 S \right\}^{Aa} \quad (5)$$

Since we have been sloppy with the boundary conditions we are going to check if this charges are conserved. Up to this end we integrate (4) respect to τ between τ_1 and τ_2 and respect to σ between 0 and π. We obtain

$$Q^{Aa}(\tau_2) - Q^{Aa}(\tau_1) = -\frac{i}{2\pi\alpha' \sqrt{2}_+} \int_{\tau_1}^{\tau_2} d\tau \left\{ \gamma_\mu [\rho^\alpha \partial_\alpha X^\mu] \rho^1 S \right\}^{Aa} \Big|_0^{\pi} \quad (6)$$

For closed strings we must impose the periodicity conditions

$$\left.\begin{aligned} X^\mu(\sigma, \tau) &= X^\mu(\sigma + \pi, \tau) \\ S(\sigma, \tau) &= S(\sigma + \pi, \tau) \end{aligned}\right\} \quad (7)$$

and then (6) implies

$$\frac{d}{d\tau} Q^{Aa}(\tau) = 0 \quad (8)$$

In the case of open strings we must first analyze which are the boundary conditions. These can be deduced from the surface terms in (10.3). For the bosonic part we have as usual

$$\frac{\partial \bar{X}^i(\sigma,\tau)}{\partial\sigma} = 0 \qquad \text{if} \quad \sigma = 0, \pi \tag{1}$$

while for the fermionic part

$$[\delta\bar{S}(\sigma,\tau)\,\gamma_-\,\beta^1\,S(\sigma\tau)] = 0 \qquad \text{if} \quad \sigma = 0,\pi \tag{2}$$

This can be written

$$[\delta\bar{S}^1(\sigma,\tau)\,\gamma_-\,S^1(\sigma\tau)] - [\delta\bar{S}^2(\sigma\tau)\,\gamma_-\,S^2(\sigma\tau)] = 0 \qquad \text{if} \quad \sigma = 0,\pi \tag{3}$$

The last equation admits several solutions

$$S^1(0,\tau) = \pm S^2(0,\tau) \quad , \quad S^1(\pi,\tau) = \pm S^2(\pi,\tau) \tag{4}$$

Fixing the relative sign between S^1 and S^2 at $\sigma = 0$ there are two possibles choices

$$\begin{aligned} S^1(0,\tau) &= S^2(0,\tau) \\ S^1(\pi,\tau) &= \begin{cases} + S^2(\pi,\tau) \\ - S^2(\pi,\tau) \end{cases} \end{aligned} \tag{5}$$

which are reminiscent of the fermionic (periodic) and bosonic (antiperiodic) sectors in the Neveu-Schwarz-Ramond model. However in our case the anti-periodic conditions are eliminated by imposing global supersymmetry. To see this consider again eq. (18.8) and using (1) we get

$$\begin{aligned} Q^{1a}(\tau_2) - Q^{1a}(\tau_1) &= -\frac{i}{2\pi\alpha'\sqrt{p_+}} \int_{\tau_1}^{\tau_2} d\tau \left\{ \gamma_\mu \frac{\partial}{\partial\tau} \partial_1 X^\mu(\sigma\tau)\, S^{1a} \right\}\Big|_0^\pi \\ Q^{2a}(\tau_2) - Q^{2a}(\tau_1) &= +\frac{i}{2\pi\alpha'\sqrt{p_+}} \int_{\tau_1}^{\tau_2} d\tau \left\{ \gamma_\mu \frac{\partial}{\partial\tau} \partial_1 X^\mu(\sigma\tau)\, S^{2a} \right\}\Big|_0^\pi \end{aligned} \tag{6}$$

Since $S^1(0,\tau) = S^2(0,\tau)$ we must add this two equations in order to obtain a conserved charge, but this is only achieved if $S^1(\pi\tau) = S^2(\pi\tau)$, i.e.

$$\frac{d}{d\tau}\left[Q^{1a}(\tau) + Q^{2a}(\tau) \right] = 0 \quad \Leftrightarrow \quad S^1(\pi,\tau) = S^2(\pi,\tau) \tag{7}$$

Thus global supersymmetry selects a particular boundary condition. Putting all that together

Equations of motion

$$\ddot{X}^i(\sigma,\tau) - X''^i(\sigma,\tau) = 0 \tag{1}$$

$$\rho^\alpha \partial_\alpha S(\sigma,\tau) = 0 \tag{2}$$

Boundary conditions

$$\left.\begin{aligned} &X'^i(0,\tau) = X'^i(\pi,\tau) = \\ &S^1(0,\tau) = S^2(0,\tau) \quad , \quad S^1(\pi,\tau) = S^2(\pi,\tau) \end{aligned}\right\} \quad \text{Open string} \tag{3}$$

$$\left.\begin{aligned} &X^\mu(\sigma,\tau) = X^\mu(\sigma+\pi,\tau) \\ &S(\sigma,\tau) = S(\sigma+\pi,\tau) \end{aligned}\right\} \quad \text{Closed string} \tag{4}$$

Conserved charges

$$Q^{Aa}(\tau) \equiv \frac{i}{2\pi\alpha'\sqrt{P_+}} \int_0^\pi d\sigma \left\{ \gamma_\mu \left[\rho^\alpha \partial_\alpha X^\mu(\sigma,\tau) \right] \rho^0 S(\sigma,\tau) \right\}^{A,a} \tag{5}$$

$$Q^a(\tau) \equiv \frac{1}{2}\left[Q^{1a}(\tau) + Q^{2a}(\tau) \right] \quad , \quad \frac{dQ^a(\tau)}{d\tau} = 0 \qquad \text{Open string} \tag{6}$$

$$\frac{dQ^{Aa}(\tau)}{d\tau} = 0 \qquad \text{closed string} \tag{7}$$

Note that as a consequence of (7.2) the undefined component $X_-(\sigma,\tau)$ does not occur in this expression.

The Open String

The bosonic equations and their boundary conditions are the same as the ones obtained in the dual models with the restriction on the number of space-time dimensions which is now ten. Therefore from (VIII-1.4)

$$X^i(\sigma,\tau) = q^i + \alpha_0^i \tau + i \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n} \alpha_n^i e^{-in\tau} \cos n\sigma \tag{8}$$

For the oscillator modes we have (VIII-5.3)

$$[\alpha_m^i \, , \, \alpha_n^j] = 2\alpha' m \, \delta_{ij} \, \delta_{m,-n} \qquad m,n \in \mathbb{Z}$$

$$\alpha_m^{i\,+} = \alpha_{-m}^i \tag{1}$$

and from (VIII - 4.5)

$$a_m^i \equiv (2m\alpha')^{-1/2} \alpha_m^i \quad , \quad a_m^{i\,+} \equiv (2m\alpha')^{-1/2} \alpha_{-m}^i \tag{2}$$

are, respectively, the destruction and creation operators.

We turn now to the fermionic degrees of freedom of the superstring. The motion equation (20.2) can be written

$$\left\{ \frac{\partial}{\partial\tau} + \frac{\partial}{\partial\sigma} \right\} S^{1a}(\sigma,\tau) = 0 \quad , \quad \left\{ \frac{\partial}{\partial\tau} - \frac{\partial}{\partial\sigma} \right\} S^{2a}(\sigma,\tau) = 0 \tag{3}$$

$$S^1(0,\tau) = S^2(0,\tau) \quad , \quad S^1(\pi,\tau) = S^2(\pi,\tau)$$

Their normal mode expansions are therefore

$$S^{1a}(\sigma,\tau) = \sum_{m=-\infty}^{+\infty} S_m^a \, e^{-im(\tau-\sigma)}$$

$$S^{2a}(\sigma,\tau) = \sum_{m=-\infty}^{+\infty} S_m^a \, e^{-im(\tau+\sigma)} \tag{4}$$

which satisfy the boundary conditions. Notice that for open superstrings the chirality of S^{1a} and S^{2a} is the same.

The canonical quantization of the S's variables proceeds as follows. We first calculate the canonical conjugate momentum of S^{Aa}:

$$\Pi_{Aa}(\sigma,\tau) = + \frac{\partial L}{\partial(\partial_\tau S^{Aa})} = \frac{i}{4\pi} \left[\bar{S}^A(\sigma,\tau) \gamma_- \right]^a \tag{5}$$

being L the Lagrangian density for the superstring. We then impose the canonical anticommutation relations

$$\{ S^{Aa}(\sigma,\tau) \, , \, \Pi_{Bb}(\sigma',\tau) \} = i \, \delta(\sigma-\sigma') \, \delta^{AB} \, \delta^{ab} \tag{6}$$

which implies

$$\{ S^{Aa}(\sigma,\tau) \, , \, [\bar{S}^B(\sigma',\tau)\gamma_-]^b \} = 4\pi \, \delta(\sigma'-\sigma) \, \delta^{AB} \, \delta^{ab} \tag{7}$$

multiplying this by $(\gamma_+)^{bc}$ we get using $\gamma_+ S = 0$

$$\{ S^{Aa}(\sigma,\tau) \ , \ \bar{S}^{Bb}(\sigma',\tau) \} = 2\pi \, \delta(\sigma-\sigma') \, \delta^{AB} (\gamma_+)^{ab} \tag{1}$$

The γ_+ - matrix accounts for the fact that $\gamma_+ S = \bar{S} \gamma_+ = 0$.

Let us remember that S^A are chiral spinors with chirality η_A. We have introduced in (2.7) the Weyl projection operator

$$h_A = \frac{1}{2} (I - \eta_A \gamma'') \tag{2}$$

Let us now multiply (1) by $(I-h_A)^{ca} (h_B)^{bd}$, then taking into account that $\bar{S}^A \gamma'' = -\eta_A S''$ and $[(I-h_A)\gamma_+ h_B]^{ab} = [\gamma_+ h_A]^{ab}$ we get

$$\{ S^{Aa}(\sigma,\tau), \ \bar{S}^{Bb}(\sigma',\tau) \} = 2\pi \, \delta(\sigma-\sigma') \, \delta^{AB} [\gamma_+ h_A]^{ab} \tag{3}$$

which is the desired result.

Let us now consider the anticommutators for the normal modes. From (21.4)

$$S_m^a = \frac{1}{2\pi} \int_0^\pi d\sigma \, [S^{1a}(\sigma,\tau) \, e^{im(\tau-\sigma)} + S^{2a}(\sigma,\tau) \, e^{+im(\tau+\sigma)}] \tag{4}$$

Hence

$$\{ S_m^a , \bar{S}_m^b \} = \frac{1}{4\pi^2} \int_0^\pi d\sigma \int_0^\pi d\sigma' \Big\{ e^{im(\tau-\sigma)} \, e^{im(\tau-\sigma')} \{ S^{1a}(\sigma,\tau) , \bar{S}^{1b}(\sigma',\tau) \}$$

$$+ e^{im(\tau+\sigma)} \, e^{im(\tau+\sigma')} \{ S^{2a}(\sigma,\tau) , \bar{S}^{2b}(\sigma',\tau) \} \Big\} =$$

$$= \frac{1}{2\pi} \int_0^\pi d\sigma \int_0^\pi d\sigma' \, \delta(\sigma-\sigma') \Big\{ e^{i(m+m)(\tau-\sigma)} [\gamma_+ h_1]^{ab} + e^{i(m+m)(\tau+\sigma)} [\gamma_+ h_2]^{ab} \Big\}$$

$$= \frac{1}{2\pi} \int_0^\pi d\sigma \, [\gamma_+ h]^{ab} \, e^{i(m+m)\tau} \, [e^{-i(m+m)\sigma} + e^{i(m+m)\sigma}] =$$

$$= \delta_{m+m,0} \, [\gamma_+ h]^{ab}$$

where we have taken into account that for the open string $h_1 = h_2 \equiv h$., i.e.

$$\{ S_m^a , \bar{S}_m^b \} = \delta_{m+m,0} \, [\gamma_+ h]^{ab} \tag{5}$$

Obviously we have also

$$[\alpha_m^i , S_m^a] = 0 \tag{6}$$

We have thus shown that the open superstring describes the vibration of an infinite number of bosonic and fermionic harmonic oscillators.

As a quantum-mechanical system there must exists a Hamiltonian H which yields the time evolution of states and operators. If $A(\tau)$ is an operator in the Heisenberg representation then its time evolution equation is given by

$$\frac{dA(\tau)}{d\tau} = \frac{\partial A(\tau)}{\partial \tau} + i\,[H, A(\tau)] \tag{1}$$

We will take for H the expression

$$H = \frac{1}{2\pi}\int_0^\pi d\sigma \left\{ \frac{1}{2\alpha'}\left[\dot{X}^i(\sigma\tau)\right]^2 + \frac{1}{2\alpha'}\left[X'^i(\sigma\tau)\right]^2 - \right.$$

$$\left. - \frac{i}{4}\,\bar{S}(\sigma,\tau)\,\gamma_-\rho^1\,\frac{\partial S(\sigma\tau)}{\partial\sigma}\right\} \tag{2}$$

The bosonic part is that of (VIII - 2.4) and gives the correct equations of motion for $X^i(\sigma,\tau)$ and $P_\tau^i(\sigma,\tau)$. Let us check the fermionic part

$$\frac{\partial S^{Aa}(\sigma\tau)}{\partial\tau} = -\frac{1}{8\pi}\int_0^\pi d\sigma' \left[S^{Aa}(\sigma\tau)\,,\, -\bar{S}^{Bb}(\sigma'\tau)\,(\gamma_-\gamma^0)^{bc}\,(\rho^0\rho^1)^{BC}\,\frac{\partial}{\partial\sigma'}\bar{S}^{Cc}(\sigma'\tau)\right] =$$

$$= -\frac{1}{8\pi}\int_0^\pi d\sigma'\,(\gamma_-\gamma_0)^{bc}(\rho^0\rho^1)^{BC}\left\{ \bar{S}^{Bb}(\sigma',\tau)\left\{ S^{Aa}(\sigma,\tau)\,,\,\frac{\partial}{\partial\sigma'}\bar{S}^{Cc}(\sigma'\tau)\right\}\right.$$

$$\left. - \left\{ S^{Aa}(\sigma,\tau)\,,\,\bar{S}^{Bb}(\sigma'\tau)\right\}\frac{\partial}{\partial\sigma'}\bar{S}^{Cc}(\sigma'\tau)\right\} =$$

$$= -\frac{1}{4}\int_0^\pi d\sigma'\,(\gamma_-\gamma_0)^{bc}\,(\rho^0\rho^1)^{BC}\left\{ \bar{S}^{Bb}(\sigma'\tau)\,\delta^{AC}\left[\gamma_+ h_A\right]^{ac}\frac{\partial}{\partial\sigma'}\delta(\sigma-\sigma')\right.$$

$$\left. - \delta^{AB}\left[\gamma_+ h_A\right]^{ab}\delta(\sigma-\sigma')\,\frac{\partial}{\partial\sigma'}\bar{S}^{Cc}(\sigma'\tau)\right\} =$$

$$= \frac{1}{4}\left\{ \left[\gamma_+ h_A\gamma_-\gamma^0\right]^{ab}(\rho^0\rho^1)^{BA}\,\frac{\partial}{\partial\sigma}\bar{S}^{Bb}(\sigma\tau) + \left[\gamma_+ h_A\gamma_-\gamma_0\right]^{ab}(\rho^0\rho^1)^{AB}\,\frac{\partial}{\partial\sigma}\bar{S}^{Bb}(\sigma\tau)\right\}$$

$$= \frac{1}{2}(-1)^{A+1}\left[\gamma_+ h_A\gamma_-\gamma^0\right]^{ab}\frac{\partial}{\partial\sigma}\bar{S}^{Ab}(\sigma\tau) = \frac{1}{2}(-1)^A\left[\gamma_+ h_A\gamma_-\right]^{ab}\frac{\partial}{\partial\sigma}\bar{S}^{Ab}(\sigma\tau)$$

$$= \frac{1}{2}(-1)^A\left[(I-h_A)\gamma_+\gamma_-\right]^{ab}\frac{\partial}{\partial\sigma}\bar{S}^{Ab}(\sigma\tau) = (-1)^A\left[1-h_A\right]^{ab}\frac{\partial}{\partial\sigma}\bar{S}^{Ab}(\sigma\tau) = (-1)^A\frac{\partial}{\partial\sigma}\bar{S}^{Aa}(\sigma\tau)$$

reproducing the desired equations (21.3).

Let us now write (2) in terms of normal modes. The bosonic part is given in (VIII - 3.3). Let us compute the fermionic one.

$$H = -\frac{i}{8\pi}\int_0^\pi d\sigma\ \bar{S}(\sigma\tau)\,\gamma_-\,(\beta^\circ\beta')\,\frac{\partial S(\sigma\tau)}{\partial\sigma} =$$

$$= -\frac{i}{8\pi}\int_0^\pi d\sigma\ \Big\{\bar{S}^1(\sigma\tau)\,\gamma_-\frac{\partial S^1(\sigma\tau)}{\partial\sigma} - \bar{S}^2(\sigma\tau)\,\gamma_-\frac{\partial S^2(\sigma\tau)}{\partial\sigma}\Big\} =$$

$$= -\frac{i}{8\pi}\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty}\int_0^\pi d\sigma\ (im)\,\bar{S}_n\,\gamma_-\,S_m\Big\{e^{-i(n+m)(\tau-\sigma)} + e^{-i(n+m)(\tau+\sigma)}\Big\}$$

$$= -\frac{i}{8\pi}\sum_{n=-\infty}^{+\infty}(-in)\,\bar{S}_n\,\gamma_-\,S_{-n}\,n\,2\pi = -\frac{1}{4}\sum_{n=-\infty}^{+\infty} n\,\bar{S}_n\,\gamma_-\,S_{-n} \qquad (1)$$

and therefore

$$H = \alpha'(p^i)^2 + \frac{1}{4\alpha'}\sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty}\alpha^i_{-n}\,\alpha^i_n + \frac{1}{4}\sum_{n=-\infty}^{+\infty} n\,\bar{S}_{-n}\,\gamma_-\,S_n \qquad (2)$$

This operator in not in normal-ordereded form ($\alpha^i_{-n} = \alpha^{i\,+}_{+n}$, $S_{-n} = S^+_n$) This requires some rearrangement of some terms in (2). The ones that must be reordered are

$$\sum_{n=1}^{\infty}\alpha^i_n\,\alpha^i_{-n} = \sum_{n=1}^{\infty}\Big\{\alpha^i_{-n}\,\alpha^i_n + [\alpha^i_n,\alpha^i_{-n}]\Big\} = \sum_{n=1}^{\infty}\alpha^i_{-n}\,\alpha^i_n + 16\alpha'\sum_{n=1}^{\infty} n \qquad (3)$$

$$\sum_{n=1}^{\infty}(-n)\,\bar{S}_n\,\gamma_-\,S_{-n} = \sum_{n=1}^{\infty} n\Big\{\bar{S}_{-n}\,\gamma_-\,S_n - \{S^a_n,\bar{S}^b_{-n}\}(\gamma_-)^{ba}\Big\} =$$

$$= \sum_{n=1}^{\infty} n\Big\{\bar{S}_{-n}\,\gamma_-\,S_n - [\gamma_+ h]^{ab}[\gamma_-]^{ba}\Big\} = \sum_{n=1}^{\infty} n\,[\bar{S}_{-n}\,\gamma_-\,S_n] - 16\sum_{n=1}^{\infty} n \qquad (4)$$

because

$$\mathrm{Tr}[\gamma_+ h_a \gamma_-] = \frac{1}{2}\mathrm{Tr}[\gamma_-\gamma_+(1-\eta\gamma'')] = \frac{1}{2}\mathrm{Tr}[\gamma_-\gamma_+] = \frac{1}{2}\,32 = 16 \qquad (5)$$

We observe the important fact that the infinite constants, proportional to $\sum_{n=1}^{\infty} n$, cancel each other in H. This is a consequence of supersymmetry: the positive vacuum energy of the bosons is balanced by the negative vacuum energy of the fermions. The normal ordered form of H is finally

$$:H: = \alpha'(p^i)^2 + \frac{1}{2\alpha'}\sum_{n=1}^{\infty}\alpha^i_{-n}\,\alpha^i_n + \frac{1}{2}\sum_{n=1}^{\infty} n\,[\bar{S}_{-n}\,\gamma_-\,S_n] \qquad (6)$$

Before going on let us now consider the corresponding formulae for the closed string.

The Closed String

The bosonic equations and their boundary conditions are the same as the ones obtained in the dual models with the restriction on the number of space time dimensions which is now ten. From (VIII-12-2)

$$X^i(\sigma,\tau) = q^i + 2\alpha_0^i \tau + \frac{i}{2}\sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \frac{1}{n}\left\{ \alpha_n^i\, e^{-2in(\tau-\sigma)} + \tilde{\alpha}_n^i\, e^{-2in(\tau+\sigma)} \right\} \tag{1}$$

and we have (VIII-13.2)

$$[\alpha_n^i, \alpha_m^j] = [\tilde{\alpha}_n^i, \tilde{\alpha}_m^j] = 2n\alpha' \delta_{n,-m}\delta_{ij}$$
$$[\alpha_n^i, \tilde{\alpha}_m^j] = 0 \tag{2}$$

For the fermionic part the equations of motion are still (21.3) with the boundary conditions (20.4)

$$S(\sigma,\tau) = S(\sigma+\pi,\tau) \tag{3}$$

and therefore we have

$$S^{1a}(\sigma,\tau) = \sum_{n=-\infty}^{+\infty} S_n^a\, e^{-2in(\tau-\sigma)}$$
$$S^{2a}(\sigma,\tau) = \sum_{n=-\infty}^{+\infty} \tilde{S}_n^a\, e^{-2in(\tau+\sigma)} \tag{4}$$

We observe again that, as in the bosonic case, the left-moving modes and the right-moving ones are independent. This will be crucial when we discuss the "heterotic" superstring. Notice furthermore that now S^1 and S^2 can have the same or opposite chiralities.

The canonical quantization of the S variables is done as before and we impose (Notice a factor 2)

$$\{ S^{Aa}(\sigma,\tau),\ \bar{S}^{Bb}(\sigma',\tau) \} = \pi\,\delta(\sigma-\sigma')\,\delta^{AB}\,[\gamma_+ h_A]^{ab} \tag{5}$$

In order to obtain the anticommuting relation for the normal modes we must take into account that (5) implies

$$S_n^a = \frac{1}{\pi}\int_0^\pi d\sigma\, S^{1a}(\sigma,\tau)\, e^{+2in(\tau-\sigma)}$$
$$\tilde{S}_n^a = \frac{1}{\pi}\int_0^\pi d\sigma\, S^{2a}(\sigma,\tau)\, e^{+2in(\tau+\sigma)} \tag{6}$$

and we obtain immediately

$$\{ S_m^a , \bar{S}_m^b \} = \delta_{m+m,0} [\gamma_+ h_1]^{ab} , \quad \{ \tilde{S}_m^a , \bar{\tilde{S}}_m^b \} = \delta_{m+m,0} [\gamma_+ h_2]^{ab}$$

$$\{ S_m^a , \tilde{S}_m^b \} = 0 \tag{1}$$

The fermionic modes commute with the bosonic modes

The Hamiltonian is given by (23.2), which in terms of normal modes is given by (see VIII-12.7)

$$H = \alpha' (p^i)^2 + \frac{1}{2\alpha'} \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} [\alpha_{-m}^i \alpha_m^i + \tilde{\alpha}_{-m}^i \tilde{\alpha}_m^i] + \frac{1}{4} \sum_{m=-\infty}^{+\infty} m [\bar{S}_{-m} \gamma_- S_m + \bar{\tilde{S}}_{-m} \gamma_- \tilde{S}_m] \tag{2}$$

When we rearrange the terms that are not normal ordered we see that the infinities cancel and we get

$$:H: = \alpha' (p^i)^2 + \frac{1}{\alpha'} \sum_{m=1}^{\infty} [\alpha_{-m}^i \alpha_m^i + \tilde{\alpha}_{-m}^i \tilde{\alpha}_m^i] +$$

$$+ \frac{1}{2} \sum_{m=1}^{\infty} m [\bar{S}_{-m} \gamma_- S_m + \bar{\tilde{S}}_{-m} \gamma_- \tilde{S}_m] \tag{3}$$

Let us return to the general case. In the transversal gauge the Hamiltonian is given also by (VIII-5.7)

$$H = 2\alpha' p_+ p_- \tag{4}$$

and therefore the mass operator M^2 is given by

$$M^2 = p^\mu p_\mu = 2 p_+ p_- - (p^i)^2 = \frac{1}{\alpha'} H - (p^i)^2 \tag{5}$$

Therefore we can write

$$M^2 = \frac{1}{\alpha'} N \qquad \text{Open string} \tag{6}$$

$$N = \frac{1}{2\alpha'} \sum_{m=1}^{\infty} \alpha_{-m}^i \alpha_m^i + \frac{1}{2} \sum_{m=1}^{\infty} m [\bar{S}_{-m} \gamma_- S_m]$$

$$M^2 = \frac{2}{\alpha'} (N + \tilde{N})$$

$$N = \frac{1}{2\alpha'} \sum_{m=1}^{\infty} \alpha_{-m}^i \alpha_m^i + \frac{1}{4} \sum_{m=1}^{\infty} m [\bar{S}_{-m} \gamma_- S_m] \qquad \text{Closed string} \tag{7}$$

$$\tilde{N} = \frac{1}{2\alpha'} \sum_{m=1}^{\infty} \tilde{\alpha}_{-m}^i \tilde{\alpha}_m^i + \frac{1}{4} \sum_{m=1}^{\infty} m [\bar{\tilde{S}}_{-m} \gamma_- S_m]$$

where N, N and $\tilde{N}$ are the so called "number" operators.

There exists however another condition which restricts the states of the closed superstring. That condition can be simply stated by the equation

$$N = \tilde{N} \qquad \text{closed superstring} \qquad (1)$$

We can give a physical meaning to this equation studying the action of the operator $N - \tilde{N}$ on the string variables

$$[N - \tilde{N}, X^i(\sigma,\tau)] = \frac{1}{2\alpha'}\Big[\sum_{m=1}^{\infty}\Big(\alpha_{-m}^i \alpha_m^i - \tilde{\alpha}_{-m}^i \tilde{\alpha}_m^i\Big),$$

$$\frac{i}{2}\sum_{\substack{m=-\infty\\ m\neq 0}}^{+\infty}\frac{1}{m}\Big(\alpha_m^i e^{-2im(\tau-\sigma)} + \tilde{\alpha}_m^i e^{-2im(\tau+\sigma)}\Big)\Big]$$

$$= \frac{1}{2\alpha'}\frac{i}{2}\sum_{m=1}^{\infty}\sum_{\substack{m=-\infty\\ m\neq 0}}^{+\infty}\frac{1}{m}\Big\{e^{-2im(\tau-\sigma)}\Big[\alpha_{-m}^j[\alpha_m^j, \alpha_m^i] + [\alpha_{-m}^j, \alpha_m^i]\alpha_m^j\Big]$$

$$- e^{-2im(\tau+\sigma)}\Big[\tilde{\alpha}_{-m}^j[\tilde{\alpha}_m^j, \tilde{\alpha}_m^i] + [\tilde{\alpha}_{-m}^j, \tilde{\alpha}_m^i]\tilde{\alpha}_m^j\Big]\Big\} =$$

$$= \frac{i}{2}\sum_{m=1}^{\infty}\sum_{\substack{m=-\infty\\ m\neq 0}}^{\infty}\frac{1}{m}\Big\{e^{-2im(\tau-\sigma)}\Big[\delta_{m+m,0}\, m\, \alpha_{-m}^i + \delta_{-m+m,0}(-m)\alpha_m^i\Big]$$

$$- e^{-2im(\tau+\sigma)}\Big[\delta_{m+m,0}\, m\, \tilde{\alpha}_{-m}^i + \delta_{-m+m,0}(-m)\tilde{\alpha}_m^i\Big]\Big\}$$

$$= \frac{i}{2}\sum_{m=1}^{\infty}\Big\{-e^{+2im(\tau-\sigma)}\alpha_{-m}^i - e^{-2im(\tau-\sigma)}\alpha_m^i + e^{2im(\tau+\sigma)}\tilde{\alpha}_{-m}^i + e^{-2im(\tau-\sigma)}\tilde{\alpha}_m^i\Big\}$$

$$= \frac{i}{2}\sum_{\substack{m=-\infty\\ m\neq 0}}^{+\infty}\Big\{-e^{-2im(\tau-\sigma)}\alpha_m^i + e^{-2im(\tau-\sigma)}\tilde{\alpha}_m^i\Big\} = \frac{i}{2}\frac{\partial X^i(\sigma,\tau)}{\partial\sigma} \qquad (2)$$

$$[N - \tilde{N}, S^a(\sigma,\tau)] = \frac{1}{4}\Big[\sum_{m=1}^{\infty} m\,[\bar{S}_{-m}\gamma_-\, S_m),\ \sum_{m=-\infty}^{+\infty} S_m^a e^{-2im(\tau-\sigma)}\Big] =$$

$$= -\frac{1}{4}\sum_{m=1}^{\infty}\sum_{m=-\infty}^{+\infty} m\,(\gamma_-\gamma^0)^{bc}\Big\{\bar{S}_{-m}^b\{\bar{S}_m^c, S_m^a\} - \{\bar{S}_{-m}^b, S_m^a\}\bar{S}_m^c\Big\}e^{-2im(\tau-\sigma)}$$

$$= -\frac{1}{2}\sum_{m=1}^{\infty}\sum_{m=-\infty}^{+\infty} m\,(\gamma_-\gamma^0)^{bc}\Big\{\bar{S}_{-m}^b\,\delta_{m+m,0}\,[\gamma_+ h_1]^{ac} - \delta_{-m+m,0}[\gamma_+ h_1]^{ab}\bar{S}_m^c\Big\}e^{-2im(\tau-\sigma)}$$

$$= -\frac{1}{2}\sum_{m=1}^{\infty} m\Big\{[\gamma_+ h_1 \gamma^{0T}\gamma_-^T]^{ab}\bar{S}_{-m}^b e^{2im(\tau-\sigma)} - [\gamma_+ h_1\gamma_-\gamma^0]^{ab}\bar{S}_m^b e^{-2im(\tau-\sigma)}\Big\}$$

$$= \frac{1}{2}\sum_{m=1}^{\infty} m\,[\gamma_+ h_1\gamma_-]^{ab}\Big\{S_{-m}^b e^{2im(\tau-\sigma)} - S_m^b e^{-2im(\tau-\sigma)}\Big\}$$

$$= -\frac{i}{2}\,[\gamma_+ h_1 \gamma_-]^{ab} \sum_{m=-\infty}^{+\infty} m\, S_m^b\, e^{-2im(\tau-\sigma)} = -\frac{1}{4i}\,[\gamma_+ h_- \gamma_-]^{ab}\,\frac{\partial S^{1b}(\sigma\tau)}{\partial\sigma}$$

$$= \frac{i}{4}\,[\gamma_+\gamma_-(I-h_1)]^{ab}\,\frac{\partial S^{1b}(\sigma\tau)}{\partial\sigma} = \frac{i}{4}\,[\gamma_+\gamma_-]^{cb}\,\frac{\partial S^{1b}(\sigma\tau)}{\partial\sigma} = \frac{i}{2}\,\frac{\partial S^{1a}(\sigma,\tau)}{\partial\sigma} \qquad (1)$$

Therefore

$$[N-\tilde N,\ X^i(\sigma,\tau)] = \frac{i}{2}\,\frac{\partial X^i(\sigma\tau)}{\partial\sigma}$$

$$[N-\tilde N,\ S^{Aa}(\sigma,\tau)] = \frac{i}{2}\,\frac{\partial S^{Aa}(\sigma,\tau)}{\partial\sigma} \qquad (2)$$

and hence

$$e^{-2i\Delta(N-\tilde N)}\,X^i(\sigma,\tau)\,e^{+2i\Delta(N-\tilde N)} = X^i(\sigma+\Delta,\tau)$$

$$e^{-2i\Delta(N-\tilde N)}\,S^{Aa}(\sigma,\tau)\,e^{+2i\Delta(N-\tilde N)} = S^{Aa}(\sigma+\Delta,\tau) \qquad (3)$$

For the closed string physics must be independent of the choice of the origin, and therefore we have an explanation of the constraint (27.1). In other words: there is not preferred point on the string. An immediate consequence of (27.1) is that the mass-shell condition (26.7) can be written as

$$M^2 = \frac{4}{\alpha'}\,N \qquad \text{Closed superstring} \qquad (4)$$

Let us now turn our attention to the construction of the supersymmetric charges that we have defined as

$$Q^{Aa}(\tau) = \frac{i}{2\pi\alpha'\sqrt{P_+}}\int_0^\pi d\sigma\,\{\gamma_\mu\,[\rho^\beta\partial_\beta X^\mu]\,\rho^0 S\}^{Aa} \qquad (5)$$

Since $\gamma_\mu \dot X^\mu = \gamma_+ X_- + \gamma_- X_+ - \gamma^i \dot X^i$ and $\gamma_+ S = 0$ and $X_+ = 2\alpha' P_+ \tau$ we can write

$$Q^{Aa}(\tau) = \frac{i}{\pi}\sqrt{P_+}\int_0^\pi d\sigma\,[\gamma_- S(\sigma\tau)]^{Aa} - \frac{i}{2\pi\alpha'\sqrt{P_+}}\int_0^\pi d\sigma\,\{\gamma^i\,[\rho^\beta\partial_\beta X^i]\,\rho^0 S\}^{Aa} \qquad (6)$$

Let us now compute first

The Open String

From (20.6)

$$Q^a = \frac{i}{2\pi}\sqrt{P_+}\int_0^\pi d\sigma\,[S^{1b}(\sigma\tau) + S^{2b}(\sigma\tau)]\,[\gamma_-]^{ab}$$

$$- \frac{i}{4\pi\alpha'}\frac{1}{\sqrt{P_+}}\int_0^\pi d\sigma\,\{[\gamma^i(\rho^\beta\partial_\beta X^i)\rho^0 S]^{1a} + [\gamma^i\rho^\beta\partial_\beta X^i\rho^0 S]^{2a}\} \qquad (1)$$

i.e.

$$Q^a = \frac{i}{2\pi}\sqrt{P_+}\,[\gamma_-]^{ab}\int_0^\pi d\sigma\,[S^{1b}(\sigma\tau) + S^{2b}(\sigma\tau)] - \frac{i}{4\pi\alpha'}\frac{1}{\sqrt{P_+}}(\gamma^i)^{ab}\cdot$$

$$\cdot\int_0^\pi d\sigma\,\left\{\frac{\partial X^i(\sigma\tau)}{\partial\tau}[S^{1b}(\sigma\tau) + S^{2b}(\sigma\tau)] - \frac{\partial X^i(\sigma\tau)}{\partial\sigma}[S^{1b}(\sigma\tau) - S^{2b}(\sigma\tau)]\right\} \qquad (2)$$

Notice

$$\int_0^\pi d\sigma\,[S^{1b}(\sigma\tau) + S^{2b}(\sigma\tau)] = \sum_{n=-\infty}^{+\infty} S_n^b\int_0^\pi d\sigma\, e^{-in\tau}[e^{in\sigma} + e^{-in\sigma}] = 2\pi S_0^b \qquad (3)$$

and

$$\int_0^\pi d\sigma\,\{\dot{X}^i(S^{1b} + S^{2b}) - X'^i(S^{1b} - S^{2b})\} =$$

$$= \sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty}\int_0^\pi d\sigma\,\{\alpha_n^i e^{-in\tau}\cos n\sigma\, S_m^b\,[e^{-im(\tau-\sigma)} + e^{-im(\tau+\sigma)}]$$

$$- i\alpha_n^i e^{-in\tau}\sin n\sigma\, S_m^b\,[-e^{-im(\tau-\sigma)} + e^{-im(\tau+\sigma)}]\}$$

$$= \sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty}\int_0^\pi d\sigma\,\alpha_n^i S_m^b\, e^{-i(n+m)\tau}[e^{i(n+m)\sigma} + e^{-i(n+m)\sigma}] = 2\pi\sum_{n=-\infty}^{+\infty} S_{-n}^b\,\alpha_n^i \qquad (4)$$

Using (2), (3) and (4)

$$Q^a = +i\sqrt{P_+}\,[\gamma_- S_0]^a - \frac{i}{2\alpha'\sqrt{P_+}}\sum_{n=-\infty}^{+\infty}[\gamma^i S_{-n}]^a\,\alpha_n^i \qquad (5)$$

which is the desired result. Since for Majora spinors $\psi^a = -(\gamma^0)^{ab}\bar{\psi}^b$ and $\bar{\psi}^a = -(\gamma^0)^{ab}\psi^b$

$$\bar{Q}^a = -i\sqrt{P_+}\,[\gamma_+\bar{S}_0]^a + \frac{i}{2\alpha'\sqrt{P_+}}\sum_{n=-\infty}^{+\infty}[\gamma^i\bar{S}_{-n}]^a\,\alpha_n^i \qquad (6)$$

<u>The closed superstring</u>

Now we have

$$Q^{Aa} = \frac{i}{\pi}\sqrt{P_+}\,[\gamma_-]^{ab}\int_0^\pi d\sigma\; S^{Ab}(\sigma,\tau)$$

$$-\frac{i}{2\pi\alpha'\sqrt{P_+}}\,[\gamma^i]^{ab}\int_0^\pi d\sigma\,\Big\{S^{Ab}(\sigma\tau)\,\frac{\partial X^i(\sigma\tau)}{\partial\tau} + (\rho^1\rho^0)^{AB}\,S^{Bb}(\sigma\tau)\,\frac{\partial X^i(\sigma\tau)}{\partial\sigma}\Big\} \qquad (1)$$

Let us now consider $A=1$. Since

$$\int_0^\pi d\sigma\; S^{1b}(\sigma\tau) = \sum_{m=-\infty}^{+\infty} S_m^b \int_0^\pi d\sigma\, e^{-2im(\tau-\sigma)} = \pi S_o^b \qquad (2)$$

$$\int_0^\pi d\sigma\,\Big\{S^{1b}(\sigma\tau)\,\frac{\partial X^i(\sigma\tau)}{\partial\tau} - S^{1b}(\sigma\tau)\,\frac{\partial X^i(\sigma\tau)}{\partial\sigma}\Big\}$$

$$= \sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty}\int_0^\pi d\sigma\; S_m^b\, e^{-2im(\tau-\sigma)}\; 2\alpha_n^i\, e^{-2in(\tau-\sigma)} = 2\pi\sum_{m=-\infty}^{+\infty} S_{-m}^b\,\alpha_m^i \qquad (3)$$

we obtain

$$Q^{1a} = i\sqrt{P_+}\,[\gamma_-\, S_o]^a - \frac{i}{\alpha'\sqrt{P_+}}\sum_{m=-\infty}^{+\infty}[\gamma^i S_{-m}]^a\,\alpha_m^i \qquad (4)$$

$$Q^{2a} = i\sqrt{P_+}\,[\gamma_-\,\tilde S_o]^a - \frac{i}{\alpha'\sqrt{P_+}}\sum_{m=-\infty}^{+\infty}[\gamma^i\tilde S_{-m}]^a\,\tilde\alpha_m^i$$

which are the conserved charges for the closed superstring.

The Open String

Let us now calculate the commutators of the charge with X^i and S^{Aa}

$$[Q^a,\, X^i(\sigma\tau)] = \frac{1}{2\alpha'\sqrt{P_+}}\sum_{n=-\infty}^{+\infty}\sum_{\substack{m=-\infty\\ m\neq 0}}^{+\infty}\Big[(\gamma^i S_{-m})^a\,\alpha_m^j,\ \frac{1}{m}\,\alpha_m^i\, e^{-im\tau}\cos m\sigma\Big]$$

$$= -\frac{1}{\sqrt{P_+}}\sum_{m=-\infty}^{+\infty}[\gamma^i S_{-m}]^a\, e^{im\tau}\cos m\sigma = -\frac{1}{2\sqrt{P_+}}\sum_{m=-\infty}^{+\infty}[\gamma^i S_m]^a\Big\{e^{-im(\tau-\sigma)} + e^{-im(\tau+\sigma)}\Big\} \Rightarrow$$

$$[Q^a,\, X^i(\sigma\tau)] = -\frac{1}{2\sqrt{P_+}}\Big\{\gamma^i S^1(\sigma\tau) + \gamma^i S^2(\sigma\tau)\Big\}^a \qquad (5)$$

We have seen before that for the open string it is useful to work with $\sigma\in[-\pi,\pi]$ where we have continued $X^\mu(\sigma,\tau)$ analytically

$$X^\mu(-\sigma,\tau) \equiv X^\mu(\sigma,\tau) \qquad \sigma\in[-\pi,+\pi] \qquad (6)$$

which is clearly compatible with the form of $X^\mu(\sigma,\tau)$ given in (20.2). Similarly

taking into account that

$$S^2(-\sigma,\tau) = S^1(\sigma,\tau) \quad , \quad S^1(-\sigma\,\tau) = S^2(\sigma,\tau) \tag{1}$$

we can define

$$S(\sigma,\tau) \equiv -\frac{1}{2}\left[S^1(\sigma\tau) + S^2(\sigma\tau)\right] = S(-\sigma,\tau) \qquad \sigma \in [-\pi, +\pi] \tag{2}$$

and therefore equation (30.5) can be written as

$$[Q^a, X^i(\sigma\tau)] = +\frac{1}{\sqrt{P_+}}[\gamma^i S(\sigma\tau)]^a \tag{3}$$

Notice that (21.4)

$$S^a(\sigma,\tau) = -\sum_{n=-\infty}^{+\infty} S_n^a\, e^{-in\tau} \cos n\sigma \tag{4}$$

then

$$\{Q^a, S^b(\sigma,\tau)\} = \left\{ i\sqrt{P_+}\,(\gamma_- S_0)^a - \frac{i}{2\alpha'\sqrt{P_+}} \sum_{n=-\infty}^{+\infty} [\gamma^i S_{-n}]^a \alpha_n^i \,,\, -\sum_{m=-\infty}^{+\infty} S_m^b\, e^{-im\tau} \cos m\sigma \right\}$$

$$= -i\sqrt{P_+}\,(\gamma_-)^{ac} \sum_{m=-\infty}^{+\infty} e^{-im(\tau-\sigma)} \{S_0^c, S_m^b\} + \frac{i}{2\alpha'\sqrt{P_+}} (\gamma^i)^{ac} \sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty} \alpha_n^i e^{-im\tau} \cos m\sigma \{S_{-n}^c, S_m^b\}$$

$$= -i\sqrt{P_+}\,[\gamma_-\gamma_+ h\gamma^0]^{ab} + \frac{i}{2\alpha'\sqrt{P_+}} [\gamma^i\gamma_+ h\gamma^0]^{ab} \sum_{n=-\infty}^{+\infty} \alpha_n^i e^{-in\tau} \cos n\sigma \tag{5}$$

but

$$\frac{\partial X^i(\sigma\tau)}{\partial\tau} = \sum_{n=-\infty}^{+\infty} \alpha_n^i e^{-in\tau} \cos n\sigma \tag{6}$$

$$X_+ = 2\alpha' P_+ \tau$$

Hence

$$\{Q^a, S^b(\sigma\tau)\} = -\frac{i}{2\alpha'\sqrt{P_+}} [\gamma_-\gamma_+ h\gamma^0]^{ab} \frac{\partial X_+(\sigma\tau)}{\partial\tau} + \frac{i}{2\alpha'\sqrt{P_+}} [\gamma^i\gamma_+ h\gamma^0]^{ab} \frac{\partial X^i(\sigma\tau)}{\partial\tau} \tag{7}$$

$$= -\frac{i}{2\alpha'\sqrt{P_+}} [\gamma_\mu\gamma_+ h\gamma^0]^{ab} \frac{\partial X^\mu(\sigma\tau)}{\partial\tau}$$

Then altogether

$$[Q^a, X^i(\sigma\tau)] = +\frac{1}{\sqrt{P_+}} [\gamma^i S(\sigma\tau)]^a \tag{8}$$

$$\{Q^a, S^b(\sigma\tau)\} = -\frac{i}{2\alpha'\sqrt{P_+}} [\gamma_\mu\gamma_+ h\gamma^0]^{ab} \frac{\partial X^\mu(\sigma\tau)}{\partial\tau}$$

Let us now introduce a parameter ϵ^a with the same chirality that S. Then

$$[(\bar\epsilon\, Q)\,,\, S^b(\sigma\tau)] = -(\gamma^0)^{ac}\epsilon^c\,\{Q^a,\, S^b(\sigma\tau)\} =$$

$$= +\frac{i}{2\alpha'\sqrt{p_+}}\,[\gamma_\mu\gamma_+\,h\,\gamma^0]^{ab}(\gamma^0)^{ac}\epsilon^c\,\dot X^\mu(\sigma\tau) = +\frac{i}{2\alpha'\sqrt{p_+}}\,\{\gamma^{0T}h^T\gamma_+^T\gamma_\mu^T\gamma^0\epsilon\}^b\,\dot X^\mu(\sigma\tau)$$

$$= -\frac{i}{2\alpha'\sqrt{p_+}}\,[\gamma_+\gamma_\mu(1-h)\epsilon]^b\,\dot X^\mu(\sigma\tau) = -\frac{i}{2\alpha'\sqrt{p_+}}\,[\gamma_+\gamma_\mu\epsilon]^b\,\dot X^\mu(\sigma\tau) \qquad (1)$$

Therefore

$$[(\bar\epsilon.Q)\,,\, X^i(\sigma,\tau)] = +\frac{1}{\sqrt{p_+}}\,[\bar\epsilon\,\gamma^i S(\sigma\tau)] \qquad (2)$$

$$[(\bar\epsilon.Q)\,,\, S^b(\sigma\tau)] = -\frac{i}{2\alpha'\sqrt{p_+}}\,[\gamma_+\gamma_\mu\,\dot X^\mu(\sigma\tau)\,\epsilon]^b$$

Since $i(\bar\epsilon.Q)$ is selfadjoint we can introduce the unitary operator

$$U(\epsilon) \equiv e^{+(\bar\epsilon.Q)} \qquad (3)$$

such that

$$U(\epsilon)\,X^i(\sigma\tau)\,U^{-1}(\epsilon) = X^i(\sigma,\tau) + \frac{1}{\sqrt{p_+}}\,[\bar\epsilon\,\gamma^i S(\sigma,\tau)] \qquad (4)$$

$$U(\epsilon)\,S^a(\sigma\tau)\,U^{-1}(\epsilon) = S^a(\sigma,\tau) - \frac{i}{2\alpha'\sqrt{p_+}}\,[\gamma_+\gamma_\mu\,\dot X^\mu(\sigma\tau)\,\epsilon]^a$$

Comparison with (10.1) tells us that $U(\epsilon)$ implements the supersymmetry transformation in the open string ($\rho^0=I$, $\rho^1=0$)

The closed string

Let us proceed as before

$$[Q^{1a}\,,\, X^i(\sigma,\tau)] = +\frac{1}{2\alpha'\sqrt{p_+}}\left\{\sum_{m=-\infty}^{+\infty}[\gamma^i S_{-m}]^a\,\alpha_m^i\,,\ \sum_{\substack{m=-\infty\\ m\neq 0}}^{+\infty}\frac{1}{m}\,\alpha_m^i\,e^{-2im(\tau-\sigma)}\right\}$$

$$= -\frac{1}{\sqrt{p_+}}\sum_{m=-\infty}^{+\infty}[\gamma^i S_{-m}]^a e^{+2im(\tau-\sigma)} = -\frac{1}{\sqrt{p_+}}\sum_{m=-\infty}^{+\infty}[\gamma^i S_m]^a e^{-2im(\tau-\sigma)} \Rightarrow$$

$$\{Q^{1a}\,,\, X^i(\sigma,\tau)] = -\frac{1}{\sqrt{p_+}}\,[\gamma^i S^1(\sigma\tau)]^a \qquad (5)$$

$$[Q^{2a}\,,\, X^i(\sigma,\tau)] = -\frac{1}{\sqrt{p_+}}\,[\gamma^i S^2(\sigma\tau)]^a$$

Similarly

$$\{ Q^{1a}, S^{2b}(\sigma,\tau) \} = \{ i\sqrt{P_+}\,[\gamma_- S_0]^a - \frac{i}{\alpha'\sqrt{P_+}} \sum_{m=-\infty}^{+\infty} [\gamma^i S_{-m}]^a \alpha_m^i \,,\, \sum_{m=-\infty}^{+\infty} S_m^b\, e^{-2im(\tau-\sigma)} \}$$

$$= i\sqrt{P_+}\,[\gamma_-\gamma_+ h_1 \gamma_0]^{ab} - \frac{i}{\alpha'\sqrt{P_+}} \sum_{m=-\infty}^{+\infty} [\gamma^i \gamma_+ h_1 \gamma_0]^{ab} \alpha_m^i\, e^{-2im(\tau-\sigma)} \Rightarrow$$

$$\{ Q^{1a}, S^{1b}(\sigma\tau) \} = i\sqrt{P_+}\,[\gamma_-\gamma_+ h_1 \gamma_0]^{ab} - \frac{i}{\alpha'\sqrt{P_+}} [\gamma^i\gamma_+ h_1 \gamma_0]^{ab} \sum_{m=-\infty}^{+\infty} \alpha_m^i\, e^{-2im(\tau-\sigma)} \qquad (1)$$

$$\{ Q^{2a}, S^{2b}(\sigma\tau) \} = i\sqrt{P_+}\,[\gamma_-\gamma_+ h_2 \gamma_0]^{ab} - \frac{i}{\alpha'\sqrt{P_+}} [\gamma^i\gamma_+ h_2 \gamma_0]^{ab} \sum_{m=-\infty}^{+\infty} \tilde\alpha_m^i\, e^{-2im(\tau+\sigma)}$$

But

$$\mathring{X}^i(\sigma,\tau) = \sum_{m=-\infty}^{+\infty} \left[\alpha_m^i\, e^{-2im(\tau-\sigma)} + \tilde\alpha_m^i\, e^{-2im(\tau+\sigma)} \right]$$

$$X'^i(\sigma\tau) = \sum_{m=-\infty}^{+\infty} \left[-\alpha_m^i\, e^{-2im(\tau-\sigma)} + \tilde\alpha_m^i\, e^{-2im(\tau+\sigma)} \right] \qquad (2)$$

$$X_+ = 2\alpha' P_+ \tau$$

Hence

$$\{ Q^{1a}, S^{1b}(\sigma,\tau) \} = \frac{i}{2\alpha'\sqrt{P_+}} [\gamma_-\gamma_+ h_1 \gamma_0]^{ab} (\mathring{X}'_+ - X'_+) - \frac{i}{2\alpha'\sqrt{P_+}} [\gamma^i\gamma_+ h_1 \gamma_0]^{ab} (\mathring{X}'^i - X'^i)$$

$$\{ Q^{2a}, S^{2b}(\sigma,\tau) \} = \frac{i}{2\alpha'\sqrt{P_+}} [\gamma_-\gamma_+ h_2 \gamma_0]^{ab} (\mathring{X}'_+ + X'_+) - \frac{i}{2\alpha'\sqrt{P_+}} [\gamma^i\gamma_+ h_2 \gamma_0]^{ab} (\mathring{X}'^i + X'^i) \qquad (3)$$

Therefore

$$\{ Q^{1a}, S^{1b}(\sigma,\tau) \} = \frac{i}{2\alpha'\sqrt{P_+}} [\gamma_\mu \gamma_+ h_1 \gamma_0]^{ab} [\mathring{X}'^\mu(\sigma\tau) - X'^\mu(\sigma\tau)]$$

$$\{ Q^{2a}, S^{2b}(\sigma,\tau) \} = \frac{i}{2\alpha'\sqrt{P_+}} [\gamma_\mu \gamma_+ h_2 \gamma_0]^{ab} [\mathring{X}'^\mu(\sigma\tau) + X'^\mu(\sigma\tau)] \qquad (4)$$

$$\{ Q^{1a}, S^{2b}(\sigma\tau) \} = \{ Q^{2a}, S^{1b}(\sigma,\tau) \} = 0$$

Let us now introduce $\epsilon^{1a}(\epsilon^{2a})$ with the same chiralities as $S^{2a}(S^{1a})$, then

$$[(\bar{\bar\epsilon}\cdot Q)\,, X'^i(\sigma\tau)] = \bar{\bar\epsilon}^{1a} \{ Q^{1a}, X'^i(\sigma\tau) \} = -\frac{1}{\sqrt{P_+}} \bar{\bar\epsilon}^{1a} [\gamma^i S(\sigma\tau)]^a \Rightarrow$$

$$i(\bar{\bar\epsilon}\cdot Q)\,, X'^i(\sigma,\tau)] = -\frac{1}{\sqrt{P_+}} [\bar{\bar\epsilon}\gamma^i S(\sigma\tau)] \qquad (5)$$

Similarly

$$[(\bar{\bar{\epsilon}}\cdot Q)\ ,\ S^{Aa}(\sigma,\tau)\ ] = \bar{\bar{\epsilon}}^{Bb}\ \{\ Q^{Bb}\ ,\ S^{Aa}(\sigma\tau)\ \} =$$

$$= (\gamma^0)^{cb}\ \Big\{ -i\,\epsilon^{1c}\ \{\ Q^{2b}\ ,\ S^{Aa}(\sigma\tau)\ \} + i\,\epsilon^{2c}\ \{\ Q^{1b}\ ,\ S^{Aa}(\sigma\tau)\ \}\ \Big\} \Rightarrow$$

$$[\ (\bar{\bar{\epsilon}}\cdot Q)\ ,\ S^{1a}(\sigma\tau)\ ] = i\,(\gamma^0)^{cb}\,\epsilon^{2c}\,\frac{i}{2\alpha'\sqrt{P_+}}\,[\gamma_\mu\gamma_+\,h_1\gamma_0]^{ba}\,[\dot{X}^\mu - X'^\mu]$$

$$= -\frac{1}{2\alpha'\sqrt{P_+}}\,[\gamma_+\gamma_\mu(I-h_1)\epsilon^2]^a\,(\dot{X}^\mu - X'^\mu) \Rightarrow$$

$$[\ (\bar{\bar{\epsilon}}\,Q)\ ,\ S^{1a}(\sigma\tau)\ ] = -\frac{1}{2\alpha'\sqrt{P_+}}\,[\gamma_+\gamma_\mu\,\epsilon^{\cdot}]^{2a}\,(\dot{X}^\mu - X'^\mu) \tag{2}$$

$$[\ (\bar{\bar{\epsilon}}\cdot Q)\ ,\ S^{2a}(\sigma\tau)\ ] = +\frac{1}{2\alpha'\sqrt{P_+}}\,[\gamma_+\gamma_\mu\,\epsilon]^{1a}\,(\dot{X}^\mu + X'^\mu)$$

Hence

$$[\ (\bar{\bar{\epsilon}}\cdot Q)\ ,\ S^{Aa}(\sigma\tau)\ ] = -\frac{i}{2\alpha'\sqrt{P_+}}\,[\gamma_+\gamma_\mu\,\rho^\beta\partial_\beta X^\mu(\sigma\tau)\ \epsilon]^{Aa} \tag{3}$$

Therefore

$$[\ (\bar{\bar{\epsilon}}\cdot Q)\ ,\ X^i(\sigma\tau)\ ] = -\frac{1}{\sqrt{P_+}}\,[\,\bar{\bar{\epsilon}}\,\gamma^i\,S(\sigma\tau)\,] \tag{4}$$

$$[\ (\bar{\bar{\epsilon}}\cdot Q)\ ,\ S^{Aa}(\sigma,\tau)\ ] = -\frac{i}{2\alpha'\sqrt{P_+}}\,[\gamma_+\gamma_\mu\,\rho^\beta\partial_\beta X^\mu(\sigma,\tau)\,\epsilon]^{Aa}$$

Since $(\bar{\bar{\epsilon}}\cdot Q)$ is anti-self-adjoint let us introduce the unitary operator

$$U(\epsilon) = e^{-(\bar{\bar{\epsilon}}\cdot Q)} \tag{5}$$

then

$$U(\epsilon)\,X^i(\sigma\tau)\,U^{-1}(\epsilon) = X^i(\sigma\tau) + \frac{1}{\sqrt{P_+}}\,[\,\bar{\bar{\epsilon}}\,\gamma^i\,S(\sigma\tau)\,]$$

$$U(\epsilon)\,S^{Aa}(\sigma\tau)\,U^{-1}(\epsilon) = S^{Aa}(\sigma\tau) + \frac{i}{2\alpha'\sqrt{P_+}}\,[\gamma_+\gamma_\mu\,\rho^\beta\partial_\beta X^\mu(\sigma\tau)\,\epsilon]^{Aa} \tag{6}$$

And easier way to compute (4) is doing directly (20.5).

$$= [\,(2)\sigma)_{,\bar{x}}\,,\,{}_{qv}\bar{\bar{\mathcal{D}}}\,] = {}_{qv}\bar{\bar{\varepsilon}}^{Ac}\,[\,\mathcal{D}^{Aa}\,,\,{}_{,\bar{x}}\,(2)\sigma)\,] = [\,(2'\sigma)_{,\bar{x}}\,,\,\bar{\bar{\mathcal{D}}}\cdot\bar{\bar{\mathcal{D}}}\,]$$

$$= -\frac{i}{2\pi\alpha'\sqrt{\bar{P}_+}}\;{}_{qv}\bar{\bar{\varepsilon}}^{Ac}\int_0^{\sigma}d\sigma'\,[\,{}_{,\sigma\rho}\,(2,\sigma)\,S^{ij}\,{}_{qv}[\,(2,\sigma)\,\dot{\bar{x}}^{i}\,,\,{}_{,\bar{x}}\,(2\sigma)\,] = [\,(2\sigma)_{,\bar{x}}\,,\,\bar{\bar{\mathcal{D}}}\cdot\bar{\bar{\mathcal{D}}}\,]$$

$$= -\frac{i}{2\pi\alpha'\sqrt{\bar{P}_+}}\;{}_{qv}\bar{\bar{\varepsilon}}^{Ac}\int_0^{\sigma}d\sigma'\,[\,{}_{,\sigma\rho}\,(2,\sigma)\,S^{ij}\,{}_{qv}[\,(2,\sigma)\,S^{kl}\,]\,[-2\pi i\,\alpha'\,p_i\,\delta(\sigma-\sigma')\,] = -\frac{1}{\sqrt{\bar{P}_+}}\,[\,\bar{\bar{\varepsilon}}^{ij}\,S^{kl}(2\sigma)\,] \qquad (1)$$

Similarly

$$[\,\bar{\bar{\mathcal{D}}}\cdot\bar{\bar{\mathcal{D}}}\,,\,S^{Ac}{}_{qv}(2\sigma)\,] = \bar{\bar{\varepsilon}}^{Bb}\,\{\,{}_{qq}\mathcal{D}^{Bb}\,,\,S^{Ac}{}_{qv}(2\sigma)\,\} = [\,(2\sigma)_{qv}S\,,\,\bar{\bar{\mathcal{D}}}\cdot\bar{\bar{\mathcal{D}}}\,]$$

$$= \{\,(2\sigma)_{qv}S\,,\,{}_{pc}S^{cc}(2'\sigma)\,\} = \frac{i}{2\pi\alpha'\sqrt{\bar{P}_+}}\;\bar{\bar{\varepsilon}}^{Bb}\int_0^{\sigma}d\sigma'\,[\,{}_{,\sigma\rho}\,\delta^{ij}\,\partial_\beta\,\bar{x}^{\mu}(2,\sigma)\,\partial_\alpha\,\bar{x}^{\nu}(2'\sigma)\,{}^{\beta}\delta^{\alpha}_{\mu\nu}\,{}^{qq\,b,cc}\,\{\,S^{cc}(2'\sigma)\,,\,S^{Aa}(2\sigma)\,\} =$$

$$= \frac{i}{2\pi\alpha'\sqrt{\bar{P}_+}}\;\bar{\bar{\varepsilon}}^{Bb}\int_0^{\sigma}d\sigma'\,[\,{}_{,\sigma\rho}\,\delta^{ij}\,\partial_\beta\,\bar{x}^{\mu}(2,\sigma)\,\partial_\alpha\,\bar{x}^{\nu}(2'\sigma)\,{}^{\beta}\delta^{\alpha}_{\mu\nu}\,{}^{qq\,b,cc}\,[\,\pi\,\delta(\sigma-\sigma')\,]\,{}_{pv}[\,\gamma^{0}\gamma^{+}\gamma^{i}\,]^{cc,Aa} =$$

$$= \frac{i}{2\alpha'\sqrt{\bar{P}_+}}\;[\,\gamma^{0}\gamma^{+}\gamma^{i}\,]^{Ac}{}_{qv} = -\frac{i}{2\alpha'\sqrt{\bar{P}_+}}\;[\,\bar{\bar{\varepsilon}}\,\gamma^{0}\gamma^{i}\gamma^{+}\gamma^{\beta}\,\partial_\beta\,\bar{x}^{\mu}(2\sigma)\,\gamma^{+}\gamma^{\alpha}\delta^{\beta}_{\mu\nu}\,\gamma^{\nu}_{L}\,\gamma^{-}_{L}\,\gamma^{\beta}_{L}\,\gamma^{+}\gamma^{-}\gamma^{\alpha}_{L}\,\gamma^{0}\,]^{Ac}{}_{qv}$$

$$= -\frac{i}{2\alpha'\sqrt{\bar{P}_+}}\;[\,\gamma^{+}\gamma^{\mu}\,\partial_\beta\,\bar{x}^{\mu}(2\sigma)\,\in\,]^{Ac}{}_{qv} \qquad (2)$$

which coincide with the above given results.

XII- COMMUTATION RELATIONS

We would like to consider the superalgebra satisfied by the Poincaré generators and the supersymmetry transformations.

The Open String

In this case the momentum components are

$$p^i = \frac{1}{2\alpha'} \alpha_0^i \qquad (1)$$

$$p_+$$

$$p_- = \frac{1}{2\alpha' p_+} \left\{ \frac{1}{4\alpha'} \sum_{m=-\infty}^{+\infty} : \alpha_{-m}^i \alpha_m^i : + \frac{1}{4} \sum_{m=-\infty}^{+\infty} m : \bar{S}_{-m} \gamma_- S_m : \right\}$$

where we have used $H = 2\alpha' p_+ p_-$. Remember

$$[\alpha_m^i , \alpha_m^j] = 2\alpha' m \, \delta_{m+m,0} \, \delta_{ij} \qquad (2)$$

$$\{ S_m^a , \bar{S}_m^b \} = \delta_{m+m,0} \, [\gamma_+ h]^{ab}$$

From all that

$$[p^\mu , p^\nu] = 0 \qquad (3)$$

Furthermore the supersymmetry generators are

$$Q^a = + i \sqrt{p_+} \, [\gamma_- S_0]^a - \frac{i}{2\alpha' \sqrt{p_+}} \sum_{m=-\infty}^{+\infty} [\gamma^i S_{-m}]^a \alpha_m^i \qquad (4)$$

Clearly

$$[Q^a , p^i] = [Q^a , p_+] = 0 \qquad (5)$$

Furthermore

$$[Q^a , p_-] = \Big| i \sqrt{p_+} \, [\gamma_- S_0]^a - \frac{i}{2\alpha' \sqrt{p_+}} \sum_{m=-\infty}^{+\infty} [\gamma^i S_{-m}]^a \alpha_m^i \, ,$$

$$\frac{1}{2\alpha' p_+} \left\{ \frac{1}{4\alpha'} \sum_{m=-\infty}^{+\infty} \alpha_{-m}^j \alpha_m^j + \frac{1}{4} \sum_{m=-\infty}^{+\infty} m \, \bar{S}_{-m} \gamma_- S_m \right\} \Big| =$$

$$= \frac{i}{8\alpha' \sqrt{p_+}} \sum_{m=-\infty}^{+\infty} m \, [[\gamma_- S_0]^a , \bar{S}_{-m} \gamma_- S_m] - \frac{i}{16 \alpha'^3 p_+^{3/2}} \sum_{m=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} [\gamma^i S_{-m}]^a [\alpha_m^i , \alpha_{-m}^j \alpha_m^j]$$

$$- \frac{i}{16 \alpha'^2 p_+^{3/2}} \sum_{m=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} m \, \alpha_m^i \, [(\gamma^i S_{-m})^a , S_{-m} \gamma_- S_m]$$

The first commutator does not contribute. Let us consider the other ones:

$$[\alpha_m^i, \alpha_{-m}^j \alpha_m^j] = \alpha_{-m}^j [\alpha_m^i, \alpha_m^j] + [\alpha_m^i, \alpha_{-m}^j] \alpha_m^j =$$

$$= 2\alpha' m \delta^{ij} \delta_{m+m,0} \alpha_{-m}^j + 2\alpha' m \delta^{ij} \delta_{m-m,0} \alpha_m^j = 2\alpha' m \alpha_m^i [\delta_{m+m,0} + \delta_{m-m,0}] \quad (1)$$

$$[(\gamma^i S_{-m})^a, S_{-m} \gamma_- S_m] = (\gamma^i)^{ab} (\gamma^0 \gamma_-)^{cd} [S_{-m}^b, S_{-m}^c S_m^d] =$$

$$= (\gamma^i)^{ab} (\gamma^0 \gamma_-)^{cd} \{ -S_{-m}^c \{ S_{-m}^b, S_m^d \} + \{ S_{-m}^b, S_{-m}^c \} S_m^d \} =$$

$$= -\delta_{m-m,0} [\gamma^i \gamma_+ h \gamma^0 \gamma_-^T \gamma^{0T} S_{-m}]^a + \delta_{m+m,0} [\gamma^i \gamma_+ h \gamma^0 \gamma^0 \gamma_- S_{-m}]^a$$

$$= (\delta_{m+m,0} + \delta_{m-m,0}) [\gamma^i \gamma_+ h \gamma_- S_{-m}]^a = 2 (\delta_{m+m,0} + \delta_{m-m,0}) [\gamma^i S_{-m}]^a \quad (2)$$

Hence

$$[Q^a, P_-] = -\frac{i}{8\alpha'^2 P_+^{3/2}} \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \{ m [\gamma^i S_{-m}]^a (\delta_{m+m,0} + \delta_{m-m,0}) \alpha_m^i$$

$$+ m [\gamma^i S_{-m}]^a (\delta_{m+m,0} - \delta_{m-m,0}) \alpha_m^i \} = 0 \quad (3)$$

and therefore

$$[Q^a, P^\mu] = 0 \quad (4)$$

Let us now proceed to the calculation of the anticommutator of two supercharges. Let us write

$$Q^a \equiv Q_1^a + Q_2^a \quad (5)$$

$$Q_1^a \equiv i\sqrt{P_+} [\gamma_- S_0]^a \quad , \quad Q_2^a \equiv -\frac{i}{2\alpha' \sqrt{P_+}} \sum_{n=-\infty}^{+\infty} [\gamma^i S_{-m}]^a \alpha_m^i$$

Now we have

$$\{ Q_1^a, \bar{Q}_1^b \} = -P_+ (\gamma^0)^{cb} (\gamma_-)^{ad} (\gamma_-)^{ce} \{ S_0^d, S_0^e \} =$$

$$= -P_+ (\gamma^0)^{cb} (\gamma_-)^{ad} (\gamma_-)^{ce} [\gamma_+ h \gamma^0]^{de} = -P_+ [\gamma_- \gamma_+ h \gamma^0 \gamma_-^T \gamma^0]^{ab} =$$

$$= P_+ [\gamma_- \gamma_+ h \gamma_-]^{cb} = P_+ [h \gamma_- \gamma_+ \gamma_-]^{ab} = 2 P_+ [h \gamma_-]^{ab} \Rightarrow$$

$$\{ Q_1^a, \bar{Q}_1^b \} = 2 P_+ [h \gamma_-]^{ab} \quad (6)$$

Furthermore

$$\{ Q_1^a, \bar{Q}_2^b \} + \{ Q_2^a, \bar{Q}_1^b \} = (\gamma^0)^{cb} \left\{ \{ Q_1^a, Q_2^c \} + \{ Q_2^a, Q_1^c \} \right\} =$$

$$= \frac{1}{2\alpha'} (\gamma^0)^{cb} \sum_{m=-\infty}^{+\infty} \alpha_m^i \left\{ (\gamma_-)^{ad} (\gamma^i)^{cf} \{ S_0^d, S_{-m}^f \} + (\gamma^i)^{ad} (\gamma_-)^{cf} \{ S_{-m}^d, S_0^f \} \right\}$$

$$= \frac{1}{2\alpha'} \alpha_0^i \left\{ [\gamma_- \gamma_+ h \gamma^0 \gamma^{iT} \gamma^0]^{cb} + [\gamma^i \gamma_+ h \gamma^0 \gamma_-^T \gamma^0] = \right.$$

$$= -\frac{1}{2\alpha'} \alpha_0^i \left\{ [h \gamma_- \gamma_+ \gamma^i]^{cb} + [h \gamma_+ \gamma_- \gamma^i]^{ab} \right\} = -\frac{1}{2\alpha'} \alpha_0^i \, 2 [h \gamma^i]^{ab} \Rightarrow$$

$$\{ Q_1^a, \bar{Q}_2^b \} + \{ Q_2^a, \bar{Q}_1^b \} = -2 p^i [h \gamma^i]^{ab} \qquad (1)$$

Let us finally consider the remaining anticommutator

$$\{ Q_2^a, \bar{Q}_2^b \} = -(\gamma^0)^{bc} \{ Q_2^a, Q_2^c \} = + \frac{1}{4\alpha'^2 p_+} (\gamma^0)^{bc} \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \{ [\gamma^i S_{-m}]^a \alpha_m^i, [\gamma^j S_{-n}]^c \alpha_n^j \}$$

$$= + \frac{1}{4\alpha'^2 p_+} (\gamma^0)^{bc} (\gamma^i)^{ad} (\gamma^j)^{cf} \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \{ S_{-m}^d \alpha_m^i, S_{-n}^f \alpha_n^j \} =$$

$$= + \frac{1}{4\alpha'^2 p_+} (\gamma^0)^{bc} (\gamma^i)^{ad} (\gamma^j)^{cf} \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \left\{ - S_{-m}^d S_{-n}^f [\alpha_m^i, \alpha_n^j] - \{ S_{-m}^d, S_{-n}^f \} \alpha_n^j \alpha_m^i \right\}$$

$$= - \frac{1}{4\alpha'^2 p_+} (\gamma^0)^{bc} (\gamma^i)^{ad} (\gamma^j)^{cf} \sum_{m=-\infty}^{+\infty} \left\{ 2\alpha' m \delta^{ij} S_{-m}^d S_m^f + \alpha_{-n}^j \alpha_m^i \, \delta_{mn} [\gamma_+ h \gamma_0]^{df} \right\}$$

$$= - \frac{1}{4\alpha'^2 p_+} i [\gamma^i \gamma_+ h \gamma_0 \gamma^{jT} \gamma^{0T}]^{ab} \sum_{m=-\infty}^{+\infty} \alpha_{-m}^j \alpha_m^i + \frac{1}{2\alpha' p_+} (\gamma^j)^{ad} (\gamma^{iT} \gamma^{0T})^{fc} \sum_{m=-\infty}^{+\infty} m S_{-m}^d S_m^f$$

$$= + \frac{1}{4\alpha'^2 p_+} [h \gamma_+ \gamma^i \gamma^j]^{ab} \sum_{m=-\infty}^{+\infty} \alpha_{-m}^j \alpha_m^i + \frac{1}{2\alpha' p_+} (\gamma^i)^{ad} (\gamma^0 \gamma^i)^{fc} \sum_{m=-\infty}^{+\infty} m S_{-m}^d S_m^f \Rightarrow$$

$$\{ Q_2^a, \bar{Q}_2^b \} = \frac{1}{4\alpha'^2 p_+} [h \gamma_+]^{ab} \sum_{m=-\infty}^{+\infty} \alpha_{-m}^i \alpha_m^i + \frac{1}{2\alpha' p_+} (\gamma^i)^{ad} (\gamma^0 \gamma^i)^{fc} \sum_{m=-\infty}^{+\infty} m S_{-m}^d S_m^f \qquad (2)$$

Let us now Wick order

$$\sum_{m=1}^{\infty} \alpha_m^i \alpha_{-m}^i = \sum_{m=1}^{\infty} \alpha_{-m}^i \alpha_m^i + 16 \alpha' \sum_{m=1}^{\infty} m \qquad (3)$$

$$\sum_{m=1}^{\infty} (-m) S_m^d S_{-m}^f = \sum_{m=1}^{\infty} m S_{-m}^f S_m - [\gamma_+ h \gamma^0]^{df} \sum_{m=1}^{\infty} m$$

Therefore

$$\{ Q_2^a , \bar{Q}_2^b \} = \frac{1}{4\alpha'^2 P_+} [h\gamma_+]^{ab} \sum_{m=-\infty}^{+\infty} : \alpha_{-m}^i \alpha_m^i : + \frac{1}{2\alpha' P_+} [\gamma^i]^{ad} [\gamma^o \gamma^i]^{fc} \sum_{m=-\infty}^{+\infty} m : S_{-m}^d S_m^f :$$

$$+ \left\{ \frac{4}{\alpha' P_+} [h\gamma_+]^{ab} - \frac{1}{2\alpha' P_+} [\gamma^i]^{ad} [\gamma^o \gamma^i]^{fc} [\gamma_+ h \gamma^o]^{df} \right\} \sum_{m=1}^{\infty} m \qquad (1)$$

But

$$[\gamma^i]^{ad} [\gamma^o \gamma^i]^{fc} [\gamma_+ h \gamma^o]^{df} = [\gamma^i \gamma_+ h \gamma^o \gamma^o \gamma^i]^{ab} = [h \gamma^i \gamma_+ \gamma^i]^{ab} = 8 [h\gamma_+]^{ab} \qquad (2)$$

and therefore the two terms in (1) cancel; therefore

$$\{ Q_2^a , \bar{Q}_2^b \} = \frac{1}{4\alpha'^2 P_+} [h\gamma_+]^{ab} \sum_{m=-\infty}^{+\infty} : \alpha_{-m}^i \alpha_m^i : + \frac{1}{2\alpha' P_+} \sum_{m=-\infty}^{+\infty} m : [\gamma^i S_{-m}]^a [\bar{S} \gamma^i_m]^b : \qquad (3)$$

or equivalently

$$\{ Q_2^a , \bar{Q}_2^b \} = 2 P_- [h\gamma_+]^{ab} + \frac{1}{2\alpha' P_+} \left\{ \sum_{m=-\infty}^{+\infty} m : [\gamma^i S_{-m}]^a [\bar{S}_m \gamma^i]^b : \right.$$

$$\left. - \frac{1}{2} [h\gamma_+]^{ab} \sum_{m=-\infty}^{+\infty} m : \bar{S}_{-m} \gamma_- S_m : \right\} \qquad (4)$$

Therefore we have obtained

$$\{ Q^a , \bar{Q}^b \} = 2 P_\mu [h \gamma^\mu]^{ab} \qquad (5)$$

if we are able to prove that

$$A^{ab} \equiv \sum_{m=-\infty}^{+\infty} m \left\{ : [\gamma^i S_{-m}]^a [\bar{S}_m \gamma^i]^b : - \frac{1}{2} [h\gamma_+]^{ab} : \bar{S}_{-m} \gamma_- S_m : \right\} \qquad (6)$$

is identically zero. Notice that this can be rewritten as

$$A^{ab} = \frac{1}{2} \sum_{m=-\infty}^{+\infty} m \; B^{ab}(m) \qquad (7)$$

$$B^{ab}(m) = : [\gamma^i S_{-m}]^a [\bar{S}_m \gamma^i]^b : - : [\gamma^i S_m]^a [\bar{S}_{-m} \gamma^i]^b : - [h\gamma_+]^{ab} : \bar{S}_{-m} \gamma_- S_m :$$

In order to prove this we consider (let us omit the time ordering)

$$C(m) \equiv [\bar{\lambda}_1 \gamma_- \gamma^i S_{-m}][\bar{S}_m \gamma^i \lambda_2] - [\bar{\lambda}_1 \gamma_- \gamma^i S_m][\bar{S}_{-m} \gamma^i \lambda_2] \quad (1)$$

where λ_1 and λ_2 are arbitrary spinors. Fierz reordering

$$C(m) = -\frac{1}{32} \sum_x [\bar{\lambda}_1 \Gamma^{(x)} \lambda_2] \{ [\bar{S}_m \gamma^i \Gamma^{(x)} \gamma_- \gamma^i S_{-m}] - [\bar{S}_{-m} \gamma^i \Gamma^{(x)} \gamma_- \gamma^i S_m] \} =$$

$$= -\frac{1}{32} \sum_x [\bar{\lambda}_1 \Gamma^{(x)} \Gamma_2] \{ [\bar{S}_m \gamma^i \Gamma^{(x)} \gamma_- \gamma^i S_{-m}] - [\bar{S}_m \gamma^0 (\gamma^i \Gamma^{(x)} \gamma_- \gamma^i)^T \gamma^0 S_{-m}] \} \quad (2)$$

Therefore the only contributing terms are those satisfying

$$\gamma^0 [\gamma^i \Gamma^{(z)} \gamma_- \gamma^i]^T \gamma^0 = -\gamma^i \Gamma^{(z)} \gamma_- \gamma^i \quad (3)$$

Let us call $\{z\}$ this set; then

$$C(m) = -\frac{1}{16} \sum_z [\bar{\lambda}_1 \Gamma^{(z)} \Gamma_2][\bar{S}_m \gamma^i \Gamma^{(z)} \gamma_- \gamma^i S_{-m}] \quad (4)$$

Let us remember that S_m and S_{-m} have the same chirality η, then

$$[\bar{S}_m \gamma^i \Gamma^{(z)} \gamma_- \gamma^i S_{-m}] = \eta^2 [\bar{S}_m \gamma^i \Gamma^{(z)} \gamma_- \gamma^i S_{-m}] = -[\bar{S}_m \gamma'' \gamma^i \Gamma^{(z)} \gamma_- \gamma^i \gamma'' S_{-m}] \quad (5)$$

and only the terms $\Gamma^{(z)}$ with even number of γ^μ can contribute; therefore

$$C(m) = -\frac{1}{16} \Big\{ [\bar{\lambda}_1 \lambda_2][\bar{S}_m \gamma^i \gamma_- \gamma^i S_{-m}] +$$

$$+ \sum_{0 \le \mu < \nu \le 9} [\bar{\lambda}_1 \gamma_\nu \gamma_\mu \lambda_2][\bar{S}_m \gamma^i \gamma^\mu \gamma^\nu \gamma_- \gamma^i S_{-m}] +$$

$$+ \sum_{0 \le \mu < \nu < \lambda < \sigma \le 9} [\bar{\lambda}_1 \gamma_\sigma \gamma_\lambda \gamma_\nu \gamma_\mu \lambda_2][\bar{S}_m \gamma^i \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma \gamma_- \gamma^i S_{-m}]$$

$$+ \sum_{0 \le \mu < \nu < \lambda < \sigma \le 9} [\bar{\lambda}_1 \gamma'' \gamma_\sigma \gamma_\lambda \gamma_\nu \gamma_\mu \lambda_2][\bar{S}_m \gamma^i \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma \gamma'' \gamma_- \gamma^i S_{-m}]$$

$$+ \sum_{0 \le \mu < \nu \le 9} [\bar{\lambda}_1 \gamma'' \gamma_\nu \gamma_\mu \lambda_2][\bar{S}_m \gamma^i \gamma^\mu \gamma^\nu \gamma'' \gamma_- \gamma^i S_{-m}]$$

$$+ [\bar{\lambda}_1 \gamma'' \lambda_2][\bar{S}_m \gamma^i \gamma'' \gamma_- \gamma^i S_{-m}] \Big\} \quad (6)$$

Remember that we must still restrict these sums to those terms satisfying equation (3). When this is done we get

$$C(m) = -\frac{1}{16}\Big\{ [\bar\lambda_1, \lambda_2][\bar S_m \gamma^i \gamma_- \gamma^i S_{-m}] +$$

$$+ [\bar\lambda_1, \gamma_1\gamma_0\lambda_2][\bar S_m \gamma^i \gamma^0\gamma^1\gamma_-\gamma^i S_{-m}] +$$

$$+ \sum_{2\le j<k<e<m\le 9} [\bar\lambda_1, \gamma_m\gamma_e\gamma_k\gamma_j\lambda_2][\bar S_m \gamma^i\gamma^j\gamma^k\gamma^e\gamma^m\gamma_-\gamma^i S_{-m}]$$

$$+ \sum_{2\le j<k<e<m\le 9} [\bar\lambda_1, \gamma''\gamma_m\gamma_e\gamma_k\gamma_j\lambda_2][\bar S_m \gamma^i\gamma^j\gamma^k\gamma^e\gamma^m\gamma''\gamma_-\gamma^i S_{-m}]$$

$$+ [\bar\lambda_1, \gamma''\gamma_1\gamma_0\lambda_2][\bar S_m \gamma^i\gamma^0\gamma^1\gamma''\gamma_-\gamma^i S_{-m}] +$$

$$+ [\bar\lambda_1, \gamma''\lambda_2][\bar S_m \gamma^i\gamma''\gamma_-\gamma^i S_{-m}]\Big\} \qquad (1)$$

The third and fourth terms do not contribute since

$$\gamma^i(\gamma^j\gamma^k\gamma^e\gamma^m)\gamma^i = 0 \qquad 2\le j<k<e<m\le 9 \qquad (2)$$

Furthermore

$$[\bar S_m \gamma^i\gamma_-\gamma^i S_{-m}] = -[\bar S_m \gamma^i\gamma^i\gamma_- S_{-m}] = 8[\bar S_m \gamma_- S_{-m}]$$

$$[\bar S_m \gamma^i\gamma^0\gamma^1\gamma_-\gamma^i S_{-m}] = 8[\bar S_m \gamma^0\gamma^1\gamma_- S_{-m}] = 8[\bar S_m (1-\gamma_+\gamma_-)\gamma_- S_{-m}] = 8[\bar S_m \gamma_- S_{-m}] \qquad (3)$$

$$[\bar S_m \gamma^i\gamma^0\gamma^1\gamma''\gamma_-\gamma^i S_{-m}] = +8\eta[\bar S_m \gamma_- S_{-m}]$$

$$[\bar S_m \gamma^i\gamma''\gamma_-\gamma^i S_{-m}] = +8\eta[\bar S_m \gamma_- S_{-m}]$$

Hence

$$C(m) = -\frac{1}{2}\Big\{[\bar\lambda_1\lambda_2] - [\bar\lambda_1\gamma^1\gamma^0\lambda_2] - \eta[\bar\lambda_1\gamma''\gamma^1\gamma^0\lambda_2] + \eta[\bar\lambda_1\gamma''\lambda_2]\Big\}[\bar S_m\gamma_- S_{-m}]$$

$$= -\frac{1}{2}\Big\{[\bar\lambda_1\gamma_-\gamma_+\lambda_2] + \eta[\bar\lambda_1\gamma''\gamma_-\gamma_+\lambda_2]\Big\}[\bar S_m\gamma_- S_{-m}] =$$

$$= -[\bar\lambda_1(I-h)\gamma_-\gamma_+\lambda_2][\bar S_m\gamma_- S_{-m}] \qquad (4)$$

Therefore

$$:[\gamma_-\gamma^i S_{-m}]^a[\bar S_m\gamma^i]^b: \; = -:[\gamma_-\gamma^i S_m]^a[\bar S_{-m}\gamma^i]^b: \; = +[(I-h)\gamma_-\gamma_+]^{ab}[\bar S_m\gamma_- S_m] \qquad (6)$$

and if we multiply by $(\gamma_+)^{ca}$

$$: [\gamma_+ \gamma_- \gamma^i S_{-m}]^a [\bar S_m \gamma^i]^b : - : [\gamma_+ \gamma_- \gamma^i S_m]^a [\bar S_{-m} \gamma^i]^b = [\gamma_+ (2-h) \gamma_- \gamma_+]^{ab} [\bar S_{-m} \gamma_- S_m] \Rightarrow$$

$$: [\gamma^i S_{-m}]^a [\bar S_m \gamma^i]^b : - : [\gamma^i S_m]^a [\bar S_{-m} \gamma^i]^b : = [h \gamma_+]^{ab} [\bar S_{-m} \gamma_- S_m] \qquad (1)$$

which is the desired equality and therefore (4.7) holds.

In order to go on we must introduce the generators of the proper Lorentz transformations $M^{\mu\nu}$. Let us remember that the momentum density is

$$P^i_\tau(\sigma,\tau) = \frac{1}{2\pi\alpha'} \sum_{m=-\infty}^{+\infty} \alpha^i_m \, e^{-im\tau} \cos m\sigma$$

$$P_{\tau+}(\sigma,\tau) = \frac{1}{\pi} P_+ \qquad (2)$$

$$P_{\tau-}(\sigma\tau) = \frac{1}{2\pi\alpha' P_+} \sum_{m=-\infty}^{+\infty} \left\{ \sum_{m=-\infty}^{+\infty} \left[\frac{1}{4\alpha'} : \alpha^i_{m-m} \alpha^i_m : + \frac{1}{4}\left(m - \frac{m}{2}\right) : \bar S_{m-m} \gamma_- S_m : \right] \right\} e^{-im\tau} \cos m\sigma$$

In order to check the last equation let us remember that from (23.2)

$$P_{\tau-}(\sigma,\tau) = \frac{1}{4\pi\alpha' P_+} \left\{ \frac{1}{2\alpha'} [\dot X^i(\sigma\tau)]^2 + \frac{1}{2\alpha'} [X'^i(\sigma\tau)]^2 - \frac{i}{4} \bar S(\sigma\tau) \gamma_- \rho' \frac{\partial S(\sigma\tau)}{\partial\sigma} \right\} \qquad (3)$$

Since

$$\dot X^i(\sigma\tau) = \sum_{m=-\infty}^{+\infty} \alpha^i_m \, e^{-im\tau} \cos m\sigma \quad , \quad X'^i(\sigma\tau) = -i \sum_{m=-\infty}^{+\infty} \alpha^i_m e^{-im\tau} \sin m\sigma \qquad (4)$$

we get

$$[\dot X^i(\sigma\tau)]^2 + [X'^i(\sigma\tau)]^2 = \sum_{m=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \left\{ \alpha^i_m \alpha^i_m \, e^{-i(m+m)\tau} \cos m\sigma \cos m\sigma \right.$$

$$\left. - \alpha^i_m \alpha^i_m \, e^{-i(m+m)\tau} \sin m\sigma \sin m\sigma \right\} =$$

$$= \sum_{m=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \alpha^i_m \alpha^i_m \, e^{-i(m+m)\tau} \cos(m+m)\sigma \Rightarrow$$

$$[\dot X^i(\sigma\tau)]^2 + [X'^i(\sigma\tau)]^2 = \sum_{m=-\infty}^{+\infty} \left\{ \sum_{m=-\infty}^{+\infty} \alpha^i_{m-m} \alpha^i_m \right\} e^{-im\tau} \cos m\sigma \qquad (5)$$

Similarly

$$S^1(\sigma\tau) = \sum_{m=-\infty}^{+\infty} S_m \, e^{-im(\tau-\sigma)} \quad , \quad S^2(\sigma\tau) = \sum_{m=-\infty}^{+\infty} S_m \, e^{-im(\tau+\sigma)} \qquad (6)$$

and

$$\frac{\partial S^1(\sigma\tau)}{\partial\sigma} = i\sum_{m=-\infty}^{+\infty} m\, S_m\, e^{-im(\tau-\sigma)}, \qquad \frac{\partial S^2(\sigma\tau)}{\partial\sigma} = -i\sum_{m=-\infty}^{+\infty} m\, S_m\, e^{-im(\tau+\sigma)} \tag{1}$$

Hence

$$\bar{S}(\sigma\tau)\,\gamma_-\,\not{p}\,\frac{\partial S(\sigma\tau)}{\partial\sigma} = \bar{S}^1(\sigma\tau)\,\gamma_-\,\not{p}\,\frac{\partial S^1(\sigma\tau)}{\partial\sigma} - \bar{S}^2(\sigma\tau)\,\gamma_-\,\not{p}\,\frac{\partial S^2(\sigma\tau)}{\partial\sigma} =$$

$$= \sum_{m=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty} m\,\bar{S}_m\,\gamma_-\,S_m\left\{ i\,e^{-im(\tau+\sigma)}\,e^{-im(\tau-\sigma)} + i\,e^{-im(\tau+\sigma)}\,e^{-im(\tau+\sigma)}\right\} =$$

$$= i\sum_{m=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty} m\,\bar{S}_m\,\gamma_-\,S_m\,e^{-i(m+m)\tau}\,2\cos(m+m)\sigma \Rightarrow$$

$$\bar{S}(\sigma\tau)\,\gamma_-\,\not{p}\,\frac{\partial S(\sigma\tau)}{\partial\sigma} = 2i\sum_{m=-\infty}^{+\infty}\left\{\sum_{m=-\infty}^{+\infty} m\,\bar{S}_{m-m}\,\gamma_-\,S_m\right\} e^{-im\tau}\cos m\tau \tag{2}$$

Hence

$$P_{\tau-}(\sigma\tau) = \frac{1}{4\pi\alpha' p_+}\sum_{m=-\infty}^{+\infty}\left\{\sum_{m=-\infty}^{+\infty}\left[\frac{1}{2\alpha'}\,\alpha^i_{m-m}\,\alpha^i_m + \frac{1}{2}\,m\,\bar{S}_{m-m}\,\gamma_-\,S_m\right]\right\} e^{-im\tau}\cos m\sigma \tag{3}$$

When we try to normal order this expression we must be careful only with the terms $m=0$, in this case

$$\sum_{m=1}^{\infty}\left\{\frac{1}{2\alpha'}\,\alpha^i_m\,\alpha^i_{-m} - \frac{1}{2}\,m\,\bar{S}_m\,\gamma_-\,S_{-m}\right\} = \sum_{m=1}^{\infty}\left\{\frac{1}{2\alpha'}\,\alpha^i_{-m}\,\alpha^i_m + \frac{1}{2}\,m\,\bar{S}_{-m}\,\gamma_-\,S_m\right\}$$

$$+\sum_{m=1}^{\infty}\left\{\frac{1}{2\alpha'}\,[\alpha^i_m,\alpha^i_{-m}] - \frac{1}{2}\,m\,(\gamma^0\gamma_-)^{cb}\,\{S^a_m, S^b_{-m}\}\right\} =$$

$$= \sum_{m=1}^{\infty}\left\{\frac{1}{2\alpha'}:\alpha^i_m\,\alpha^i_{-m}: - \frac{1}{2}\,m:\bar{S}_m\,\gamma_-\,S_{-m}:\right\} + \sum_{m=1}^{\infty}\left[m\,\delta^{ii} - \frac{1}{4}\,m\,\mathrm{Tr}(I)\right] \tag{4}$$

and the last two terms cancel. Therefore

$$P_{\tau-}(\sigma\tau) = \frac{1}{4\pi\alpha' p_+}\sum_{m=-\infty}^{+\infty}\left\{\sum_{m=-\infty}^{+\infty}\left[\frac{1}{2\alpha'}:\alpha^i_{m-m}\,\alpha^i_m: + \frac{1}{2}\,m:\bar{S}_{m-m}\,\gamma_-\,S_m:\right]\right\} e^{-im\tau}\cos m\sigma \tag{5}$$

and since

$$:\bar{S}_{m-m}\,\gamma_-\,S_m: = -:\bar{S}_m\,\gamma_-\,S_{m-m}: \tag{6}$$

we can write

$$P_{\tau-}(\sigma\tau) = \frac{1}{4\pi\alpha' p_+}\sum_{m=-\infty}^{+\infty}\left\{\sum_{m=-\infty}^{+\infty}\left[\frac{1}{2\alpha'}:\alpha^i_{m-m}\,\alpha^i_m: + \frac{1}{2}\left(m-\frac{m}{2}\right):\bar{S}_{m-m}\,\gamma_-\,S_m:\right]\right\} e^{-im\tau}\cos m\sigma \tag{7}$$

Then introducing

$$\alpha_{m,-} \equiv \sum_{m=-\infty}^{+\infty} \left\{ \frac{1}{4\alpha'} : \alpha^i_{m-m} \alpha^i_m : + \frac{1}{4} \left(m - \frac{m}{2}\right) : \bar{S}_{m-m} \gamma_- S_m : \right\} \tag{1}$$

we can write

$$P^i_\tau(\sigma,\tau) = \frac{1}{2\pi\alpha'} \sum_{m=-\infty}^{+\infty} \alpha^i_m e^{-im\tau} \cos m\sigma$$

$$P_{\tau+}(\sigma,\tau) = \frac{1}{\pi} P_+ \tag{2}$$

$$P_{\tau-}(\sigma,\tau) = \frac{1}{2\pi\alpha' P_+} \sum_{m=-\infty}^{+\infty} \alpha_{m,-} e^{-im\tau} \cos m\sigma$$

and hence

$$X^i(\sigma\tau) = q^i + \alpha^i_0 \tau + i \sum_{\substack{m=-\infty \\ m\neq 0}}^{+\infty} \frac{1}{m} \alpha^i_m e^{-im\tau} \cos m\sigma$$

$$X_+(\sigma,\tau) = 2\alpha' P_+ \tau \tag{3}$$

$$X_-(\sigma\tau) = q_- + \frac{1}{P_+} \alpha_{0,-} \tau + \frac{i}{P_+} \sum_{\substack{m=-\infty \\ m\neq 0}}^{+\infty} \frac{1}{m} \alpha_{m,-} e^{-im\tau} \cos m\sigma$$

Before going on let us compute the commutation relations of α_{m-}, α_{m-}.

$$[\alpha_{m-}, \alpha_{m-}] = \sum_{\ell=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} \left\{ \frac{1}{16\alpha'^2} [\alpha^i_{m-\ell} \alpha^i_\ell , \alpha^j_{m-k} \alpha^j_k] + \right.$$

$$\left. + \frac{1}{16} \left(\ell - \frac{m}{2}\right)\left(k - \frac{m}{2}\right) (\gamma^0\gamma_-)^{ab} (\gamma^0\gamma_-)^{cd} [S^a_{m-\ell} S^b_\ell , S^c_{m-k} S^d_k] \right\} =$$

$$= \sum_{\ell=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} \left\{ \frac{1}{16\alpha'^2} [\alpha^i_{m-\ell} \alpha^j_{m-k} [\alpha^i_\ell, \alpha^j_k] + \alpha^i_{m-\ell} [\alpha^i_\ell, \alpha^j_{m-k}] \alpha^j_k + \right.$$

$$+ \alpha^j_{m-k} [\alpha^i_{m-\ell}, \alpha^j_k] \alpha^i_\ell + [\alpha^i_{m-\ell}, \alpha^j_{m-k}] \alpha^j_k \alpha^i_\ell]$$

$$+ \frac{1}{16} \left(\ell - \frac{m}{2}\right)\left(k - \frac{m}{2}\right) (\gamma^0\gamma_-)^{ab} (\gamma^0\gamma_-)^{cd} [- S^a_{m-\ell} S^c_{m-k} \{S^b_\ell, S^d_k\} + S^a_{m-\ell} \{S^b_\ell, S^c_{m-k}\} S^d_k$$

$$\left. - S^c_{m-k} \{S^a_{m-\ell}, S^d_k\} S^b_\ell + \{S^a_{m-\ell}, S^c_{m-k}\} S^d_k S^b_\ell] \right\}$$

$$= \sum_{\ell=-\infty}^{+\infty} \left\{ \frac{1}{8\alpha'} \left[\ell\, \alpha^i_{m-\ell}\, \alpha^i_{m+\ell} + \ell\, \alpha^i_{m-\ell}\, \alpha^i_{m+\ell} + (m-\ell)\, \alpha^i_{m+m-\ell}\, \alpha^i_{\ell} + (m-\ell)\, \alpha^i_{m+m-\ell}\, \alpha^i_{\ell} \right] \right.$$

$$+ \frac{1}{16} \left[2\left(\ell - \frac{m}{2}\right)\left(\ell + \frac{m}{2}\right) \bar{S}_{m-\ell}\, \gamma_-\, S_{m+\ell} + 2\left(\ell - \frac{m}{2}\right)\left(\ell + \frac{m}{2}\right) \bar{S}_{m-\ell}\, \gamma_-\, S_{m+\ell} \right.$$

$$\left.\left. + 2\left(\ell - \frac{m}{2}\right)\left(\ell - m - \frac{m}{2}\right) \bar{S}_{m+m-\ell}\, \gamma_-\, S_{\ell} - 2\left(\ell - \frac{m}{2}\right)\left(\ell - m - \frac{m}{2}\right) \bar{S}_{m+m-\ell}\, \gamma_-\, S_{\ell} \right] \right\} \Rightarrow$$

$$[\alpha_{m,-}, \alpha_{m,-}] = \sum_{\ell=-\infty}^{+\infty} \left\{ \frac{1}{4\alpha'} \left[\ell\, \alpha^i_{m-\ell}\, \alpha^i_{m+\ell} + (m-\ell)\, \alpha^i_{m+m-\ell}\, \alpha^i_{\ell} \right] \right.$$

$$\left. + \frac{1}{4} \left[\left(\ell - \frac{m}{2}\right)\left(\ell + \frac{m}{2}\right) \bar{S}_{m-\ell}\, \gamma_-\, S_{m+\ell} - \left(\ell - \frac{m}{2}\right)\left(\ell - m - \frac{m}{2}\right) \bar{S}_{m+m-\ell}\, \gamma_-\, S_{\ell} \right] \right\} \qquad (1)$$

If $m+m \neq 0$ (1) is still valid with ordered product and then we can write

$$[\alpha_{m,-}, \alpha_{m,-}] = \sum_{\ell=-\infty}^{+\infty} \left\{ \frac{1}{4\alpha'} \left[(\ell - m) + (m-\ell) \right] : \alpha^i_{m+m-\ell}\, \alpha^i_{\ell} : + \right.$$

$$\left. + \frac{1}{4} \left[\left(\ell - m - \frac{m}{2}\right)\left(\ell - \frac{m}{2}\right) - \left(\ell - \frac{m}{2}\right)\left(\ell - m - \frac{m}{2}\right) \right] : \bar{S}_{m+m-\ell}\, \gamma_-\, S_{\ell} : \right\} =$$

$$= (m-m) \sum_{\ell=-\infty}^{+\infty} \left\{ \frac{1}{4\alpha'} : \alpha^i_{m+m-\ell}\, \alpha^i_{\ell} : + \frac{1}{4} \left(\ell - \frac{m+m}{2}\right) : \bar{S}_{m+m-\ell}\, \gamma_-\, S_{\ell} : \right\} \Rightarrow$$

$$[\alpha_{m,-}, \alpha_{m,-}] = (m-m)\, \alpha_{m+m,0} \qquad m+m \neq 0. \qquad (2)$$

Let us now consider the case $m = -m > 0$ in this case in order to be able to substitute this expression by its time ordered one we must subtract the following quantity

$$A = \frac{1}{4\alpha'} \left\{ \sum_{\ell=-\infty}^{m} \ell\, [\alpha^i_{m-\ell}, \alpha^i_{-m+\ell}] + \sum_{\ell=-\infty}^{0} (m-\ell)\, [\alpha^i_{-\ell}, \alpha^i_{\ell}] \right\}$$

$$+ \frac{1}{4} \left\{ \sum_{\ell=-\infty}^{m} \left(\ell - \frac{m}{2}\right)\left(\ell - \frac{m}{2}\right) (\gamma^0 \gamma_-)^{ab} \left\{ S^a_{m-\ell}, S^b_{-m+\ell} \right\} - \sum_{\ell=-\infty}^{0} \left(\ell - \frac{m}{2}\right)\left(\ell - \frac{m}{2}\right) (\gamma^0 \gamma_-)^{ab} \left\{ S^a_{-\ell}, S^b_{\ell} \right\} \right\}$$

$$= \frac{1}{4\alpha'} \left\{ \sum_{\ell=-\infty}^{m} \ell(m-\ell)\, 2\alpha'\, \delta^{ii} + \sum_{\ell=-\infty}^{0} (m-\ell)(-\ell)\, 2\alpha'\, \delta^{ii} \right\}$$

$$+ \frac{1}{4} \left\{ \sum_{\ell=-\infty}^{m} \left(\ell - \frac{m}{2}\right)^2 \frac{1}{2}\, \mathrm{Tr}(I) - \sum_{\ell=-\infty}^{0} \left(\ell - \frac{m}{2}\right)^2 \frac{1}{2}\, \mathrm{Tr}(I) \right\} =$$

$$= 4 \left\{ \sum_{\ell=1}^{m} \ell(m-\ell) \right\} - 4 \left\{ \sum_{\ell=1}^{m} \left(\ell - \frac{m}{2}\right)^2 \right\} = -m^2 \sum_{\ell=1}^{m} 1 = -m^3 \qquad (3)$$

And therefore

$$[\alpha_{n,-}\,,\,\alpha_{m,-}] = (n-m)\,\alpha_{n+m,-} + n^3\,\delta_{n+m,0}$$

$$[\alpha^i_n\,,\,\alpha_{m,-}] = n\,\alpha^i_{n+m} \tag{1}$$

where the second is proved immediately

Let us now continue the study of the generators $M^{\mu\nu}$. Let us begin considering M^{ij}

$$M^{ij} = \int_0^\pi d\sigma\; M^{ij}_\tau(\sigma,\tau)$$

$$M^{ij}_\tau(\sigma\tau) = X^i(\sigma\tau)\,P^j_\tau(\sigma\tau) - X^j(\sigma\tau)\,P^i_\tau(\sigma\tau) + \tag{2}$$

$$+ \frac{1}{16\pi}\,\bar{\bar{S}}(\sigma\tau)\,\gamma_-\,\sigma^{ij}\,\rho^0\,S(\sigma\tau)$$

$$\sigma^{\mu\nu} \equiv \frac{i}{2}\,[\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu]$$

Therefore

$$\int_0^\pi d\sigma\,[X^i(\sigma\tau)\,P^j_\tau(\sigma\tau) - (i\leftrightarrow j)] =$$

$$= \int_0^\pi d\sigma\,\Big\{\Big[q^i + \alpha^i_0\tau + i\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{1}{n}\,\alpha^i_n\,e^{-in\tau}\cos n\sigma\Big]\frac{1}{2\pi\alpha'}\sum_{m=-\infty}^{+\infty}\alpha^j_m\,e^{-im\tau}\cos m\sigma - (i\leftrightarrow j)\Big\} \tag{3}$$

But

$$\int_0^\pi d\sigma\,\cos n\sigma\,\cos m\sigma = \frac{\pi}{2}\,[\delta_{n+m,0} + \delta_{n-m,0}] \tag{4}$$

hence

$$\int_0^\pi d\sigma\,[X^i(\sigma\tau)\,P^j_\tau(\sigma\tau) - X^j(\sigma\tau)\,P^i_\tau(\sigma\tau)] =$$

$$= (q^i P^j - q^j P^i) + \frac{i}{4\alpha'}\sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty}\frac{1}{n}\,[\alpha^i_n\,\alpha^j_{-n} - \alpha^j_n\,\alpha^i_{-n}] \tag{5}$$

or equivalently

$$\int_0^\pi d\sigma\,[X^i(\sigma\tau)\,P^j_\tau(\sigma\tau) - X^j(\sigma\tau)\,P^i_\tau(\sigma\tau)] = q^i P^j - q^j P^i$$

$$- \frac{i}{2\alpha'}\sum_{n=1}^{\infty}\frac{1}{n}\,[\alpha^i_{-n}\,\alpha^j_n - \alpha^j_{-n}\,\alpha^i_n] \tag{6}$$

Similarly

$$\int_0^\pi d\sigma\ \bar{S}(\sigma\tau)\,\gamma_-\,\sigma^{ij}\,\partial^0 S(\sigma\tau) = \int_0^\pi d\sigma \left\{ \bar{S}^1(\sigma\tau)\,\gamma_-\,\sigma^{ij}\,S^1(\sigma\tau) + \bar{S}^2(\sigma\tau)\,\gamma_-\,\sigma^{ij}\,S^2(\sigma\tau) \right\} =$$

$$= \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} (\bar{S}_m\,\gamma_-\,\sigma^{ij}\,S_m) \int_0^\pi d\sigma\, e^{-i(n+m)\tau} \left\{ e^{+i(n+m)\sigma} + e^{-i(n+m)\sigma} \right\} \Rightarrow$$

$$\int_0^\pi d\sigma\ \bar{S}(\sigma\tau)\,\gamma_-\,\sigma^{ij}\,\partial^0 S(\sigma\tau) = 2\pi \sum_{m=-\infty}^{+\infty} (\bar{S}_{-m}\,\gamma_-\,\sigma^{ij}\,S_m) \qquad (1)$$

Therefore

$$M^{ij} = L^{ij} + K_0^{ij}$$

$$L^{ij} \equiv \ell^{ij} - \frac{i}{2\alpha'} \sum_{m=1}^{\infty} \frac{1}{m} \left[\alpha_{-m}^i \alpha_m^j - \alpha_{-m}^j \alpha_m^i \right] \equiv \ell^{ij} + \hat{L}^{ij}$$

$$K_m^{ij} \equiv \frac{1}{8} \sum_{m=-\infty}^{+\infty} \left[\bar{S}_{n-m}\,\gamma_-\,\sigma^{ij}\,S_m \right] \qquad (2)$$

$$\sigma^{\mu\nu} \equiv \frac{i}{2} \left(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu \right)$$

$$\ell^{\mu\nu} \equiv q^\mu p^\nu - q^\nu p^\mu$$

In order for the theory to be Poincare invariant you expect that

$$[\, p^\mu,\ p^\nu \,] = 0$$

$$[\, M^{\lambda\rho},\ p^\mu \,] = -i \left(\eta^{\lambda\mu} p^\rho - \eta^{\rho\mu} p^\lambda \right) \qquad (3)$$

$$[\, M^{\lambda\rho},\ M^{\mu\nu} \,] = -i \left(\eta^{\lambda\mu} M^{\rho\nu} + \eta^{\rho\nu} M^{\lambda\mu} - \eta^{\rho\mu} M^{\lambda\nu} - \eta^{\lambda\nu} M^{\rho\mu} \right)$$

and we have checked the first one. Let us check for instance that

$$[\, M^{23},\ M^{34} \,] = i\, M^{42} \qquad (4)$$

Since ℓ^{ij}, $\hat{L}^{ij}$, K_0^{ij} commute among themselves each one must satisfy the last equation. Taking into account that

$$[AB, CD] = AC\,[B, D] + A\,[B, C]\,D + C\,[A, D]\,B + [A, C]\,DB$$

$$= -AC\,\{B, D\} + A\,\{B, C\}\,D - C\,\{A, D\}\,B + \{A, C\}\,DB \qquad (5)$$

and that

$$[\, q^i,\ p^j \,] = i\,\delta_{ij} \tag{1}$$

we obtain

$$[\, p^{23},\ p^{34} \,] = [\, q^2 p^3,\ q^3 p^4 \,] - [\, q^2 p^3,\ q^4 p^3 \,] - [\, q^3 p^2,\ q^3 p^4 \,] + [\, q^3 p^2,\ q^4 p^3 \,] =$$

$$= q^2 [\, p^3,\ q^3 \,] p^4 + q^4 [\, q^3,\ p^3 \,] p^2 = i\,(q^4 p^2 - q^2 p^4) = i\, p^{42} \tag{2}$$

$$[\, \hat{L}^{23},\ \hat{L}^{34} \,] = -\frac{1}{4\alpha'^2} \sum_{m=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m\,m} \Big\{ [\, \alpha^2_{-m}\, \alpha^3_m,\ \alpha^3_{-m}\, \alpha^4_m \,] - [\, \alpha^2_{-m}\, \alpha^3_m,\ \alpha^4_{-m}\, \alpha^3_m \,]$$

$$+ [\, \alpha^3_{-m}\, \alpha^2_m,\ \alpha^3_{-m}\, \alpha^4_m \,] + [\, \alpha^3_{-m}\, \alpha^2_m,\ \alpha^4_{-m}\, \alpha^3_m \,] \Big\} =$$

$$= -\frac{1}{4\alpha'^2} \sum_{m=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m\,m} \Big\{ \alpha^2_{-m}\, [\, \alpha^3_m,\ \alpha^3_{-m} \,]\, \alpha^4_m + \alpha^4_{-m}\, [\, \alpha^3_{-m},\ \alpha^3_m \,]\, \alpha^2_m \Big\} =$$

$$= -\frac{1}{2\alpha'} \sum_{m=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m\,m} \Big\{ m\,\delta_{m,m}\, \alpha^2_{-m}\, \alpha^4_m - m\,\delta_{m,m}\, \alpha^4_{-m}\, \alpha^2_m \Big\} =$$

$$= -\frac{1}{2\alpha'} \sum_{m=1}^{\infty} \frac{1}{m} [\, \alpha^2_{-m}\, \alpha^4_m - \alpha^4_{-m}\, \alpha^2_m \,] = i\, \hat{L}^{42} \tag{3}$$

$$[\, K_0^{23},\ K_0^{34} \,] = -\frac{1}{64} \sum_{m=-\infty}^{\infty} \sum_{m=-\infty}^{+\infty} (\gamma^0\gamma_-\gamma^2\gamma^3)^{ab} (\gamma^0\gamma_-\gamma^3\gamma^4)^{cd} [\, S^a_{-m}\, S^b_m,\ S^c_{-m}\, S^d_m \,]$$

$$= -\frac{1}{64} (\gamma^0\gamma_-\gamma^2\gamma^3)^{cb} (\gamma^0\gamma_-\gamma^3\gamma^4)^{cd} \sum_{m=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \Big\{ -S^a_{-m}\, S^c_{-m} \{\, S^b_m,\ S^d_m \,] +$$

$$+ S^a_{-m} \{\, S^b_m,\ S^c_{-m} \}\, S^d_m - S^c_{-m} \{\, S^a_{-m},\ S^d_m \}\, S^b_m + \{\, S^a_{-m},\ S^c_{-m} \}\, S^d_m\, S^b_m \Big\} =$$

$$= -\frac{1}{64} (\gamma^0\gamma_-\gamma^2\gamma^3)^{ab} (\gamma^0\gamma_-\gamma^3\gamma^4)^{cd} \sum_{m=-\infty}^{+\infty} \Big\{ -[\, \gamma_+ h \gamma^0 \,]^{bd}\, S^a_{-m}\, S^c_m +$$

$$+ [\, \gamma_+ h \gamma^0 \,]^{bc}\, S^a_{-m}\, S^d_m - [\, \gamma_+ h \gamma^0 \,]^{ad}\, S^c_{-m}\, S^b_m + [\, \gamma_+ h \gamma^0 \,]^{ac}\, S^d_{-m}\, S^b_m \Big\} =$$

$$= -\frac{1}{64} \sum_{m=-\infty}^{+\infty} \bar{S}_{-m} \Big[-\gamma^0\gamma^0\gamma_-\gamma^2\gamma^3\gamma_+ h \gamma^0 \gamma^{4T} \gamma^{3T} \gamma_-^T \gamma^{0T} + \gamma^0\gamma^0\gamma_-\gamma^2\gamma^3\gamma_+ h \gamma^0\gamma^0\gamma_-\gamma^3\gamma^4$$

$$- \gamma^0\gamma^0\gamma_-\gamma^3\gamma^4\gamma^{0T} h \gamma_+^T \gamma^0\gamma_-\gamma^2\gamma^3 + \gamma^0\gamma^{4T}\gamma^{3T}\gamma_-^T\gamma^{0T}\gamma^{0T} h \gamma_+^T \gamma^0\gamma_-\gamma^2\gamma^3 \Big] S_m =$$

$$= -\frac{1}{64} \sum_{m=-\infty}^{+\infty} \bar{S}_{-m} \{ -2\gamma_-\gamma^2\gamma^4 - 2\gamma_-\gamma^2\gamma^4 - 2\gamma_-\gamma^2\gamma^4 - 2\gamma_-\gamma^2\gamma^4 \} S_m = \frac{1}{8} \sum_{m=-\infty}^{+\infty} \bar{S}_{-m}\, \gamma_-\gamma^2\gamma^4\, S_m$$

$$= \frac{i}{8} \sum_{m=-\infty}^{+\infty} \bar{S}_{-m}\, \gamma_-\, \sigma^{42}\, S_m = i\, K_0^{42} \tag{4}$$

For the generators of the proper Lorentz group we take

$$M^{ij} = L^{ij} + K_0^{ij} \quad , \quad M_{+-} = \ell_{+-}$$

$$M^{i}_{-} = L^{i}_{-} + K^{i}_{-} \quad , \quad M^{i}_{+} = \ell^{i}_{+}$$

$$L^{ij} = \ell^{ij} - \frac{i}{2\alpha'} \sum_{m=1}^{\infty} \frac{1}{m} \left[\alpha^{i}_{-m} \alpha^{j}_{m} - \alpha^{j}_{-m} \alpha^{i}_{m} \right] \equiv \ell^{ij} + \hat{L}^{ij}$$

$$L^{i}_{-} = \ell^{i}_{-} - \frac{i}{2\alpha' p_+} \sum_{m=1}^{\infty} \frac{1}{m} \left[\alpha^{i}_{-m} \alpha_{m,-} - \alpha_{-m,-} \alpha^{i}_{m} \right] \equiv \ell^{i}_{-} + \hat{L}^{i}_{-} \qquad (1)$$

$$K^{i}_{-} = \frac{1}{2\alpha' p_+} \sum_{m=-\infty}^{+\infty} K^{ij}_{-m} \alpha^{j}_{m}$$

$$\ell^{\mu\nu} = q^{\mu} p^{\nu} - q^{\nu} p^{\mu}$$

$$K^{ij}_{m} = \frac{1}{8} \sum_{m=-\infty}^{+\infty} \left[\bar{S}_{n-m} \gamma_{-} \sigma^{ij} S_{m} \right]$$

$$\sigma^{\mu\nu} = \frac{i}{2} \left[\gamma^{\mu} , \gamma^{\nu} \right]$$

and now it is necessary to check the relations (§2.3). We have already check the first one. Let us consider the second.

$$\left[M^{ij} , p^{k} \right] = \left[\ell^{ij} , p^{k} \right] = \left[q^{i} p^{j} - q^{j} p^{i} , p^{k} \right] =$$

$$= \left[q^{i} , p^{k} \right] p^{j} - \left[q^{j} , p^{k} \right] p^{i} = i \delta^{ik} p^{j} - i \delta^{jk} p^{i} \qquad (2)$$

$$\left[M^{ij} , p_{+} \right] = \left[\ell^{ij} , p_{+} \right] = 0 \qquad (3)$$

$$\left[M^{ij} , p_{-} \right] = \left[\ell^{ij} , p_{-} \right] + \left[\hat{L}^{ij} , p_{-} \right] + \left[K^{ij}_{0} , p_{-} \right] \qquad (4)$$

$$\left[\ell^{ij} , p_{-} \right] = \frac{1}{2\alpha'} \left[q^{i} \alpha^{j}_{0} - q^{j} \alpha^{i}_{0} , p_{-} \right] = 0$$

$$\left[\hat{L}^{ij} , p_{-} \right] = - \frac{i}{16 \alpha'^{3} p_{+}} \sum_{m=-\infty}^{+\infty} \sum_{m=1}^{\infty} \frac{1}{m} \left[\alpha^{i}_{-m} \alpha^{j}_{m} - \alpha^{j}_{-m} \alpha^{i}_{m} , \alpha^{k}_{-m} \alpha^{k}_{m} \right] =$$

$$= - \frac{i}{16 \alpha'^{3} p_{+}} \sum_{m=-\infty}^{+\infty} \sum_{m=1}^{\infty} \frac{1}{m} \left\{ \alpha^{i}_{-m} \alpha^{k}_{-m} \left[\alpha^{j}_{m} , \alpha^{k}_{m} \right] + \alpha^{i}_{-m} \left[\alpha^{j}_{m} , \alpha^{k}_{-m} \right] \alpha^{k}_{m} \right.$$

$$\left. + \alpha^{k}_{-m} \left[\alpha^{i}_{-m} , \alpha^{k}_{m} \right] \alpha^{j}_{m} + \left[\alpha^{i}_{-m} , \alpha^{k}_{-m} \right] \alpha^{k}_{m} \alpha^{j}_{m} - (i \rightarrow j) \right\} =$$

$$= -\frac{i}{8\alpha' P_+}\sum_{m=1}^{\infty}\left\{\alpha^i_{-m}\alpha^j_m + \alpha^i_{-m}\alpha^j_m - \alpha^i_{-m}\alpha^j_m - \alpha^i_{-m}\alpha^j_m - (i\leftrightarrow j)\right\} = 0 \qquad (1)$$

$$[K^{ij}_0, P_-] = \frac{1}{64\alpha' P_+}\sum_{m=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty}[\bar S_{-m}\gamma_-\sigma^{ij}S_m, m\bar S_{-m}\gamma_- S_m] =$$

$$= \frac{1}{64\alpha' P_+}[\gamma^0\gamma_-\sigma^{ij}]^{ab}[\gamma^0\gamma_-]^{cd}\sum_{m=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty} m[S^a_{-m}S^b_m, S^c_{-m}S^d_m] =$$

$$= \frac{i}{64\alpha' P_+}[\gamma^0\gamma_-\gamma^i\gamma^j]^{ab}[\gamma^0\gamma_-]^{cd}\sum_{m=-\infty}^{+\infty}\sum_{m=-\infty}^{+\infty} m\{-S^a_{-m}S^c_{-m}\{S^b_m, S^d_m\}$$

$$+ S^a_{-m}\{S^b_m, S^c_{-m}\}S^d_m - S^c_{-m}\{S^a_{-m}, S^d_m\}S^b_m + \{S^a_{-m}, S^c_{-m}\}S^d_m S^b_m\} =$$

$$= \frac{i}{64\alpha' P_+}[\gamma^0\gamma_-\gamma^i\gamma^j]^{cb}[\gamma^0\gamma_-]^{cd}\sum_{m=-\infty}^{+\infty}\{m[\gamma_+ h\gamma^0]^{bd}S^a_{-m}S^c_m$$

$$+ m[\gamma_+ h\gamma^0]^{bc}S^a_{-m}S^d_m - m[\gamma_+ h\gamma^0]^{cd}S^c_{-m}S^b_m + m[\gamma_+ h\gamma^0]^{ac}S^d_{-m}S^b_m\} =$$

$$= \frac{i}{64\alpha' P_+}\sum_{m=-\infty}^{+\infty} m\bar S_{-m}[\gamma^0\gamma^0\gamma_-\gamma^i\gamma^j\gamma_+ h\gamma^0\gamma_-^T\gamma_0^T + \gamma^0\gamma^0\gamma_-\gamma^i\gamma^j\gamma_+ h\gamma^0\gamma^0\gamma_-$$

$$- \gamma^0\gamma^0\gamma_-\gamma^{0T}h\gamma_+^T\gamma^0\gamma_-\gamma^i\gamma^j - \gamma^0\gamma_-^T\gamma^{0T}\gamma^{0T}h\gamma_+^T\gamma^0\gamma_-\gamma^i\gamma^j]S_m =$$

$$= \frac{i}{64\alpha' P_+}\sum_{m=-\infty}^{+\infty} m\bar S_{-m}[2\gamma_-\gamma^i\gamma^j + 2\gamma_-\gamma^i\gamma^j - 2\gamma_-\gamma^i\gamma^j - 2\gamma_-\gamma^i\gamma^j] = 0 \qquad (1)$$

Hence

$$[M^{ij}, P^k] = -i(\eta^{ik}P^j - \eta^{jk}P^i) \qquad (2)$$

$$[M^{ij}, P_+] = [M^{ij}, P_-] = 0$$

in agreement with (12.3).

Let us now consider the commutators of the $M^{\mu\nu}$ with the charges (2.5), (14.1)

$$[M^{ij}, Q^a] = [\ell^{ij} + \hat L^{ij} + K_0^{ij}, Q_1^a + Q_2^a] \qquad (3)$$

Then

$$[\ell^{ij}, Q^a] = -\frac{1}{2\alpha' P_+}\{[\gamma^j S_0]^a\gamma^i - [\gamma^i S_0]^a\gamma^j\} \qquad (4)$$

Furthermore

$$[\hat{L}_0^{ij}, Q_1^a + Q_2^a] = [\hat{L}_0^{ij}, Q_2^a] =$$

$$= -\frac{1}{4\alpha'^2\sqrt{P_+}} \sum_{m=1}^{\infty} \sum_{m=-\infty}^{+\infty} \frac{1}{m} [\alpha_{-m}^i \alpha_m^j - \alpha_{-m}^j \alpha_m^i, [\gamma^k S_{-m}]^a \alpha_m^k]$$

$$= -\frac{1}{4\alpha'^2\sqrt{P_+}} \sum_{m=1}^{\infty} \sum_{m=-\infty}^{+\infty} \frac{1}{m} [\gamma^k S_{-m}]^a \{ \alpha_{-m}^i [\alpha_m^j, \alpha_m^k] + [\alpha_{-m}^i, \alpha_m^k] \alpha_m^j -$$

$$- \alpha_{-m}^j [\alpha_m^i, \alpha_m^k] - [\alpha_{-m}^j, \alpha_m^k] \alpha_m^i \} =$$

$$= -\frac{1}{2\alpha'\sqrt{P_+}} \sum_{m=1}^{\infty} \{ [\gamma^j S_m]^a \alpha_{-m}^i - [\gamma^i S_{-m}]^a \alpha_m^j - [\gamma^i S_m]^a \alpha_{-m}^j + [\gamma^j S_{-m}]^a \alpha_m^i \}$$

$$= -\frac{1}{2\alpha'\sqrt{P_+}} \sum_{\substack{m=-\infty \\ m\neq 0}}^{+\infty} \{ [\gamma^j S_{-m}]^a \alpha_m^i - [\gamma^i S_{-m}]^a \alpha_m^j \} \qquad (1)$$

$$[K_0^{ij}, Q_1^a] = \frac{i}{8}\sqrt{P_+} \sum_{m=-\infty}^{+\infty} [\bar{S}_{-m} \gamma_- \sigma^{ij} S_m, [\gamma_- S_0]^a] =$$

$$= \frac{i}{8}\sqrt{P_+} [\gamma^0\gamma_- \sigma^{ij}]^{cd} [\gamma_-]^{ab} \sum_{m=-\infty}^{+\infty} [S_{-m}^c S_m^d, S_0^b] =$$

$$= \frac{i}{8}\sqrt{P_+} [\gamma^0\gamma_- \sigma^{ij}]^{cd} [\gamma_-]^{cb} \sum_{m=-\infty}^{+\infty} \{ [\gamma_+ h \gamma^0]^{db} \delta_{m,0} S_0^c - [\gamma_+ h \gamma^0]^{cb} \delta_{m0} S_0^d \}$$

$$= \frac{i}{8}\sqrt{P_+} \{ [\gamma_- \gamma^{0T} h \gamma_+^T (\sigma^{ij})^T \gamma_-^T \gamma^{0T} S_0]^a - [\gamma_- \gamma^{0T} h \gamma_+^T \gamma^0 \gamma_- \sigma^{ij} S_0]^a \}$$

$$= \frac{i}{8}\sqrt{P_+} \{ -[\gamma_- \gamma_+ \sigma^{ij} \gamma_- S_0]^a - [\gamma_- \gamma_+ \gamma_- \sigma^{ij} S_0]^a \}$$

$$= -\frac{i}{2}\sqrt{P_+} [\sigma^{ij} \gamma_- S_0]^a = -\frac{1}{2} [\sigma^{ij} Q_1]^a \qquad (2)$$

$$[K_0^{ij}, Q_2^a] = -\frac{i}{16\alpha'\sqrt{P_+}} \sum_{m=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} [\bar{S}_{-m} \gamma_- \sigma^{ij} S_m, [\gamma^k S_{-m}]^a \alpha_m^k]$$

$$= -\frac{i}{16\alpha'\sqrt{P_+}} [\gamma^0\gamma_- \sigma^{ij}]^{cd} [\gamma^k]^{ab} \sum_{m=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} [S_{-m}^c S_m^d, S_{-m}^b] \alpha_m^k =$$

$$= -\frac{i}{16\alpha'\sqrt{P_+}} [\gamma^0\gamma_- \sigma^{ij}]^{cd} [\gamma^k]^{ab} \sum_{m=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \{ S_{-m}^c [\gamma_+ h \gamma^0]^{ab} \delta_{m,m} \alpha_m^k -$$

$$- [\gamma_+ h \gamma^0]^{cb} \delta_{m,-m} S_m^d \alpha_m^k \} =$$

$$= -\frac{i}{16\alpha'\sqrt{P_+}} \sum_{m=-\infty}^{+\infty} \{ [\gamma^k \gamma^{0T} h \gamma_+^T (\sigma^{ij})^T \gamma_-^T \gamma^{0T} S_{-m}]^a \alpha_m^k$$

$$- [\gamma^k \gamma^{0T} h \gamma_+^T \gamma^0 \gamma_- \sigma^{ij} S_m]^a \alpha_{-m}^k \} =$$

$$=$$

$$= - \frac{i}{16\alpha' \sqrt{P_+}} \sum_{m=-\infty}^{+\infty} \Big\{ - 2 \, [\gamma^k \sigma^{ij} S_{-m}]^a \alpha_m^k - 2 \, [\gamma^k \sigma^{ij} S_m]^a \alpha_{-m}^k \Big\}$$

$$= + \frac{i}{4\alpha' \sqrt{P_+}} \sum_{m=-\infty}^{+\infty} [\gamma^k \sigma^{ij} S_{-m}]^a \alpha_m^k \qquad (1)$$

But

$$\gamma^k \sigma^{ij} = \sigma^{ij} \gamma^k - 2i \delta^{ki} \gamma^j + 2i \delta^{kj} \gamma^i \qquad (2)$$

and hence

$$[K_0^{ij}, Q_2^a] = \frac{i}{4\alpha' \sqrt{P_+}} \sum_{m=-\infty}^{+\infty} \Big\{ [\sigma^{ij} \gamma^k S_{-m}]^a \alpha_m^k - 2i \, [\gamma^j S_{-m}]^a \alpha_m^i + 2i \, [\gamma^i S_{-m}]^a \alpha_m^j \Big\}$$

$$= - \frac{1}{2} [\sigma^{ij} Q_2]^a + \frac{1}{4\alpha' \sqrt{P_+}} \sum_{m=-\infty}^{+\infty} \Big\{ [\gamma^j S_{-m}]^a \alpha_m^i - [\gamma^i S_{-m}]^a \alpha_m^j \Big\} \qquad (3)$$

Now using (15.4), (16.1), (16.2) and (3)

$$[M^{ij}, Q^a] = - \frac{1}{2} [\sigma^{ij} Q]^a \qquad (4)$$

Furthermore we are going to prove that

$$[M^{\mu\nu}, Q^a] = - \frac{1}{2} [\sigma^{\mu\nu} Q]^a \qquad (5)$$

Let us begin considering

$$[M_{+-}, Q^a] = [\ell_{+-}, Q^a] = [q_+ P_- - q_- P_+, Q^a] \qquad (6)$$

since

$$[q_-, P_+] = [q_+, P_-] = -i \qquad (7)$$

we get

$$[M_{+-}, Q^a] = + i P_+ \frac{\partial}{\partial P_+} Q^a = + \frac{i}{2} \left(Q_1^a - Q_2^a \right) \qquad (8)$$

But

$$[\sigma_{+-} Q_1]^a = \frac{i}{2} \{ [\gamma_+ \gamma_- - \gamma_- \gamma_+] Q_1 \}^a = - i Q_1^a \qquad (9)$$

$$[\sigma_{+-} Q_2]^a = + i Q_2^a$$

and hence

$$[M_{+-},\ Q^a] = -\frac{1}{2}\,[\sigma_{+-}\,Q]^a \tag{1}$$

Similarly

$$[M^i_{+},\ Q^a] = [\ell^i_{+},\ Q^a] = [q^i p_+ - q_+ p^i,\ Q^a] = i\,p_+\frac{\partial}{\partial p^i}\,Q^a$$

$$= 2\alpha' i\,p_+\frac{\partial}{\partial \alpha_0^i}\,Q^a = \sqrt{p_+}\,[\gamma^i S_0]^a \tag{2}$$

but

$$[\sigma^i_{+}\,Q_1]^a = \frac{i}{2}\,[(\gamma^i\gamma_+ - \gamma_+\gamma^i)\,Q_1]^a = -2\sqrt{p_+}\,[\gamma^i S_0]^a \tag{3}$$

$$[\sigma^i_{+}\,Q_2]^a = 0$$

and therefore

$$[M^i_{+},\ Q^a] = -\frac{1}{2}\,[\sigma^i_{+}\,Q]^a \tag{4}$$

Now we must calculate $[M^i_{-},\ Q^a]$. We need some previous results.

$$[q^i p_- - q_- p^i,\ Q_1^a] = -p^i\frac{\partial}{\partial p_+}\sqrt{p_+}\,[\gamma_- S_0]^a = -\frac{1}{2}\frac{1}{\sqrt{p_+}}\,p^i[\gamma_- S_0]^a \Rightarrow$$

$$[\ell^i_{-},\ Q_1^a] = -\frac{1}{4\alpha'\sqrt{p_+}}\,[\gamma_- S_0]^a\,\alpha_0^i \tag{5}$$

Furthermore

$$[\alpha_{m,-},\ S_m^a] = \frac{1}{4}\sum_{k=-\infty}^{+\infty}(k-\frac{m}{2})\,[\bar S_{m-k}\gamma_- S_k,\ S_m^a] =$$

$$= \frac{1}{4}\,(\gamma^0\gamma_-)^{bc}\sum_{k=-\infty}^{+\infty}(k-\frac{m}{2})\,[S^b_{m-k}S^c_k,\ S^a_m] =$$

$$= \frac{1}{4}\,(\gamma^0\gamma_-)^{bc}\sum_{k=-\infty}^{+\infty}(k-\frac{m}{2})\,\{S^b_{m-k}\,[\gamma_+ h^{(1)}_0]^{ca}\,\delta_{k+m,0} - S^c_k\,[\gamma_+ h^{(1)}_0]^{ba}\,\delta_{m-k+m,0}\}$$

$$= \frac{1}{4}\,\{[\gamma_0^T h\,\gamma_+^T\gamma_-^T\gamma^{0T}S_{m+m}]^a(-m-\tfrac{m}{2}) - [\gamma^{0T}h\,\gamma_+^T\gamma^0\gamma_- S_{m+m}]^a(m+m-\tfrac{m}{2})\}$$

$$= \frac{1}{4}\,\{(-m-\tfrac{m}{2})\,2\,S^a_{m+m} - (m+\tfrac{m}{2})\,2\,S^a_{m+m}\} \Rightarrow$$

$$[\alpha_{m-},\ S_m^a] = -(m+\frac{m}{2})\,S^a_{m+m} \tag{6}$$

$$\left[-\frac{i}{2\alpha' P_+} \sum_{m=1}^{\infty} \frac{1}{m} \left[\alpha^i_{-m} \alpha_{m,-} - \alpha_{-m,-} \alpha^i_m \right] , Q_1^a \right] =$$

$$= \frac{1}{2\alpha' \sqrt{P_+}} \sum_{m=1}^{\infty} \frac{1}{m} \left[\left[\alpha^i_{-m} \alpha_{m-} - \alpha_{-m,-} \alpha^i_m \right] , (\gamma_-)^{ab} S_0^b \right] =$$

$$= \frac{1}{2\alpha' \sqrt{P_+}} (\gamma_-)^{ab} \sum_{m=1}^{\infty} \frac{1}{m} \left\{ \alpha^i_{-m} \left[\alpha_{m,-} , S_0^b \right] - \alpha^i_m \left[\alpha_{-m,-} , S_0^b \right] \right\}$$

$$= \frac{1}{2\alpha' \sqrt{P_+}} (\gamma_-)^{ab} \sum_{m=1}^{\infty} \frac{1}{m} \left\{ - \alpha^i_{-m} \frac{m}{2} S_m^b - \alpha^i_m \frac{m}{2} S^b_{-m} \right\} =$$

$$= - \frac{1}{4\alpha' \sqrt{P_+}} (\gamma_-)^{ab} \sum_{m=1}^{\infty} \left[\alpha^i_{-m} S_m^b + \alpha^i_m S^b_{-m} \right] \Rightarrow$$

$$\left[-\frac{i}{2\alpha' P_+} \sum_{m=1}^{\infty} \frac{1}{m} \left[\alpha^i_{-m} \alpha_{m,-} - \alpha_{-m,-} \alpha^i_m \right] , Q_1^a \right] = - \frac{1}{4\alpha' \sqrt{P_+}} \sum_{\substack{m=-\infty \\ m \neq 0}}^{+\infty} \left[\gamma_- S_m \right]^a \alpha^i_{-m} \qquad (1)$$

Also

$$\left[K^{ij}_{-m} , S_0^a \right] = \frac{1}{8} \sum_{m=-\infty}^{+\infty} \left[\bar{S}_{-m-m} \gamma_- \sigma^{ij} S_m , S_0^a \right] =$$

$$= \frac{1}{8} (\gamma^0 \gamma_- \sigma^{ij})^{bc} \sum_{m=-\infty}^{+\infty} \left[S^b_{-m-m} S^c_m , S_0^a \right] =$$

$$= \frac{1}{8} (\gamma^0 \gamma_- \sigma^{ij})^{bc} \sum_{m=-\infty}^{+\infty} \left\{ S^b_{-m-m} \left[\gamma_+ h \gamma^0 \right]^{ca} \delta_{m,0} - \left[\gamma_+ h \gamma^0 \right]^{ba} \delta_{m+m,0} S^c_m \right\}$$

$$= \frac{1}{8} \left\{ \left[\gamma^{0T} h \gamma_+^T (\sigma^{ij})^T \gamma_-^T \gamma^{0T} S_{-m} \right]^a - \left[\gamma^{0T} h \gamma_+^T \gamma^0 \gamma_- \sigma^{ij} S_{-m} \right]^a \right\}$$

$$= \frac{1}{8} \left\{ - 2 \left[(\sigma^{ij}) S_{-m} \right]^a - 2 \left[(\sigma^{ij}) S_{-m} \right]^a \right\} \Rightarrow$$

$$\left[K^{ij}_{-m} , S_0^a \right] = - \frac{1}{2} \left[\sigma^{ij} S_{-m} \right]^a \qquad (2)$$

and

$$\left[\frac{1}{2\alpha' P_+} \sum_{m=-\infty}^{+\infty} K^{ij}_{-m} \alpha^j_m , Q_1^a \right] = \frac{i}{2\alpha' \sqrt{P_+}} (\gamma_-)^{ab} \sum_{m=-\infty}^{+\infty} \left[K^{ij}_{-m} \alpha^j_m , S_0^b \right] =$$

$$= - \frac{i}{4\alpha' \sqrt{P_+}} (\gamma_-)^{ab} \sum_{m=-\infty}^{+\infty} \left[\sigma^{ij} S_{-m} \right]^b \alpha^j_m \Rightarrow$$

$$\left[\frac{1}{2\alpha' P_+} \sum_{m=-\infty}^{+\infty} K^{ij}_{-m} \alpha^j_m , Q_1^a \right] = - \frac{i}{4\alpha' \sqrt{P_+}} \sum_{m=-\infty}^{+\infty} \left[\gamma_- \sigma^{ij} S_{-m} \right]^a \alpha^j_m \qquad (3)$$

Then

$$[M^i_-, Q^a_1] = -\frac{1}{4\alpha'\sqrt{P_+}} [\gamma_- S_0]^a \alpha_0^i - \frac{1}{4\alpha'\sqrt{P_+}} \sum_{\substack{m=-\infty \\ m\neq 0}}^{+\infty} [\gamma_- S_{-m}]^a \alpha^i_m$$

$$- \frac{i}{4\alpha'\sqrt{P_+}} \sum_{m=-\infty}^{+\infty} [\gamma_- \sigma^{ij} S_{-m}]^a \alpha_m^j =$$

$$= -\frac{1}{4\alpha'\sqrt{P_+}} \sum_{m=-\infty}^{+\infty} \left\{ [\gamma_- S_{-m}]^a \alpha_m^i + i [\gamma_- \sigma^{ij} S_{-m}]^a \alpha_m^j \right\}$$

$$= +\frac{1}{4\alpha'\sqrt{P_+}} \sum_{m=-\infty}^{+\infty} [\gamma_- \gamma^i \gamma^j S_{-m}]^a \alpha_m^j \quad \Rightarrow$$

$$[M^i_-, Q^a_1] = \frac{1}{4\alpha'\sqrt{P_+}} \sum_{m=-\infty}^{+\infty} [\gamma_- \gamma^i \gamma^j S_{-m}]^a \alpha_m^j \tag{1}$$

Notice

$$[\sigma^i_- Q]^a = \frac{i}{2} \left\{ [\gamma^i \gamma_- Q]^a - [\gamma_- \gamma^i Q]^a \right\} =$$

$$= \frac{1}{4\alpha'\sqrt{P_+}} \sum_{m=-\infty}^{+\infty} \left\{ [\gamma^i \gamma_- \gamma^j S_{-m}]^a \alpha_m^j - [\gamma_- \gamma^i \gamma^j S_{-m}]^a \alpha_m^j \right\} =$$

$$= -\frac{1}{2\alpha'\sqrt{P_+}} \sum_{m=-\infty}^{+\infty} [\gamma_- \gamma^i \gamma^j S_{-m}]^a \alpha_m^j \tag{2}$$

The comparison of (1) and (2) gives

$$[M^i_-, Q^a_1] = -\frac{1}{2} [\sigma^i_- Q]^a \tag{3}$$

In order to prove the desired result

$$[M^i_-, Q^a] = -\frac{1}{2} [\sigma^i_- Q]^a \tag{4}$$

we must prove that

$$[M^i_-, Q^a_2] = 0 \tag{5}$$

In order to prove (5) let us proceed in several steps

$$[\ell^i_-, Q^a_2] = [q^i \ell_- - q_- p^i, Q^a_2] = \left(\ell_- i \frac{\partial}{\partial \ell_i} + i p^i \frac{\partial}{\partial \ell_+} \right) \frac{-i}{2\alpha'\sqrt{P_+}} \sum_{m=-\infty}^{+\infty} [\gamma^j S_{-m}]^a \alpha_m^j$$

$$= \frac{1}{\sqrt{P_+}} \ell_- [\gamma^i S_0]^a - \frac{1}{8\alpha'^2} \frac{1}{P_+^{3/2}} \alpha_0^i \sum_{m=-\infty}^{+\infty} [\gamma^j S_{-m}]^a \alpha_m^j \tag{6}$$

Hence

$$\left[\ell^i_-, Q_2^a \right] = \frac{1}{2\alpha' p_+^{3/2}} \alpha_{0,-} \left[\gamma^i S_0 \right]^a - \frac{1}{8\alpha'^2 p_+^{3/2}} \alpha_0^i \sum_{m=-\infty}^{+\infty} \left[\gamma^j S_{-m} \right]^a \alpha_m^j \qquad (1)$$

Furthermore

$$\left[-\frac{i}{2\alpha' p_+} \sum_{m=1}^{\infty} \frac{1}{m} \left[\alpha_{-m}^i \alpha_{m,-} - \alpha_{-m,-} \alpha_m^i \right], Q_2^a \right] =$$

$$= -\frac{1}{4\alpha'^2 p_+^{3/2}} (\gamma^j)^{ab} \sum_{m=1}^{\infty} \sum_{m=-\infty}^{+\infty} \frac{1}{m} \left[\alpha_{-m}^i \alpha_{m,-} - \alpha_{-m,-} \alpha_m^i, S_{-m}^b \alpha_m^j \right] =$$

$$= -\frac{1}{4\alpha'^2 p_+^{3/2}} (\gamma^j)^{ab} \sum_{m=1}^{\infty} \sum_{m=-\infty}^{+\infty} \frac{1}{m} \Big\{ \alpha_{-m}^i S_{-m}^b \left[\alpha_{m,-}, \alpha_m^j \right] + \alpha_{-m}^i \left[\alpha_{m,-}, S_{-m}^b \right] \alpha_m^j +$$

$$+ S_{-m}^b \left[\alpha_{-m}^i, \alpha_m^j \right] \alpha_{m,-} - \alpha_{-m,-} S_{-m}^b \left[\alpha_m^i, \alpha_m^j \right] - S_{-m}^b \left[\alpha_{-m,-}, \alpha_m^j \right] \alpha_m^i - \left[\alpha_{-m,-}, S_{-m}^b \right] \alpha_m^j \alpha_m^i \Big\}$$

$$= -\frac{1}{4\alpha'^2 p_+^{3/2}} (\gamma^j)^{ab} \sum_{m=1}^{\infty} \sum_{m=-\infty}^{+\infty} \frac{1}{m} \Big\{ -m \alpha_{-m}^i S_{-m}^b \alpha_{m+m}^j + \left(m - \frac{m}{2} \right) \alpha_{-m}^i S_{m-m}^b \alpha_m^j$$

$$- 2\alpha' m \delta_{ij} \delta_{m,m} S_{-m}^b \alpha_{m-} - 2\alpha' m \delta_{ij} \delta_{m,-m} \alpha_{-m,-} S_m^b + m S_{-m}^b \alpha_{m-m}^j \alpha_m^i$$

$$+ \left(m + \frac{m}{2} \right) S_{-m-m}^b \alpha_m^j \alpha_m^i \Big\} =$$

$$= \frac{1}{2\alpha' p_+^{3/2}} (\gamma^i)^{ab} \sum_{m=1}^{\infty} \left[S_{-m}^b \alpha_{m,-} + \alpha_{-m,-} S_m^b \right] - \frac{1}{4\alpha'^2 p_+^{3/2}} (\gamma^j)^{ab} \sum_{m=1}^{\infty} \sum_{m=-\infty}^{+\infty} \frac{1}{m}$$

$$\cdot \Big\{ -m \alpha_{-m}^i S_{-m}^b \alpha_{m+m}^j + \left(m - \frac{m}{2} \right) \alpha_{-m}^i S_{m-m}^b \alpha_m^j + m S_{-m}^b \alpha_m^i \alpha_{m-m}^j$$

$$+ \left(m + \frac{m}{2} \right) S_{-m-m}^b \alpha_m^i \alpha_m^j - m^2 \alpha' \delta_{ij} S_0^b \delta_{m,-m} \Big\} =$$

$$= \frac{1}{2\alpha' p_+^{3/2}} (\gamma^i)^{ab} \sum_{m=1}^{\infty} \left[\alpha_{m,-} S_{-m}^b + \alpha_{-m,-} S_m^b \right] - \frac{1}{4\alpha' p_+^{3/2}} \sum_{m=1}^{\infty} m^2 \left[\gamma^i S_0 \right]^a$$

$$- \frac{1}{4\alpha'^2 p_+^{3/2}} (\gamma^j)^{ab} \sum_{m=1}^{\infty} \sum_{m=-\infty}^{+\infty} \frac{1}{m} \Big\{ -m \alpha_{-m}^i S_{-m}^b \alpha_{m+m}^j + m \alpha_m^i S_{-m}^b \alpha_{m-m}^j$$

$$+ \left(m + \frac{m}{2} \right) \alpha_{-m}^i S_{-m}^b \alpha_{m+m}^j - \left(m - \frac{m}{2} \right) \alpha_m^i S_{-m}^b \alpha_{m-m}^j \Big\} + \frac{1}{4\alpha' p_+^{3/2}} \sum_{m=1}^{\infty} m^2 \left[\gamma^i S_0 \right]^a$$

$$= \frac{1}{2\alpha' p_+^{3/2}} \sum_{\substack{m=-\infty \\ m \neq 0}}^{+\infty} \alpha_{m,-} \left[\gamma^i S_{-m} \right]^a - \frac{1}{8\alpha'^2 p_+^{3/2}} \sum_{\substack{m=-\infty \\ m \neq 0}}^{+\infty} \sum_{m=-\infty}^{+\infty} \alpha_{-m}^i \left[\gamma^j S_{-m} \right]^a \alpha_{m+m}^j \qquad (2)$$

and from (1) and (2)

$$[L^i_{-}, Q^a_2] = \frac{1}{2\alpha' p_+^{3/2}} \sum_{m=-\infty}^{+\infty} \alpha_{m-} [\gamma^i S_{-m}]^a - \frac{1}{8\alpha'^2 p_+^{3/2}} \sum_{m=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \alpha^i_{-m} [\gamma^j S_{-m}]^a \alpha^j_{m+m} \quad (1)$$

Let us now consider

$$[K^i_{-}, Q^a_2] = -\frac{i}{4\alpha'^2 p_+^{3/2}} (\gamma^k)^{ab} \sum_{m=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} [K^{ij}_{-m} \alpha^j_m, S^b_{-m} \alpha^k_m]$$

$$= -\frac{i}{4\alpha'^2 p_+^{3/2}} (\gamma^k)^{ab} \sum_{m=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \{ K^{ij}_{-m} S^b_{-m} [\alpha^j_m, \alpha^k_m] + [K^{ij}_{-m}, S^b_{-m}] \alpha^k_m \alpha^j_m \} \quad (2)$$

but

$$[K^{ij}_m, S^a_m] = \frac{1}{8} \sum_{k=-\infty}^{+\infty} [\gamma^0 \gamma_- \sigma^{ij}]^{bc} [S^b_{m-k} S^c_k, S^a_m] =$$

$$= \frac{1}{8} [\gamma^0 \gamma_- \sigma^{ij}]^{bc} \sum_{k=-\infty}^{+\infty} \{ S^b_{m-k} \} S^c_k, S^a_m \} - \{ S^b_{m-k}, S^a_m \} S^c_k \} =$$

$$= \frac{1}{8} [\gamma^0 \gamma_- \sigma^{ij}]^{bc} \sum_{k=-\infty}^{+\infty} \{ S^b_{m-k} [\gamma_+ h \gamma^0]^{ca} \delta_{m+k,0} - S^c_k [\gamma_+ h \gamma^0]^{ba} \delta_{m+m-k,0} \} =$$

$$= \frac{1}{8} \{ [\gamma^{0T} h \gamma_+^T (\sigma^{ij})^T \gamma_-^T \gamma^{0T} S_{m+m}]^a - [\gamma^{0T} h \gamma_+^T \gamma^0 \gamma_- \sigma^{ij} S_{m+m}]^a \} =$$

$$= \frac{1}{8} \{ -2 [\sigma^{ij} S_{m+m}]^a - 2 [\sigma^{ij} S_{m+m}]^a \} \Rightarrow$$

$$[K^{ij}_m, S^a_m] = -\frac{1}{2} [\sigma^{ij} S_{m+m}]^a \quad (3)$$

Now from (2)

$$[K^i_{-}, Q^a_2] = -\frac{i}{2\alpha' p_+^{3/2}} \sum_{m=-\infty}^{+\infty} m K^{ij}_m [\gamma^j S_{-m}]^a + \frac{i}{8\alpha'^2 p_+^{3/2}} \sum_{m=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} [\gamma^k \sigma^{ij} S_{-m-m}]^c \alpha^k_m \alpha^j_m \quad (4)$$

Then from (1) and (4)

$$[M^i_{-}, Q^a_2] = \frac{1}{8\alpha'^2 p_+^{3/2}} \sum_{m=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \{ \alpha^j_{m-m} \alpha^j_m [\gamma^i S_{-m}]^a - \alpha^i_{-m} [\gamma^j S_{-m}]^a \alpha^j_{m+m}$$

$$+ i [\gamma^k \sigma^{ij} S_{-m-m}]^a \alpha^k_m \alpha^j_m \} + \frac{1}{2\alpha' p_+^{3/2}} \sum_{m=-\infty}^{+\infty} \{ V_m [\gamma^i S_{-m}]^a - im K^{ij}_m [\gamma^j S_{-m}]^a \} \quad (5)$$

where

$$V_m \equiv \frac{1}{4} \sum_{m=-\infty}^{+\infty} (m - \frac{m}{2}) \bar{S}_{m-m} \gamma_- S_m \quad (6)$$

where we have omitted :: in both terms since both infinite parts cancel.

Let us now consider the first term in the r.h.s of (22.5)

$$A = \sum_{m=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \Big\{ \alpha^j_{n-m}\, \alpha^j_m : [\gamma^i S_{-n}]^a - \alpha^i_m\, \alpha^j_{n-m}\, [\gamma^j S_{-n}]^a + i\, \alpha^k_m\, \alpha^j_{n-m}\, [\gamma^k \sigma^{ij} S_{-n}]^a \Big\} =$$

$$= \sum_{m=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \Big\{ \alpha^j_{n-m}\, \alpha^j_m\, [\gamma^i S_{-n}]^a - \alpha^i_m\, \alpha^j_{n-m}\, [\gamma^j S_{-n}]^a + \alpha^j_m\, \alpha^i_{n-m}\, [\gamma^j S_{-n}]^a + \alpha^k_m\, \alpha^j_{n-m}\, [\gamma^k \gamma^j \gamma^i S_{-n}]^a \Big\} =$$

$$= \sum_{m=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \Big\{ \alpha^j_{n-m}\, \alpha^j_m\, [\gamma^i S_{-n}]^a - \alpha^i_m\, \alpha^j_{n-m}\, [\gamma^j S_{-n}]^a + \alpha^j_{n-m}\, \alpha^i_m\, [\gamma^j S_{-n}]^a - \alpha^j_{n-m}\, \alpha^j_m\, [\gamma^i S_{-n}]^a - i\, \alpha^k_m\, \alpha^j_{n-m}\, [\sigma^{kj} \gamma^i S_{-n}]^a \Big\} =$$

$$= \sum_{m=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \Big\{ [\alpha^j_{n-m}\, \alpha^i_m - \alpha^i_m\, \alpha^j_{n-m}]\, [\gamma^j S_{-n}]^a - \frac{i}{2} [\alpha^k_m\, \alpha^j_{n-m} - \alpha^j_{n-m}\, \alpha^k_m]\, [\sigma^{kj} \gamma^i S_{-n}]^a \Big\}$$

$$= 2\alpha' \sum_{m=-\infty}^{+\infty} \Big\{ -m\, [\gamma^j S_0]^a \Big\} = 0 \qquad (1)$$

and hence

$$[M^{i-}, Q_2^a] = \frac{1}{2\alpha' p_+^{3/2}} \sum_{m=-\infty}^{+\infty} \Big\{ V_m\, [\gamma^i S_{-m}]^a - im\, K_m^{ij}\, [\gamma^j S_{-m}]^a \Big\} \qquad (2)$$

Taking into account the expressions for V_m and K_m^{ij} this sum can be written as

$$B^a = \sum_{m=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \Big\{ \frac{1}{4} \left(m - \frac{m}{2}\right) [\bar{S}_{m-m} \gamma^- S_m]\, [\gamma^i S_{-m}]^a - \frac{i}{8} m\, [\bar{S}_{m-m} \gamma^- \sigma^{ij} S_m]\, [\gamma^j S_{-n}]^a \Big\} \qquad (3)$$

This is a complicated cubic expression in S^a_m. A direct proof that this expression is zero would require an inspired choice of Fierz transformation. A simpler approach, which is completely general, is to show that the anticommutator of this expression with S vanishes.

XIII - SPECTRUM OF THE OPEN STRING

We must first discuss the ground state of the open superstring. We need to consider equations (XI-26.6)

$$\alpha' M^2 = N \tag{1}$$

$$N = \frac{1}{2\alpha'} \sum_{m=1}^{\infty} \alpha_{-m}^{i} \alpha_m^{i} + \frac{1}{2} \sum_{m=1}^{\infty} m \left[\bar{S}_{-m} \gamma_- S_m \right]$$

being α_m^i and S_m^a the annihilation operators. The ground states are characterized by

$$\alpha_m^i |g.s.\rangle = S_m^a |g.s\rangle = 0 \qquad m = 1, 2, \ldots \tag{2}$$

From (1) it follows that

$$M^2 |g.s.\rangle = 0 \tag{3}$$

i.e. the ground state is massless, avoiding thus the tachyon nature of this state typical of the old string theory. Notice that the fact that $P^2 = 0$ does not mean that $P^\mu = 0$.

As the theory is supersymmetric this ground state should contain the states of a supermultiplet in $D = 10$ and consist of a massless vector and a massless Majorana-Weyl spinor. Let us begin by representing the eight physical transverse-polarization states of the vector by ket vectors $|i\rangle$, $i = 2, 3, \ldots, 9$ with the normalization condition

$$\langle i | j \rangle = \delta_{ij} \qquad i, j = 2, 3, \ldots, 9 \tag{4}$$

Acting on these with the operator S_0^a gives fermion states, which are also massless since

$$[S_0^a , N] = 0 \tag{5}$$

Thus we define the spinor states $|a\rangle$ by

$$|a\rangle = \frac{i}{8} [\gamma^i S_0]^a |i\rangle \qquad a = 1, 2, \ldots, 32 \tag{6}$$

where the repeated transverse index is summed as usual. Notice that an

immediate consequence is that

$$[\gamma_+]^{ab}\,|b\rangle = [h]^{ab}\,|b\rangle = 0 \tag{1}$$

and therefore we have only 8 independent degrees of freedom in the fermionic sector as expected.

Furthermore

$$S_0^a\,\bar{S}_0^b = \frac{1}{2}\left\{S_0^a, \bar{S}_0^b\right\} + \frac{1}{2}\left[S_0^a, \bar{S}_0^b\right\} =$$

$$= \frac{1}{2}\,[\gamma_+ h]^{ab} + \frac{1}{2}\left[S_0^a, \bar{S}_0^b\right] \tag{2}$$

In order to compute the second term let us consider

$$A \equiv [\bar{\lambda}_1 \gamma_-]^a\,\lambda_2^b\,\left[S_0^a \bar{S}_0^b - \bar{S}_0^b S_0^a\right] =$$

$$= [\bar{\lambda}_1 \gamma_- S_0]\,[\bar{S}_0 \lambda_2] + [\bar{S}_0 \lambda_2]\,[\bar{\lambda}_1 \gamma_- S_0] =$$

$$= -[\bar{S}_0 \gamma_- \lambda_1]\,[\bar{\lambda}_2 S_0] + [\bar{S}_0 \lambda_2]\,[\bar{\lambda}_1 \gamma_- S_0]$$

$$= \frac{1}{32}\sum_x \left\{[\bar{S}_0 \gamma_- \Gamma^{(x)} S_0]\,[\bar{\lambda}_2 \Gamma^{(x)} \lambda_1] - [\bar{S}_0 \Gamma^{(x)} \gamma_- S_0]\,[\bar{\lambda}_1 \Gamma^{(x)} \lambda_2]\right\}$$

$$= \frac{1}{32}\sum_x \left\{[\bar{S}_0 \gamma_- \Gamma^{(x)} S_0]\,[\bar{\lambda}_2 \Gamma^{(x)} \lambda_1] - [\bar{S}_0 \Gamma^{(x)} \gamma_- S_0]\,[\bar{\lambda}_2 \gamma^0 \Gamma^{(x)T} \gamma^0 \lambda_1]\right\} \tag{3}$$

Since

$$[\bar{S}_0 \gamma_- \Gamma^{(x)} S_0] = \eta^2\,[\bar{S}_0 \gamma_- \Gamma^{(x)} S_0] = -[\bar{S}_0 \gamma'' \gamma_- \Gamma^{(x)} \gamma'' S_0] \tag{4}$$

it is clear that we need to consider $\Gamma^{(x)}$ with even number of γ. Clearly the terms I and γ'' do not contribute. Hence

$$A = \frac{1}{32}\left\{\sum_{0\le\mu<\nu\le 9}\left\{[\bar{S}_0 \gamma_- \gamma^\mu \gamma^\nu S_0]\,[\bar{\lambda}_2 \gamma_\nu \gamma_\mu \lambda_1] - [\bar{S}_0 \gamma^\mu \gamma^\nu \gamma_- S_0]\,[\bar{\lambda}_2 \gamma_\mu \gamma_\nu \lambda_1]\right\} +\right.$$

$$+ \sum_{0\le\mu<\nu<\lambda<\sigma\le 9}\left\{[\bar{S}_0 \gamma_- \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma S_0]\,[\bar{\lambda}_2 \gamma_\sigma \gamma_\lambda \gamma_\nu \gamma_\mu \lambda_1] - [\bar{S}_0 \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma \gamma_- S_0]\,[\bar{\lambda}_2 \gamma_\mu \gamma_\nu \gamma_\lambda \gamma_\sigma \lambda_1]\right\} +$$

$$+ \sum_{0\le\mu<\nu<\lambda<\sigma\le 9}\left\{[\bar{S}_0 \gamma_- \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma \gamma'' S_0]\,[\bar{\lambda}_2 \gamma'' \gamma_\sigma \gamma_\lambda \gamma_\nu \gamma_\mu \lambda_1] + [\bar{S}_0 \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma \gamma'' \gamma_- S_0]\,[\bar{\lambda}_2 \gamma_\mu \gamma_\nu \gamma_\lambda \gamma_\sigma \gamma'' \lambda_1]\right\} +$$

$$\left. + \sum_{0\le\mu<\nu\le 9}\left\{[\bar{S}_0 \gamma_- \gamma^\mu \gamma^\nu \gamma'' S_0]\,[\bar{\lambda}_2 \gamma'' \gamma_\nu \gamma_\mu \lambda_1] + [\bar{S}_0 \gamma^\mu \gamma^\nu \gamma'' \gamma_- S_0]\,[\bar{\lambda}_2 \gamma_\mu \gamma_\nu \gamma'' \lambda_1]\right\}\right\} \tag{5}$$

$$A = \frac{1}{16}\Big\{ \sum_{0\le\mu<\nu\le 9} [\bar S_0 (\gamma_- \gamma^\mu \gamma^\nu + \gamma^\mu \gamma^\nu \gamma_-) S_0][\bar\lambda_2 (I-h)\gamma_\nu \gamma_\mu \lambda_1] +$$

$$+ \sum_{0\le\mu<\nu<\lambda<\sigma\le 9} [\bar S_0 (\gamma_- \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma + \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma \gamma_-) S_0][\bar\lambda_2 (I-h)\gamma_\sigma \gamma_\lambda \gamma_\nu \gamma_\mu \lambda_1]\Big\} \qquad (1)$$

And hence

$$A = \frac{1}{16}\Big\{ \sum_{2\le i<j\le 9} 2[\bar S_0 \gamma_- \gamma^i \gamma^j S_0][\bar\lambda_2 (I-h)\gamma^j \gamma^i \lambda_1] +$$

$$+ \sum_{2\le i<j\le 9} (-2)[\bar S_0 \gamma_- \gamma^i \gamma^j S_0][\bar\lambda_2 (I-h)\gamma^j \gamma^i (I - \gamma_+\gamma_-)\lambda_1] =$$

$$= \frac{1}{8} \sum_{2\le i<j\le 9} [\bar S_0 \gamma_- \gamma^i \gamma^j S_0][\bar\lambda_2 (I-h)\gamma^j \gamma^i \gamma_+ \gamma_- \lambda_1] =$$

$$= \frac{1}{8} \sum_{2\le i<j\le 9} [\bar S_0 \gamma_- \gamma^i \gamma^j S_0][\bar\lambda_1 \gamma_- \gamma_+ \gamma^i \gamma^j h \lambda_2] =$$

$$= -\frac{1}{16}[\bar S_0 \gamma_- \sigma^{ij} S_0][\bar\lambda_1 \gamma_- \gamma_+ \sigma^{ij} h \lambda_2] \qquad (2)$$

Hence

$$[\gamma_- S_0]^a \bar S_0^b - \bar S_0^b [\gamma_- S_0]^a = -\frac{1}{16}[\gamma_- \gamma_+ \sigma^{ij} h]^{ab}[\bar S_0 \gamma_- \sigma^{ij} S_0] \qquad (3)$$

and multiplying by $[\gamma_+]^{ca}$

$$[\gamma_+ \gamma_- S_0]^a \bar S_0^b - \bar S_0^b [\gamma_+ \gamma_- S_0]^a = -\frac{1}{16}[\gamma_+ \gamma_- \gamma_+ \sigma^{ij} h]^{ab}[\bar S_0 \gamma_- \sigma^{ij} S_0] \qquad (4)$$

and hence

$$S_0^a \bar S_0^b - \bar S_0^b S_0^a = -\frac{1}{16}[\gamma_+ \sigma^{ij} h]^{ab}[\bar S_0 \gamma_- \sigma^{ij} S_0] \qquad (5)$$

And using (2.2)

$$S_0^a \bar S_0^b = \frac{1}{2}[\gamma_+ h]^{ab} - \frac{1}{4}[\gamma_+ \sigma^{ij} h]^{ab} R_0^{ij} \qquad (6)$$

$$R_0^{ij} = \frac{1}{8}[\bar S_0 \gamma_- \sigma^{ij} S_0]$$

Comparing this with (XII-14.2) we see that R_0^{ij} is the generator of the transverse rotations in the massless sector. Therefore

$$[R_o^{ij}, R_o^{ke}] = i\left(\delta_{ij} R^{je} + \delta_{je} R^{ik} - \delta_{jk} R^{ie} - \delta_{ie} R^{jk}\right) \tag{1}$$

Furthermore

$$[R_o^{ij}, S_o^a] = \frac{1}{8} [\gamma^0\gamma_- \sigma^{ij}]^{bc} [S_o^b S_o^c, S_o^a]$$

$$= \frac{1}{8} [\gamma^0\gamma_- \sigma^{ij}]^{bc} \left\{ [\gamma_+ h \gamma^0]^{ca} S_o^b - [\gamma_+ h \gamma^0]^{ba} S_o^c \right\} =$$

$$= \frac{1}{8} \left\{ [\gamma^{0T} h \gamma_+^T (\sigma^{ij})^T \gamma_-^T \gamma^{0T} S_o]^a - [\gamma^{0T} h \gamma_+^T \gamma^0 \gamma_- \sigma^{ij} S_o]^a \right\} \Rightarrow$$

$$[R_o^{ij}, S_o^a] = -\frac{1}{2} [\sigma^{ij} S_o]^a \tag{2}$$

Let us denote by $R(\delta w_{ij})$ (i,j = fixed) the orthogonal matrix representing an infinitesimal active rotation in the plane (ij), then their matrix elements are

$$[R(\delta w_{ij})]_{ke} = \delta_{ke} - \delta w_{ij} \delta_{ik} \delta_{je} + \delta w_{ij} \delta_{ie} \delta_{jk} \tag{3}$$

then under this rotation vectors transform as

$$A^k \xrightarrow{R(\delta w_{ij})} A^k - \delta w_{ij} \delta_{ik} A^j + \delta w_{ij} \delta_{jk} A^i \tag{4}$$

The unitary operator implementing this transformation is

$$\hat{R}(\delta w_{ij}) = I + i \delta w_{ij} M^{ij} \tag{5}$$

and hence

$$R_o^{ij} |k\rangle = i \left\{ \delta_{ik} |j\rangle - \delta_{jk} |i\rangle \right\} \tag{6}$$

$$R_o^{ij} |a\rangle = \frac{1}{2} [\sigma^{ij}]^{ab} |b\rangle \tag{7}$$

These equations imply that the states $|i\rangle$ and $|a\rangle$ form an irreducible representation of the algebra of S_o "oscillators".

We would like to invert (1.6). Multiplying equations (3.5) and (3.6) by (γ_-^{ca}) and taking traces we get

$$(\gamma_- S_o)^a \bar{S}_o^a = [\bar{S}_o \gamma_- S_o]$$

$$(\gamma_- S_o)^a \bar{S}_o^a = 8 \tag{8}$$

and this implies

$$[\bar{S}_0 \gamma_- S_0] = 8 \tag{1}$$

Now it is immediate to prove that

$$|i\rangle = \frac{i}{8} [\bar{S}_0 \gamma_- \gamma^i]^a |a\rangle \tag{2}$$

Proof:

$$\frac{i}{8} [\bar{S}_0 \gamma_- \gamma^i]^a |a\rangle = -\frac{1}{64} [\bar{S}_0 \gamma_- \gamma^i \gamma^j S_0] |j\rangle =$$

$$= \frac{1}{64} [\bar{S}_0 \gamma_- (\delta_{ij} + i\sigma^{ij}) S_0] |j\rangle = \frac{1}{64} (\bar{S}_0 \gamma_- S_0) |i\rangle + \frac{i}{8} R_0^{ij} |j\rangle =$$

$$= \frac{1}{8} |i\rangle - \frac{1}{8} [\delta_{ij} |j\rangle - \delta_{jj} |i\rangle] = \frac{1}{8} |i\rangle - \frac{1}{8} |i\rangle + |i\rangle = |i\rangle \quad \text{Q.E.D.}$$

Let us now compute the scalar product of fermionic states. From (3.5) and (3.6)

$$\bar{S}_0^b S_0^a = \frac{1}{2} [\gamma_+ h]^{ab} + \frac{1}{4} [\gamma_+ \sigma^{ke} h]^{ab} R_0^{ke} \tag{3}$$

Hence

$$\langle i | \bar{S}_0^b S_0^a | j \rangle = \delta_{ij} \frac{1}{2} [\gamma_+ h]^{ab} + \frac{1}{4} [\gamma_+ \sigma^{ke} h]^{ab} \langle i | R_0^{ke} | j \rangle =$$

$$= \delta_{ij} \frac{1}{2} [\gamma_+ h]^{ab} + \frac{i}{4} [\gamma_+ \sigma^{ke} h]^{ab} (\delta_{kj} \delta_{ei} - \delta_{ki} \delta_{ej}) =$$

$$= \delta_{ij} \frac{1}{2} [\gamma_+ h]^{ab} + \frac{i}{2} [\gamma_+ \sigma^{ji} h]^{ab} \Rightarrow$$

$$\langle i | \bar{S}_0^b S_0^a | j \rangle = -\frac{1}{2} [\gamma_+ \gamma^j \gamma^i h]^{ab} \tag{4}$$

Therefore, contracting with $(\gamma^i)^{bc} (\gamma^j)^{da}$, we get

$$\langle i | [\bar{S}_0 \gamma^i]^c [\gamma^j S_0]^d | j \rangle = -\frac{1}{2} [\gamma^j \gamma_+ \gamma^j \gamma^i h \gamma^i]^{dc} =$$

$$= +\frac{1}{2} [h \gamma_+ \gamma^j \gamma^j \gamma^i \gamma^i]^{dc} = 32 [h \gamma_+]^{dc} \tag{5}$$

and hence

$$\langle a | b \rangle \equiv \langle i | [\bar{S}_0 \gamma^i]^a \frac{(-i)}{8} \frac{i}{8} [\gamma^j S_0]^b | j \rangle = \frac{1}{2} [h \gamma_+]^{ba} \tag{6}$$

Finally the bosonic states $|i\rangle$ transforms as the 8_v representation of $SO(8)$ and therefore has one state of helicity 1, one of helicity -1 and six of helicity 0 in $SO(1,3)$. The fermionic states $|a\rangle$ transforms as 8_s, and contains four states of helicity $+1/2$ and four of helicity $-1/2$. Note that the 8_s representation of the fermionic states is different from the 8_c representation corresponding to the fermionic operators. The zero-mode operator S_0^a must transform the ground state bosons into the ground state fermions and viceversa. Since $8_c \otimes 8_v$ contains 8_s, it is necessary for the weights of $|a\rangle$ and S_m^b to be in the two different congruency classes. From all that

$$\begin{array}{lll} |i\rangle \ , \ \alpha_m^i & \text{belong to} & 8_v \\ |a\rangle & \text{belongs to} & 8_s \\ S_m^a & \text{belong to} & 8_c \end{array} \qquad (1)$$

Let us now consider the level $N=1$, i.e. $\alpha' M^2 = 1$. Now we have

Bosonic states

$$\alpha_{-1}^i |j\rangle \qquad 8_v \otimes 8_v = (1)_s \oplus (28)_a \oplus (35_v)_s$$

$$S_{-1}^a |b\rangle \qquad 8_c \otimes 8_s = 8_v \oplus 56_v \qquad 128 \text{ - states} \qquad (2)$$

Fermionic states

$$\alpha_{-1}^i |a\rangle \qquad 8_v \otimes 8_s = 8_c \oplus 56_c$$

$$S_{-1}^a |i\rangle \qquad 8_c \otimes 8_v = 8_s \oplus 56_s \qquad 128 \text{ - states} \qquad (3)$$

As they are massive states they must fall into irreducible representations of the little group for massive representations of $SO(1,9)$, i.e. $SO(9)$. In fact

$$\begin{array}{lll} \text{Bosons} & (128) & = 44 \oplus 84 \\ \text{Fermions} & (128) & = 128 \end{array} \qquad (4)$$

Since $SO(9) \supset SO(8)$ with

$$44 = 1 \oplus 8_v \oplus 35_v \quad , \quad 84 = 28 \oplus 56_v$$

$$128 = 8_c \oplus 8_s \oplus 56_c \oplus 56_s \tag{1}$$

Furthermore in the decomposition of $SO(9) \supset SO(3) \otimes SO(6)$ we find

$$44 = (1,1) \oplus (1,20') \oplus (3,6) \oplus (5,1)$$

$$84 = (1,1) \oplus (1,10) \oplus (1,\bar{10}) \oplus (3,6) \oplus (3,15) \tag{2}$$

$$128 = (2,4) \oplus (2,\bar{4}) \oplus (2,20) \oplus (2,\bar{20}) \oplus (4,4) \oplus (4,\bar{4})$$

and therefore we have the following massive fields

1	of spin	2	8	of spin	3/2
27	of spin	1	48	of spin	1/2
42	of spin	0			

(3)

In the level $N=2$, i.e. $\alpha' M^2 = 2$ we have the following states

i) Bosons

$\alpha_{-1}^i \alpha_{-1}^j |k\rangle$ $\quad (8_v \otimes 8_v) \otimes 8_v = [(1)_S \oplus (35_v)_S] \times 8_v = 8_v \oplus 8_v \oplus$

$\oplus 112_v \oplus 160_v$

$S_{-1}^a S_{-1}^b |i\rangle$ $\quad (8_c \otimes 8_c) \otimes 8_v = [(28)_S] \otimes 8_v = 8_v \oplus 56_v \oplus 160_v$ (4)

$\alpha_{-2}^i |j\rangle$ $\quad 8_v \otimes 8_v = 1 \oplus 28 \oplus 35_v$

$\alpha_{-1}^i S_{-1}^a |b\rangle$ $\quad 8_v \otimes 8_c \otimes 8_s = 1 \oplus 28 \oplus 28 \oplus 35_v \oplus 35_s \oplus 35_c \oplus 350$

$S_{-2}^a |b\rangle$ $\quad 8_c \otimes 8_s = 8_v \oplus 56_v$

i.e. a total 1152 states. Notice that in the first row in $(8_v \otimes 8_v)$ we need to consider only the symmetric part and in the second row in $(8_c \otimes 8_c)$ only the antisymmetric one.

This massive states belong to the $SO(9)$ representations

$$9 \oplus 36 \oplus 126 \oplus 156 \oplus 231 \oplus 594 \tag{5}$$

ii) Fermions

$\alpha_{-1}^i S_{-1}^a |j\rangle$ $\quad 8_v \otimes 8_c \otimes 8_v = 8_c \oplus 8_c \oplus 56_c \oplus 56_c \oplus 160_c \oplus 224_{vc}$

$S^a_{-\frac{1}{2}} |i\rangle$ $\qquad 8_c \otimes 8_v = 8_s \otimes 56_s$

$\alpha^i_{-1}\, \alpha^j_{-1} |a\rangle$ $\qquad (8_v \otimes 8_v) \otimes 8_s = (1 \oplus 35_v) \otimes 8_s = 8_s \oplus 56_s \oplus 224_{vs}$ (1)

$\alpha^i_{-2} |a\rangle$ $\qquad 8_v \otimes 8_s = 8_c \otimes 56_c$

$S^a_{-1}\, S^b_{-1} |c\rangle$ $\qquad (8_c \otimes 8_c) \otimes 8_s = 28 \otimes 8_s = 8_s + 56_s + 160_s$

We have therefore also 1152 states that fall in the SO(9) representation

$$16 \otimes 128 \otimes 432 \otimes 576 \qquad (2)$$

A general feature of the spectrum is that we have the same number of bosom and fermion states at each level. In the context of superstring is very simple to stablish this equality at each level. Indeed let us denote by B (F) a combination of α^i_{-m} and S^a_{-m} operators which is a bosonic (fermionic) operator, and let N_B (N_F) the total number of those operators at a given level. Then it follows that the spectrum of states is given by

$$\left.\begin{array}{l} B\,|i\rangle \\ F\,|a\rangle \end{array}\right\} \quad 8\,(N_B + N_F) \qquad \text{bosonic states} \qquad (3)$$

$$\left.\begin{array}{l} B\,|a\rangle \\ F\,|i\rangle \end{array}\right\} \quad 8\,(N_B + N_F) \qquad \text{fermionic states}$$

Finally let us mention that the open string states at each level correspond to an irreducible representation of the superalgebra of $N=1$ and $SO(1,9)$. The open string theories are called Type I.

XIII - A - SPIN IN 10 DIMENSIONS

For $D = 10$, the Lorentz group is $SO(1,9)$. The problem of finding the representations of the 10-dimensional inhomogeneous Lorentz (Poincaré) group is not very different that it is in 4 dimensions. We recall that the unitary representations of a non-compact real Lie algebra are infinite dimensional. Nevertheless, the spin states are finite in number. The infinity arises from the fact that the momentum label of the states is a continuous variable, and can be changed by arbitrarily small amounts.

We generalize the method used by Wigner for $D = 4$. He considers the stability subgroup of one of three classes of vectors: timelike, lightlike and spacelike. In the first two cases, the vector is left unchanged by the compact subgroups $SO(D-1)$ and $SO(D-2)$, respectively. We do not need the spacelike case, where the stability group is still noncompact $SO(1, D-2)$. The representations of the Poincaré of the Poincaré group induced by the stability group of the timelike vector are called massive representations, and those from the stability group of the lightlike vector are called massless representations (in D dimensions).

In the massive case, there is a state in the Hilbert space that describes a particle at rest $|M, S; \vec{0}, \sigma\rangle$, where the mass M and spin S are the Casimir invariants of the Poincaré group (now in D dimensions), the zero entry is the spatial momentum $\vec{p} = 0$, and σ is the spin projection, which is most conveniently taken along the direction of motion (when we give the state momentum by boosting it). The general state is defining by boosting the rest state with $L(\vec{p})$:

$$|M, S; \vec{p}, \sigma\rangle = L(\vec{p})\, |M, S; \vec{0}, \sigma\rangle \tag{1}$$

A unitary Lorentz transformation on this state $U(\Lambda)$ is then given by

$$U(\Lambda)\, |M, S; p\, \sigma\rangle = \sum_{\sigma'} D^{(S)}_{\sigma', \sigma} \left(L^{-1}(\Lambda \vec{p})\, \Lambda\, L(\vec{p}) \right) |M, S; \Lambda \vec{p}, \sigma'\rangle \tag{2}$$

The crucial point is that the argument of the representation matrix D is the stability group $SO(D-1)$ for a massive particle. Thus, the label $(\vec{S})$ on $D^{(\vec{S})}$ is an irreducible representation of the compact group $SO(D-1)$, and the component labels σ are weights in the finite-dimensional representation

The reference lightlike vector $(1, 1, 0, \ldots 0)$ so $P^2 = 0$ which is left invariant under $SO(D-2)$ transformations. This is to be compared with the $SO(2)$ stability group of $(1, 1, 0, 0)$ in four dimensions, where the one-dimensional irreducible representations of $SO(2)$ label the helicity. (Actually, the group that leaves a light-like vector invariant is slightly larger since it includes translations in the plane orthogonal to the reference vector. Thus, in $D=4$ there are two additional translation generators. However, it is only the compact subgroup $SO(2)$ of the light-cone symmetry that is used in identifying quantum numbers). If the 4-dimensional helicity is nonzero, the CPT invariance of the theory implies that helicities come in pairs, λ and $-\lambda$. In $D=10$ the problem is simpler, because the self-conjugacy of the irreducible representations of $SO(8)$ already satisfies this requirement of local quantum field theory.

The physical degrees of freedom of a massless field in $D=10$ belong to an irreducible representation of $SO(8)$, which has highest weight, (a_1, a_2, a_3, a_4). The physical degrees of a massive state are in an $SO(8)$ irreducible representation, and are also identified by 4 non-negative integers

We now examine some of the subgroups of the Lorentz group. We have already seen the simplest example, where the light cone $SO(D-2)$ is a subgroup of $SO(D-1)$. This is obvious, because the transformations of $SO(D-2)$ that leave $(1, 1, 0, \ldots 0)$ invariant are a subset of those of $SO(D-1)$ that leave $(1, 0, 0, \ldots 0)$ invariant. Thus, a vector of $SO(9)$ "9" contains the vector of $SO(8)$ "8_v" according to the branching rule

$$9 = 1 \oplus 8_v \qquad (1)$$

Thus, just as the helicity states of a massive 4-dimensional state can be gathered together to form an SO(3) irreducible representation, the helicity SO(8) irreducible representations of a massive state in D = 10 can be gathered together to form an SO(9) irreducible representation.

A list of SO(8) or SO(9) irreducible representations may not appear very informative, since the spin content of a field is more easily interpreted in 4 dimensions. Thus, it is necessary to find out how the SO(1,3) of four dimensions is embedded in SO(1, D-1), and then to learn how to find the branching rules for SO(9) and SO(8) irreducible representations into SO(3) or SO(2) irreducible representations. We can see how this is done by dividing the Minkowski metric as

$$\eta = \begin{vmatrix} 1 & & & & & \\ & -1 & & & & \\ & & -1 & & & \\ & & & -1 & & \\ & & & & -1 & \\ & & & & & \ddots \\ & & & & & & -1 \end{vmatrix} \qquad (1)$$

Now it is easy to see how the 4-dimensional Lorentz group is embedded into the 10-dimensional group. The 10 dimensional vector can be written as one 4-dimensional vector and 6 singlets under SO(1,3). However, the Lorentz group for the extra dimensions is larger; it is SO(6), and its embedding in the 10-dimensional Lorentz algebra is $SO(1,9) \supset SO(1,3) \otimes SO(6)$, defined by the branching rule

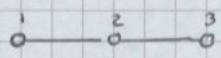
$$10 = (4,1) \oplus (1,6) \qquad (2)$$

This can now be restated for the irreducible representations labeled by the stability groups. For massless irreducible representations, we have the embedding $SO(8) \supset SO(2)^{\text{helicity}} \otimes SO(6)$, where the embedding is specified by

$$8_V = 1(1) \oplus 1(-1) + 6(0)$$

$$8_S = 4(1/2) \oplus \bar{4}(-1/2) \qquad (3)$$

where the 4-dimensional helicity is contained in the brackets

SU(6). 

Rank 3 Every irreducible representation is uniquely defined by a set of integers (a_1, a_2, a_3), $a_i \geq 0$, and each set is a highest weight of one and only one irreducible representation. (Dynkin labels)

Congruency : The four congruency classes of SU(6) are labelled by Z_4

SU(6) Irreducible Representations

Congruency 0		Congruency 1		Congruency -1		Congruency 2	
(0 0 0)	1	(1 0 0)	4	(0 0 1)	$\bar{4}$	(0 1 0)	6
(1 0 1)	15	(0 1 1)	20	(1 1 0)	$\bar{20}$	(2 0 0)	10
(0 2 0)	20'	(0 0 3)	20''	(3 0 0)	$\bar{20}''$	(0 0 2)	$\bar{10}$
(4 0 0)	35	(2 0 1)	36	(1 0 2)	$\bar{36}$	(0 3 0)	50
(0 0 4)	$\bar{35}$	(5 0 0)	56	(0 0 5)	$\bar{56}$	(1 1 1)	64
(2 1 0)	45	(1 2 0)	60	(0 2 1)	$\bar{60}$	(3 0 1)	70
(0 1 2)	$\bar{45}$	(3 1 0)	84'	(0 1 3)	$\bar{84}'$	(1 0 3)	$\bar{70}$
(2 0 2)	84	(1 0 4)	120	(4 0 1)	$\bar{120}$	(6 0 0)	84''
(0 4 0)	105	(0 0 7)	120'	(7 0 0)	$\bar{120}'$	(0 0 6)	$\bar{84}''$
(8 0 0)	165	(1 1 2)	140	(2 1 1)	$\bar{140}$	(2 2 0)	126
(0 0 8)	$\bar{165}$	(0 3 1)	140'	(1 3 0)	$\bar{140}'$	(0 2 2)	$\bar{126}$
(1 2 1)	175	(3 0 2)	160	(2 0 3)	$\bar{160}$	(4 1 0)	140''
						(0 1 4)	$\bar{140}''$

SU(6) Tensor products

$4 \otimes 4 = 6_a \oplus 10_s$

$4 \otimes \bar{4} = 1 \oplus 15$

$6 \otimes \bar{4} = 4 \oplus 20$

$6 \otimes 6 = 1_s \oplus 15_a + 20'_s$

$\bar{10} \otimes \bar{4} = 20 \oplus 20''$

$10 \otimes \bar{4} = 4 \oplus 36$

$10 \otimes 6 = 15 \oplus 45$

$10 \otimes 10 = 20'_s \oplus 35_s \oplus 45_a$

$10 \otimes \bar{10} = 1 \oplus 15 \oplus 84$

$15 \otimes 4 = 4 \oplus 20 \oplus 36$

$15 \otimes 6 = 6 \oplus 10 \oplus \overline{10} \oplus 64$

$15 \otimes 10 = 6 \oplus 10 \oplus 64 \oplus 70$

$15 \otimes 15 = 1_s \oplus 15_s \oplus 15_a \oplus 20'_s \oplus 45_a \oplus \overline{45}_a \oplus 84_s$

$\overline{20} \otimes 4 = 15 \oplus 20' \oplus 45$

$20 \otimes 4 = 6 \oplus \overline{10} \oplus 64$

$\overline{20} \otimes 6 = 4 \oplus 20 \oplus 36 \oplus 60$

$\overline{20} \otimes 10 = 20 \oplus 36 \oplus 60 \oplus 84'$

$\overline{20} \otimes \overline{10} = 4 \oplus 20 \oplus 36 \oplus 140$

$20 \otimes 15 = 4 \oplus 20 \oplus 20 \oplus 20'' \oplus 36 \oplus 60 \oplus 140$

$20 \otimes 20 = 6_q \oplus 10_s \oplus \overline{10}_s \oplus 50_a \oplus 64_s \oplus 64_a \oplus 70_a + \overline{126}_s$

$20 \otimes \overline{20} = 1 \oplus 15 \oplus 15 \oplus 20' \oplus 45 \oplus \overline{45} \oplus 84 \oplus 175$

$20' \otimes 4 = 20 \oplus 60$

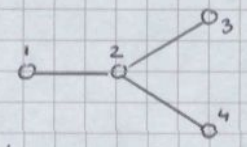
$20' \otimes 6 = 6 \oplus 50 \oplus 64$

$20' \otimes 10 = 10 \oplus 64 \oplus 126$

$20' \otimes 15 = 15 \oplus 20' \oplus 45 \oplus \overline{45} \oplus 175$

$20' \otimes 20 = 4 \oplus 20 \oplus 36 \oplus 60 \oplus 140 \oplus 140'$

$20' \otimes 20' = 1_s \oplus 15_a \oplus 20'_s \oplus 84_s \oplus 105_s + 175_a$

SO(8) 

Rank 4

Congruency : The congruency class is labeled $Z_2 \times Z_2$.

SO(8) Irreducible Representations . The subindex designates the congruency class except in cases like 35_c with congruency (0,0)

Congruency $0 \equiv (0,0)$		Congruency $S \equiv (1,0)$		Congruency $C \equiv (0,1)$		Congruency $V \equiv (1,1)$	
(0 0 0 0)	1	(0 0 0 1)	8_S	(0 0 1 0)	8_C	(1 0 0 0)	8_V
(0 1 0 0)	28	(1 0 1 0)	56_S	(1 0 0 1)	56_C	(0 0 1 1)	56_V
(2 0 0 0)	35_V	(0 0 0 3)	112_S	(0 0 3 0)	112_C	(3 0 0 0)	112_V
(0 0 0 2)	35_S	(0 1 0 1)	160_S	(0 1 1 0)	160_C	(1 1 0 0)	160_V
(0 0 2 0)	35_C	(2 0 0 1)	224_{VS}	(2 0 1 0)	224_{VC}	(1 0 0 2)	224_{SV}
(4 0 0 0)	294_V	(0 0 2 1)	224_{CS}	(0 0 1 2)	224_{SC}	(1 0 2 0)	224_{CV}
(0 0 0 4)	294_S	(3 0 1 0)	672_{VS}	(3 0 0 1)	672_{VC}	(0 0 1 3)	672_{SV}
(0 0 4 0)	294_C	(1 0 3 0)	672_{CS}	(1 0 0 3)	672_{SC}	(0 0 3 1)	672_{CV}
(0 2 0 0)	300	(0 0 0 5)	$672'_S$	(0 0 5 0)	$672'_{SC}$	(5 0 0 0)	$672'_V$
(1 0 1 1)	350	(1 1 1 0)	840_S	(1 1 0 1)	840_C	(0 1 1 1)	840_V
(2 1 0 0)	567_V	(1 0 1 2)	1296_S	(1 0 2 1)	1296_C	(2 0 1 1)	1296_V
(0 1 0 2)	567_S						
(0 1 2 0)	567_C						
(0 0 2 2)	$840'_V$						
(2 0 2 0)	$840'_S$						
(2 0 0 2)	$840'_C$						

SO(8) Tensor products

$8_V \otimes 8_V = (1)_S \oplus (28)_A \oplus (35_V)_S$

$8_C \otimes 8_S = 8_V \oplus 56_V$

$28 \otimes 8_V = 8_V \oplus 56_V \oplus 160_V$

$28 \otimes 28 = (1)_S \oplus (28)_A \oplus (35_V)_S \oplus (35_C)_S \oplus (35_S)_S \oplus (300)_S \oplus (350)_A$

$35_V \otimes 8_V = 8_V \oplus 160_V \oplus 112_V$

$35_C \otimes 8_V = 56_V \oplus 224_{CV}$

$35_V \otimes 28 = 28 \oplus 35_V \oplus 350 \oplus 567_V$

$35_v \otimes 35_s = 35_c \oplus 350 \oplus 840'_c$

$35_v \otimes 35_v = (1)_s \oplus (28)_a \oplus (35_v)_s \oplus (294_v)_s \oplus (300)_s \oplus (567_v)_a$

$56_v \otimes 8_v = 28 \oplus 35_c \oplus 35_s \oplus 350$

$56_s \otimes 8_c = 8_v \oplus 56_v \oplus 160_v \oplus 224_{cv}$

$56_v \otimes 28 = 8_v \oplus 56_v \oplus 56_v \oplus 160_v \oplus 224_{sv} \oplus 224_{cv} \oplus 840_v$

$56_v \otimes 35_v = 56_v \oplus 160_v \oplus 224_{sv} \oplus 224_{cv} \oplus 1296_v$

$56_v \otimes 35_c = 8_v \oplus 56_v \oplus 160_v \oplus 224_{cv} \oplus 672_{cv} \oplus 840_v$

$56_v \otimes 56_v = 1 \oplus 28 \oplus 28 \oplus 35_v \oplus 35_s \oplus 35_c \oplus 300 \oplus$
$350 \oplus 350 \oplus 567_s \oplus 567_c \oplus 840'_v$

$56_c \otimes 56_s = 8_v \oplus 56_v \oplus 56_v \oplus 112_v \oplus 160_v \oplus 160_v \oplus 224_{sv} \oplus$
$224_{cv} \oplus 840_v \oplus 1296_v$

$160_v \otimes 8_v = 28 \oplus 35_v \oplus 300 \oplus 350 \oplus 567_v$

$160_c \otimes 8_s = 56_v \oplus 160_v \oplus 224_{cv} \oplus 840_v$

$160_v \otimes 28 = 8_v \oplus 56_v \oplus 112_v \oplus 160_v \oplus 160_v \oplus 224_{sv} \oplus 224_{cv} \oplus$
$840_v \oplus 1296_v \oplus 1400_v$

$160_v \otimes 35_v = 8_v \oplus 56_v \oplus 112_v \oplus 160_v \oplus 160_v \oplus 840_v \oplus 1296_v \oplus$
$1400_v \oplus 1581_v$

$160_v \otimes 35_c = 56_v \oplus 160_v \oplus 224_{sv} \oplus 224_{cv} \oplus 840_v \oplus 1296_v \oplus 2800_v$

$160_v \otimes 56_v = 28 \oplus 35_v \oplus 35_s \oplus 35_c \oplus 300 \oplus 350 \oplus 350 \oplus$
$350 \oplus 567_v \oplus 567_s \oplus 567_c \oplus 840'_s \oplus 840'_c \oplus 4096$

In this case many products are related by congruency and only an example is given. For instance from

$$8_c \otimes 8_s = 8_v \oplus 56_v$$

we deduce

$$8_s \otimes 8_v = 8_c \oplus 56_c$$

$$8_c \otimes 8_v = 8_s \oplus 56_s$$

$SO(8) \supset SO(6) \otimes SO(2)$

$U(1)$ eigenvalue normalized to 4-dimensional helicity

$1 = 1\,(0)$

$8_v = 1\,(1) \oplus 6\,(0) \oplus 1\,(-1)$

$8_s = 4\,(1/2) \oplus 4\,(-1/2)$

$8_c = \bar{4}\,(1/2) \oplus \bar{4}\,(-1/2)$

$28 = 6\,(1) \oplus 1\,(0) \oplus 15\,(0) \oplus 6\,(-1)$

$35_V = 1\,(2) \oplus 6\,(1) \oplus 1\,(0) \oplus 20'\,(0) \oplus 6\,(-1) + 1\,(-2)$

$35_S = 10\,(1) \oplus 15\,(0) \oplus \overline{10}\,(-1)$

$35_C = \overline{10}\,(1) \oplus 15\,(0) + 10\,(-1)$

$56_V = 15\,(1) \oplus 6\,(0) \oplus 10\,(0) \oplus \overline{10}\,(0) + 15\,(-1)$

$56_S = \overline{4}\,(3/2) \oplus 4\,(1/2) + 20\,(1/2) + \overline{4}\,(-1/2) + \overline{20}\,(-1/2) + 4\,(3/2)$

$56_C = 4\,(3/2) \oplus \overline{4}\,(1/2) \oplus \overline{20}\,(1/2) \oplus 4\,(-1/2) \oplus 20\,(-1/2) \oplus \overline{4}\,(-3/2)$

$112_V = 1\,(3) \oplus 6\,(2) \oplus 1\,(1) \oplus 20'\,(1) \oplus 6\,(0) \oplus 50\,(0) + 1\,(-1) \oplus 20'\,(-1) \oplus$
$\oplus 6\,(-2) \oplus 1\,(-3)$

$112_S = \overline{20}''\,(3/2) \oplus 36\,(1/2) \oplus \overline{36}\,(-1/2) \oplus 20''\,(-3/2)$

$112_C = 20''\,(3/2) \oplus \overline{36}\,(1/2) \oplus 36\,(-1/2) \oplus \overline{20}''\,(-3/2)$

$160_V = 6\,(2) \oplus 1\,(1) \oplus 15\,(1) \oplus 20'\,(1) \oplus 6\,(0) \oplus 6\,(0) \oplus 64\,(0) + 1\,(-1) \oplus$
$\oplus 15\,(-1) \oplus 20'\,(-1) \oplus 6\,(-2)$

$160_S = \overline{20}\,(3/2) \oplus 4\,(1/2) + 20\,(1/2) + 36\,(1/2) + \overline{4}\,(-1/2) + \overline{20}\,(-1/2) + \overline{36}\,(-1/2) + 20\,(-3/2)$

$160_C = 20\,(3/2) \oplus \overline{4}\,(1/2) + \overline{20}\,(1/2) \oplus \overline{36}\,(1/2) + 4\,(-1/2) \oplus 20\,(-1/2) + 36\,(-1/2) + \overline{20}\,(-3/2)$

$224_{SV} = 10\,(2) \oplus 15\,(1) \oplus 45\,(1) \oplus 10\,(0) + \overline{10}\,(0) + 64\,(0) + 15\,(-1) + \overline{45}\,(-1) + \overline{10}\,(-2)$

$224_{CV} = \overline{10}\,(2) \oplus 15\,(1) \oplus \overline{45}\,(1) \oplus 10\,(0) + \overline{10}\,(0) \oplus 64\,(0) + 15\,(-1) \oplus 45\,(-1) + 10\,(-2)$

$224_{VS} = 4\,(5/2) + \overline{4}\,(3/2) + \overline{20}\,(3/2) \oplus 4\,(1/2) + 20\,(1/2) + 60\,(1/2) + \overline{4}\,(-1/2) + \overline{20}\,(-1/2)$
$+ \overline{60}\,(-1/2) + 4\,(-3/2) + 20\,(-3/2) + \overline{4}\,(-5/2)$

$224_{CS} = \overline{36}\,(3/2) \oplus 20\,(1/2) + 20''\,(1/2) \oplus 36\,(1/2) + \overline{20}\,(-1/2) + \overline{20}''\,(-1/2) + \overline{36}\,(-1/2) + 36\,(-3/2)$

$224_{VC} = \overline{4}\,(5/2) \oplus 4\,(3/2) \oplus 20\,(3/2) \oplus \overline{4}\,(1/2) + \overline{20}\,(1/2) \oplus \overline{60}\,(1/2) \oplus 4\,(-1/2) + 20\,(-1/2) \oplus$
$\oplus 60\,(-1/2) + \overline{4}\,(-3/2) + \overline{20}\,(-3/2) \oplus 4\,(-5/2)$

$224_{SC} = 36\,(3/2) + \overline{20}\,(1/2) + \overline{20}''\,(1/2) \oplus \overline{36}\,(1/2) + 20\,(-1/2) + 20''\,(-1/2) + 36\,(-1/2) + \overline{36}\,(-3/2)$

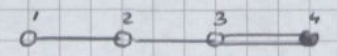
$294_S = 35\,(2) + 70\,(1) + 84\,(0) + \overline{70}\,(-1) + \overline{35}\,(-2)$

$294_C = \overline{35}\,(2) + \overline{70}\,(1) + 84\,(0) + 70\,(-1) + 35\,(-2)$

$294_V = 1\,(4) \oplus 6\,(3) \oplus 1\,(2) \oplus 20'\,(2) \oplus 6\,(1) \oplus 50\,(1) + 1\,(0) + 20'\,(0) + 105\,(0)$
$+ 6\,(-1) + 50\,(-1) + 1\,(-2) + 20'\,(-2) + 6\,(-3) + 1\,(-4)$

$300 = 20'\,(2) \oplus 6\,(1) \oplus 64\,(1) + 1\,(0) + 15\,(0) + 20'\,(0) + 84\,(0) + 6\,(-1) \oplus$
$\oplus 64\,(-1) + 20'\,(-2)$

$350 = 15\,(2) \oplus 6\,(1) \oplus 10\,(1) + \overline{10}\,(1) + 64\,(1) + 15\,(0) + 15\,(0) + 20'\,(0) + 45\,(0)$
$+ \overline{45}\,(0) + 6\,(-1) + 10\,(-1) + \overline{10}\,(-1) + 64\,(-1) + 15\,(-2)$

SO(9) 

Rank 4

Congruency: The congruency classes are labeled by Z_2.

SO(9) Irreducible Representations

Congruency 0		Congruency 1 (spinors)	
(0 0 0 0)	1	(0 0 0 1)	16
(1 0 0 0)	9	(1 0 0 1)	128
(0 1 0 0)	36	(0 1 0 1)	432
(2 0 0 0)	44	(2 0 0 1)	576
(0 0 1 0)	84	(0 0 0 3)	672
(0 0 0 2)	126	(0 0 1 1)	768
(3 0 0 0)	156	(3 0 0 1)	1920
(1 1 0 0)	231	(1 1 0 1)	2560
(4 0 0 0)	450		
(0 2 0 0)	495		
(1 0 1 0)	594		
(2 1 0 0)	910		
(1 0 0 2)	924		
(5 0 0 0)	1122		
(0 1 1 0)	1650		
(0 0 2 0)	1980		
(2 0 1 0)	2457		
(6 0 0 0)	2508		
(1 2 0 0)	2574		

SO(9) Tensor Products

$9 \otimes 9 = 1_s \oplus 36_a \oplus 44_s$

$16 \otimes 9 = 16 \oplus 128$

$16 \otimes 16 = 1_s \oplus 9_s \oplus 36_a \oplus 84_a \oplus 126_s$

$36 \otimes 9 = 9 \oplus 84 \oplus 231$

$36 \otimes 16 = 16 \oplus 128 \oplus 432$

$36 \otimes 36 = 1_s \oplus 36_a \oplus 44_s \oplus 495_s \oplus 594_a$

$44 \otimes 9 = 9 \oplus 156 \oplus 231$

$44 \otimes 16 = 128 \oplus 576$

$44 \otimes 36 = 36 \otimes 44 \otimes 594 \oplus 910$

$44 \otimes 44 = 1_s \oplus 36_a \oplus 44_s \oplus 450_s \oplus 495_s + 910_a$

$84 \otimes 9 = 36 \otimes 126 \otimes 594$

$84 \otimes 16 = 16 \otimes 128 \otimes 432 \otimes 768$

$84 \otimes 36 = 9 \otimes 84 \otimes 126 \otimes 231 \otimes 924 \otimes 1650$

$84 \otimes 44 = 84 \otimes 231 \otimes 924 \otimes 2457$

$84 \otimes 84 = 1_s \oplus 36_a \oplus 44_s \oplus 84_a \oplus 126_s \oplus 495_s \oplus 594_a \oplus 924_s \oplus 1980_s \oplus 2772_a$

$126 \otimes 9 = 84 \otimes 126 \otimes 924$

$126 \otimes 16 = 16 \otimes 128 \otimes 432 \otimes 672 \otimes 768$

$126 \otimes 36 = 36 \otimes 84 \otimes 126 \otimes 594 \otimes 924 \otimes 2772$

$126 \otimes 44 = 126 \otimes 594 \otimes 924 \otimes 3900$

$126 \otimes 84 = 9 \otimes 36 \otimes 84 \otimes 126 \otimes 231 \otimes 594 \otimes 924 \otimes 1650 \otimes 2772 \otimes 4158$

$126 \otimes 126 = 1_s \oplus 9_s \oplus 36_a \oplus 44_s \oplus 84_a \oplus 126_s \oplus 231_a \oplus 495_s \oplus 594_a \oplus 924_s \oplus$
$\oplus 1650_s \oplus 1980_s \oplus 2772_a \oplus 2772'_s \oplus 4158_a.$

$SO(9) \supset SO(8)$

$9 = 1 \otimes 8_v$

$16 = 8_c \otimes 8_s$

$36 = 8_v \otimes 28$

$44 = 1 \otimes 8_v \otimes 35_v$

$84 = 28 \otimes 56_v$

$126 = 35_c \otimes 35_s \otimes 56_v$

$128 = 8_c \otimes 8_s \otimes 56_c \otimes 56_s$

$156 = 1 \otimes 8_v \otimes 35_v \otimes 112_v$

$231 = 8_v \otimes 28 \otimes 35_v \otimes 160_v$

$432 = 56_c \otimes 56_s \otimes 160_c \otimes 160_s$

$450 = 1 \otimes 8_v \otimes 35_v \otimes 112_v \otimes 294_v$

$495 = 35_v \otimes 160_v \otimes 300$

$576 = 8_c \otimes 8_s \otimes 56_c \otimes 56_s \otimes 224_{vc} \otimes 224_{vs}$

$594 = 28 \otimes 56_v \otimes 160_v \otimes 350$

$SO(9) \supset SO(3) \otimes SO(6)$

$1 = (1, 1)$

$9 = (3, 1) + (1, 6)$

$16 = (2, 4) \oplus (2, \bar{4})$

$36 = (3, 1) \oplus (1, 15) \oplus (3, 6)$

$44 = (1, 1) \oplus (1, 20') \oplus (3, 6) \oplus (5, 1)$

$84 = (1, 1) \oplus (3, 6) \oplus (1, 10) \oplus (1, \bar{10}) \oplus (3, 15)$

$126 = (1, 6) \oplus (1, 15) \oplus (3, 10) \oplus (3, \bar{10}) \oplus (3, 15)$

$128 = (2, 4) \oplus (2, \bar{4}) \oplus (4, 4) \oplus (4, \bar{4}) \oplus (2, 20) \oplus (2, \bar{20})$

$156 = (1, 6) \oplus (1, 50) \oplus (3, 1) \oplus (3, 20') \oplus (5, 6) \oplus (7, 1)$

$231 = (1, 6) \oplus (1, 64) \oplus (3, 6) \oplus (3, 15) \oplus (3, 20') \oplus (3, 1) \oplus (5, 1) \oplus (5, 6)$

$432 = (2, 4) \oplus (2, \bar{4}) \oplus (2, 20) \oplus (2, \bar{20}) \oplus (2, 36) \oplus (2, \bar{36}) \oplus (4, 4) \oplus (4, \bar{4})$
$\oplus (4, 20) \oplus (4, \bar{20})$

XIV - SPECTRUM OF THE CLOSED STRING

We must first discuss the ground state of the closed superstring We need to consider the equations.

$$\alpha' M^2 = 4N$$

$$N = \tilde{N} \tag{1}$$

$$N = \frac{1}{2\alpha'} \sum_{m=1}^{\infty} \alpha_{-m}^{i} \alpha_{m}^{i} + \frac{1}{4} \sum_{m=1}^{\infty} m \left[\bar{S}_{-m} \gamma_{-} S_{m} \right]$$

$$\tilde{N} = \frac{1}{2\alpha'} \sum_{m=1}^{\infty} \tilde{\alpha}_{-m}^{i} \tilde{\alpha}_{m}^{i} + \frac{1}{4} \sum_{m=1}^{\infty} m \left[\bar{\tilde{S}}_{-m} \gamma_{-} \tilde{S}_{m} \right]$$

being α_m^i, $\tilde{\alpha}_m^i$, S_m^a and $\tilde{S}_m^a$ the annihilation operators. The ground states are characterized by

$$\alpha_m^i \, | g.s. \rangle = \tilde{\alpha}_m^i \, | g.s. \rangle = S_m^a \, | g.s. \rangle = \tilde{S}_m^a \, | g.s. \rangle = 0 \qquad m = 1, 2, 3 \ldots \tag{2}$$

From (1) it follows that for the ground state

$$N = 0 \iff \alpha' M^2 = 0 \tag{3}$$

i.e. the ground state is massless, avoiding thus the tachyon motive of this state typical of the old string theory.

The Fock space of the states can be defined as the direct product of two subspaces, where act independently the operators without tilde (α_m^i, S_m^a) and with tilde ($\tilde{\alpha}_m^i$, $\tilde{S}_m^a$). Each of these subspaces is analogous to the Fock space of the open superstring. The symmetry relations among these operators form an $N = 2$, $D = 10$ supersymmetry.

This time there are two choices for the ground state. The ground states bosons are $|ij\rangle$ and $|ab\rangle$, and the fermions are $|ia\rangle$ and $|bj\rangle$. The question is whether the first and second fermion operators have the same set o a different set of weights that the first.

Type IIA theory : One set transforms as 8_S and the other as 8_C (4)

Type IIB theory : Both set transform in the same form, let say 8_S

Let us now consider both cases

Type II A

$$|ij\rangle \qquad 8_v \otimes 8_v = (1)_s \oplus (28)_c \oplus (35_v)_s$$
$$|ab\rangle \qquad 8_s \otimes 8_c = 8_v \oplus 56_v$$

} 128 bosonic states

(1)

$$|ia\rangle \qquad 8_v \otimes 8_c = 8_s \oplus 56_s$$
$$|bi\rangle \qquad 8_s \otimes 8_v = 8_c \oplus 56_c$$

} 128 fermionic states

Let us now consider the helicity content of this SO(8) irreducible representation of massless particles and let us introduce the associated field

$(1)_s = 1(0)$ ϕ scalar field

$(28)_c = 6(1) \oplus 1(0) \oplus 15(0) \oplus 6(-1)$ A^{ij} second order antisymmetric tensor

$(35)_{v\,s} = 1(2) \oplus 6(1) \oplus 1(0) \oplus 20'(0) \oplus 6(-1) \oplus 1(-2)$ g^{ij} traceless symmetric tensor

$8_v = 1(1) \oplus 6(0) \oplus 1(-1)$ A^i vector field

$56_v = 15(1) \oplus 6(0) \oplus 10(0) \oplus \bar{10}(0) \oplus 15(-1)$ A^{ijk} third order fully antisymmetric tensor.

$8_s = 4(1/2) \oplus 4(-1/2)$ λ^a_R right-handed Majorana-Weyl spinor (2)

$8_c = \bar{4}(1/2) \oplus \bar{4}(-1/2)$ λ^a_L left-handed Majorana-Weyl spinor

$56_s = \bar{4}(3/2) \oplus 4(1/2) \oplus 20(1/2) \oplus \bar{20}(-1/2) \oplus \bar{4}(-1/2) + 4(-3/2)$ ψ^{ia}_L left-handed spin 3/2 field (Rarita-Schwinger)

$56_c = 4(3/2) \oplus \bar{4}(1/2) + \bar{20}(1/2) \oplus 20(-1/2) \oplus 4(-1/2) \oplus \bar{4}(-3/2)$ ψ^{ia}_R right-handed spin 3/2 field (Rarita-Schwinger)

The field g^{ij} is the "graviton" in $D = 10$. ϕ is the scalar field that is called the "dilaton".

This Type II A theory can be obtained from $D = 11$ supergravity through the method known as "dimensional reduction", which consist in giving a trivial dependence of the fields on the "extra" dimension:

$D = 11$ Supergravity $\xrightarrow{\text{Dimensional reduction}}$ Massless sector of Type IIA Superstring (1)

$$I = \{ i = 1, \dots 9, 10 \} \qquad A = \{ a = 1, \dots 9, 10 \}$$

$$g^{IJ} = \{ g^{ij}, \; g^{i\,10} = A^i, \; g^{10\,10} = \phi \} \qquad (2)$$

$$A^{IJK} = \{ A^{ijK}, \; A^{ij\,10} = A^{ij} \}$$

$$\psi^{IA} = \{ \psi^{ia}_R, \; \psi^{ia}_L, \; \lambda^a_R, \; \lambda^a_L \}$$

In odd dimensions there is no notion of chirality, since does not exists γ^{D+1} (D = odd). This explains the non-chiral nature of the type IIA superstring theory.

Type II B

$|i,j\rangle \qquad 8_V \otimes 8_V = (1)_S + (28)_a + (35_V)_S$

$|a,b\rangle \qquad 8_S \otimes 8_S = (1)_S + (28)_a + (35_S)_S$

} 128 bosonic states

$|i\,a\rangle \qquad 8_V \otimes 8_S = 8_c + 56_c$

$|a\,i\rangle \qquad 8_S \otimes 8_V = 8_c + 56_c$

} 128 fermionic states

The fields are now

$(1)_S = 1\,(0)$ ϕ scalar field

$(28)_a = 6\,(1) + 1\,(0) + 15\,(0) + 6\,(-1)$ A^{ij} second order antisymmetric tensor

$(35_V)_S = 1\,(2) + 6\,(1) + 1\,(0) + 20'\,(1) + 6\,(+1) + 1\,(-2)$ g^{ij} traceless symmetric tensor

$(1)_S = 1\,(0)$ ϕ' scalar field

$(28)_a = 6\,(1) + 1\,(0) + 15\,(0) + 6\,(-1)$ A'^{ij} second order antisymmetric tensor

$(35_S)_S = 10\,(1) + 15\,(0) + \bar{10}\,(-1)$ A^{ijke} antisymmetric fourth order tensor

$(8)_c = 8\,(1/2) + \bar{8}\,(-1/2)$ λ^a_L, λ'^{ia}_L Majorana - Weyl fields

$(56)_c = 8\,(3/2) + \bar{8}\,(1/2) + \bar{20}\,(1/2) + 20\,(-1/2) + 8\,(-1/2) + \bar{8}\,(-3/2)$ ψ^{ia}_R, ψ'^{ia}_R spin $3/2$ fields (Rarita - Schwinger)

This type IIB theory cannot be obtained from D=11, so that it is a "genuine" D=10 supergravity theory.

Apart from the type II theories there is another possible theory of closed superstrings known as type I in which one restricts the spectrum of the states to those that are symmetric under the interchange of the tilde and non tilde spaces. This requires that S_m^a and $\tilde{S}_m^a$ belong to the same representation. The massless states are

$\|ij\rangle + \|ji\rangle$	$1 + 35_v$	Φ, g^{ij}	}	64 bosonic states
$\|ab\rangle - \|ba\rangle$	28	A^{ij}		
$\|ia\rangle + \|ai\rangle$	$8_c + 56_c$	λ_L^a, $\psi^{ia}{}_R$		64 fermionic states

(4)

One must consider only the symmetric superchange $Q_L = \frac{1}{2}(Q_L^1 + Q_L^2)$, which maps the states of type I into themselves, so that this theory has only N=1 supersymmetry. It is for this reason that the open superstrings are said to be of type I since they also posses N=1 supersymmetry. Type I superstrings consist of open and closed sectors.

XV - THE HETEROTIC STRING

In this section we shall present the operator formulation of the free heterotic string in light-cone gauge. Before proceeding let us review the philosophy behind the construction.

Noninteracting strings are constructed by first quantization of an action given by the invariant area of the world sheet swept out by the string, or by its supersymmetric generalization. The reparametrization invariance of the classical action permits one to choose a "gauge" in which the timelike parameter of the world sheet, τ, is related to $X^+ = (X^0 + X^9)/\sqrt{2} = 2\alpha' p^+ \tau$. In this light-cone gauge the theory reduces to a two-dimensional free field theory of the physical degrees of freedom (the 24 transverse coordinates, X^i, of the bosonic string, or the 8 transverse coordinates, X^i, and their fermionic partners, of the superstring) subject to constraints. The procedure is only valid, however, in the critical dimension of 26 for the bosonic string and 10 for the superstring.

In the critical dimension the physical degrees of freedom, being massless two-dimensional free fields, can be decomposed into right and left movers, i.e. functions of $\tau - \sigma$ and $\tau + \sigma$ respectively. If we consider only closed strings then the right- and left-moving modes never mix. The separation of these modes will persist even in the presence of interactions, as long we consider orientable world surfaces. This is because interactions between closed strings are constructed, order by order in perturbation theory, by modifying the topology of the two-dimensional world-sheet on which the string propagate. In terms of the first quantized two-dimensional theory, however, no interaction is introduced; the right and left movers still propagate independently. Thus there is no obstacle to treating the right movers and the left movers differently, as long as each sector is separately consistent. This is the idea behind the construction of the heterotic string, which combines the right movers of the superstring with the left movers of the bosonic string. It is necessarily a theory of closed orientable strings, since open string boundary conditions would mix right and left movers, and since one can clearly define the orientation of such a string.

The physical degrees of freedom of the right-moving sector of the fermionic superstring consists of 8 transverse coordinates $X^i(\tau-\sigma)$ ($i=1,\dots 8$) and 8 Majorana-Weyl fermionic coordinates $S^a(\tau-\sigma)$. The physical degrees of freedom of the left-moving sector of the bosonic string consist of 24 transverse coordinates $X^i(\tau+\sigma)$ and $X^I(\tau+\sigma)$ ($i=1,\dots 8$, $I=1,\dots 16$). Together they compose the physical degrees of freedom of the heterotic string. The eight transverse right- and left-moving coordinates combine with the longitudinal light-cone coordinates $X_\pm(\sigma,\tau)$ to describe the position of the string embedded in ten-dimensional space-time.

The dynamics of the free heterotic string is governed by the light-cone gauge action

$$S = \int d\tau \int_0^\pi d\sigma \left\{ -\frac{1}{4\pi\alpha'}\partial_\alpha X^i \partial^\alpha X^i - \frac{1}{4\pi\alpha'}\partial_\alpha X^I \partial_\alpha X^I - \frac{i}{4\pi}\bar{S}\gamma_-\left(\partial_\tau S + \partial_\sigma S\right)\right\} \qquad (1)$$

where S is a 32-dimensional Majorana-Weyl spinor, with components S^a, satisfying the constraints

$$\gamma_+ S = 0 \qquad h S = 0 \qquad (2)$$

$$h \equiv \frac{1}{2}\left[I + \gamma''\right]$$

The coordinates X^i will contain both right movers, $X^i(\tau-\sigma)$, and left movers, $X^i(\tau+\sigma)$; however, we must impose a constraint restricting X^I to consist of left-moving modes alone. This can be achieved by demanding that

$$\Phi^I \equiv (\partial_\tau - \partial_\sigma) X^I(\sigma,\tau) = 0 \qquad I = 1,\dots 16 \qquad (3)$$

Alternatively, following W. Siegel [Nucl. Phys. B238, 307 (1984)], we can add to the action a term proportional to $\lambda^I [(\partial_\tau - \partial_\sigma) X^I]^2$, which has the same effect. Siegel has shown that the resulting action possesses a local gauge symmetry which allows the Lagrange multipliers λ^I to be gauged away, leaving one with the constraint (3).

The above action is invariant under a supersymmetry transformation that relates right-moving bosons and fermions, while the left-

moving bosons remain unchanged :

$$\delta X^i = \frac{i}{\sqrt{P_+}}\ \bar{\epsilon}\ \gamma^i\ S \tag{1}$$

$$\delta S = \frac{1}{2\alpha'\sqrt{P_+}}\ \gamma_+\ \gamma_\mu\ (\partial_\tau - \partial_\sigma)\ X^\mu\ \epsilon$$

(This coincides with (X-10.1) for $\epsilon^1 \equiv 0$, $\epsilon^2 = \epsilon$). In terms of two-dimensional field theory we have $N = 1/2$ supersymmetry.

Canonical quantization of the transverse and fermionic coordinates is straight-forward. Since we are dealing with closed strings, the fields are periodic functions of $0 \le \sigma \le \pi$ and can be expanded

$$X^i(\tau-\sigma) = \frac{1}{2}\ q^i + \alpha' P^i(\tau-\sigma) + \frac{i}{2} \sum_{\substack{m=-\infty \\ m\neq 0}}^{+\infty} \frac{1}{m}\ \alpha_m^i\ e^{-2im(\tau-\sigma)}$$

$$X^i(\tau+\sigma) = \frac{1}{2}\ q^i + \alpha' P^i(\tau+\sigma) + \frac{i}{2} \sum_{\substack{m=-\infty \\ m\neq 0}}^{+\infty} \frac{1}{m}\ \tilde{\alpha}_m^i\ e^{-2im(\tau+\sigma)} \tag{2}$$

$$S^a(\tau-\sigma) = \sum_{m=-\infty}^{+\infty} S_m^a\ e^{-2im(\tau-\sigma)}$$

where

$$[\,q^i\,,\ P^j\,] = i\,\delta^{ij}\ ,$$

$$[\,\alpha_m^i\,,\ \alpha_m^j\,] = [\,\tilde{\alpha}_m^i\,,\ \tilde{\alpha}_m^j\,] = 2\alpha' m\ \delta_{m+m,0}\ \delta^{ij} \tag{3}$$

$$\{\,S_m^a\,,\ \bar{S}_m^b\,\} = i\,\gamma_+\,h\,]^{ab}\ \delta_{m+m,0}$$

In the case of the left-moving internal coordinates, $X^I(\tau+\sigma)$, we must introduce Dirac brackets to take into account the second class constraints $\Phi^I = 0$. Thus the Poisson bracket of X^I with its canonical momentum P^I is

$$\{\,X^I(\sigma,\tau)\,,\ P^J(\sigma',\tau)\,\} = \delta^{IJ}\,\delta(\sigma-\sigma')\ , \qquad P^J(\sigma,\tau) \equiv \frac{1}{2\pi\alpha'}\ \dot{X}^J(\sigma,\tau) \tag{4}$$

is modified to yield the Dirac bracket

$$\{\,X^I(\sigma,\tau),\ P^J(\sigma',\tau)\,\}_{DB} = \{\,X^I(\sigma,\tau)\ ,\ P^J(\sigma',\tau)\,\}$$

$$- \int d\sigma''\,d\sigma'''\ \{\,X^I(\sigma,\tau)\ ,\ \Phi^K(\sigma'',\tau)\,\}\ C_{KK'}(\sigma'',\sigma''')\ \{\,\Phi^{K'}(\sigma''',\tau\ ,\ P^J(\sigma',\tau)\,\} \tag{5}$$

where $C_{KK'}(\sigma,\sigma')$ is the inverse of the Poisson bracket of $\Phi^K(\sigma,\tau)$ and $\Phi^{K'}(\sigma',\tau)$. Since

$$\{A(\sigma,\tau),B(\sigma',\tau)\} = \sum_L \int d\sigma'' \left\{ \frac{\delta A(\sigma\tau)}{\delta X^L(\sigma''\tau)} \frac{\delta B(\sigma',\tau)}{\delta P_\tau^L(\sigma''\tau)} - \frac{\delta B(\sigma'\tau)}{\delta X^L(\sigma''\tau)} \frac{\delta A(\sigma\tau)}{\delta P_\tau^L(\sigma''\tau)} \right\} \quad (1)$$

we get

$$\{\Phi^K(\sigma,\tau),\Phi^{K'}(\sigma',\tau)\} = -4\pi\alpha' \delta_{KK'} \partial_\sigma \delta(\sigma-\sigma')$$

$$\{X^i(\sigma,\tau),\Phi^K(\sigma',\tau)\} = 2\pi\alpha' \delta_{iK} \delta(\sigma-\sigma') \quad (2)$$

$$\{\Phi^K(\sigma\tau),P_\tau^J(\sigma',\tau)\} = -\delta_{KJ} \partial_\sigma \delta(\sigma-\sigma')$$

and hence

$$C_{KK'}(\sigma,\sigma'') = \frac{1}{4\pi\alpha'} \delta_{KK'} (\partial_\sigma)^{-1} \delta(\sigma-\sigma') \quad (3)$$

Then

$$\{X^I(\sigma\tau),P_\tau^J(\sigma',\tau)\} = \frac{1}{2} \delta_{IJ} \delta(\sigma-\sigma') \quad (4)$$

Therefore $X^I(\sigma,\tau)$ has the expansion

$$X^I(\sigma,\tau) = q^I + 2\alpha' p^I(\tau+\sigma) + \frac{i}{2} \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} \frac{1}{n} \tilde{\alpha}_n^I e^{-2in(\tau+\sigma)} \quad (5)$$

where upon quantization

$$[q^I, p^J] = \frac{i}{2} \delta_{IJ}$$

$$[\tilde{\alpha}_m^I, \tilde{\alpha}_m^J] = 2\alpha' m \delta_{IJ} \delta_{m+m,0} \quad (6)$$

The factor of $1/2$ in the commutator of q^I and p^I, the center-of-mass position and momenta of the internal space, arises due to the constraint that ensures that X^I is function of $\tau+\sigma$ It means that $2p^I$, $i=1,\dots,16$ are the generators of translations of the internal space.

XV - LIGHT-CONE-GAUGE VERTICES

In the previous sections we have described some properties of the superstrings. The true test of whether they are able to provide potential useful theories only comes with the formulation and analysis of interactions. There is a uniquely defined procedure for introducing interaction into each of the string theories that physically corresponds to the splitting and joining of strings. Thus even though strings are extended objects, the interactions are local. The strength of interaction is characterized by a coupling constant that represents the second arbitrary parameter (α' was the first one) appearing in a string theory. In the case of open-string interactions the coupling constant, called g, is a Yang-Mills coupling constant in the sense that it characterizes the interaction of three massless vector states and is the coupling constant of the Yang-Mills theory that survives in the $\alpha' \to 0$ limit. Similarly, the coupling constant of closed-string states is called κ. It gives the three graviton interaction, in particular, and is therefore identified as a gravitational coupling constant. In theories that contain both open and closed strings the fact that open strings can join their ends to give closed strings implies that κ is proportional to g^2. The only other parameters that occur in the string theories are length scales characterizing the size of compactified dimensions.

All the states of a string theory are on an equal footing inasmuch as they just correspond to the various different oscillation modes of the string. However, the massless states are somewhat special. All other states have masses proportional to $(\alpha')^{-1/2}$, which would be as large as the Planck mass. We need to analyze the scattering amplitudes for massless particles.

This section describes the construction of vertices for emission of on-mass-shell massless particles from an arbitrary string configuration.

Let us return to the old bosonic string. The Veneziano model spectrum contains a massless vector with momentum k^μ and polarization $\zeta^\mu(k)$ described by the Fock-space state

$$[\,\xi \cdot \alpha_{-1}\,]\,|\,0, k\rangle \qquad (1)$$

The wave equation (VII-17.2), with $\alpha(0) = 1$, shows that this state has zero mass

$$0 = (L_0 - 1)\,[\,\xi \cdot \alpha_{-1}\,]\,|\,0, k\rangle = \left\{ -\alpha' p^2 - \frac{1}{2\alpha'} \sum_{m=1}^{\infty} \alpha_{-m} \cdot \alpha_m - 1 \right\} [\,\xi \cdot \alpha_{-1}\,]\,|\,0, k\rangle$$

$$= \left\{ -\alpha' k^2 - \frac{1}{2\alpha'} (\alpha_{-1} \cdot \alpha_1) - 1 \right\} [\,\xi \cdot \alpha_{-1}\,]\,|\,0, k\rangle =$$

$$= \left\{ -\alpha' k^2 + 1 - 1 \right\} [\,\xi \cdot \alpha_{-1}\,]\,|\,0, k\rangle = -\alpha' k^2\,[\,\xi \cdot \alpha_{-1}\,]\,|\,0, k\rangle \qquad (2)$$

The L_1 subsidiary condition implies that the polarization vector ξ^μ is transverse (no spin zero part):

$$L_1\,[\,\xi \cdot \alpha_{-1}\,]\,|\,0, k\rangle = -\frac{1}{2\alpha'} (\alpha_1 \cdot \alpha_0)\,[\,\xi \cdot \alpha_{-1}\,]\,|\,0, k\rangle =$$

$$= [\,\xi \cdot \alpha_0\,]\,|\,0, k\rangle = 2\alpha'\,[\,\xi \cdot p\,]\,|\,0, k\rangle = 2\alpha'\,[\,\xi \cdot k\,]\,|\,0, k\rangle \qquad (3)$$

The motivation may not yet be obvious, but we choose to seek a vertex operator $V_\mu(k; z)$ to describe the emission of this state at any place along a tree configuration. We require that $k^2 = 0$ and that $V_\mu(k; z)$ transforms as a Lorentz vector. Most important, in order to guarantee the usual subsidiary conditions and duality properties we must require that $V_\mu(k; z)$ have conformal or generalized projection spin 1.

A solution for the transverse components of $V_\mu(k; z)$ is relatively easy

$$[\,\xi \cdot V(k; z)\,] = [\,\xi \cdot p(z)\,]\; e^{-i k \cdot Q(z)} \qquad (4)$$

has the required properties when $\xi \cdot k = 0$. The first factor has conformal spin 1 (see VII-A-3-1) and the second has conformal spin 0. Furthermore, from (VII-A-2.3), $\xi \cdot k = 0$ ensures that the two factors commute. Therefore no normal ordering is required. These circumstances are sufficient to conclude that the conformal spin of the product is the sum of the conformal spins of the two factors.

The operator in (4) is not sufficient to do all that we want. It pays to ask a more ambitious question: can we find an operator $V_\mu(k; z)$ with conformal spin 1, agreeing with (4) for trans-

verse polarizations, but without restrictions to such polarizations? A natural first choice for V_μ is

$$V^{(1)\mu}(k;z) = :p^\mu(z)\exp\left(-ik\cdot Q(z)\right):$$

$$= \left(\frac{1}{2}p^{(0)\mu}(z) + p^{(+)\mu}(z)\right)V(k;z) + V(k;z)\left(\frac{1}{2}p^{(0)\mu}(z) + p^{(-)\mu}(z)\right) \tag{1}$$

where we have used the notation of (VII-A). The definition of normal ordering has been extended for (1) to include a symmetrization of the zero modes. Because of normal ordering it is not obvious if this operator has conformal spin (1). Let us see if this is so. Since $k^2=0$ from (VII-A-4.3)

$$[L_m, V(k;z)] = z^{m+1}\frac{dV(k;z)}{dz} \tag{2}$$

Furthermore

$$[L_m, p^{(\alpha)\mu}(z)] = \left\langle -\frac{1}{4\alpha'}z'^m :p^2(z'):, p^{(\alpha)\mu}(z)\right] \rangle =$$

$$= \left\langle -\frac{1}{2\alpha'}z'^m p_\nu(z')[p^\nu(z'), p^{(\alpha)\mu}(z)]\right\rangle \tag{3}$$

The only non zero commutators are

$$[p^{(+)\nu}(z'), p^{(-)\mu}(z)] = (2\alpha')\eta^{\nu\mu}\sum_{\ell=1}^{\infty}\ell\frac{z'^\ell}{z^\ell}$$

$$[p^{(-)\nu}(z'), p^{(+)\mu}(z)] = -(2\alpha')\eta^{\mu\nu}\sum_{\ell=1}^{\infty}\ell\frac{z^\ell}{z'^\ell} \tag{4}$$

and since

$$p_\nu(z') = \sum_{r=-\infty}^{+\infty}\alpha^\mu_{-r}z'^r \tag{5}$$

we obtain

$$[L_m, p^{(0)\mu}(z)] = 0$$

$$[L_m, p^{(+)\mu}(z)] = \sum_{\ell=1}^{\infty}\ell\,\alpha^\mu_{m-\ell}z^\ell$$

$$[L_m, p^{(-)\mu}(z)] = -\sum_{\ell=1}^{\infty}\ell\,\alpha^\mu_{m+\ell}z^{-\ell}$$

Then

$$[L_m, V^{(1)\mu}(k;z)] =$$

$$= \left\{ \frac{1}{2} p^{(0)\mu} + p^{(+)\mu}(z) \right\} [L_m, V(k,z)] + [L_m, V(k,z)] \left\{ \frac{1}{2} p^{(0)\mu} + p^{(-)\mu}(z) \right\}$$

$$+ [L_m, p^{(+)\mu}(z)] V(k,z) + V(k;z) [L_m, p^{(-)\mu}(z)] =$$

$$= z^{m+1} \left\{ \left(\frac{1}{2} p^{(0)\mu} + p^{(+)\mu}(z) \right) \frac{dV(k;z)}{dz} + \frac{dV(k;z)}{dz} \left(\frac{1}{2} p^{(0)\mu} + p^{(-)\mu}(z) \right) \right\}$$

$$+ [L_m, p^{(+)\mu}(z)] V(k;z) + V(k;z) [L_m, p^{(-)\mu}(z)] \qquad (1)$$

Then

$$S_m^\mu \equiv [L_m, V^{(1)\mu}(k;z)] - z^m \left(z \frac{d}{dz} + m \right) V^{(1)\mu}(k;z) =$$

$$= \left\{ [L_m, p^{(+)\mu}(z)] - z^{m+1} \frac{dp^{(+)\mu}(z)}{dz} - m z^m \left(\frac{1}{2} p^{(0)\mu} + p^{(+)\mu}(z) \right) \right\} V(k;z) +$$

$$+ V(k;z) \left\{ [L_m, p^{(-)\mu}(z)] - z^{m+1} \frac{dp^{(-)\mu}(z)}{dz} - m z^m \left(\frac{1}{2} p^{(0)\mu}(z) + p^{(-)\mu}(z) \right) \right\} \qquad (2)$$

To be precise let us assume $m>0$, then

$$S_m^\mu = \left\{ +m\alpha' z^m p^\mu + \sum_{e=1}^{m-1} e\, \alpha_{m-e}^\mu z^e \right\} V(k,z) + V(k;z) \left\{ m\alpha' z^m p^\mu - \sum_{e=1}^{m-1} e\, \alpha_{m-e}^\mu z^e \right\} \Rightarrow$$

$$S_m^\mu = m\alpha' z^m [p^\mu, V(k;z)] + \sum_{e=1}^{m-1} e z^e [\alpha_{m-e}^\mu, V(k;z)] \qquad (3)$$

But

$$[\alpha_s^\mu, e^{\lambda(k.\alpha_r)}] = -2\alpha' \lambda s k^\mu e^{\lambda(k.\alpha_r)} \delta_{s+r,0} \qquad (4)$$

Since

$$V(k;z) = \exp\left\{ -\sum_{m=1}^{\infty} \frac{z^m}{m} (k.\alpha_{-m}) \right\} \exp\left\{ -ik.q - (2\alpha')(k.p) \ln z \right\} \exp\left\{ +\sum_{m=1}^{\infty} \frac{z^{-m}}{m} (k.\alpha_m) \right\} \qquad (5)$$

we obtain

$$[p^\mu, V(k;z)] = + k^\mu V(k,z)$$

$$[\alpha_{m-e}^\mu, V(k,z)] = + 2\alpha' k^\mu z^{m-e} V(k,z) \qquad (6)$$

and

$$S_m^\mu = + \left[m\alpha' z^m k^\mu + \sum_{\ell=1}^{m-1} 2\alpha' \ell\, k^\mu z^{m-\ell} \right] V(k;z) = \alpha' m^2 z^{m-1} k^\mu V(k;z) \qquad (1)$$

Therefore

$$\left[L_m , V^{(1)\mu}(k;z) \right] = z^m \left(z \frac{d}{dz} + m \right) V^{(1)\mu}(k;z) + m^2 \alpha' z^{m-1} k^\mu e^{-ik\cdot Q(z)} \qquad (2)$$

The second term shows the extend by which $V^{(1)\mu}(k;z)$ fails to have conformal spin 1. As expected this term drops out on contraction with a transverse polarization vector. In order to find the adequate vertex let us compute

$$\left[L_m , -z \frac{d}{dz} \ln \left[-k\cdot \ell(z) \right] \right] =$$

$$= \left\langle \frac{1}{4\alpha'} z^{1-m} \left[: \ell^2(z') : , z \frac{d}{dz} \ln \left[-k\cdot \ell(z) \right] \right] \right\rangle \equiv R_m \qquad (2)$$

From (VII-A-2.5)

$$\left[\ell^\mu(z') , \ell^\nu(z) \right] = -4\pi\alpha' i \eta^{\mu\nu} \frac{\partial \delta(\theta'-\theta)}{\partial \theta'} \qquad (3)$$

and hence

$$\left[\ell^\mu(z') , F(\ell^\nu(z')) \right] = -4\pi\alpha' i \frac{\partial \delta(\theta'-\theta)}{\partial \theta'} \frac{dF(\ell^\nu(z'))}{d\ell^\nu(z')} \eta^{\mu\nu} \qquad (4)$$

Therefore

$$R_m = \frac{1}{8\pi\alpha'} \int_{-\pi}^{+\pi} d\theta' \left(-i \frac{\partial}{\partial\theta} \right) e^{im\theta'} (-4\pi\alpha' i) \frac{\partial \delta(\theta'-\theta)}{\partial \theta'} \cdot$$

$$\left\{ k\cdot \ell(z') \frac{1}{k\ell(z)} + \frac{1}{k\ell(z)} k\cdot\ell(z') \right\} =$$

$$= + \frac{1}{2} \frac{d}{d\theta} \int_{-\pi}^{+\pi} d\theta' \, \delta(\theta'-\theta) \frac{\partial}{\partial\theta'} \left\{ e^{im\theta'} k\cdot\ell(z') \frac{1}{k\cdot\ell(z)} + e^{im\theta'} \frac{1}{k\ell(z)} k\cdot\ell(z') \right\}$$

$$= \frac{d}{d\theta} \left\{ im\, e^{im\theta} + e^{im\theta} \frac{d}{d\theta} \ln \left[-k\cdot\ell(z) \right] \right\} =$$

$$= e^{im\theta} \left(-i \frac{d}{d\theta} + m \right) i \frac{d}{d\theta} \ln \left[-k\cdot\ell(z) \right] - m^2 e^{im\theta} \Rightarrow$$

$$\left[L_m , -z \frac{d}{dz} \ln \left[-k . \xi(z) \right] \right] =$$

$$= z^m \left(z \frac{d}{dz} + m \right) \left\{ -z \frac{d}{dz} \ln \left[-k . \xi(z) \right] \right\} - m^2 z^m \qquad (1)$$

Hence it follows that the required operator is

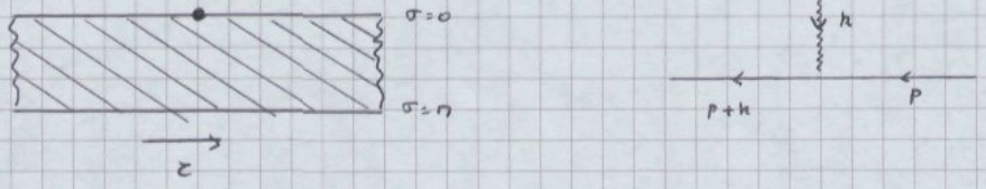
$$V^\mu(k;z) = : \xi^\mu(z) \, e^{-ik . Q(z)} : -$$

$$- \alpha' k^\mu z \frac{d}{dz} \left[\ln (-k . \xi(z)) \right] \, e^{-ik . Q(z)} , \qquad (2)$$

This result is unique only up to the addition of an arbitrary multiple of the trivial term

$$k^\mu z \frac{d}{dz} \, e^{-ik . Q(z)} = - k^\mu \left[k . \xi(z) \right] e^{-ik Q(z)} \qquad (3)$$

Let us return to our problem. The problem addressed here is the construction of an operator $V(\zeta, k, \tau)$ describing the emission of a massless vector with momentum k^μ ($\mu = 0, \ldots 25$) and polarization vector $\zeta^\mu(k)$ at time τ from $\sigma = 0$ end of a string, as depicted in next figure



(4)

Emission of a point particle from the $\sigma = 0$ boundary of a world sheet and kinematics of a vertex

We consider only physical on-mass-shell emissions satisfying

$$k . k = \zeta . k = 0 \qquad (5)$$

as this is sufficient to construct S-matrix elements, the objects of primary physical interest. The vector-emission vertex has been given above (3). Particularizing it to (5) and the $\sigma = 0$ end of the world sheet we get the covariant vertex

$$V(\xi, k, \tau) = -\frac{\pi}{2\alpha'} g\, \xi \cdot p_c(\tau)\, e^{-ik\cdot X(\tau)} \qquad (1)$$

where we have taken into account (VII-126)

$$Q^\mu(z = e^{i\tau}) = X^\mu(0,\tau) \equiv X^\mu(\tau) \qquad (2)$$

$$p^\mu(z = e^{i\tau}) = \dot{X}^\mu(0,\tau) + X'^\mu(0,\tau) = \dot{X}^\mu(0,\tau) = 2\pi\alpha' P_c^\mu(0,\tau) = 2\pi\alpha' P_c^\mu(\tau)$$

In the light-cone gauge formalism $X_-(\tau)$ is very complicated and does not commute with the other components, so that eq (1) becomes ill-defined. Also the fact that $X_+(\tau)$ is so simple

$$X_+(\tau) = q_+ + 2\alpha' p_+ \tau \qquad (3)$$

means effectively that $a_{m,+} = 0$. This formalism requires that the emission vertex be described in the gauge

$$\xi_-(k) = 0 \qquad (4)$$

The transverse polarization states $\xi^i(k)$ are unrestricted and $\xi_+(k)$ is determined from $\xi(k)\cdot k = 0$, unless k_- vanishes, in which case $\xi_-(k)$ is an independent mode and the transverse modes have one restriction.

Consider the special case in which the only non zero component of k^μ is k_-. In this case (1) reduces to

$$V(\xi, k_-, \tau) = \frac{\pi}{2\alpha'} g\, p_c^i(\tau)\, e^{-ik_- X_+(\tau)} \qquad (5)$$

$$X_+(\tau) = q_+ + 2\alpha' p_+ \tau$$

$$p_c^i(\tau) = \frac{1}{2\pi\alpha'} \sum_{m=-\infty}^{+\infty} \alpha_m^i\, e^{-im\tau}$$

The approach taken here is to postulate (5) as a starting point and to deduce the general case by a Lorentz transformation.

Let us remember that the mass shell condition for bosonic open strings is $L_0 = 1$ and the τ dependence of any operator $A(\tau)$ is

$$A(\tau) = e^{iH\tau} A(0)\, e^{-iH\tau}, \qquad H = L_0 - 1 \qquad (6)$$

or equivalently

$$A(\tau) = e^{iL_0\tau} A(0)\, e^{-iL_0(\tau)} \qquad (7)$$

Note that a matrix element of $A(\tau)$ between any pair of physical on mass shell states is independent of τ. Correspondingly, the τ derivative of any operator has vanishing matrix elements. Such operators never contribute to any physical calculation and may always be dropped.

The generalization of eq (7.5) to an arbitrary massless momentum vector k^μ is obtained by a Lorentz transformation that

$$(0, k_-, 0) \longrightarrow \left(\frac{k^i k^i}{2k_-}, k_-, k^i \right) \tag{1}$$

Since

$$[M^i_-, P_+] = i P^i \quad , \quad [M^i_-, P_-] = 0 \quad , \quad [M^i_-, P^j] = i \delta_{ij} P_- \tag{2}$$

we get

$$\begin{aligned} &\exp\left(i a^i M^i_-\right) P_+ \exp\left(-i a^i M^i_-\right) = P_+ - a^i P^i + \frac{1}{2} a^i a^j P_- \\ &\exp\left(i a^i M^i_-\right) P_- \exp\left(-i a^i M^i_-\right) = P_- \\ &\exp\left(i a^i M^i_-\right) P^j \exp\left(-i a^i M^i_-\right) = P^j - a^j P_- \end{aligned} \tag{3}$$

where we used

$$e^{+A} B e^{-A} = B + \frac{1}{1!}[A, B] + \frac{1}{2!}[A, [A, B]] + \cdots \tag{4}$$

Then the appropiate choice of vertex is

$$V(\xi, k, \tau) = U(k)\, V(\xi, k_-, \tau)\, U^{-1}(k) \quad , \quad U(k) \equiv \exp\left\{ i \frac{k^i}{k_-} M^i_- \right\} \tag{5}$$

where M^i_- is given in (VIII - 7.4). The form of the answer is not very useful for explicit calculations. The expansion of (5) in multiple commutators is reasonably tractable for the case $k_+ = 0$. The mass-shell condition in such a frame is $k^i k^i = 0$, which forces us to consider complex values (which is no problem) in order not to be restricted to the trivial case $k^i = 0$. Vertices with $k_+ \neq 0$ can also be evaluated, but they are rather messy. The special vertices $k_+ = 0$ are sufficiently general for a complete determination of amplitudes with 26 or fewer external lines.

Let us therefore consider the vertex with $k_+ = 0$, but k^i and k_- non-zero. It is given by

$$V(\zeta, k, \tau) = -\frac{\pi}{2\alpha'} g \, \zeta . P_\tau(\tau) \, e^{-ik.X(\tau)} \quad , \quad k_+ = 0 \; , \; \zeta_- = 0 \tag{1}$$

where

$$P_{\tau -}(\tau) = P_- + \frac{1}{2\pi\alpha' P_+} \sum_{\substack{m=-\infty \\ m\neq 0}}^{+\infty} \alpha_{m,-} \, e^{-im\tau} \tag{2}$$

A convenient method of verifying this result is to use the group property for $M^i{}_-$ transformation to argue that it is sufficient to check the effect in an infinitesimal change $k^i \to k^i + \delta^i$. Specifically one should show that

$$\left[i \frac{\delta^i}{k_-} M^i{}_- , \; V(\zeta, k, \tau) \right] = -\frac{\pi}{2\alpha'} g \, [\zeta . P_\tau(\tau)] \, i \, \delta^i X^i(\tau) \, e^{-ik.X(\tau)} +$$

$$+ \; \tau \text{ derivative term} \tag{3}$$

where

$$\delta^i k^i = \delta^i \zeta^i = 0 \tag{4}$$

in order to maintain the physical state conditions (6.5) to first order in δ. It is quite straightforward to verify (3)

$$A \equiv \left[i \frac{\delta^i}{k_-} M^i{}_- , \; V(\zeta, k, \tau) \right] =$$

$$= -\frac{i}{k_-} \frac{\pi}{2\alpha'} g \left[\delta^i M^i_- , \; \zeta . P_\tau(\tau) \, e^{-ik X(\tau)} \right] =$$

$$= -\frac{i}{k_-} \frac{\pi}{2\alpha'} g \left\{ [\delta^i M^i_- , \; \zeta . P_\tau(\tau)] \, e^{-ik.X(\tau)} + (\zeta . P_\tau(\tau)) \, [\delta^i M^i_- , \; e^{-ik.X(\tau)}] \right\}$$

But up to time derivative terms (rembember (8.3))

$$[\delta^i M^i{}_- , \; \zeta P_\tau(\tau)] = [\delta^i M^i{}_- , \; \zeta_+ P_{\tau -}(\tau) - \zeta^j P^j(\tau)] = 0$$

$$[\delta^i M^i{}_- , \; -ik.X(\tau)] = -i \, [\delta^i M^i{}_- , \; k_- X_+(\tau) - k^j X^j(\tau)] = k_- \delta^i X^i(\tau) \tag{5}$$

$$[\delta^i M^i{}_- , \; (-ik X(\tau))^m] = m \, (k_- \delta^i X^i(\tau)) \, (-ik X(\tau))^{m-1}$$

and hence

$$A = -\frac{i}{k_-} \frac{\pi}{2\alpha'} g \, (\zeta . P_\tau(\tau)) \, k_- \delta^i X^i(\tau) \, e^{-ik X(\tau)} \qquad \text{Q.E.D.}$$

The τ derivative term appears because the commutation rule $[J^i{}_-, J^j{}_-] = 0$ is only satisfied on mass-shell state, whereas initial and final string

configurations are not so restricted. Of course, as has already been explained, the τ derivative term does not contribute to matrix elements between on mass-shell states. Later on we will explain why derivative terms can also be dropped when calculating tree amplitudes, even though the internal lines are not on shell.

Let us now calculate the coupling of three massless vector particles. The first issue is how to relate a vector state described by a polarization vector $\xi^\mu(k)$ to the states $|i\rangle \equiv \alpha^i_{-1}|0\rangle$. For a vector the formalism requires the gauge $\xi_- = 0$ and the transversality condition for a physical state then reduces to $k_-\xi_+ = k^i\xi^i$. The appropriate definition is then

$$|\xi\rangle = |i\rangle\left(\xi^i - \frac{k^i}{k_+}\xi_+\right) \tag{1}$$

The quantity that we wish to calculate is

$$\langle\xi_1|V(\xi_2)|\xi_3\rangle \equiv \xi_1^i\langle 0;k_1|\alpha^i_{+1}\,(+2\pi\alpha' g)\,(\xi_2^R P_z^R(0))\,e^{+i\vec{k}_2\cdot\vec{X}(0)}\,\alpha^j_{-1}|0,k_3\rangle\xi_3^j$$

$$= \frac{\pi}{2\alpha'}g\,\xi_1^i\xi_2^R\xi_3^j\,\langle 0|e^{+ik_1q}\alpha^i_{+1}P_z^R(0)e^{+i\vec{k}_2\vec{X}(0)}\alpha^j_{-1}e^{-ik_3q}|0\rangle \tag{1'}$$

where we have taken all polarization vectors as well as k_2 transversal. Once this result is obtained, it can be uniquely extended to a covariant expression, which is guaranteed to exist due to the Lorentz-invariance built into the formalism by eq. (8.5). Remember

$$P_z^R(0) = \frac{1}{2\pi\alpha'}\sum_{n=-\infty}^{+\infty}\alpha_n^R\,, \qquad X^\ell(0) = q^\ell + i\sum_{\substack{m=-\infty\\ m\neq 0}}^{+\infty}\frac{1}{m}\alpha_m^\ell \tag{2}$$

Since $[\alpha^i_{+i}, P_z^k(0)] = \delta^{ik}/\pi$

$$A^{ikj} \equiv \langle 0|e^{ik_1q}\alpha^i_{+1}P_z^k(0)e^{+i\vec{k}_2\vec{X}(0)}\alpha^j_{-1}e^{-ik_3q}|0\rangle =$$

$$= \langle 0|e^{ik_1q}P_z^k(0)\alpha^i_{+i}e^{+i\vec{k}_2\vec{X}(0)}\alpha^j_{-1}e^{-ik_3q}|0\rangle + \frac{1}{\pi}\delta^{ik}\langle 0|e^{ik_1q}e^{i\vec{k}_2\vec{X}(0)}\alpha^j_{-1}e^{-ik_3q}|0\rangle \tag{3}$$

Notice $m\neq 0$

$$[\alpha_m^i\,,\ i\vec{k}_2\vec{X}(0)] = 2\alpha' k_2^i$$

$$[\alpha_m^i\,,\ e^{i\vec{k}_2\vec{X}(0)}] = 2\alpha' k_2^i\,e^{i\vec{k}_2\vec{X}(0)} \tag{4}$$

Then

$$\langle 0|\, e^{ik_1 q}\, e^{i\vec{k}_2\cdot\vec{X}(0)}\, \alpha_{-1}^{j}\, e^{-ik_3 q}\,|0\rangle = 2\alpha' k_2^{j}\,\langle 0|\, e^{ik_1 q}\, e^{i\vec{k}_2\vec{q}}\, e^{-ik_3 q}\,|0\rangle = 2\alpha' k_2^{j} \qquad (1)$$

where $k_1 = k_2 + k_3$. Therefore

$$A^{ijk} = \frac{1}{\pi}\, 2\alpha'\, \delta^{ik} k_2^{j} + \langle 0|\, e^{ik_1 q}\, p_z^{k}(0)\, e^{+i\vec{k}_2\vec{X}(0)}\, \alpha_{+1}^{i}\,\alpha_{-1}^{j}\, e^{-ik_3 q}\,|0\rangle$$

$$+ \langle 0|\, e^{ik_1 q}\, p_z^{k}(0)\, e^{i\vec{k}_2\cdot\vec{X}(0)}\, \alpha_{-1}^{j}\, e^{-ik_3 q}\,|0\rangle\, 2\alpha' k_2^{i} =$$

$$= \frac{2\alpha'}{\pi}\,\delta^{ik} k_2^{j} + 2\alpha'\delta^{ij}\,\langle 0|\, e^{+ik_1 q}\, p_z^{k}(0)\, e^{i\vec{k}_2\cdot\vec{X}(0)}\, e^{-ik_3 q}\,|0\rangle$$

$$+ 2\alpha' k_2^{i}\,\langle 0|\, e^{ik_1 q}\, p_z^{k}(0)\, e^{i\vec{k}_2\cdot\vec{X}(0)}\, \alpha_{-1}^{j}\, e^{-ik_3 q}\,|0\rangle =$$

$$= \frac{2\alpha'}{\pi}\,\delta^{ik} k_2^{j} + \frac{2\alpha'}{\pi}\,\delta^{ij} k_1^{k} + \frac{1}{\pi}\,\delta^{ij}\sum_{m=1}^{\infty}\langle 0|\, e^{ik_1 q}\,\alpha_m^{k}\, e^{i\vec{k}_2\cdot\vec{X}(0)}\, e^{-ik_3 q}\,|0\rangle$$

$$+ \frac{2\alpha'}{\pi}\, k_2^{i} k_1^{k}\,\langle 0|\, e^{ik_1 q}\, e^{i\vec{k}_2\cdot\vec{X}(0)}\,\alpha_{-1}^{j}\, e^{-ik_3 q}\,|0\rangle$$

$$+ \frac{1}{\pi}\, k_2^{i}\sum_{m=1}^{\infty}\langle 0|\, e^{ik_1 q}\,\alpha_m^{k}\, e^{i\vec{k}_2\cdot\vec{X}(0)}\,\alpha_{-1}^{j}\, e^{-ik_3 q}\,|0\rangle =$$

$$= \frac{2\alpha'}{\pi}\,\delta^{ik} k_2^{j} + \frac{2\alpha'}{\pi}\,\delta^{ij} k_1^{k} + \frac{2\alpha'}{\pi}\,\delta^{ij} k_2^{k} - \frac{4\alpha'^2}{\pi}\, k_2^{i} k_1^{k} k_2^{j}$$

$$+ \frac{1}{\pi}\, k_2^{i}\sum_{m=1}^{\infty}\langle 0|\, e^{ik_1 q}\, e^{i\vec{k}_2\cdot\vec{X}(0)}\,\alpha_m^{k}\,\alpha_{-1}^{j}\, e^{-ik_3 q}\,|0\rangle +$$

$$+ \frac{2\alpha'}{\pi}\, k_2^{j} k_2^{k}\sum_{m=1}^{\infty}\langle 0|\, e^{ik_1 q}\, e^{i\vec{k}_2\cdot\vec{X}(0)}\,\alpha_{-1}^{j}\, e^{-ik_3 q}\,|0\rangle \qquad (11)$$

Since $\vec{k}_2\cdot\vec{\xi}_2 = 0$

$$A^{ikj} = \frac{2\alpha'}{\pi}\,\delta^{ik} k_2^{j} + \frac{2\alpha'}{\pi}\,\delta^{ij} k_1^{k} - \frac{4\alpha'^2}{\pi}\, k_2^{i} k_1^{k} k_2^{j} + \frac{2\alpha'}{\pi}\, k_2^{i}\,\delta^{kj} \qquad (2)$$

Hence

$$\langle \xi_1|V(\xi_2)|\xi_3\rangle = g\,\{\,(\vec{\xi}_1\vec{\xi}_2)(\vec{k}_2\vec{\xi}_3) + (\vec{\xi}_1\vec{\xi}_3)(\vec{k}_1\vec{\xi}_2) + (\vec{\xi}_2\vec{\xi}_3)(\vec{k}_2\vec{\xi}_1)$$

$$+ 2\alpha'(\vec{\xi}_1\vec{k}_2)(\vec{\xi}_2\vec{k}_1)(\vec{k}_2\vec{\xi}_3)\,\} \qquad (3)$$

and written in an invariant way

$$\langle \xi_1|V(\xi_2)|\xi_3\rangle = g\,\{\,(\xi_1\cdot k_2)(\xi_2\cdot\xi_3) + (\xi_2\cdot k_3)(\xi_1\cdot\xi_3) + (\xi_3\cdot k_1)(\xi_1\cdot\xi_2)$$

$$+ 2\alpha'(\xi_1\cdot k_2)(\xi_2\cdot k_3)(\xi_3\cdot k_1)\,\} \qquad (4)$$

The first three terms are the usual Yang-Mills result, and the last term is an $O(2\alpha')$ correction.

Massless Vertices for SST I

After this introduction let us consider the vertices in a superstring theory with open superstring. The first issue is how to relate a vector state described by a polarization vector $\xi^\mu(k)$ or the spinor state described by a Dirac wave function $u^a(k)$ to the states $|i\rangle$ and $|a\rangle$. For a vector the formalism requires

$$\xi_-(k) = 0 \tag{1}$$

The transversality condition for a physical state then reduces to

$$k_- \xi_+(k) = k^i \xi^i(k) \tag{2}$$

The only restriction on a spinor $u^a(k)$ is that it satisfy the Dirac equation

$$\gamma\cdot k\, u(k) = 0 \tag{3}$$

The appropiate definitions are then

$$|\xi\rangle \equiv |i\rangle \left(\xi^i - \frac{k^i}{k_+} \xi_+ \right) \tag{4}$$

$$|u\rangle \equiv -\frac{2}{\sqrt{k_+}} \bar{u}^a(k) |a\rangle = -\frac{i}{4\sqrt{k_+}} \bar{u}(k) \gamma^i S_0 |i\rangle$$

As the first test of these definitions let us consider the supersymmetry transformations

$$(\bar\epsilon\cdot Q)|\xi\rangle = \left\{ i\sqrt{p_+}\,(\bar\epsilon\gamma_- S_0) - \frac{i}{2\alpha'\sqrt{p_+}} \sum_{m=-\infty}^{+\infty} (\bar\epsilon \gamma^e S_{-m})\alpha^e_m \right\} |\xi\rangle =$$

$$= \frac{i}{\sqrt{p_+}} \left\{ p_+ (\bar\epsilon\gamma_- S_0) - (\bar\epsilon\gamma^e S_0) p^e \right\} |\xi\rangle = \frac{i}{\sqrt{k_+}} [\bar\epsilon\gamma\cdot k S_0]\,|\xi\rangle$$

$$= \frac{i}{\sqrt{k_+}} [\bar\epsilon\gamma\cdot k S_0]\,|i\rangle \left(\xi^i - \frac{k^i}{k_+}\xi_+\right) =$$

$$= -\frac{i}{8\sqrt{k_+}} [\bar\epsilon\gamma\cdot k \gamma^i]^a [\gamma^i S_0]^a |i\rangle \left(\xi^i - \frac{k^i}{k_+}\xi_+\right) =$$

$$= -\frac{1}{\sqrt{k_+}} [\bar\epsilon\gamma\cdot k \gamma^i]^a |a\rangle \left(\xi^i - \frac{k^i}{k_+}\xi_+\right) \tag{5}$$

Identifying the right-hand side of this equation as a $|u\rangle$ state gives the relation

$$u(k) = \frac{1}{2}\left(\Sigma^i - \frac{k^i}{k_+}\Sigma_+\right)\gamma^i(\gamma\cdot k)\epsilon + \gamma_+ w^- \qquad (1)$$

where the term with w^- has appeared because $\gamma_+|a\rangle = 0$. In order to find w^- let us impose that $u(k)$ is a solution of the Dirac equation

$$\not{k}\, u(k) = \frac{1}{2}\left(\Sigma^i - \frac{k^i}{k_+}\Sigma_+\right)\not{k}\gamma^i\not{k}\epsilon + \not{k}\gamma_+ w^- = 0 \qquad (2)$$

Since

$$(\not{k}\gamma^i\not{k}) = \gamma^\mu\gamma^\nu\gamma^\nu k_\mu k_\nu = [2\eta^{\mu i} - \gamma^i\gamma^\mu]\gamma^\nu k_\mu k_\nu = 2k^i\not{k} \qquad (3)$$

we have

$$(k^i\Sigma^i - \frac{k^ik^i}{k_+}\Sigma_+)\not{k}\epsilon + \not{k}\gamma_+ w^- = 0 \Rightarrow \not{k}\gamma_+ w^- = k_-\Sigma_+\not{k}\epsilon \Rightarrow$$

$$\gamma_-\not{k}\gamma_+ w^- = k_-\Sigma_+\gamma_-\not{k}\epsilon \Rightarrow +k^i\gamma^i\gamma_-\gamma_+ w^- = k_-\Sigma_+\gamma_-\not{k}\epsilon$$

$$-k^ik^i\gamma_-\gamma_+ w^- = k_-\Sigma_+(k^i\gamma^i)\gamma_-\not{k}\epsilon \Rightarrow -2k_+(2-\gamma_+\gamma_-)w^- = \Sigma_+(k^i\gamma^i)\gamma_-\not{k}\epsilon$$

$$-4\gamma_+ k_+ w^- = \Sigma_+\gamma_+(k^i\gamma^i)\gamma_-\not{k}\epsilon \Rightarrow -4k_+\gamma_+ w^- = -\Sigma_+\gamma_+\gamma_-(k^i\gamma^i)\not{k}\epsilon \Rightarrow$$

$$-4k_+\gamma_+ w^- = -\Sigma_+(\gamma_+\gamma_-)[\not{k} - k_+\gamma_- - k_-\gamma_+]\not{k}\epsilon = +\Sigma_+\gamma_+\gamma_-\gamma_+\not{k}\not{k}\epsilon = 2\Sigma_+k_-\gamma_+\not{k}\epsilon$$

$$\gamma_+ w^- = -\frac{k_-}{2k_+}\Sigma_+\gamma_+\not{k}\epsilon \qquad (4)$$

Hence

$$u(k) = \frac{1}{2}\left(\Sigma^i - \frac{k^i}{k_+}\Sigma_+\right)\gamma^i\not{k}\epsilon - \frac{k_-}{2k_+}\Sigma_+\gamma_+\not{k}\epsilon =$$

$$= \frac{1}{2}\left\{\Sigma^i\gamma^i - \frac{\Sigma_+}{k_+}(k^i\gamma^i - k_-\gamma_+)\right\}\not{k}\epsilon = \frac{1}{2}\left\{\Sigma^i\gamma^i - \frac{\Sigma_+}{k_+}(k_+\gamma_- - \not{k})\right\}\not{k}\epsilon$$

$$= -\frac{1}{2}\left\{\Sigma_+\gamma_- - \Sigma^i\gamma^i\right\}\not{k}\epsilon = -\frac{1}{2}\not{k}\not{k}\epsilon = +i\frac{1}{2}\sigma^{\mu\nu}\Sigma_\mu k_\nu\epsilon \Rightarrow$$

$$u(k) = \frac{i}{2}\sigma^{\mu\nu}\Sigma_\mu k_\nu\epsilon \qquad (5)$$

Similarly

$$(\bar\epsilon\, Q)|u\rangle = \left\{i\sqrt{P_+}(\bar\epsilon\gamma_- S_0) - \frac{i}{2\alpha'\sqrt{P_+}}\sum_{\substack{n=-\infty\\ n\neq 0}}^{+\infty}\frac{1}{n}(\bar\epsilon\gamma^e\Sigma_m)\alpha_m^e\right\}|u\rangle$$

$$= \frac{i}{\sqrt{k_+}} \left[\bar{\epsilon} \not{k} S_0 \right] |u\rangle = \frac{1}{4k_+} \left[\bar{\epsilon} \not{k} S_0 \right] \left[\bar{u}(k) \gamma^i S_0 \right] |i\rangle = \frac{1}{4k_+} \left[\bar{\epsilon} \not{k} S_0 \right] \left[\bar{S}_0 \gamma^i u(k) \right] |i\rangle$$

$$= \frac{1}{8k_+} \left\{ \left[\bar{\epsilon} \not{k} \gamma_+ h \gamma^i u(k) \right] - \frac{1}{2} \left[\bar{\epsilon} \not{k} \gamma_+ \sigma^{kj} h \gamma^i u(k) \right] R_0^{kj} \right\} |i\rangle$$

$$= \frac{1}{8k_+} \left\{ \left[\bar{\epsilon} \not{k} \gamma_+ \gamma^i u(k) \right] + i \left[\bar{\epsilon} \not{k} \gamma_+ \sigma^{ij} \gamma^j u(k) \right] \right\} |i\rangle$$

$$= \frac{1}{k_+} \left[\bar{\epsilon} \not{k} \gamma_+ \gamma^i u(k) \right] |i\rangle \qquad (1)$$

But

$$\not{k} \gamma_+ \gamma^i u(k) = (k_+ \gamma_- - k^j \gamma^j) \gamma_+ \gamma^i u(k) = k_+ (2 - \gamma_+ \gamma_-) \gamma^i u(k) + k^j \gamma_+ \gamma^j \gamma^i u(k)$$

$$= \left\{ 2k_+ \gamma^i - k_+ \gamma_+ \gamma_- \gamma^i + k^j \gamma_+ (-2\delta_{ij} - \gamma^i \gamma^j) \right\} u(k)$$

$$= \left\{ 2k_+ \gamma^i - k_+ \gamma_+ \gamma_- \gamma^i - 2k^i \gamma_+ - \gamma_+ \gamma^i (-\not{k} + \gamma_- k_+) \right\} u(k)$$

$$= \left\{ -k_+ \gamma_+ \gamma_- \gamma^i - \gamma_+ \gamma^i \gamma_- k_+ + 2k_+ \gamma^i - 2k^i \gamma_+ \right\} u(k) = 2k_+ \left(\gamma^i - \frac{k^i}{k_+} \gamma_+ \right) u(k) \qquad (2)$$

Hence

$$(\bar{\epsilon} Q) |u\rangle = 2 \left[\bar{\epsilon} \left(\gamma^i - \frac{k^i}{k_+} \gamma_+ \right) u(k) \right] |i\rangle \qquad (3)$$

which should be identified with a state $|S\rangle$ with $S_- = (k.S) = 0$

$$S^i - \frac{k^i}{k_+} S_+ = 2 \left[\bar{\epsilon} \left(\gamma^i - \frac{k^i}{k_+} \gamma_+ \right) u(k) \right] \Rightarrow \qquad (4)$$

$$k^i S^i - \frac{k^i k^i}{k_+} S_+ = 2 \left[\bar{\epsilon} \left(k^i \gamma^i - \frac{k^i k^i}{k_+} \gamma_+ \right) u(k) \right] \Rightarrow$$

$$k_- S_+ - 2k_- S_+ = 2 \left[\bar{\epsilon} \left(\gamma_+ k_- + \gamma_- k_+ - 2k_- \gamma_+ \right) u(k) \right] \Rightarrow$$

$$S_+ = 2 \bar{\epsilon} \left(\gamma_+ - \frac{k_+}{k_-} \gamma_- \right) u(k) \qquad (5)$$

Substituting this in (4)

$$S^i - \frac{k^i}{k_+} 2 \bar{\epsilon} \left(\gamma_+ - \frac{k_+}{k_-} \gamma_- \right) u(k) = 2 \bar{\epsilon} \left(\gamma^i - \frac{k^i}{k_+} \gamma_+ \right) u(k) \Rightarrow$$

$$S^i = 2 \bar{\epsilon} \left(\gamma^i - \frac{k^i}{k_-} \gamma_- \right) u(k) \qquad (6)$$

And hence

$$S^\mu(k) = 2 \bar{\epsilon} \left\{ \gamma^\mu - \frac{k^\mu}{k_-} \gamma_- \right\} u(k) \qquad (7)$$

Therefore

$$\bar{\epsilon}\cdot Q\,|\xi\rangle = |u\rangle \qquad u(k) = \frac{i}{2}\,\sigma^{\mu\nu}\,\xi_\mu\,k_\nu\,\epsilon$$

$$\bar{\epsilon}\cdot Q\,|u\rangle = |\xi\rangle \qquad \xi^\mu(k) = 2\,\bar{\epsilon}\left[\gamma^\mu - \frac{k^\mu}{k_-}\gamma_-\right]u(k) \tag{1}$$

These equations are exactly the transformation laws that one would deduce from the field theory in the light-cone gauge. It is also possible to prove that the states $|\xi\rangle$ and $|u\rangle$, transform under each of the Lorentz generators in the appropriate covariant fashion.

Now we must construct vertex operators describing the couplings of the massless states described above with an arbitrary pair of states of the theory. These vertices serve to define the interacting theory completely since, as we will see later on, they allow the calculation of all tree graphs with massless external states. Furthermore, they have simple supersymmetry transformation properties that allow supersymmetry relations among S-matrix elements to be deduced.

Our method consists first of constructing the vertices in a particular frame where the formulas are specially simple. The result in a general frame is then deduced by applying a Lorentz rotation. The special frame that turns out to be most convenient is one in which the emitted massless state has its momentum in the minus direction only, that is $k^i = k_+ = 0$, $k_- \neq 0$. In the case of a vector particle this implies $\xi_+ = 0$, whereas ξ^i is unrestricted and $\xi_- = 0$ is a gauge choice. In this case we "guess" the formula

$$V_B(\xi^i, k_-, z) = +g\,\frac{\pi}{2\alpha'}\;\xi^i\,P^i_L(z)\;e^{-ik_-\bar{X}_+(z)}$$

$$\bar{X}_+(z) = q_+ + 2\alpha'\,p_+\,z \tag{2}$$

$$P^i_L(z) = \frac{1}{2\pi\alpha'}\sum_{n=-\infty}^{+\infty}\alpha^i_n\,e^{-inz}$$

Let us now compute

$$[\bar{\epsilon}\cdot Q\,,\; V_B(\xi^i, k_-, z)] = \Big[\,i\sqrt{p_+}\,(\bar{\epsilon}\,\gamma_-\,S_0) - \frac{i}{2\alpha'\sqrt{p_+}}\sum_{n=-\infty}^{+\infty}(\bar{\epsilon}\,\gamma^i\,S_{-n})\,\alpha^i_n\,,\; g\,\frac{\pi}{2\alpha'}\,\xi^i P^i_L(z)\,e^{-ik_-\bar{X}_+(z)}\,\Big]$$

$$= -i \frac{g}{(2\alpha')^3 \sqrt{P_+}} \zeta^j e^{-ik_- \bar{X}_+(\tau)} \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} (\epsilon \gamma^i S_m) e^{-in\tau} [\alpha_m^i, \alpha_n^j]$$

$$= -i \frac{g}{(2\alpha')^2 \sqrt{P_+}} \zeta^i e^{-ik_- \bar{X}_+(\tau)} \sum_{m=-\infty}^{+\infty} m (\epsilon \gamma^i S_m) e^{im\tau} \qquad (1)$$

Hence

$$[\bar{\epsilon} Q, V_B(\zeta^i, k_-, \tau)] = \frac{g}{(2\alpha')^2 \sqrt{P_+}} i \bar{\epsilon} \gamma \cdot \zeta \dot{S}(\tau)] e^{-ik_- \bar{X}_+(\tau)} \qquad (2)$$

$$S(\tau) \equiv \sum_{m=-\infty}^{+\infty} S_m e^{im\tau}$$

Since

$$\frac{d}{d\tau} \left\{ \frac{1}{\sqrt{P_+}} S(\tau) e^{-ik_- \bar{X}_+(\tau)} \right\} = \frac{1}{\sqrt{P_+}} \dot{S}(\tau) e^{-ik_- \bar{X}_+(\tau)} - i 2\alpha' k_- \sqrt{P_+} S(\tau) e^{-ik_- \bar{X}_+(\tau)} \qquad (3)$$

Therefore

$$[\bar{\epsilon} Q, V_B(\zeta^i, k_-, \tau)] = + i \frac{g}{2\alpha'} \sqrt{P_+} k_- [\bar{\epsilon} \gamma \cdot \zeta S(\tau)] e^{-ik_- \bar{X}_+(\tau)}$$

$$+ \frac{d}{d\tau} \left\{ \frac{g}{(2\alpha')^2 \sqrt{P_+}} [\bar{\epsilon} \gamma \cdot \zeta S(\tau)] e^{-ik_- \bar{X}_+(\tau)} \right\} \qquad (4)$$

The derivative term in this equation may be dropped for reasons frequently invoked in the old days of dual string theory under the rubric of "the cancelled-propagator argument". The idea is that any operator $A(\tau)$ satisfies

$$\frac{d}{d\tau} A(\tau) = i [L_0, A(\tau)] \qquad (5)$$

Therefore a derivative has vanishing matrix elements between an arbitrary pair of physical on-mass-shell states. Moreover, when a derivative is sandwiched between a pair of propagators, or a propagator and a physical state, a propagator is cancelled or the state is annihilated in each of the two terms resulting from substitution of the commutator. By a simple analyticity argument a term in which all poles are cancelled in a particular invariant energy variable must be identically zero because of the Regge asymptotic behavior of string amplitudes. This justifies dropping the derivative in eq. (4).

We wish to identify the right-hand side of eq (16.4) as the fermionic vertex $V_F(u, k_-, \tau)$ with u given by equation (15.1), which in the present kinematical context is

$$u(k) = \frac{i}{2}\,\sigma^{\mu\nu}\,\xi_\mu\,k_\nu\,\epsilon = -\frac{1}{2}\,k_-\,\gamma_+\,\gamma^i\,\xi^i\,\epsilon \qquad (1)$$

Therefore

$$k_-\,[\,\bar\xi(\tau)\,\gamma^i\,\xi^i\,\epsilon\,] = -\,\bar\xi(\tau)\,\gamma_-\,u(k) \qquad (2)$$

and we conclude

$$V_F(u, k_-, \tau) = +\,i\,\frac{g}{2\alpha'}\,\sqrt{p_+}\;\;\bar\xi(\tau)\,\gamma_-\,u(k)\;e^{-ik_-\,x_+(\tau)} \qquad (3)$$

The Dirac equation in this frame is just $\gamma_+\,u(k) = 0$. We may also transform back using

$$[\,\bar\epsilon\,Q\,,\,V_F(u, k_-, \tau)\,] = \Big[\,i\,\sqrt{p_+}\;(\bar\epsilon\,\gamma_-\,\xi_0) - \frac{i}{2\alpha'\sqrt{p_+}}\sum_{m=-\infty}^{+\infty}(\bar\epsilon\,\gamma^i\,\xi_{-m})\,\alpha^i_m\,,\,+\,i\,\frac{g}{2\alpha'}\,\sqrt{p_+}\;\bar\xi(\tau)\,\gamma_-\,u(k)\,e^{-ik_-x_+(\tau)}\Big]$$

$$= -\frac{g}{2\alpha'}\;p_+\;(\bar\epsilon\,\gamma_-)^a\,[\gamma_-\,u(k)]^b\;e^{-ik_-\,x_+(\tau)}\sum_{m=-\infty}^{+\infty}\{\,\xi_0^a\,,\,\bar\xi_m^b\,\}\,e^{-im\tau}$$

$$+\;\frac{g}{(2\alpha')^2}\;(\bar\epsilon\,\gamma^i)^a\,[\gamma_-\,u(k)]^b\;e^{-ik_-\,x_+(\tau)}\sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty}\{\,\xi^a_{-m}\,,\,\bar\xi^b_n\,\}\,\alpha^i_n\,e^{-i(m+n)\tau}$$

$$= -\frac{g}{2\alpha'}\;p_+\;[\,\bar\epsilon\,\gamma_-\,\gamma_+\,h\,\gamma_-\,u(k)\,]\;e^{-ik_-\,x_+(\tau)}$$

$$+\;\frac{g}{(2\alpha')^2}\;[\,\bar\epsilon\,\gamma^i\,\gamma_+\,h\,\gamma_-\,u(k)\,]\;e^{-ik_-\,x_+(\tau)}\;\;2\pi\alpha'\,p^i_2(\tau) \qquad \Rightarrow$$

$$[\,\bar\epsilon\,Q\,,\,V_F(u, k_-, \tau)\,] = +\,\pi\,\frac{g}{\alpha'}\,[\,\bar\epsilon\,\gamma^i\,u(k)\,]\,p^i_2(\tau)\;e^{-ik_-\,x_+(\tau)}$$

$$+\;\frac{d}{d\tau}\left\{+\frac{i\,g}{2\alpha'^2}\;\frac{1}{k_-}\;[\,\bar\epsilon\,\gamma_-\,u(k)\,]\;e^{-ik_-\,x_+(\tau)}\right\} \qquad (4)$$

This corresponds to $V_B(J, k, \tau)$ with

$$J^i = +\,2\,[\,\bar\epsilon\,\gamma^i\,u(k)\,]\,, \qquad J_+ = J_- = 0 \qquad (5)$$

in agreement with (15.1)

Therefore

$$[\bar\epsilon Q, \quad V_B(\xi^i, k_-, \tau)] = V_F(u, k_-, \tau)$$

$$[\bar\epsilon Q, \quad V_F(u, k_-, \tau)] = V_B(\xi^i, k_-, \tau)$$

$$V_B(\xi^i, k_-, \tau) = g\frac{\pi}{2\alpha'}\,\xi^i P_L^i(\tau)\, e^{-ik_-X_+(\tau)} \tag{1}$$

$$V_F(u, k_-, \tau) = +ig\frac{1}{2\alpha'}\sqrt{P_+}\;\bar S(\tau)\,\gamma_-\,u(k)\,e^{-ik_-X_+(\tau)}$$

$$u(k) = \frac{i}{2}\,\sigma^{\mu\nu}\xi_\mu k_\nu\,\epsilon \qquad , \qquad \xi^\mu(k) = 2\bar\epsilon\left[\gamma^\mu - \frac{k^\mu}{k_-}\gamma_-\right]u(k)$$

Thus the emitted states are transformed under supersymmetry in the same way as in (15.1)

To obtain the vertices in an arbitrary frame we use

$$V(k^\mu, \tau) = U(k)\,V(k_-, \tau)\,U^{-1}(k) \tag{2}$$

$$U(k) = \exp\left\{i\frac{k^i}{k_-}M^i_-\right\}$$

as we have discussed on pag 8. Of course, eq (2) is correct as it stands, but not very handy for explicit calculations of scattering amplitudes. Therefore evaluation of the multiple commutators is of some value. We have done this for the special case $k_+ = 0$, but with k^i, k_-, ξ^μ, and u otherwise only restricted by the physical state conditions

$$k^ik^i = k\cdot\xi = \not{k}\,u = 0 \tag{3}$$

(Note $k^ik^i = 0$ does not imply $k^i = 0$, since we allow complex values).

$$V_B(\xi, k, \tau) = U(k)\,V(\xi^i, k_-, \tau)\,U^{-1}(k)$$

$$= \exp\left\{i\frac{k^i}{k_-}M^i_-\right\}\, g\frac{\pi}{2\alpha'}\,\xi^j P_L^j(\tau)\, e^{-ik_-X_+(\tau)}\, \exp\left\{-i\frac{k^i}{k_-}M^i_-\right\} =$$

$$= g\frac{\pi}{2\alpha'}\exp\left\{i\frac{k^i}{k_-}M^i_-\right\}(\xi^j P_L^j(\tau))\exp\left\{-i\frac{k^i}{k_-}M^i_-\right\}\cdot$$

$$\exp\left\{i\frac{k^i}{k_-}M^i_-\right\}\, e^{-ik_-X_+(\tau)}\, \exp\left\{-i\frac{k^i}{k_-}M^i_-\right\} \tag{5}$$

Let us remember

$$e^{A}\, B\, e^{-A} = \sum_{m=0}^{\infty} \frac{1}{m!} B_m \qquad (1)$$

$$B_0 \equiv B \;, \qquad B_m = [A, B_{m-1}] \qquad m \geq 1$$

and

$$e^{A}\, e^{B}\, e^{-A} = \exp\left\{ \sum_{m=0}^{\infty} \frac{1}{m!} B_m \right\} \qquad (2)$$

Let us now consider

$$A \equiv i \frac{k^i}{k_-} M^i{}_- \;, \qquad B \equiv -i k_- X_+(\tau) \qquad (3)$$

$$B_0 = -i k_- X_+(\tau)$$

$$B_1 \equiv [A, B_0] = k^i [M^i{}_-, q_+ + 2\alpha' p_+ \tau]$$

$$[e^i{}_-, q_+ + 2\alpha' p_+ \tau] = [q^i p_- - q_- p^i, q_+ + 2\alpha' \tau p_+] =$$

$$= [q^i p_-, q_+] - 2\alpha'\tau [q_- p^i, p_+] = i q^i + 2\alpha'\tau i p^i = i(q^i + 2\alpha' \tau p^i)$$

$$[\hat{L}^i{}_-, q_+ + 2\alpha' p_+ \tau] = [K^i{}_-, q_+ + 2\alpha' p_+ \tau] = 0$$

$$B_1 = i (k^i q^i + 2\alpha' \tau k^i p^i) \qquad (4)$$

$$B_2 = -\frac{k^i k^j}{k_-} [M^i{}_-, q^j + 2\alpha' \tau p^j]$$

$$e^{A}\, B\, e^{-A} = \sum_{m=0}^{\infty} \frac{1}{m!}\, B_m$$

$$e^{A}\, e^{B}\, e^{-A} = \exp\left\{ \sum_{m=0}^{\infty} \frac{1}{m!}\, B_m \right\} \qquad (1)$$

$$B_0 \equiv B \quad , \quad B_m \equiv [A, B_{m-1}] \qquad m \geq 1$$

Let us choose

$$A \equiv i\, \frac{k^i}{k_-}\, M^i_- \quad , \qquad B = \xi^j \beta^j_\xi(z) \qquad (2)$$

$$B_0 = \xi^j \beta^j_\xi(z)$$

$$B_1 = \frac{i\, k^i \xi^j}{k_-}\, [M^i_-, \beta^j_\xi(z)] = \frac{i\, k^i \xi^j}{2\pi\alpha' k_-} \sum_{m=-\infty}^{+\infty} e^{-2\pi i m z}\, [M^i_-, \alpha^j_m]$$

$$\sum_{m=-\infty}^{+\infty} e^{-2\pi i m z}\, [\ell^i_-, \alpha^j_m] = 2\alpha' [\ell^i_-, \beta^j] = i\, 2\alpha'\, \delta_{ij}\, \beta_-$$

$$\sum_{m=-\infty}^{+\infty} e^{-2\pi i m z}\, [\hat{L}^i_-, \alpha^j_m] = -\frac{i}{2\alpha' \beta_+} \sum_{m=-\infty}^{+\infty} \sum_{n=1}^{\infty} \frac{e^{-2\pi i m z}}{m}\, [\alpha^i_{-m}\, \alpha_{m,-} - \alpha_{-m,-}\, \alpha^i_m\, ,\, \alpha^j_m]$$

$$= -\frac{i}{2\alpha' \beta_+} \sum_{m=-\infty}^{+\infty} \sum_{n=1}^{\infty} \frac{e^{-2\pi i m z}}{m} \left\{ \alpha^i_{-m}\, [\alpha_{m,-}, \alpha^j_m] + [\alpha^i_{-m}, \alpha^j_m]\, \alpha_{m,-} \right.$$

$$\left. - \alpha_{-m,-}\, [\alpha^i_m, \alpha^j_m] - [\alpha_{-m,-}, \alpha^j_m]\, \alpha^i_m \right\} =$$

$$= -\frac{i}{2\alpha' \beta_+} \sum_{m=-\infty}^{+\infty} \sum_{n=1}^{\infty} \frac{e^{-2\pi i m z}}{m} \left\{ -m\, \alpha^i_{-m}\, \alpha^j_{n+m} - 2\alpha' m\, \delta_{ij}\, \delta_{n,m}\, \alpha_{m,-} \right.$$

$$\left. - 2\alpha' m\, \delta_{ij}\, \delta_{m,-n}\, \alpha_{-m,-} + m\, \alpha^j_{n-m}\, \alpha^i_m \right\} =$$

$$= \frac{i}{\beta_+}\, \delta_{ij} \sum_{\substack{m=-\infty \\ m \neq 0}}^{+\infty} \alpha_{m,-}\, e^{-2\pi i m z} + \frac{2\beta}{i\beta} \left\{ \frac{1}{2\alpha' \beta_+} \sum_{m=-\infty}^{+\infty} \sum_{n=1}^{\infty} \frac{e^{-2\pi i m z}}{m}\, [-\alpha^i_{-m}\, \alpha^j_{n+m} + \alpha^j_{n-m}\, \alpha^i_m] \right\} =$$

$$= \frac{i}{\beta_+}\, \delta_{ij} \sum_{\substack{m=-\infty \\ m \neq 0}}^{+\infty} \alpha_{m,-}\, e^{-2\pi i m z}$$

$$\sum_{m=-\infty}^{+\infty} e^{-2\pi i m z}\, [\hat{K}^i_-, \alpha^j_m] = \frac{1}{2\alpha' \beta_+} \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} e^{-2\pi i m z}\, [\beta_j K^i_{-n}\, \alpha^j_{n+m}, \alpha^j_m] =$$

$$= -\frac{1}{\beta_+} \sum_{m=-\infty}^{+\infty} m\, e^{-2\pi i m z}\, K^{ij}_{-m} + \frac{2\beta}{i\beta} \left[-\frac{i}{\beta_+} \sum_{m=-\infty}^{+\infty} e^{-2\pi i m z}\, K^{ij}_{-m} \right] = 0$$

$$B_1 = -\frac{k^i \xi^i}{2\pi\alpha' k_-} \left\{ 2\alpha' \beta_- + \frac{1}{\beta_+} \sum_{\substack{m=-\infty \\ m \neq 0}}^{+\infty} \alpha_{m,-}\, e^{-2\pi i m z} \right\} = -\frac{\xi_+}{2\pi\alpha'}\, \beta_{z-}(z)\, 2\pi\alpha' =$$