PERIODIC SOLUTIONS OF SECOND-ORDER DIFFERENTIAL INCLUSIONS SYSTEMS WITH p–LAPLACIAN

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ABSTRACT. Some existence results are obtained for periodic solutions of nonautonomous second-order differential inclusions systems with p–Laplacian.

1. INTRODUCTION

Consider the second order system

$$\ddot{u}(t) = \nabla F(t, u(t)) \text{ a.e. } t \in [0, T]$$
$$u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0$$

(1)

where $T > 0$ and $F : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ satisfies the following assumption:

(A) $F(t, x)$ is measurable in t for each $x \in \mathbb{R}^n$ and continuously differentiable in x for a.e. $t \in [0, T]$, and there exist $a \in C(\mathbb{R}_+, \mathbb{R}_+)$, $b \in L^1(0, T; \mathbb{R}_+)$ such that

$$|F(t, x)| \leq a(\|x\|)b(t), \qquad \|\nabla F(t, x)\| \leq a(\|x\|)b(t)$$

for all $x \in \mathbb{R}^n$ and a.e. $t \in [0, T]$.

In the last years many authors starting with Mawhin and Willem (see [2]) proved the existence of solutions for problem (1) under suitable conditions on the potential F (see [6]-[17]). Also in a series of papers (see [3]-[5]) we have generalized some of these results for the case when the potential F is just locally Lipschitz in the second variable x not continuously differentiable.

The aim of this paper is to consider the problem (1) in a more general sense. More exactly our results represent the extensions to systems with p–Laplacian and also with discontinuity (we consider the generalized gradients unlike continuously gradient in classical results).

Consider the second order differential inclusions system

$$(2) \qquad \frac{d}{dt}\big(\|\dot{u}(t)\|^{p-2}\dot{u}(t)\big) \in \partial F(t, u(t)) \text{ a.e. } t \in [0, T],$$

$$u(0) = u(T), \dot{u}(0) = \dot{u}(T),$$

where $p > 1$, $T > 0$, $F : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ and ∂ denotes the Clarke subdifferential.

We suppose that $F = F_1 + F_2$ and F_1, F_2 satisfy the following assumption:

(A') F_1, F_2 are measurable in t for each $x \in \mathbb{R}^n$, at least F_1 or F_2 are strictly differentiable in x and there exist $k_1, k_2 \in L^q(0, T; \mathbb{R})$ such that

$$|F_1(t, x_1) - F_1(t, x_2)| \le k_1(t)\|x_1 - x_2\|,$$
$$|F_2(t, x_1) - F_2(t, x_2)| \le k_2(t)\|x_1 - x_2\|,$$

for all $x_1, x_2 \in \mathbb{R}^n$ and all $t \in [0, T]$.

The corresponding functional $\varphi : W_T^{1,p} \to \mathbb{R}$ is given by

$$\varphi(u) = \frac{1}{p}\int_0^T \|\dot{u}(t)\|^p dt + \int_0^T F(t, u(t))dt.$$

Definition 1. *A function $G : \mathbb{R}^n \to \mathbb{R}$ is called to be (λ, μ)–subconvex if*

$$G(\lambda(x + y)) \le \mu(G(x) + G(y))$$

for some $\lambda, \mu > 0$ and all $x, y \in \mathbb{R}^n$.

Remark 1. *When $\lambda = \mu = \frac{1}{2}$, a function $(\frac{1}{2}, \frac{1}{2})$–subconvex is called convex. When $\lambda = \mu = 1$, a function $(1, 1)$–subconvex is called subadditive. When $\lambda = 1$, $\mu > 0$, a function $(1, \mu)$–subconvex is called μ–subadditive.*

2. Main results

Theorem 1. *Assume that $F = F_1 + F_2$, where F_1, F_2 satisfy assumption (A') and the following conditions:*

(i) *$F_1(t, \cdot)$ is (λ, μ)–subconvex with $\lambda > 1/2$ and $0 < \mu < 2^{p-1}\lambda^p$ for a.e. $t \in [0, T]$;*

(ii) *there exist $c_1, c_2 > 0$ and $\alpha \in [0, p - 1)$ such that*

$$\zeta \in \partial F_2(t, x) \Rightarrow \|\zeta\| \le c_1\|x\|^\alpha + c_2$$

for all $x \in \mathbb{R}^n$ and a.e. $t \in [0, T]$;

(iii) *for $q = \frac{p}{p-1}$,*

$$\frac{1}{\|x\|^{q\alpha}}\Big[\frac{1}{\mu}\int_0^T F_1(t, \lambda x)dt + \int_0^T F_2(t, x)dt\Big] \to \infty, \text{ as } \|x\| \to \infty.$$

Then the problem (2) has at least one solution which minimizes φ on $W_T^{1,p}$.

Theorem 2. *Assume that $F = F_1 + F_2$, where F_1, F_2 satisfy assumption (A') and the following conditions:*

(iv) *$F_1(t, \cdot)$ is (λ, μ)-subconvex for a.e. $t \in [0, T]$, and there exists $\gamma \in L^1(0, T; \mathbb{R})$, $h \in L^1(0, T; \mathbb{R}^n)$ with $\int_0^T h(t)dt = 0$ such that*

$$F_1(t, x) \geq \langle h(t), x \rangle + \gamma(t)$$

for all $x \in \mathbb{R}^n$ and a.e. $t \in [0, T]$;

(v) *there exist $c_1 > 0$, $c_0 \in \mathbb{R}$ such that*

$$\zeta \in \partial F_2(t, x) \Rightarrow \|\zeta\| \leq c_1$$

for all $x \in \mathbb{R}^n$ and all $t \in [0, T]$, and

$$\int_0^T F_2(t, x)dt \geq c_0$$

for all $x \in \mathbb{R}^n$;

(vi)

$$\frac{1}{\mu} \int_0^T F_1(t, \lambda x)dt + \int_0^T F_2(t, x)dt \to \infty, \ \text{as } \|x\| \to \infty.$$

Then the problem (2) has at least one solution which minimizes φ on $W_T^{1,p}$.

Theorem 3. *Assume that $F = F_1 + F_2$, where F_1, F_2 satisfy assumption (A') and the following conditions:*

(vii) *$F_1(t, \cdot)$ is (λ, μ)-subconvex for a.e. $t \in [0, T]$, and there exists $\gamma \in L^1(0, T; \mathbb{R})$, $h \in L^1(0, T; \mathbb{R}^n)$ with $\int_0^T h(t)dt = 0$ such that*

$$F_1(t, x) \geq \langle h(t), x \rangle + \gamma(t)$$

for all $x \in \mathbb{R}^n$ and a.e. $t \in [0, T]$;

(viii) *there exist $c_1, c_2 > 0$ and $\alpha \in [0, p - 1)$ such that*

$$\zeta \in \partial F_2(t, x) \Rightarrow \|\zeta\| \leq c_1 \|x\|^\alpha + c_2$$

for all $x \in \mathbb{R}^n$ and a.e. $t \in [0, T]$;

(ix) *for $q = \frac{p}{p-1}$,*

$$\frac{1}{\|x\|^{q\alpha}} \int_0^T F_2(t, x)dt \to \infty, \ \text{as } \|x\| \to \infty.$$

Then the problem (2) has at least one solution which minimizes φ on $W_T^{1,p}$.

Remark 2. *Theorems 1, 2 and 3, generalizes the corresponding Theorems 2.1, 2.3 and 2.4 of [5]. In fact, it follows from these theorems letting $p = 2$.*

3. The preliminary results

We introduce some functional spaces. Let $[0, T]$ be a fixed real interval $(0 < T < \infty)$ and $1 < p < \infty$. We denote by $W_T^{1,p}$ the Sobolev space of functions $u \in L^p(0, T; \mathbb{R}^n)$ having a weak derivative $\dot{u} \in L^p(0, T; \mathbb{R}^n)$. The norm over $W_T^{1,p}$ is defined by

$$\|u\|_{W_T^{1,p}} = \Big(\int_0^T \|u(t)\|^p dt + \int_0^T \|\dot{u}(t)\|^p dt \Big)^{\frac{1}{p}}.$$

We recall that

$$\|u\|_{L^p} = \Big(\int_0^T \|u(t)\|^p dt \Big)^{\frac{1}{p}} \text{ and } \|u\|_\infty = \max_{t \in [0,T]} \|u(t)\|.$$

For our aims it is necessary to recall some very well know results (for proof and details see [2]).

Proposition 4. *If $u \in W_T^{1,p}$ then*

$$\|u\|_\infty \leq c\|u\|_{W_T^{1,p}}.$$

If $u \in W_T^{1,p}$ and $\int_0^T u(t)dt = 0$ then

$$\|u\|_\infty \leq c\|\dot{u}\|_{L^p}.$$

Proposition 5. *If the sequence $(u_k)_k$ converges weakly to u in $W_T^{1,p}$, then $(u_k)_k$ converges uniformly to u on $[0, T]$.*

Let X be a Banach space. Now follows [1], for each $x, v \in X$, we define the *generalized directional derivative* at x in the direction v of a given $f \in Lip_{loc}(X, \mathbb{R})$ as

$$f^0(x; v) = \limsup_{y \to x, \lambda \searrow 0} \frac{f(y + \lambda v) - f(y)}{\lambda}$$

and we denote by

$$\partial f(x) = \{x^* \in X^* : f^0(x; v) \geq \langle x^*, v \rangle, \text{ for all } v \in X\}$$

the *generalized gradient* of f at x (the Clarke subdifferential).

We recall the *Lebourg's mean value theorem* (see [1], Theorem 2.3.7).

Theorem 6. *Let x and y be points in X, and suppose that f is Lipschitz on open set containing the line segment [x,y]. Then there exists a point u in (x,y) such that*

$$f(y) - f(x) \in \langle \partial f(u), y - x \rangle.$$

Clarke consider in [1] the following abstract framework:

- let $(T, \mathcal{T}, \mu)$ be a positive complete measure space with $\mu(T) < \infty$, and let Y be a separable Banach space;
- let Z be a closed subspace of $L^p(T; Y)$ (for some p in $[1, \infty)$), where $L^p(T; Y)$ is the space of p- integrable functions from T to Y;
- define a functional f on Z via

$$f(x) = \int_T f_t(x(t))\mu(dt),$$

  where $f_t : Y \to \mathbb{R}$, $(t \in T)$ is a given family of functions;
- suppose that for each y in Y the function $t \to f_t(y)$ is measurable, and that x is a point at which $f(x)$ is defined (finitely).

Hypothesis 1: There is a function k in $L^q(T, \mathbb{R})$, $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$ such that, for all $t \in T$,

$$|f_t(y_1) - f_t(y_2)| \leq k(t)\|y_1 - y_2\|_Y \text{ for all } y_1, y_2 \in Y$$

Hypothesis 2: Each function f_t is Lipschitz (of some rank) near each point of Y, and for some constant c, for all $t \in T$, $y \in Y$, one has

$$\zeta \in \partial f_t(y) \Rightarrow \|\zeta\|_{Y^*} \leq c\{1 + \|y\|_Y^{p-1}\}.$$

Under this conditions described above Clarke prove (see [1], Theorem 2.7.5):

Theorem 7. *Under the conditions described above, under either of Hypothesis 1 or 2, f is uniformly Lipschitz on bounded subsets of Z, and one has*

$$\partial f(x) \subset \int_T \partial f_t(x(t))\mu(dt).$$

Further, if each f_t is regular at $x(t)$ then f is regular at x and equality holds.

Remark 3. *f is globally Lipschitz on Z when Hypothesis 1 hold.*

Now we can prove the following result.

Theorem 8. *Let $F : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ such that $F = F_1 + F_2$ where F_1, F_2 are measurable in t for each $x \in \mathbb{R}^n$, and there exist $k_1 \in L^q(0, T; \mathbb{R})$, $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1(0, T; \mathbb{R}^+)$, $c_1, c_2 > 0$ and $\alpha \in [0, p-1)$ such that*

$$|F_1(t, x_1) - F_1(t, x_2)| \leq k_1(t)\|x_1 - x_2\|$$

$$|F_2(t, x)| \leq a(\|x\|)b(t)$$

$$(3) \qquad \zeta \in \partial F_2(t, x) \Rightarrow \|\zeta\| \leq c_1\|x\|^\alpha + c_2$$

for all $t \in [0,T]$ and all x, x_1, $x_2 \in \mathbb{R}^n$. We suppose that $L : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, is given by $L(t,x,y) = \frac{1}{p}\|y\|^p + F(t,x)$.
Then, the functional $f : Z \in \mathbb{R}$, where

$$Z = \left\{ (u,v) \in L^p(0,T;Y) : u(t) = \int_0^t v(s)ds + c, c \in \mathbb{R}^n \right\}$$

given by $f(u,v) = \int_0^T L(t,u(t),v(t))dt$, is uniformly Lipschitz on bounded subsets of Z and one has

$$\partial f(u,v) \subset \int_0^T \{\partial F_1(t,u(t)) + \partial F_2(t,u(t))\} \times \{\|v(t)\|^{p-2}v(t)\}dt.$$

Proof. Let $L_1(t,x,y) = F_1(t,x)$, $L_2(t,x,y) = \frac{1}{p}\|y\|^p + F_2(t,x)$ and $f_1, f_2 : Z \to \mathbb{R}$ given by $f_1(u,v) = \int_0^T L_1(t,u(t),v(t))dt$, $f_2(u,v) = \int_0^T L_2(t,u(t),v(t))dt$. For f_1 we can apply Theorem 7 under Hypothesis 1, with the following cast of characters:

- $(T,\mathcal{T}, \mu)=[0,\text{T}]$ with Lebesgue measure, $Y = \mathbb{R}^n \times \mathbb{R}^n$ be the Hilbert product space (hence is separable);
- $p > 1$ and

$$Z = \left\{ (u,v) \in L^p(0,T;Y) : u(t) = \int_0^t v(s)ds + c, c \in \mathbb{R}^n \right\}$$

  be a closed subspace of $L^p(0,T;Y)$;
- $f_t(x,y) = L_1(t,x,y) = F_1(t,x)$; in our assumptions it results that the integrand $L_1(t,x,y)$ is measurable in t for a given element (x,y) of Y and there exists $k \in L^q(0,T;\mathbb{R})$ such that

$$|L_1(t,x_1,y_1) - L_1(t,x_2,y_2)| = |F_1(t,x_1) - F_1(t,x_2)| \le$$

(4) $$\le k_1(t)\|x_1 - x_2\| \le k_1(t)\big(\|x_1 - x_2\| + \|y_1 - y_2\|\big) =$$

$$= k_1(t)\|(x_1,y_1) - (x_2,y_2)\|_Y$$

  for all $t \in [0,T]$ and all (x_1,y_1), $(x_2,y_2) \in Y$. Hence f_1 is uniformly Lipschitz on bounded subsets of Z and one has

$$\partial f_1(u,v) \subset \int_0^T \partial L_1(t,u(t),v(t))dt.$$

For f_2 we can apply Theorem 7 under Hypothesis 2 with the same cast of characters, but now $f_t(x,y) = L_2(t,x,y) = \frac{1}{p}\|y\|^p + F_2(t,x)$. In our assumptions it results that the integrand $L_2(t,x,y)$ is measurable in t for a given element (x,y) of Y and locally Lipschitz in (x,y) for each $t \in [0,T]$.

Proposition 2.3.15 from [1] implies

$$\partial L_2(t,x,y) \subset \partial_x L_2(t,x,y) \times \partial_y L_2(t,x,y) = \partial F_2(t,x) \times \{\|y\|^{p-2}y\}.$$

Using (3) and (4), if $\zeta = (\zeta_1, \zeta_2) \in \partial L_2(t, x, y)$ it results $\zeta_1 \in \partial F_2(t, x)$ and $\zeta_2 = \|y\|^{p-2} y$, and hence

$$\|\zeta\| = \|\zeta_1\| + \|\zeta_2\| \leq c_1 \|x\|^\alpha + c_2 + \|y\|^{p-1} \leq \tilde{c}\{1 + \|(x, y)\|^{p-1}\}$$

for each $t \in [0, T]$, since $\alpha < p-1$ and $p > 1$. Hence f_2 is uniformly Lipschitz on bounded subsets of Z and one has

$$\partial f_2(u, v) \subset \int_0^T \partial L_2(t, u(t), v(t)) dt.$$

It follows that $f = f_1 + f_2$ is uniformly Lipschitz on the bounded subsets of Z.

Proposition 2.3.3 and Proposition 2.3.15 from [1] implies

$$\partial f(u, v) \subset \partial f_1(u, v) + \partial f_2(u, v) \subset$$

$$\subset \int_0^T \Big[\partial L_1(t, u(t), v(t)) + \partial L_2(t, u(t), v(t)) \Big] dt \subset$$

$$\subset \int_0^T \Big[\Big(\partial_x L_1(t, u(t), v(t)) \times \partial_y L_1(t, u(t), v(t)) \Big) +$$

$$+ \Big(\partial_x L_2(t, u(t), v(t)) \times \partial_y L_2(t, u(t), v(t)) \Big) \Big] dt \subset$$

$$\subset \int_0^T \Big[\Big(\partial_x L_1(t, u(t), v(t)) + \partial_x L_2(t, u(t), v(t)) \Big) \times$$

$$\times \Big(\partial_y L_1(t, u(t), v(t)) + \partial_y L_2(t, u(t), v(t)) \Big) \Big] dt =$$

$$= \int_0^T \Big(\partial F_1(t, u(t)) + \partial F_2(t, u(t)) \Big) \times \{\|v(t)\|^{p-2} v(t)\} dt.$$

Moreover, Corollary 1 of Proposition 2.3.3 from from [1] imply that, if at least of the functions F_1, F_2 is strictly differentiable in x for all $t \in [0, T]$ then

$$(5) \qquad \partial f(u, v) \subset \int_0^T \partial F(t, u(t)) \times \{\|v(t)\|^{p-2} v(t)\} dt.$$

$\square$

Remark 4. *The interpretation of expression (5) is as follows: if (u_0, v_0) is an element of Z (so that $v_0 = \dot{u}_0$) and if $\zeta \in \partial f(u_0, v_0)$, we deduce the existence of a measurable function $(q(t), p(t))$ such that*

(6) * $q(t) \in \partial F(t, u_0(t))$ and $p(t) = \|v_0(t)\|^{p-2} v_0(t)$ a.e. on $[0, T]$*

and for any (u, v) in Z, one has

$$\langle \zeta, (u, v) \rangle = \int_0^T \{ \langle q(t), u(t) \rangle + \langle p(t), v(t) \rangle \} dt.$$

In particular, if $\zeta = 0$ (so that u_0 is critical point for $\varphi(u) = \int_0^T \left[\frac{1}{p} \|\dot{u}(t)\|^p + F(t, u(t)) \right] dt$), it then follows easily that $q(t) = \dot{p}(t)$ a.e., or taking into account (6)

$$\frac{d}{dt} \left(\|\dot{u_0}(t)\|^{p-2} \dot{u_0}(t) \right) \in \partial F(t, u_0(t)) \ \ a.e. \ \ on \ [0, T],$$

so that u_0 satisfies the inclusions system (2).

Remark 5. *Of course if $p = 2$ and F is continuously differentiable in x, then the system (2) becomes system (1).*

4. Proofs of the Theorems

Proof of Theorem 1. From (A') it follows immediately there exist $a \in C(\mathbb{R}_+, \mathbb{R}_+)$, $b \in L^1(0, T; \mathbb{R}_+)$ such that

$$|F_1(t, x)| \leq a(\|x\|) b(t),$$

for all $x \in \mathbb{R}^n$ and all $t \in [0, T]$. Like in [11] we obtain

$$F_1(t, x) \leq (2\mu \|x\|^\beta + 1) a_0 b(t)$$

for all $x \in \mathbb{R}^n$ and all $t \in [0, T]$, where $\beta < p$ and $a_0 = \max_{0 \leq s \leq 1} a(s)$.
For $u \in W_T^{1,p}$, let $\bar{u} = \frac{1}{T} \int_0^T u(t) dt$ and $\tilde{u} = u - \bar{u}$. From Lebourg's mean value theorem it follows that for each $t \in [0, T]$ there exist $z(t)$ in $(\bar{u}, u(t))$ and $\zeta \in \partial F_2(t, z(t))$ such that $F_2(t, u(t)) - F_2(t, \bar{u}) = \langle \zeta, \tilde{u}(t) \rangle$. It follows from *(ii)* and Hölder's inequality that

$$\left| \int_0^T [F_2(t, u(t)) - F_2(t, \bar{u})] dt \right| \leq \int_0^T |F_2(t, u(t)) - F_2(t, \bar{u})| dt \leq$$

$$\leq \int_0^T \|\zeta\| \|\tilde{u}(t)\| dt \leq \int_0^T \left[2c_1(\|\bar{u}\|^\alpha + \|\tilde{u}(t)\|^\alpha) + c_2 \right] \|\tilde{u}(t)\| dt \leq$$

$$\leq C_1 \|\tilde{u}\|_\infty^{\alpha+1} + C_2 \|\tilde{u}\|_\infty \|\bar{u}\|^\alpha + C_3 \|\tilde{u}\|_\infty \leq$$

$$\leq C_4 \|\dot{u}\|_{L^p}^{\alpha+1} + \frac{1}{2p} \|\dot{u}\|_{L^p}^p + C_5 \|\dot{u}\|_{L^p} + C_6 \|\bar{u}\|^{q\alpha}$$

for all $u \in W_T^{1,p}$ and some positive constants C_4, C_5 and C_6. Hence we have

$$\varphi(u) \geq \frac{1}{p} \int_0^T \|\dot{u}(t)\|^p dt + \frac{1}{\mu} \int_0^T F_1(t, \lambda \bar{u}) dt - \int_0^T F_1(t, -\tilde{u}(t)) dt +$$

$$+ \int_0^T F_2(t, \bar{u})dt + \int_0^T [F_2(t, u(t)) - F_2(t, \bar{u})]dt \geq$$

$$\geq \frac{1}{2p}\|\dot{u}\|_{L^p}^p - C_4\|\dot{u}\|_{L^p}^{\alpha+1} - C_5\|\dot{u}\|_{L^p} - C_6\|\bar{u}\|^{q\alpha} - (2\mu\|\tilde{u}\|_\infty^\beta + 1)\int_0^T a_0 b(t)dt +$$

$$+ \frac{1}{\mu}\int_0^T F_1(t, \lambda\bar{u})dt + \int_0^T F_2(t, \bar{u})dt \geq \frac{1}{2p}\|\dot{u}\|_{L^p}^p - C_4\|\dot{u}\|_{L^p}^{\alpha+1} - C_5\|\dot{u}\|_{L^p} -$$

$$-C_7\|\dot{u}\|_{L^p}^\beta - C_8 + \|\bar{u}\|^{q\alpha}\left\{ \frac{1}{\|\bar{u}\|^{q\alpha}}\left[\frac{1}{\mu}\int_0^T F_1(t, \lambda\bar{u})dt + \int_0^T F_2(t, \bar{u})dt \right] - C_6 \right\}$$

for all $u \in W_T^{1,p}$, which implies that $\varphi(u) \to \infty$ as $\|u\| \to \infty$ by *(iii)* because $\alpha < p - 1$, $\beta < p$, and the norm $\|u\| = (\|\bar{u}\|^p + \|\dot{u}\|_{L^p}^p)^{\frac{1}{p}}$ is an equivalent norm on $W_T^{1,p}$. Now we write $\varphi(u) = \varphi_1(u) + \varphi_2(u)$ where

$$\varphi_1(u) = \frac{1}{p}\int_0^T \|\dot{u}(t)\|^p dt \text{ and } \varphi_2(u) = \int_0^T F(t, u(t))dt.$$

The function φ_1 is weakly lower semi-continuous (w.l.s.c.) on $W_T^{1,p}$. From *(i)*, *(ii)* and Theorem 7, taking to account Remark 3 and Proposition 5, it follows that φ_2 is w.l.s.c. on $W_T^{1,p}$. By Theorem 1.1 in [2] it follows that φ has a minimum u_0 on $W_T^{1,p}$. Evidently $Z \simeq W_T^{1,p}$ and $\varphi(u) = f(u, v)$ for all $(u, v) \in Z$. From Theorem 8, it results that f is uniformly Lipschitz on bounded subsets of Z, and therefore φ possesses the same properties relative to $W_T^{1,p}$. Proposition 2.3.2 in [1] implies that $0 \in \partial\varphi(u_0)$ (so that u_0 is critical point for φ). Now from Theorem 8 and Remark 4 it follows that the problem (2) has at least one solution $u \in W_T^{1,p}$. ∎

Proof of Theorem 2. Let (u_k) be a minimizing sequence of φ. It follows from *(iv)*, *(v)*, Lebourg's mean value theorem and Proposition 4 that

$$\varphi(u_k) \geq \frac{1}{p}\|\dot{u}_k\|_{L^p}^p + \int_0^T \langle h(t), u_k(t)\rangle dt + \int_0^T \gamma(t)dt +$$

$$+ \int_0^T F_2(t, \bar{u}_k)dt - \int_0^T \|\zeta\|\|\tilde{u}_k(t)\|dt \geq \frac{1}{p}\|\dot{u}_k\|_{L^p}^p -$$

$$- \|\tilde{u}_k\|_\infty \int_0^T \|h(t)\|dt + \int_0^T \gamma(t)dt - c_1\|\tilde{u}_k\|_\infty + c_0 \geq$$

$$\geq \frac{1}{p}\|\dot{u}_k\|_{L^p}^p - c_2\|\dot{u}_k\|_{L^p} - c_3$$

for all k and some constants c_2, c_3, which implies that $(\tilde{u}_k)$ is bounded. On the other hand, in a way similar to the proof of Theorem 1, one has

$$\left| \int_0^T [F_2(t, u(t)) - F_2(t, \bar{u})]dt \right| \leq \frac{1}{2p}\|\dot{u}\|_{L^p}^p + C_1\|\dot{u}\|_{L^p}$$

for all k and some positive constant C_1, which implies that

$$\varphi(u_k) \geq \frac{1}{p}\|\dot{u}_k\|_{L^p}^p + \frac{1}{\mu}\int_0^T F_1(t, \lambda\bar{u}_k)dt - \int_0^T F_1(t, -\tilde{u}_k(t))dt +$$

$$+ \int_0^T F_2(t, \bar{u}_k)dt + \int_0^T [F_2(t, u(t)) - F_2(t, \bar{u}_k)]dt \geq$$

$$\geq \frac{1}{2p}\|\dot{u}_k\|_{L^p}^p - a(\|\tilde{u}_k\|_\infty)\int_0^T b(t)dt - C_1\|\dot{u}_k\|_{L^p} +$$

$$+ \frac{1}{\mu}\int_0^T F_1(t, \lambda\bar{u}_k)dt + \int_0^T F_2(t, \bar{u}_k)dt$$

for all k and some positive constant C_1. It follows from *(vi)* and the boundedness of $(\tilde{u}_k)$ that $(\bar{u}_k)$ is bounded. Hence φ has a bounded minimizing sequence (u_k). Now Theorem 2 follows like Theorem 1. ∎

Proof of Theorem 3. From *(vii)*, *(viii)* and Proposition 4 it follows that

$$\varphi(u) \geq \frac{1}{p}\|\dot{u}\|_{L^p}^p + \int_0^T \langle h(t), u(t)\rangle dt +$$

$$+ \int_0^T \gamma(t)dt + \int_0^T F_2(t, \bar{u})dt + \int_0^T [F_2(t, u(t)) - F_2(t, \bar{u})]dt \geq$$

$$\geq \frac{1}{2p}\|\dot{u}\|_{L^p}^p - \|\tilde{u}\|_\infty \int_0^T \|h(t)\|dt +$$

$$+ \int_0^T \gamma(t)dt - C_1\|\dot{u}\|_{L^p}^{\alpha+1} - C_2\|\dot{u}\|_{L^p} + \int_0^T F_2(t, \bar{u})dt - C_3\|\bar{u}\|^{q\alpha} \geq$$

$$\geq \frac{1}{2p}\|\dot{u}\|_{L^p}^p - C_1\|\dot{u}\|_{L^p}^{\alpha+1} - C_4(\|\dot{u}\|_{L^p}+1) + \|\bar{u}\|^{q\alpha}\left[\frac{1}{\|\bar{u}\|^{q\alpha}}\int_0^T F_2(t, \bar{u})dt - C_3\right]$$

for all $u \in W_T^{1,p}$ and some positive constants C_1, C_3 and C_4. Now follows like in the proof of Theorem 1 that φ is coercive by *(ix)*, which completes the proof. ∎

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