GLOBAL ERIODICITY AND COMPLETE INTEGRABILITY OF DISCRETE DYNAMICAL SYSTEMS

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Abstract. Consider the discrete dynamical system generated a map F. It is said that it is *globally periodic* if there exists a natural number p such that $F^p(x) \equiv x$. On the other hand it is called *completely integrable* if it has as many functionally independent first integrals as the dimension of the phase space. In this paper we relate both concepts. We also give a large list of globally periodic dynamical systems together with a complete set of their first integrals, emphasizing in the ones coming from difference equations.

1. Introduction and Main Result

Let $F : \mathcal{U} \subseteq \mathbb{K}^k \to \mathcal{U}$, be a map where $\mathcal{U}$ is an open set of $\mathbb{K}^k$ and $\mathbb{K}$ denotes either $\mathbb{R}$ or $\mathbb{C}$. This paper concerns with some features of the dynamical system generated by F. We start recalling some definitions.

We say that a map F is *globally periodic* if it exists some $p \in \mathbb{N}$ such that $F^p(x) = x$ for all $x \in \mathcal{U}$, where $F^p = F \circ \overset{p)}{\cdots} \circ F$. In particular notice that globally periodic maps have to be bijective and $F^{-1} = F^{p-1}$. This name comes from the fact that if F is globally periodic then the discrete dynamical system (DDS for short) generated by F has all its orbits as periodic orbits. The study of globally periodic maps is a current subject of interest of the dynamical systems community, see for instance [1, 2, 3, 4, 5, 8, 18, 22]. Indeed, the functional equation $F^p = \mathrm{Id}$ is also known in the literature as Babbage equation, see [17] and it is one of the oldest iterative functional equations ever discussed.

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A discrete dynamical system generated by a map $F:\mathcal{U}\subseteq\mathbb{K}^k\to\mathcal{U}$ is called *completely integrable* if there exists k functionally independent first integrals of the DDS. Recall that a first integral of the DDS generated by F is a non constant $\mathbb{K}$–valued function H which is constant on the orbits of the DDS. That is,

$$H(F(\mathbf{x}))=H(\mathbf{x}),\quad \text{for all}\quad \mathbf{x}\in\mathcal{U}.$$

Integrable dynamical systems have a distinguished role as models of natural phenomena and are one of the main subjects of contemporaneous nonlinear science, from the theoretic and the applied view point. Nowadays, discrete–time integrable systems is also an area of a deep research activity, see for instance [7, 14, 15]. Note that there is a minor, but important, difference between the definition of complete integrable discrete and complete integrable continuous dynamical systems. For the later case the number of functionally independent first integrals has to be just $k-1$.

In this paper we relate the globally periodicity of a map with the complete integrability of its associated DDS, see the Theorem 1 below. Somehow, our result extents to DDS a common knowledge result in continuous dynamics, namely that the existence of several continua of periodic solutions is related with the existence of several first integrals. Along the paper we only consider the positive iterates of F. Thus, indeed we deal with discrete semi–dynamical systems. Nevertheless, for the sake of simplicity, we keep the name of discrete dynamical system.

The main result of this paper is the following Theorem:

Theorem 1. *Let $F:\mathcal{U}\subseteq\mathbb{K}^k\to\mathcal{U}$ an injective differentiable map defined in an open set $\mathcal{U}$. The following statements hold:*

(a) *If the DDS generated by F is completely integrable and there exists a set of k of its first integrals $H_1,H_2,\cdots,H_k$ such that*

$$\operatorname{Card}\left(\bigcap_{i=1}^{k}\{H_i=c_i\}\cap\mathcal{U}\right)\leq K,\ \textit{for all } c_i\in\mathbb{K}, \tag{1}$$

being K a given positive integer, then F is globally periodic.

(b) *If F is globally periodic, then the DDS generated by F is completely integrable.*

Statement (a) of Theorem 1 is not difficult to be proved. Its proof is given in Section 2. We also present a complete integrable DDS (see Example 2.1) showing that the uniform finiteness condition (1) is necessary in order to get global periodicity. Statement (b) is the most difficult part of the Theorem and it is proved in Section 3. Our proof is constructive, *i.e.* it provides a method for obtaining a complete set of first integrals for DDS generated by global periodic maps. We end Section 3 with several examples which illustrate the procedure.

The key idea to prove part (b) of Theorem 1 is the following: Assume $F^p =$Id and let $\mathcal{F} : \mathcal{U} \to \mathbb{K}^{kp}$ be defined by $\mathcal{F}(x) = (x, F(x), \ldots, F^{p-1}(x))$. Since $\mathcal{F}(F(x)) = (F(x), F^2(x), \ldots, F^{p-1}(x), x)$, each $S : (\mathbb{K}^k)^p \to (\mathbb{K}^k)^p$ satisfying $S(v_1, v_2, \ldots, v_p) = S(v_2, v_3, \ldots, v_p, v_1)$ gives a first integral of the DDS generated by F, $S \circ \mathcal{F}$, whenever it is not a constant function. In order to prove the existence of many of such first integrals (functionally independent if some minimum regularity conditions are fulfilled) we can consider as first integrals the ones obtained when the functions S are chosen from a basis of the symmetric polynomials in $kp = N$ variables. The difficult part is to see that among all these N functions, k of them are functionally independent.

Observe that, essentially, Theorem 1 establishes the equivalence between complete integrability (with the uniform finiteness condition (1)) and global periodicity. There is only one small point missing: when F is globally periodic we know that its associated DDS is completely integrable but condition (1) is not guaranteed for a given set of first integrals. Indeed, in Example 3.3, we show that there are completely integrable maps being globally periodic but having sets of functionally independent first integrals which do not fulfill condition (1). In the particular case of the example we also show that there is another set of functionally independent first integrals which satisfies condition (1). We think that this will be the situation when we deal with F being polynomial or rational maps, but for general F it seems much more difficult to be proved. We do not consider this problem in the paper.

In Section 4 we adapt and improve the general method to DDS which came from difference equations. A full collection of periodic recurrences, especially rational, are presented.

Section 5 is devoted to consider the case where F is non–smooth or even discontinuous. From the proof of Theorem 1 it is easy to see that for a F globally periodic, the condition on the regularity of F given in its statement is only needed to ensure the existence of exactly k functionally independent first integrals. Without this condition in general we can only prove the existence of some first integrals of the DDS given by F. We end this section with two examples to illustrate how the method works in the non–smooth or the discontinuos cases.

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2. Complete integrability implies global periodicity

Proof of Theorem 1 (a). Consider $c_i \in \mathbb{K}$, $i = 1, \ldots, k$ such that $\mathcal{L} = \bigcap_{i=1}^{k}\{H_i = c_i\} \cap \mathcal{U} \neq \emptyset$. Thus $\mathcal{L}$ is an invariant non–empty finite set of points for the DDS generated by F and $\mathrm{card}(\mathcal{L}) = K_{\mathcal{L}} \leq K$. Hence the dynamics over it decomposes into subsystems which are either periodic or pre–periodic (the ones which, with a finite number of iterates, reach a periodic orbit as a ω–limit set). Since F is injective this last option can be discarded. Furthermore the period of the periodic subsystems in $\mathcal{L}$ has to be a divisor of $K_{\mathcal{L}}!$. Thus a period p (not necessarily minimal) which satisfies the statement is $p = K!$. □

Notice that the above proof does not use the regularity of F. As we will see this hypothesis is included in the statement of the Theorem to enunciate part (b); see also Section 5.

Next example shows that the uniform finiteness condition (1) is necessary in statement (a) of Theorem 1.

Example 2.1: *Consider the map* $F : \mathbb{R}^2 \to \mathbb{R}^2$,

$$F(x, y) = (x + 2\pi, y).$$

This map F is bijective in the whole $\mathbb{R}^2$ and it is clearly not p–periodic for any $p \in \mathbb{N}$. Moreover, the DDS generated by it is completely integrable. In fact $H_1(x, y) = y - \sin(x)$ and $H_2(x, y) = y$ are two functionally independent first integrals. Finally, it is clear that for some k_1 and k_2 the finiteness condition is no satisfied, i.e. the set $\{H_1(x, y) = k_1\} \cap \{H_2(x, y) = k_2\}$ has infinite many points.

Recall that if a set of r smooth functions, $H_1, H_2, \ldots, H_r$ from $\mathcal{W} \subset \mathbb{K}^k$ into $\mathbb{K}$ is such that the rank of the matrix $(DH)_x$ is r for almost all $x \in \mathcal{W}$ (here $H = (H_1, H_2, \ldots, H_r)$) then these functions are functionally independent, see [13, 26]. This property together with the assumption that the first integral are rational functions allow to obtain alternative hypotheses to (1) in Theorem 1 (a). The results are stated in the following corollary:

Corollary 2. *If $F : \mathcal{U} \subseteq \mathbb{K}^k \to \mathcal{U}$ is an injective map on $\mathcal{U}$ and there exists k rational first integrals $H_1, H_2, \cdots, H_k$ for the DDS generated by F such that*

$$\det(\nabla H_1, \nabla H_2, \ldots, \nabla H_k) \neq 0 \text{ in } \mathcal{U}, \tag{2}$$

then F is globally periodic.

Proof. Set $H_i = P_i/Q_i$ and $d_i = \max\limits_{c \in \mathbb{K}}\{\deg(P_i - cQ_i)\}$. Using Bézout Theorem we know that $\bigcap_{i=1}^{m}\{H_i = c_i\}$ contains either a number less or equal than $\prod_{i=1}^{n} d_i$ points or an infinite number of them. Condition (2) prevents this second possibility. Thus, the result is obtained applying Theorem 1 (a), with $K = \prod_{i=1}^{n} d_i$. □

Remark 3. *The above results obtained by taking rational first integrals can also be extended to Khovansky–type first integrals. The reason is that the maximum number of solutions of systems of equations given by such type of functions can also be uniformly bounded in terms of some generalized degrees, see* [16, 24].

Recall that a map $f : \mathbb{R}^k \to \mathbb{R}^k$ *is of Khovansky–type (and write* $f \in \mathrm{K}^k_{m,n}$ *where* $m = (m_1, \ldots, m_k)$*) if* $f = (f_1, \ldots, f_k)$ *where each* $f_i \in \mathbb{R}[x, y]$ *is a polynomial with degree* m_i*,* $x \in \mathbb{R}^k$*,* $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ *and* $y_i = e^{\langle a_i, x\rangle}$ *with* $a_i \in \mathbb{R}^k$*. Note that when* $n = 0$*,* $\mathrm{K}^k_{m,0}$ *is the set of polynomial maps.*

For instance, for $f \in \mathrm{K}^2_{m,n}$ *(planar Khovansky maps) the maximum number of solutions of a system defined by* $f = p$ *(for any* $p \in \mathbb{R}^2$*), whenever it is finite, is bounded by*

$$K = m_1 m_2 (1 + m_1 + m_2)^n 2^{n(n-1)/2}.$$

3. Global periodicity implies complete integrability

Proposition 4. *Let* $F : \mathcal{U} \subset \mathbb{K}^k \to \mathcal{U}$ *be a globally* p*–periodic map on* U*. Let*

$$\Phi : \mathcal{U}^p = \overbrace{\mathcal{U} \times \mathcal{U} \times \cdots \times \mathcal{U}}^{p} \longrightarrow \mathbb{K}$$

be invariant by the permutation of the indices $S(y_1, y_2, \ldots, y_p) = (y_2, y_3, \ldots, y_p, y_1)$*, where each* $y_i \in \mathcal{U}$*, i.e.* $\Phi(S(y)) = \Phi(y)$*. Then, whenever it is not a constant function,*

$$H_\phi(x) = \Phi(x, F(x), \ldots, F^{p-1}(x))$$

is a first integral of the DDS generated by F*.*

Proof. Since F is p–periodic, and Φ is invariant under the index permutation S we get:

$$\begin{aligned} H_\Phi(F(x)) &= \Phi(F(x), F^2(x), \ldots, F^p(x)) = \\ &= \Phi(F(x), F^2(x), \ldots, F^{p-1}(x), x) = \\ &= \Phi(S(x, F(x), \ldots, F^{p-1}(x))) = \\ &= \Phi(x, F(x), \ldots, F^{p-1}(x)) = H_\Phi(x). \end{aligned}$$

□

Remark 5. *The case where the above procedure gives rise to a constant function* H_Φ *can not be discarded, as the following example shows:* $F(x, y) = (y, c/(xy))$*,* $F^2(x, y) = (c/(xy), x)$*,* $F^3(x, y) = (x, y)$*,* $\Phi(y_1, y_2, y_3) = y_1 y_2 y_3$ *and* $H_\phi(F(x, y)) = c$*.*

From the above result, each function Φ invariant by the permutation S gives rise to a candidate to be a first integral of the DDS generated by F.

As we will see, to our purposes it will suffice to consider Φ being symmetric polynomials in $N = kp$ variables.

The set of all symmetric polynomials in N variables forms an algebra of dimension N. There are several homogeneous basis for the symmetric polynomials. We shall use two of them: the so called elementary symmetric polynomials in N variables:

$$\begin{aligned}
\sigma_1(y_1,\ldots,y_N) &:= y_1+\cdots+y_N,\\
\sigma_2(y_1,\ldots,y_N) &:= y_1y_2+y_1y_3+\cdots+y_{N-1}y_N,\\
\vdots\quad &\ \vdots\qquad \vdots\\
\sigma_{N-1}(y_1,\ldots,y_N) &:= y_1y_2\ldots y_{N-1}+y_1y_2\ldots y_{N-2}y_N+\cdots+y_2y_3\ldots y_N,\\
\sigma_N(y_1,\ldots,y_N) &:= y_1y_2\ldots y_N.
\end{aligned}$$

and the power sums $S_i(y_1,y_2,\ldots,y_N)=\sum_{j=1}^N y_j^i$, *i.e.*,

$$\begin{aligned}
S_1(y_1,\ldots,y_N) &:= y_1+\cdots+y_N,\\
S_2(y_1,\ldots,y_N) &:= y_1^2+y_2^2+\cdots+y_N^2,\\
\vdots\quad &\ \vdots\qquad \vdots\\
S_N(y_1,\ldots,y_N) &:= y_1^N+y_2^N+\cdots+y_N^N.
\end{aligned}$$

The functions σ_i are the coefficients (modulus a signum) of the polynomial $p(w)=(w-y_1)\cdot\cdots\cdot(w-y_N)$, in fact

$$p(w)=w^N+\cdots+(-1)^{N-i}\sigma_i(y_1,\ldots,y_N)w^{N-i}+\cdots+(-1)^N\sigma_N(y_1,\ldots,y_N).$$

Set $\mathbf{x}=(x_1,\ldots,x_k)\in\mathbb{K}^k$, and introduce the notation $(F_1^i(\mathbf{x}),F_2^i(\mathbf{x}),\ldots,F_k^i(\mathbf{x})):=F^i(\mathbf{x})$. We also have the following Corollary of Proposition 4.

Corollary 6. *Let $F:\mathcal{U}\to\mathcal{U}$ be a globally p–periodic map on a set $\mathcal{U}\subset\mathbb{K}^k$ and let $\Psi:\mathcal{U}\to\mathbb{K}$ be an arbitrary function. Then, whenever they are not constant functions,*

$$H_{\sigma_i,\Psi}(x)=\sigma_i(\Psi(\mathbf{x}),\Psi(F(\mathbf{x})),\ldots,\Psi(F^{p-1}(\mathbf{x})))$$

and

$$H_{S_i,\Psi}(x)=S_i(\Psi(\mathbf{x}),\Psi(F(\mathbf{x})),\ldots,\Psi(F^{p-1}(\mathbf{x}))) \tag{3}$$

are first integrals of the DDS generated by F.

Proof. It follows directly from Proposition 4, taking $\Phi(y_1,\ldots,y_p)=\sigma_i(\Psi(y_1),\ldots,\Psi(y_n))$, or $\Phi(y_1,\ldots,y_p)=S_i(\Psi(y_1),\ldots,\Psi(y_n))$. □

The following result proves part (b) of Theorem 1.

Theorem 7. *Let $F:\mathcal{U}\subseteq\mathbb{K}^k\to\mathcal{U}$ be a globally p–periodic differentiable map. Then DDS generated by F is completely integrable. Furthermore the set of k functionally independent first integrals of F is constructed from the* *set of functions (3) given in Corollary 6 with $\Psi = Id$. As a consequence they are analytic (respectively, rational) if F is analytic (respectively, rational).*

Proof. Set $N = kp$ and consider $\mathcal{F}(x) := (x, F(x), F^2(x), \ldots, F^{p-1}(x))$. For each $i = 1, 2, \ldots, N$ let y_i be defined through

$$(y_1, y_2, \ldots, y_N) := (x, F(x), F^2(x), \ldots, F^{p-1}(x)).$$

It can occur that for some indices r, s, $y_r \equiv y_s$. Assume that among the N components $y_1, y_2, \ldots, y_N$ there are ℓ which are different. Let $Y_1, Y_2, \ldots, Y_\ell$ denote these components. Clearly $\ell \geq k$. For each $i = 1, 2, \ldots, \ell$ let n_i be the number of variables in $(y_1, \ldots, y_N)$ which coincides with Y_i.

Now consider the homogeneous symmetric polynomials $S_1, S_2, \ldots, S_N$ introduced before $S_i(y_1, y_2, \ldots, y_N) = y_1^i + y_2^i + \ldots + y_N^i$. Among these N polynomials choose the ℓ first ones and consider $S_1, S_2, \ldots, S_\ell$. Also for $i = 1, 2, \ldots, \ell$, let $H_i(x)$ be defined by

$$H_i(x) = S_i(\mathcal{F}(x)).$$

Then, $S = (S_1, S_2, \ldots, S_\ell)$ and $H = (H_1, H_2, \ldots, H_\ell)$ define some maps:

$$S : \mathbb{K}^{pk} \to \mathbb{K}^\ell, \quad H : \mathcal{U} \subset \mathbb{K}^k \to \mathbb{K}^\ell.$$

Recall that from Proposition 4 the functions H_i, whenever they are not constant, are first integrals of the DDS generated by F. In order to prove that they are functionally independent it suffices to show that the matrix $(DH)_x$ has rank k for almost all $x \in \mathcal{U}$. To this end we describe the map H as $\bar{S} \circ Y \circ \mathcal{F}$ where

$$\mathcal{F} : \mathcal{U}^k \to \mathbb{K}^{kp} \quad , \quad Y : \mathbb{K}^{kp} \to \mathbb{K}^\ell \quad , \quad \bar{S} : \mathbb{K}^\ell \to \mathbb{K}^\ell$$

and Y and $\bar{S}$ are defined through: $Y(y_1, y_2, \ldots, y_N) = (Y_1, Y_2, \ldots, Y_\ell)$ and $\bar{S} = (\bar{S}_1, \bar{S}_2, \ldots, \bar{S}_\ell)$ with $\bar{S}_i(w_1, w_2, \ldots, w_\ell) = n_1 w_1^i + n_2 w_2^i + \cdots + n_\ell w_\ell^i$. Then by the chain rule we have that for each $x \in \mathcal{U}$:

$$(DH)_x = (D\bar{S})_{(Y \circ \mathcal{F}(x))} \, (DY)_{\mathcal{F}(x)} \, (D\mathcal{F})_x.$$

The matrix $(D\mathcal{F})_x$ has the maximum rank equals to k for all $x \in \mathcal{U}$, because the first k components of $\mathcal{F}$ gives us the identity map. The matrix of (DY) has maximum rank ℓ at each point of $\mathbb{K}^k$.

We claim that $(D\bar{S})_{(Y \circ \mathcal{F}(x))}$ has also the maximum rank equal ℓ for almost all $x \in \mathcal{U}$. If the claim is true, since $\ell \geq k$ the result follows.

In order to prove the claim notice that for $i = 1, 2, \ldots, \ell$,

$$\bar{S}_i(w_1, w_2, \ldots, w_n) = n_1 w_1^i + n_2 w_2^i + \cdots + n_\ell w_\ell^i,$$

and so

$$(DS)_{(w_1,w_2,\ldots,w_n)} = \begin{pmatrix} n_1 & n_2 & \cdots & n_\ell \\ 2n_1w_1 & 2n_2w_2 & \cdots & 2n_\ell w_\ell \\ 3n_1w_1^2 & 3n_2w_2^2 & \cdots & 3n_\ell w_\ell^3 \\ \vdots & \vdots & \vdots & \vdots \\ \ell n_1 w_1^{l-1} & \ell n_2 w_2^{\ell-1} & \cdots & \ell n_\ell w_\ell^{\ell-1} \end{pmatrix}.$$

To compute the determinant of this matrix we observe that

$$det(D\bar{S})_{(w_1,w_2,\ldots,w_n)} = \ell!\, n_1 n_2 \cdots n_\ell \det(V)$$

where

$$det(V) = \prod_{i<j,\, j=1,\ldots,\ell} (w_i - w_j)$$

is the known Vandermonde's determinant. Taking $(w_1, w_2, \ldots, w_n) = (Y \circ \mathcal{F})(x)$, since among the components of $Y \circ \mathcal{F}$ there are no repetitions, this determinant does not vanish identically and the result follows. □

Remark 8. *The Montgomery-Bochner Theorem, see* [23] *asserts that if* $x^* \in \mathbb{K}^k$ *is a fixed point of a* $\mathcal{C}^k$, $k \geq 1$ *or analytic map* F *such that* $F^p = Id$ *then there is a neigbourhood of* x^* *where the DDS generated by* F *is* $\mathcal{C}^k$ *or analytic conjugated to the DDS generated by its linear part* $L(x) := DF(x^*)\, x$ *near* 0*. Thus the existence of a complete set of integrals for the DDS generated by* F *(only in a neigbourhood of* x^**) is also a consequence of the study of the same problem for the linear DDS generated by* $L(x)$ *near* 0.

In order to illustrate the method developed in Theorem 7 we begin with a trivial but pedagogical example.

Example 3.1: *Consider the 3-periodic linear map:* $F(x, y, z) = (y, z, x)$. *Here* $k = 3$, $p = 3$ *and* $N = 9$. *Since*

$$\mathcal{F}(x, y, z) = (x, y, z, y, z, x, z, x, y),$$

ℓ *is just* 3*. Following the proof of the above theorem we have to consider* S_1, S_2 *and* S_3 *of* 9 *coordinates and do the appropriate substitutions. We obtain the following set of functionally independent first integrals in* $\mathcal{U} = \mathbb{K}^3$:

$$\begin{aligned} H_1(x, y, z) &= 3(x + y + z), \\ H_2(x, y, z) &= 3\left(x^2 + y^2 + z^2\right), \\ H_3(x, y, z) &= 3\left(x^3 + y^3 + z^3\right). \end{aligned}$$

Example 3.2: *Let F be given by:*

$$F(x,y)=\left(\frac{y}{1+x+y},\frac{-x}{1+x+y}\right).$$

It is easy to check that $F^4=Id$ in $\mathcal{U}=\mathbb{K}^2\setminus\{(1+x+y)(1+2y)(1-x+y)=0\}$ and that

$$\mathcal{F}(x,y)= \\ =\left(x,y,\frac{y}{1+x+y},\frac{-x}{1+x+y},\frac{-x}{1+2y},\frac{-y}{1+2y},\frac{-y}{1-x+y},\frac{x}{1-x+y}\right).$$

Thus a system of two independent first integrals is:

$$H_1(x,y) = (x+y)\frac{2y}{1+2y}+(x-y)\frac{2x}{(1-x+y)(1+x+y)},$$

$$H_{\sigma_8}(x,y) = \left(\frac{x^2y^2}{(1-x+y)\,(1+2y)\,(1+x+y)}\right)^2,$$

where $H_{\sigma_8}(x)=\sigma_8(\mathcal{F}(x))$.

The above example is given by the flow at time $2\pi/4$ of the planar isochronous center $\dot{x}=-y+x^2$, $\dot{y}=x(1+y)$, see [11]. Recall that given an ordinary differential equation defined in an invariant open set $\mathcal{U}\subset\mathbb{R}^k$ it is called *isochronous, or $T-$isochronous* in $\mathcal{U}$ if there exists $T>0$ such that $\varphi(T,x)\equiv x$ for all $x\in\mathcal{U}$, where φ is the flow associated to the ordinary differential equation. Thus isochronous flows give rise to a full collection of $p-$periodic maps in $\mathcal{U}$; it suffices to consider, for any $p\in\mathbb{N}$, $F_p(x):=\varphi(T/p,x)$ and by the flow property $F_p^p(x)\equiv x$ for all $x\in\mathcal{U}$. For instance another 4-periodic map is given by

$$F(x,y)=\left(\frac{-y}{\sqrt[3]{1+4y^3-4x^3}},\frac{x}{\sqrt[3]{1+4y^3-4x^3}}\right).$$

It cames from the flow at time $(2\pi)/4$ of the $2\pi-$isochronous center $\dot{x}=-y+4x^2y^2$, $\dot{y}=x+4xy^3$. Similar computations to the ones used in the above example give a complete set of integrals for the DDS generated by F.

From the above considerations and Theorem 7 we get the following corollary:

Corollary 9. *Let $F_p(x):=\varphi(T/p,x)$ be a map defined from $\mathcal{U}$ into itself, where $\varphi(t,x)$ is the flow of a $T-$isochronous ordinary differential equation defined in $\mathcal{U}$ and $p\in\mathbb{N}$. Then the DDS generated by F_p is completely integrable.*

Remark 10. *The method given by Theorem 7 gives a constructive way for obtaining k functionally independent first integrals $H_1,\ldots,H_k$ for the*

DDS associated to F. Recall that any function of these first integrals is, whenever it is not a constant function, again a new first integral $H = \Phi(H_1, H_2, \ldots, H_k)$. *The fact that the proposed method is algorithmic is its main advantage; on the other hand it does not necessarily give the simpler first integrals. For instance, for Example 3.2 a new first integral is* $H(x, y) = (x^2 + y^2)/(1 + y)^2$. *This first integral can be obtained, for instance, by using the Darboux–type theory of integrability introduced in* [11].

We end this section with an example in $\mathbb{R}^2$ showing that not all the complete sets of first integrals of global periodic maps satisfy the uniform finiteness condition (1).

Example 3.3: *Consider the map* $F : \mathbb{R}^2 \to \mathbb{R}^2$ *be defined by*

$$F(x, y) = (-x, y).$$

This map F *is bijective in the whole* $\mathbb{R}^2$, *it is clearly* 2*–periodic and the DDS generated by it is completely integrable, being* $H_1(x, y) = y + \cos(x)$ *and* $H_2(x, y) = y$ *two functionally independent first integrals. It is also clear that for some* k_1, k_2 *the set* $\{H_1(x, y) = k_1\} \cap \{H_1(x, y) = k_1\}$ *has infinitely many points. On the other hand the two functions* $\tilde{H}_1(x, y) = x^2$ *and* $H_2(x, y) = y$ *give another complete set of first integrals which satisfy condition (1).*

4. Invariants for globally periodic recurrences

In this section we particularize and improve the results of previous sections to difference equations of order k :

$$x_{n+k} = f(x_n, x_{n+1}, x_{n+2}, \ldots, x_{n+k-1}), \tag{4}$$

where $\mathcal{U}$ is an open set of $\mathbb{K}^k$ and $f : \mathcal{U} \to \mathbb{K}$, is such that for any $(x_1, x_2, \ldots, x_k) \in \mathcal{U}$, $(x_2, x_3, \ldots, f(x_1, x_2, \ldots, x_k)) \in \mathcal{U}$.

As it is known each difference equation has an associated DDS given by the map

$$F(x_1, \ldots, x_k) = (x_2, x_3, \ldots, x_k, f(x_1, x_2, \ldots, x_k)), \tag{5}$$

and the projection of each element of $\mathbb{K}^k$ onto the x_1-axis, maps the orbit $\{\mathbf{x}, F(\mathbf{x}),\ F^2(\mathbf{x}), \ldots\}$ onto the sequence $\{x_1, x_2, x_3, x_4, \ldots\}$ defined by (4), where $\mathbf{x} = (x_1, x_2, \ldots, x_k)$ is the vector of initial conditions.

The difference equation (4) is said to be globally $p-$periodic or simply $p-$periodic if $x_{n+p} = x_n$ for all $n \in \mathbb{N}$, and for all $(x_1, x_2, \ldots, x_k) \in \mathcal{U}$. Clearly p must be greater than or equal to k. Note that this property is equivalent to the fact that the DDS generated by (5) is globally $p-$periodic, *i.e.* $F^p(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \mathcal{U}$.

Recall also that an invariant for the difference equation (4) is a non-constant function $I : \mathcal{U} \longrightarrow \mathbb{K}$, satisfying

$$\begin{aligned} & I(x_{n+1}, x_{n+2}, \ldots, x_{n+k}) = \\ = \;& I(x_{n+1}, x_{n+2}, \ldots, f(x_n, x_{n+1}, x_{n+2}, \ldots, x_{n+k-1})) = \\ = \;& I(x_n, x_{n+1}, x_{n+2}, \ldots, x_{n+k-1}). \end{aligned}$$

Note that the invariants for the difference equation (4) can be seen as first integrals for the DDS induced by (5).

Next theorem adapts and simplifies the method given in Theorem 7 to difference equations. The point is that, in this case, the special form of $F(x)$ makes that in the set

$$\mathcal{F}(x) = (x, F(x), F^2(x), \ldots, F^{p-1}(x)),$$

a lot of repetitions appear. Using this observation, the dimension of the space of symmetric polynomials which give us the corresponding invariants of the periodic difference equations, can be reduced.

Theorem 11. *Consider a differentiable difference equation (4) and assume that it is globally p–periodic. Then it has k functionally independent invariants. Furthermore these invariants can be chosen among the ones of the form*

$$I_{S_i}(x_1, x_2, \ldots, x_k) = S_i(x_1, x_2, \ldots, x_p)\,,\; i = 1, 2, \ldots, p,$$

where the set of functions S_i is the basis of elementary symmetric polynomials in p variables introduced in Section 3.

Proof. Given the p–periodic difference equation (4), consider the associated map (5). Now take the symmetric polynomials in $N = kp$ variables. Consider $\mathcal{F}$ and ℓ as in the proof of Theorem (7). From that theorem we know that among the first ℓ symmetric polynomials in N variables $\bar{S}_1, \bar{S}_2, \ldots, \bar{S}_\ell$, there are k which are functionally independent, namely $\bar{S}_{i_1}, \bar{S}_{i_2}, \ldots, \bar{S}_{i_k}$. These first integrals are of the form $(\bar{S}_{i_n} \circ \mathcal{F})(x)$. If the p-periodic orbit (4) is $x_1, x_2, \ldots, x_p$ then from the form of F and by the periodicity of the sequence $\{x_n\}_n$ we see that for $i = 1, 2, \ldots, p$ each x_i appears at least k times in the expression of $\mathcal{F}(x)$. Hence,

$$(\bar{S}_{i_n} \circ \mathcal{F})(x) = kx_1^{i_n} + kx_2^{i_n} + \cdots + kx_p^{i_n}.$$

Since $\bar{S}_{i_1}, \bar{S}_{i_2}, \ldots, \bar{S}_{i_k}$ are functionally independent invariants, the same is true for

$$S_{i_n}(x_1, x_2, \ldots, x_p) = x_1^{i_n} + x_2^{i_n} + \cdots + x_p^{i_n},$$

considering $n = 1, 2, \ldots, k$. □

Recall that the Newton–Girard formulae relate the two basis of the symmetric polynomials introduced in Section 3, see for instance [21]. Hence, instead of using the invariants I_{S_i}'s we sometimes also consider the I_{σ_i}'s given by

$$I_{\sigma_i}(x_1,x_2,\ldots,x_k)=\sigma_i(x_1,x_2,\ldots,x_p)\,,\; i=1,2,\ldots,p.$$

Remark 12. *Note that from the above Theorem we get that globally $p-$periodic rational (respectively, analytic) difference equations of order k have k rational (respectively, analytic) functionally independent invariants.*

From now one this section will be devoted to use Theorem 11 to present a complete set of rational invariants for several examples of globally $p-$periodic rational recurrences.

Example 4.1: *Consider the globally $3-$periodic difference equation of order two:*

$$x_{n+2}=\frac{c}{x_nx_{n+1}}\text{ with c}\neq 0.$$

Two functionally independent invariants of its associated DDS are

$$\begin{aligned} I_{S_1}(x,y)=S_1\left(x,y,c/xy\right) &= x+y+\frac{c}{xy},\\ I_{S_2}(x,y)=S_2\left(x,y,c/xy\right) &= x^2+y^2+\frac{c^2}{x^2y^2}. \end{aligned}$$

Next results give a complete set of rational independent first integrals for all the known globally periodic rational difference equations of order k of the form

$$x_{n+k}=\frac{a_1x_n+a_2x_{n+1}+\cdots+a_kx_{n+k-1}+a_0}{b_1x_n+b_2x_{n+1}+\cdots+b_kx_{n+k-1}+b_0},$$

with real coefficients. Indeed, the characterization of the globally periodic recurrences of the above type is still an open problem, see [5, 8, 18] or [20].

Firstly, we list to our knowledge, all the known periodic recurrences, modulus equivalences [5, 8] and minimality (see the definitions below). They are:

(i) The trivial ones: $x_{n+1}=x_n$ and $x_{n+1}=-x_n$,
(ii) The ones given by the Möbius transformations $x_{n+1}=(ax_n+b)/(x_n+d)$ such that $\Delta:=(d-a)^2+4b\neq 0$, and $(d+a-\sqrt{\Delta})/(d+a+\sqrt{\Delta})$ is a p-root of the unity,
(iii) The Lyness–type ones:

$$x_{n+2}=\frac{x_{n+1}}{x_n},\qquad x_{n+2}=\frac{x_{n+1}+1}{x_n},$$
$$x_{n+3}=\frac{x_{n+1}+x_{n+2}+1}{x_n},\text{ and }\quad x_{n+3}=\frac{x_{n+1}-x_{n+2}-1}{x_n}.$$

Recall that given a $p-$periodic recurrence a change of scale of the variables can give a different, but *equivalent* $p-$periodic recurrence. On the other hand given a p–periodic difference equation of order k:

$$x_{n+k} = f(x_n, x_{n+1}, x_{n+2}, \ldots, x_{n+k-1}) \tag{6}$$

it *generates* a rp–periodic difference equation of order rk given by

$$x_{n+kr} = f(x_n, x_{n+r}, x_{n+2r}, \ldots, x_{n+(k-1)r}). \tag{7}$$

We call the one with minimum order, among all the ones obtained by using the above procedure, a *minimal* difference equation. For instance the equivalent difference equations to the Lyness equation $x_{n+2} = (1 + x_{n+1})/x_n$, which is $5-$periodic and minimal, are the ones given by $x_{n+1} = (a^2 + ax_{n+1})/x_n$, where $a \neq 0$, and the ones generated by it are the $5r-$periodic difference equations $x_{n+2r} = (1+x_{n+r})/x_n$, where r is any positive integer.

Some invariants of the trivial minimal periodic difference equations $x_{n+1} = x_n$ and $x_{n+1} = -x_n$ are $I(x) = x$ and $I(x) = x^2$, respectively. It is well known that all the periodic Möbius transformations $x_{n+1} = (ax_n + b)/(cx_n + d)$ (with real or complex coefficients) have the invariant $I(x) = [(2x - a + d - \sqrt{\Delta})/(2x - a + d + \sqrt{\Delta})]^p$, (see, for instance, [6]).

Next result gives the a complete set of rational invariants for the Lyness type periodic recurrences.

Proposition 13. *(i)The 6-periodic recurrence $x_{n+2} = x_{n+1}/x_n$ has the two functionally independent invariants*

$$\begin{aligned} I_{S_1}(x,y) &= [x^2y + xy^2 + y^2 + y + x + x^2]/(xy) \quad \text{and} \\ I_{S_2}(x,y) &= [x^4y^2 + x^2y^4 + y^4 + y^2 + x^2 + x^4]/(x^2y^2). \end{aligned}$$

(ii) The 5-periodic recurrence $x_{n+2} = (1 + x_{n+1})/x_n$ has the two functionally independent invariants

$$I_{\sigma_2}(x,y) =$$
$$= [y^4x+(x^3+x^2+2x+1)y^3+(x^3+5x^2+3x+2)y^2+(x^4+2x^3+3x^2+3x+1)y+x^3+2x^2+x]/(x^2y^2)$$

and

$$I_{\sigma_5}(x,y) = [(1+x)(1+y)(1+x+y)]/(xy).$$

(iii) The 8–periodic recurrence $x_{n+3} = (1 + x_{n+1} + x_{n+2})/x_n$, has the three functionally independent invariants

$$\begin{aligned} H(x,y,z) &= [(x+1)(y+1)(z+1)(1+x+y+z)]/(xyz), \\ I(x,y,z) &= [(1+y+z)(1+x+y)(1+x+y+z+xz)]/(xyz) \quad \text{and} \\ I_{\sigma_2}(x,y,z) &= [p_4(y,z)x^4+p_3(y,z)x^3+p_2(y,z)x^2+p_1(y,z)x+p_0(y,z)]/(x^2y^2z^2), \end{aligned}$$

where

$$\begin{aligned}
p_4(y,z) &= y\big(yz+1+2\,z+z^2\big),\\
p_3(y,z) &= \big(z^2+2\,z\big)y^3+\big(3+2\,z^2+5\,z+z^3\big)y^2+\big(5\,z^2+7\,z+2\,z^3+4\big)y+3\,z^2+1+3\,z+z^3,\\
p_2(y,z) &= y^4z+\big(3+2\,z^2+5\,z+z^3\big)y^3+\big(2\,z^3+8+12\,z^2+12\,z\big)y^2+\\
&\quad+\big(12\,z^2+5\,z^3+z^4+7+15\,z\big)y+8\,z^2+7\,z+3\,z^3+2,\\
p_1(y,z) &= (z+y+1)\big(\big(2\,z+z^2+1\big)y^3+\big(3+2\,z^2+5\,z+z^3\big)y^2+\big(5\,z^2+3+7\,z+2\,z^3\big)y+3\,z^2+4\,z+1\big),\\
p_0(y,z) &= z\big(yz+1+y^2+2\,y\big)(z+y+1)^2.
\end{aligned}$$

(iv) The 8*–periodic recurrence* $x_{n+3}=(-1+x_{n+1}+x_{n+2})/x_n$*, has the three functionally independent invariants*

$$\begin{aligned}
&I_{\sigma_1}(x,y,z)=\\
&\quad=[(xy-y+1+x)z^2+\big(y^2-3y+xy^2+x^2y-x^2-3x\big)z-1-3xy-xy^2+y^2+x^2y+x^2]/(xyz),\\
&I_{\sigma_5}(x,y,z)=\\
&\quad=(-1+y-z)(1-x-y+z+xz)(1-x+y+z+xz)(-1+x-z-xy-xz+y^2-yz)(1-x+y)/(x^2y^2z^2)\\
&I_{\sigma_2}(x,y,z)=\\
&\quad=(p_4(x,y)z^4+p_3(x,y)z^3+p_2(x,y)z^2+p_1(x,y)z+p_0(x,y))/(x^2y^2z^2),
\end{aligned}$$

where

$$\begin{aligned}
p_4(x,y) &= -xy^2+\big(x^2-1\big)y,\\
p_3(x,y) &= y^3x^2+\big(3-x+x^3\big)y^2+\big(-x^2+x-2\big)y-3x-3x^2-x^3-1,\\
p_2(x,y) &= y^4x+\big(-3+x^3-x\big)y^3+\big(4-12x^2\big)y^2+\big(-x^3-x^4+3x+1\big)y-2+3x^3+4x^2-x,\\
p_1(x,y) &= \big(-x^2+1\big)y^4+\big(-2-x^2\big)y^3+\big(-x^3+x^4+5x\big)y^2+\big(2+3x^2-x^3\big)y-3x^3+3x+x^2-1,\\
p_0(x,y) &= -y^4x+\big(3x^2-2x\big)y^3+\big(4x^2-3x^3\big)y^2+\big(2x-2x^3+x^4-x^2\big)y+x-2x^2+x^3.
\end{aligned}$$

Proof. (i) For this $6-$periodic difference equation its corresponding sequence is

$$x_1,x_2,\frac{x_2}{x_1},\frac{1}{x_1},\frac{1}{x_2},\frac{x_1}{x_2},x_1,x_2,\ldots$$

To construct the invariants it is enough to consider

$$I_{S_1}(x,y)=S_1\left(x,y,\frac{y}{x},\frac{1}{x},\frac{1}{y},\frac{x}{y}\right)=x+y+\frac{y}{x}+\frac{1}{x}+\frac{1}{y}+\frac{x}{y},$$

and

$$I_{S_2}(x,y)=S_2\left(x,y,\frac{y}{x},\frac{1}{x},\frac{1}{y},\frac{x}{y}\right)=x^2+y^2+\left(\frac{y}{x}\right)^2+\frac{1}{x^2}+\frac{1}{y^2}+\left(\frac{x}{y}\right)^2,$$

which are the two independent first integrals that appear in the statement.

(ii) For the planar Lyness two difference equation its corresponding sequence is

$$x_1,x_2,\frac{1+x_2}{x_1},\frac{1+x_1+x_2}{x_1x_2},\frac{1+x_1}{x_2},x_1,x_2,\ldots$$

In this case we proceed as in the previous one but taking here the symmetric polynomials $S_1 = \sigma_1$ and $\sigma_5(x,y,z,t,u) = xyztu$. Since

$$\det(\nabla I_{\sigma_1}, \nabla I_{\sigma_5}) = \\ = -\frac{(xy-y-1)(y-x)\left(-x^2+1+y\right)(xy-1-x)\left(y^2-x-1\right)}{x^4y^4},$$

we get that both rational functions are functionally independent.

(iii) This difference equation, also known as Todd's equation, is 8−periodic. It is well known that $H(x,y)$ is an invariant for it. Using σ_8, our method gives:

$$I_{\sigma_8}(x,y,z) = \left(\frac{(1+y+z)(1+x+y)(1+x+y+z+xz)}{xyz}\right)^2.$$

It can be checked that

$$I(x,y,z) = \frac{(1+y+z)(1+x+y)(1+x+y+z+xz)}{xyz},$$

is also an invariant. In fact this last invariant is a particular case of a second invariant for the k^{th}–order Lyness equation that has also been found in [12]. Finally, by using again the same method we obtain the invariant I_{σ_2} given in the statement.

Another tedious computation shows that

$$\det(\nabla H, \nabla I, \nabla I_{\sigma_2}) = (f_1\, f_2\, f_3\, f_4\, f_5)/(x^5y^5z^5),$$

where

$$\begin{aligned} f_1 =\ & (-z+x), \quad f_2 = (x+1+y-yz), \quad f_3 = (xy-1-y-z), \\ f_4 =\ & (xyz-xz-x-1-y-z) \\ f_5 =\ & \left(z^2x^2+zx^2-xy-y-y^2x-2y^2-y^3+xz-yz-y^2z+z^2x\right) \\ f_6 =\ & \big(-1-3y+z^2xy-y^2xz+x^2yz-2yz+2z^2x-x^2y+z^2x^2-yz^2 \\ & -2y^2z+x^2-2xy+z^2+2xz-2y^2x+2zx^2-3y^2-y^3\big) \end{aligned}$$

Thus the three rational invariants give a complete systems of invariants for the Todd's equation.

(iv) The first integrals have been obtained using again Theorem 11. Another trivial computation shows that

$$\det(\nabla I_{\sigma_1}, \nabla I_{\sigma_2}, \nabla I_{\sigma_5}) = -\frac{P_{12}(x,y,z)}{x^4y^4z^3},$$

where $P_{12}(x,y,z)$ is an ugly nonzero polynomial of degree 12. Thus we have proved that the three invariant are functionally independent, as desired. □

4.1. **Invariants for non-minimal recurrences.** Given a difference equation having an invariant it is very easy to extend this invariant to any equivalent difference equation to the given one: it suffices to perform the change of variables which transforms one equation into the other in the given invariant. This section is devoted to explain how to extend the invariant of a difference equation (6) to the difference equations generated by a given one (7).

Proposition 14. *Consider the difference equation (7):*

$$x_{n+kr} = f(x_n, x_{n+r}, x_{n+2r}, \ldots, x_{n+(k-1)r}),$$

and let $I : \mathbb{K}^k \to \mathbb{K}$ *be an invariant of the generating difference equation*

$$x_{n+k} = f(x_n, x_{n+1}, x_{n+2}, \ldots, x_{n+k-1}),$$

then

$$J(\mathbf{x}) = I(\phi(\mathbf{x}))I(\phi \circ F(\mathbf{x})) \cdots I(\phi \circ F^{r-1}(\mathbf{x}))$$

is an invariant of (7), where $\mathbf{x} = (x_1, \ldots, x_{kr}) \in \mathbb{K}^{kr}$, $\phi : \mathbb{K}^{kr} \to \mathbb{K}^k$ *is the projection map given by*

$$\phi(y_1, y_2, \ldots, y_{kr}) = (y_1, y_{1+r}, y_{1+2r}, \ldots, y_{1+(k-1)r}),$$

and $F : \mathbb{K}^{kr} \to \mathbb{K}^{kr}$ *is the map which generates the DDS associated to (7)*

$$F(y_1, \ldots, y_{kr}) = (y_2, y_3, \ldots, y_{kr-1}, y_{kr}, f(y_1, y_{1+r}, y_{1+2r}, \ldots, y_{1+(k-1)r})).$$

Proof. Observe that $I(\phi \circ F^r(\mathbf{x})) = I(\phi(\mathbf{x}))$, since

$$\begin{aligned} I(\phi \circ F^r(\mathbf{x})) &= I(\phi(x_{1+r}, x_{2+r}, x_{3+r}, \ldots, x_{kr+r})) = \\ &= I(x_{1+r}, x_{1+2r}, \ldots, x_{1+kr}) = \\ &= I(x_1, x_{1+r}, x_{1+2r}, \ldots, x_{1+(k-1)r}) = I(\phi(\mathbf{x})). \end{aligned}$$

Therefore

$$\begin{aligned} J(F(\mathbf{x})) &= I(\phi \circ F(\mathbf{x})) \cdots I(\phi \circ F^{r-1}(\mathbf{x}))I(\phi \circ F^r(\mathbf{x})) = \\ &= I(\phi \circ F(\mathbf{x})) \cdots H(\phi \circ F^{r-1}(\mathbf{x}))I(\phi(\mathbf{x})) = J(\mathbf{x}). \end{aligned}$$

□

We end this subsection with some applications of the above result.

Example 4.2: (i) Setting $r = 2$, the 6–periodic recurrence (i) of Proposition 13, generates a 12–periodic recurrence $x_{n+4} = x_{n+2}/x_n$, which has the invariants

$$J_i(x_n, x_{n+1}, x_{n+2}, x_{n+3}) = I_{S_i}(x_n, x_{n+2})I_{S_i}(x_{n+1}, x_{n+3}), \quad i = 1, 2,$$

where the functions I_{S_i} are given in (i) of Proposition 13.

(ii) Taking $r = 3$ the 5–periodic Lyness recurrence, generates a 15–periodic recurrence given by $x_{n+6} = (1 + x_{n+3})/x_n$. One of its invariants is

$$\begin{aligned} J(x_n, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}, x_{n+5}) = \\ = \quad I_{\sigma_5}(x_n, x_{n+3}) I_{\sigma_5}(x_{n+1}, x_{n+4}) I_{\sigma_5}(x_{n+2}, x_{n+5}), \end{aligned}$$

where $I_{\sigma_5}(x, y) = (1 + x)(1 + y)(1 + x + y)/(xy)$.

(iii) The Möbius recurrence

$$z_{n+1} = \frac{(-1 + i)z_n + 2i}{z_n - 1 + i} =: f(z_n)$$

is a 4–periodic recurrence with invariant given by $I(z_n) = \left(\frac{z_n - 1 - i}{z_n + 1 + i}\right)^4$. Taking $r = 3$, we obtain that the 12–periodic recurrence $z_{n+3} = f(z_n)$ has the invariant $J(z_n, z_{n+1}, z_{n+2}) = I(z_n)I(z_{n+1})I(z_{n+2})$.

5. The non–differentiable case

In the case that the map F is not differentiable or even discontinuous, part (b) of Theorem 1 does not apply. However the following result shows that if F is globally periodic then the DDS generated by F has at least k (some of them maybe functionally dependent) non constant first integrals of the form H_{σ_i}.

Theorem 15. *Let $F : \mathcal{U} \subseteq \mathbb{K}^k \to \mathcal{U}$ be a globally p–periodic map. Define the set of functions $\Sigma := \{H_{\sigma_i}\}_{\{i=1,\ldots,kp\}}$, where $H_{\sigma_i}(x) = \sigma_i(x, F(x), \ldots, F^{p-1}(x))$ and σ_i is the basis of the symmetric polynomials in kp variables given in Section 3. Then, among all the elements of Σ there are at least k non constant first integrals for the DDS generated by F.*

Proof. By Theorem 7 we already know that all the non-constant functions in Σ are first integrals of the DDS generated by F. Let us prove that at least k of them are effectively non constant.

Set $\mathbf{x} = (x_1, \cdots, x_k)$. Notice that, modulus a sign, the functions in Σ are the coefficients of the polynomial

$$\begin{aligned} P(w, \mathbf{x}) := \quad & (w - x_1) \ldots (w - x_k)(w - F_1(\mathbf{x})) \ldots (w - F_k(\mathbf{x}))(w - F_1^2(\mathbf{x}))\cdot \\ & \ldots (w - F_k^2(\mathbf{x})) \ldots (w - F_1^{p-1}(\mathbf{x})) \ldots (w - F_k^{p-1}(\mathbf{x})) \\ & = w^{kp} + \cdots + (-1)^i H_{\sigma_i}(\mathbf{x}) w^{kp-i} + \cdots + (-1)^{kp} H_{\sigma_{nk}}(\mathbf{x}). \end{aligned}$$

First we prove that that at least there is one non constant function in Σ. Assume, by contradiction, that all the functions in Σ are constant functions. Then $P(w, \mathbf{x}) = P(w)$ is a polynomial of degree kp with constant coefficients having infinitely many zeros (for instance $w = x_1$ is a zero for any value of

x_1.). Thus P has to be identically zero, in contradiction with the fact that its coefficient of the $(kp)^{th}$–order term is 1.

We continue again by contradiction. Assume that the statement is not true and let

$$H_{\sigma_{i_1}}, H_{\sigma_{i_2}}, \ldots, H_{\sigma_{i_{k-1}}}$$

(with $i_\ell \neq i_s$ if $\ell \neq s$) such that can depend on $\mathbf{x}$, while the others coefficients of $P(w, \mathbf{x})$ are constants. For the sake of simplicity, write $\tilde{H}_i(x) = (-1)^i H_{\sigma_i}(x)$.

Since $P(w, \mathbf{x})$ vanishes for $w = x_i$ for all $i = 1, 2, \ldots, k$ we can consider the system L_1 of equations $P(x_\ell, \mathbf{x}) = 0$ for $\ell = 1, 2, \ldots, k-1$, *i.e.*,

$$L_1 : \begin{cases} \tilde{H}_{i_1}(\mathbf{x}) x_\ell^{kp-i_1} + \tilde{H}_{i_2}(\mathbf{x}) x_\ell^{kp-i_2} + \cdots + \tilde{H}_{i_{k-1}}(\mathbf{x}) x_\ell^{kp-i_{k-1}} = \\ \qquad = -\sum_{j \notin \{i_1, i_2, \ldots, i_{k-1}\}} c_j x_\ell^{kp-j} \end{cases},$$

looking at $\tilde{H}_{i_r}(\mathbf{x})$, $r = 1, \ldots, k-1$, as the unknowns. The determinant of the matrix of the coefficients of this linear system is:

$$\begin{vmatrix} x_1^{kp-i_1} & x_1^{kp-i_2} & \ldots & x_1^{kp-i_{k-1}} \\ x_2^{kp-i_1} & x_2^{kp-i_2} & \ldots & x_2^{kp-i_{k-1}} \\ \vdots & \vdots & \vdots & \vdots \\ x_{k-1}^{kp-i_1} & x_{k-1}^{kp-i_2} & \ldots & x_{k-1}^{kp-i_{k-1}} \end{vmatrix} := D(x_1, x_2, \ldots, x_{k-1}).$$

The monomials of the polynomial $D(x_1, x_2, \ldots, x_{k-1})$ have 1 or -1 as a coefficient and two of these monomials can not coincide (it is because not two of the exponents $i_1, i_2, \ldots, i_{k-1}$ coincide). In particular $D(x_1, x_2, \ldots, x_{k-1})$ is not identically zero and so, for almost all the values of $x_1, x_2, \ldots, x_{k-1}$, it is possible to isolate $\tilde{H}_{i_r}(\mathbf{x})$ for $r = 1, 2, \ldots, k-1$ in terms of $x_1, x_2, \ldots, x_{k-1}$. It implies that in fact each $\tilde{H}_{i_r}(\mathbf{x})$, $r = 1, 2, \ldots, k-1$ does not depend on x_k.

Now we can repeat the above process taking the systems of equations :

$$L_r : \{P(x_\ell, \mathbf{x}) = 0 \quad \text{for} \quad \ell \in \{1, 2, \ldots, k\} \quad \text{but} \quad \ell \neq k - r + 1\}$$

for each $r \in \{1, 2, \ldots, k\}$.

System L_r implies that each $\tilde{H}_{i_j}(\mathbf{x})$, for $j = 1, 2, \ldots, k-1$ does not depend on x_{k+1-r}. Hence, for each $j = 1, 2, \ldots, k-1$, $\tilde{H}_{i_j}(\mathbf{x})$ does not depend on $x_1, x_2, \ldots, x_k$, *i.e.* it is constant. Therefore all the functions in Σ are constant, which is contradiction with our first step, where we have shown that at least one of the elements in Σ has to be a non constant function. Thus we can ensure that among all the elements of Σ at least k of them are non constant functions, as desired. □

Following the guidelines of Theorem 15 and the spirit of Section 4, we give several invariants of two non–smooth globally periodic difference equations.

Note that in the non-differentiable case it is a difficult task to elucidate whether several functions are functionally independent or not. We do not consider this question in the study of the following examples.

In [2], Abu–Saris and Al–Hassan, introduced the following second order globally 4–periodic difference equation:

$$x_{n+2} = g(x_{n+1})/x_n, \text{ where } g(x) = \begin{cases} a & \text{if } x \in \mathbb{Q}, \\ b & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \end{cases} \tag{8}$$

and $a, b \in \mathbb{Q} \setminus \{0\}$. We obtain the following result.

Proposition 16. *The following functions are invariants for the difference equation (8):*

$I_{\sigma_1}(x_n, x_{n+1}) = x_n + x_{n+1} + \dfrac{g(x_{n+1})}{x_n} + \dfrac{g(x_n)}{x_{n+1}}$.

$I_{\sigma_2}(x_n, x_{n+1}) = x_n x_{n+1} + g(x_n) + g(x_{n+1}) + \dfrac{x_{n+1}g(x_{n+1})}{x_n} + \dfrac{x_n g(x_n)}{x_{n+1}} + \dfrac{g(x_n)g(x_{n+1})}{x_n x_{n+1}}$.

$I_{\sigma_3}(x_n, x_{n+1}) = x_n g(x_n) + x_{n+1}g(x_{n+1}) + \dfrac{g(x_n)g(x_{n+1})}{x_n} + \dfrac{g(x_n)g(x_{n+1})}{x_{n+1}}$.

$I_{\sigma_4}(x_n, x_{n+1}) = g(x_n)g(x_{n+1})$.

Proof. The proof is an immediate check. It suffices to note that $g(g(x_{n+1})/x_n) = g(x_n)$ by using a case by case study. Thus, by using Theorem 15 we obtain that

$$\begin{aligned} I_{\sigma_k}(x_n, x_{n+1}) &= \sigma_k(x_n, x_{n+1}, x_{n+2}, x_{n+3}) \\ &= \sigma_k\left(x_n, x_{n+1}, \frac{g(x_{n+1})}{x_n}, \frac{g(x_n)}{x_{n+1}}\right), \end{aligned}$$

for $k = 1, \ldots, 4$ are invariants of the difference equation (8). □

In [8], Csörnyei and Laczkovich studied second order globally 5–periodic recurrence:

$$x_{n+2} = \max(0, x_{n+1}) - x_n, \tag{9}$$

which has an associated non–smooth dynamical system

$$F(x, y) = \begin{cases} (y, y - x) & \text{if } y \geq 0, \\ (y, -x) & \text{if } y < 0. \end{cases}$$

Observe that, according to [25], equation (9) is the ultradiscrete version of the Lyness recurrence.

Define the following subsets of $\mathbb{R}^2$:

$$\begin{array}{ll} D_1 = \{(x,y)\,:\, x\geq 0, y\geq 0, y\leq x\}, & D_2 = \{(x,y)\,:\, x\geq 0, y\leq 0\},\\ D_3 = \{(x,y)\,:\, x\leq 0, y\leq 0\}, & D_4 = \{(x,y)\,:\, x\leq 0, y\geq 0\},\\ D_5 = \{(x,y)\,:\, x\geq 0, y\geq 0, y\geq x\}. & \end{array}$$

Note that the DDS generated by F behaves as

$$D_1 \xrightarrow{F} D_2 \xrightarrow{F} D_3 \xrightarrow{F} D_4 \xrightarrow{F} D_5 \xrightarrow{F} D_1,$$

and each time that F actues is indeed a linear map. By using again Theorem 11 with the symmetric polynomials S_i with 5 indeterminates, we obtain a set of continuous invariants of the difference equation (9), which naturally have different expressions on each different subset D_i.

Proposition 17. *The functions $I_{\sigma_k}(x,y)$ listed below, are invariants for the difference equation (9).*

$$I_{\sigma_1}(x,y) = \begin{cases} x & \text{if } (x,y)\in D_1\\ x-y & \text{if } (x,y)\in D_2,\\ -x-y & \text{if } (x,y)\in D_3,\\ y-x & \text{if } (x,y)\in D_4,\\ y & \text{if } (x,y)\in D_5. \end{cases}$$

$$I_{\sigma_2}(x,y) = \begin{cases} -x^2+2xy-2y^2 & \text{if } (x,y)\in D_1,\\ -2x^2+2xy-y^2 & \text{if } (x,y)\in D_5,\\ -x^2-y^2 & \text{otherwise}. \end{cases}$$

$$I_{\sigma_3}(x,y) = \begin{cases} -x(x^2-2xy+2y^2) & \text{if } (x,y)\in D_1,\\ -x^3+x^2y-xy^2+y^3 & \text{if } (x,y)\in D_2,\\ x^3+x^2y+xy^2+y^3 & \text{if } (x,y)\in D_3,\\ x^3-x^2y+y^2x-y^3 & \text{if } (x,y)\in D_4,\\ -y(y^2-2xy+2x^2) & \text{if } (x,y)\in D_5. \end{cases}$$

$$I_{\sigma_4}(x,y) = \begin{cases} -x^2+2xy-2y^2 & \text{if } (x,y)\in D_1,\\ -2x^2+2xy-y^2 & \text{if } (x,y)\in D_5,\\ x^2y^2 & \text{otherwise}. \end{cases}$$

$$I_{\sigma_5}(x,y) = \begin{cases} xy^2(x-y)^2 & \text{if } (x,y)\in D_1,\\ x^2y^2(x-y) & \text{if } (x,y)\in D_2,\\ -x^2y^2(x+y) & \text{if } (x,y)\in D_3,\\ x^2y^2(y-x) & \text{if } (x,y)\in D_4,\\ x^2y(x-y)^2 & \text{if } (x,y)\in D_5. \end{cases}$$

Proof. Once again the proof is only a (tedious) check. We give the details for obtaining the first invariant I_{σ_1}. Setting $x_n := x$ and $x_{n+1} := y$, we obtain $x_{n+2} = \max(0, y) - x$, $x_{n+3} = \max(0, \max(0, y) - x) - y$ and $x_{n+4} = \max(0, \max(0, \max(0, y) - x) - y) - \max(0, y) + x$. Thus, by using Theorem 11 we have the following invariant:

$$\begin{aligned} I_{\sigma_1}(x_n, x_{n+1}) &= \sigma_1(x_n, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}) = \\ &= x_n + x_{n+1} + x_{n+2} + x_{n+3} + x_{n+4}, \end{aligned}$$

which can be written as

$$I_{\sigma_1}(x, y) = x + \max\left(0, \max\left(0, y\right) - x\right) + \max\left(0, \max\left(0, \max\left(0, y\right) - x\right) - y\right).$$

By studying the above function in each sector D_i, $i = 1, \ldots, 5$ we obtain the expression given in the statement.

The other four invariants can be obtained by following the same procedure but using $\sigma_i, i = 2, \ldots, 5$ instead of σ_1. □

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