THE AUTOMORPHISM GROUP OF A FREE-BY-CYCLIC GROUP IN RANK 2

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ABSTRACT. Let ϕ be an automorphism of a free group F_2 of rank 2 and let $M_\phi = F_2 \rtimes_\phi \mathbb{Z}$ be the corresponding mapping torus of ϕ. We prove that the group $Out(M_\phi)$ is usually virtually cyclic. Moreover, we classify the cases when this group is finite depending on the conjugacy class of the image of ϕ in $GL_2(\mathbb{Z})$.

1. INTRODUCTION

Let F_n be the free group of rank n freely generated by $x_1, \ldots, x_n$, and let us denote automorphisms $\phi \in Aut(F_n)$ as acting on the right, $x \mapsto x\phi$. In this paper we consider extensions of finitely generated free groups by the infinite cyclic group (*[f.g. free]-by-$\mathbb{Z}$* groups, for short). More precisely, for any given $\phi \in Aut(F_n)$, we consider the mapping torus, $M_\phi = F \rtimes_\phi \mathbb{Z}$, of ϕ i.e. the extension of F_n presented by

$$M_\phi = \langle x_1, \ldots, x_n, \, t \mid t^{-1} x_i t = x_i \phi \rangle.$$

The aim of the paper is to study the automorphism group of such groups, $Aut(M_\phi)$. We shall give partial results for arbitrary rank n, and a complete description for the cases $n = 1, 2$.

To help avoiding possible confusions, we will use greek letters (such as ϕ or ψ) to denote automorphisms of F_n, and capital greek letters (such as Φ or Ψ) to denote automorphisms of M_ϕ. Accordingly, for every word $w \in F_n$, we shall write γ_w to denote the inner automorphism of F_n given by right conjugation by w, $x\gamma_w = w^{-1}xw$. And, for every element $g \in M_\phi$, we shall write Γ_g to denote the inner automorphism of M_ϕ given by right conjugation by g, $x\Gamma_g = g^{-1}xg$. As usual, $Inn(G)$ denotes the group of inner automorphisms of a group G, and $Out(G) = Aut(G)/Inn(G)$.

Although [f.g. free]-by-$\mathbb{Z}$ groups have received a great deal of attention in recent years, there has been no real systematic study of their automorphisms. In fact, it still seems to be an open question whether or not they have finitely generated or finitely presented automorphism groups. Having said that, there are certain cases in which the automorphism group is understood. For instance, when M_ϕ is word hyperbolic, it is known to have finite outer automorphism group (this can be deduced from papers [1, 3]). However, note that in the rank 2 case, the group M_ϕ can never be hyperbolic. In fact, by a result of Nielsen (see Proposition 5.1 in [4]), $([x_1, x_2])\phi$ and so $t^{-1}[x_1, x_2]t$, must be conjugate to $[x_1, x_2]^{\pm 1}$ in F_2. Hence, M_ϕ contains a free abelian subgroup of rank 2 implying that M_ϕ is not hyperbolic.

It is straightforward to verify (see Lemma 2.1 below) that the isomorphism type of M_ϕ depends only on the conjugacy class of the outer automorphism $[\phi]^{\pm 1} \in Out(F_n) = Aut(F_n)/Inn(F_n)$ determined by $\phi^{\pm 1}$. While this characterises the isomorphism class of M_ϕ when $n = 2$, it seems also possible, for $n > 2$, that two [f.g. free]-by-$\mathbb{Z}$ groups are isomorphic even if they do not arise from conjugate or conjugate-inverse outer automorphisms.

In Section 2, we analyse $Aut(M_\phi)$ for arbitrary n but under certain technical restrictions for ϕ (see Theorems 2.4 and 2.5). After dedicating Section 3 to recall a standard classification of 2×2 matrices, the main result comes in Section 4, where we analyse $Aut(M_\phi)$ in the case when the underlying free group has rank $n = 2$, and without conditions on ϕ. The rank 2 case is doubtless the easiest to deal with, but we believe that some of our methods may be of general interest. For instance, our detailed look at the parabolic case is certain to be of use in the more general UPG case (the definition of UPG automorphisms can be found in [2]). The information obtained there is summarised in the following theorem, which is the main result of the paper.

Theorem 1.1. *Let $F_2 = \langle a, b \rangle$ be a free group of rank 2, let $\phi \in Aut(F_2)$, and consider the mapping torus $M_\phi = F_2 \rtimes_\phi \mathbb{Z}$. Let $\phi^{\mathrm{ab}} \in GL_2(\mathbb{Z})$ be the map induced by ϕ on $F_2^{\mathrm{ab}} \cong \mathbb{Z}^2$ (written in row form with respect to $\{a, b\}$).*

- i) *If $\phi^{\mathrm{ab}} = I_2$, then $Out(M_\phi) \cong (\mathbb{Z}^2 \rtimes C_2) \rtimes GL_2(\mathbb{Z})$, where C_2 is the cyclic group of order 2 acting on $\mathbb{Z}^2$ by sending u to $-u$, $u \in \mathbb{Z}^2$, and where $GL_2(\mathbb{Z})$ acts trivially on C_2 and naturally on $\mathbb{Z}^2$ (thinking vectors as columns there).*
- ii) *If $\phi^{\mathrm{ab}} = -I_2$, then $Out(M_\phi) \cong PGL_2(\mathbb{Z}) \times C_2$.*
- iii) *If $\phi^{\mathrm{ab}} \neq -I_2$ and does not have 1 as an eigenvalue then $Out(M_\phi)$ is finite.*
- iv) *If ϕ^{ab} is conjugate to $\left(\begin{smallmatrix} 1 & k \\ 0 & -1 \end{smallmatrix}\right)$ for some integer k, then $Out(M_\phi)$ has an infinite cyclic subgroup of finite index.*
- v) *If ϕ^{ab} is conjugate to $\left(\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right)$ for some $k \neq 0$, then $Out(M_\phi)$ has an infinite cyclic subgroup of finite index.*

Furthermore, for every $\phi \in Aut(F_2)$, ϕ^{ab} fits into exactly one of the above cases.

Note that for $n = 2$, M_ϕ is always the fundamental group of a 3-manifold, namely the corresponding mapping torus of a once punctured torus, say M. When $\phi^{\,\mathrm{ab}}$ is hyperbolic, M is a hyperbolic manifold and it is known that, in this case, $Out(\pi_1 M) = Out(M_\phi)$ is finite. This fact is contained in the statement of (iii), which will be proven later using algebraic methods.

Finally, we remark that the case of rank 1 is straightforward to analyse.

Proposition 1.2. *Let $F_1 = \langle a \rangle$ be a free group of rank 1, let $\phi \in Aut(F_1)$, and consider the mapping torus $M_\phi = F_1 \rtimes_\phi \mathbb{Z}$.*

i) *If $\phi = Id$ then $Out(M_\phi) = GL_2(\mathbb{Z})$.*
ii) *If ϕ is the inversion then $Out(M_\phi)$ is finite.* □

2. Results for general rank

Using the defining relations of M_ϕ under the form $wt = t(w\phi)$ and $wt^{-1} = t^{-1}(w\phi^{-1})$, $w \in F_n$, it is clear that in any element of M_ϕ we can push all the t letters to one side. That is, we have a normal form in M_ϕ whereby we can write elements uniquely in the form $t^k w$, where k is an integer and $w \in F_n$.

Our first observation is that the isomorphism type of M_ϕ depends only on the outer automorphism determined by $\phi^{\pm 1}$, up to conjugacy in $Out(F_n)$.

Lemma 2.1. *Let F_n be a free group of rank n, let $\phi, \psi \in Aut(F_n)$ and consider*

$$M_\phi = \langle x_1, \ldots, x_n,\, t \mid t^{-1}x_i t = x_i\phi \rangle$$

and

$$M_\psi = \langle x_1, \ldots, x_n,\, s \mid s^{-1}x_i s = x_i\psi \rangle.$$

If the automorphisms ϕ and ψ are conjugate or conjugate-inverse to each other in $Out(F_n)$, then M_ϕ and M_ψ are isomorphic. More precisely, if $\chi \in Aut(F_n)$ is such that $\chi^{-1}\phi\chi = \psi^\epsilon\gamma_w$, for some $\epsilon = \pm 1$ and $w \in F_n$, then the map $\Omega\colon M_\phi \to M_\psi$, $x_i \mapsto x_i\chi$, $t \mapsto s^\epsilon w$ extends to an isomorphism.

Proof. The map Ω is extended multiplicatively and one needs to show that it is well defined. In order to do this, it is enough to show that the relators in M_ϕ are all sent to the trivial element in M_ψ:

$$\begin{array}{rcl} t^{-1}x_i t(x_i\phi)^{-1} & \mapsto & w^{-1}s^{-\epsilon}(x_i\chi)s^\epsilon w(x_i\phi\chi)^{-1} \\ & = & w^{-1}(x_i\chi\psi^\epsilon)w(x_i\phi\chi)^{-1} \\ & = & (x_i\chi\psi^\epsilon\gamma_w)(x_i\phi\chi)^{-1} \\ & = & (x_i\phi\chi)(x_i\phi\chi)^{-1} \\ & = & 1 \in M_\psi. \end{array}$$

Thus, we have a well defined homomorphism $\Omega\colon M_\phi \to M_\psi$, which is surjective by inspection. Moreover, if an element $t^k x \in M_\phi$ is in the kernel of Ω, we immediately deduce that $k = 0$ and $x\chi = 1$. Since χ is an automorphism of F_n, we have $x = 1$ and Ω has trivial kernel. □

We continue with some basic facts about the automorphism group of M_ϕ.

Lemma 2.2. *Let F_n be a free group of rank n, and let $\phi \in Aut(F_n)$. The group $M_\phi = F_n \rtimes_\phi \mathbb{Z}$ has non-trivial centre if and only if $\phi^k = \gamma_w$ for some $k \neq 0$ and some $w \in F_n$. If this equation holds and $n \geqslant 2$, then $w\phi = w$.*

Proof. The result is clear for $n = 0, 1$. So, we may assume that $n \geqslant 2$.

A straightforward calculation shows that the element $t^k w^{-1} \in M_\phi$ commutes with every $x \in F_n$ if and only if $\phi^k = \gamma_w$. Similarly, $t^k w^{-1}$ commutes with t if and only if $w\phi = w$. So, $t^k w^{-1}$ is central in M_ϕ if and only if $\phi^k = \gamma_w$ and $w\phi = w$. Hence, M_ϕ has non-trivial centre if and only if $\phi^k = \gamma_w$ and $w\phi = w$ for some integer k and some $w \in F_n$ such that $(k, w) \neq (0, 1)$.

Now, using the fact $n \geqslant 2$, we can simplify this. Assume the equation $\phi^k = \gamma_w$ holds. Since, $\gamma_w = \phi^{-1}\gamma_w\phi = \gamma_{w\phi}$ we have $w\phi = w$. Also, note that $k = 0$ implies $w = 1$ (because F_n has trivial centre for $n \geqslant 2$). Thus, M_ϕ has non-trivial centre if and only if $\phi^k = \gamma_w$ for some integer $k \neq 0$ and some $w \in F_n$. □

Let $\Psi \in Aut(M_\phi)$ and suppose it leaves F_n invariant. In this situation, its restriction to F_n, $\psi = \Psi|_{F_n}$, is an endomorphism of F_n that will be seen in the next proposition to be always an automorphism. On the other hand, factorising by the normal and Ψ-invariant subgroup F_n, we get an automorphism $\overline{\Psi}$ of $M_\phi/F_n \cong \mathbb{Z}$. If $\overline{\Psi} = Id$ we shall say that Ψ is a *positive* automorphism of M_ϕ. Otherwise, $\overline{\Psi}$ is the inversion of $\mathbb{Z}$ and we say that Ψ is *negative*. In any case, $t\Psi = t^\epsilon w$ for some $w \in F$, where $\epsilon = \pm 1$ is the *signum* of Ψ.

Proposition 2.3. *Let F_n be a free group of rank n, let $\phi \in Aut(F_n)$ and consider $M_\phi = F_n \rtimes_\phi \mathbb{Z}$. Let $\Psi \in Aut(M_\phi)$ be such that $F_n\Psi \leqslant F_n$, and denote by $\psi\colon F_n \to F_n$ its restriction to F_n. Then,*

i) *ψ is an automorphism of F_n,*
ii) *there exists $w \in F_n$ such that $\phi\psi = \psi\phi^\epsilon\gamma_w$, where ϵ is the signum of Ψ. Furthermore, if $n \geqslant 2$ then the word w is unique and satisfies the equation $t\Psi = t^\epsilon w$.*

Proof. Since $F_n\Psi \leqslant F_n$, we must have that $t\Psi = t^{\pm 1}w$ for some $w \in F_n$ (otherwise, t would not be in the image of Ψ). Now, clearly, $F_n\Psi$ is a normal subgroup of $M_\phi = M_\phi\Psi = \langle F_n\psi, t^{\pm 1}w\rangle$. Hence any element $g \in M_\phi$ can be written in the form $g = (v\psi)(t^{\pm 1}w)^k$ for some $v \in F_n$ and $k \in \mathbb{Z}$. And here $g \in F_n$ if and only if $k = 0$. Thus, $F_n\Psi = F_n$ and Ψ induces an automorphism on F_n. This proves (i).

Let $\epsilon = \pm 1$ be the signum of Ψ, that is, $t\Psi = t^{\epsilon} w$ for some $w \in F_n$. Applying Ψ to both sides of the equality $x\phi = t^{-1}xt$ we get

$$x\phi\psi = w^{-1}t^{-\epsilon}(x\psi)t^{\epsilon}w = x\psi\phi^{\epsilon}\gamma_w,$$

for all $x \in F_n$. Hence, $\phi\psi = \psi\phi^{\epsilon}\gamma_w$. Furthermore, if $n \geqslant 2$, this last equation can only happen for a unique $w \in F_n$. This proves (ii). □

In the following result, we impose certain hypothesis on ϕ to ensure that every automorphism of M_ϕ leaves the free part invariant. Under these circumstances, computing $Out(M_\phi)$ is fairly straightforward.

Theorem 2.4. *Let F_n be a free group of rank n, let $\phi \in Aut(F_n)$ and consider $M_\phi = F_n \rtimes_\phi \mathbb{Z}$. Let M_ϕ^{ab} denote the abelianisation of M_ϕ, and $F_n^{\mathrm{ab}} \cong \mathbb{Z}^n$ the abelianisation of F_n (which is not in general the image of $F_n \leqslant M_\phi$ in M_ϕ^{ab}). Let $\phi^{\mathrm{ab}} \in GL_n(\mathbb{Z})$ be the map induced by ϕ on F_n^{ab}, and $[\phi]$ be the class of ϕ in $Out(F_n)$. The following are equivalent:*

a) *M_ϕ^{ab} is the direct sum of an infinite cyclic group and a finite abelian group,*
b) *the matrix ϕ^{ab} has no eigenvalue 1.*

Furthermore, if these conditions hold then every automorphism of M_ϕ leaves F_n invariant,

$$Aut^+(M_\phi) = \{\Psi \in Aut(M_\phi) \mid \Psi \text{ is positive}\}$$

is a normal subgroup of $Aut(M_\phi)$ of index at most 2 and moreover, if $n \geqslant 2$, its image $Out^+(M_\phi)$ in $Out(M_\phi)$ is also normal, of index at most two, and isomorphic to $C([\phi])/\langle[\phi]\rangle$, where $C([\phi])$ denotes the centraliser of $[\phi]$ in $Out(F_n)$.

Proof. To prove the equivalence of (a) and (b), note that $M_\phi^{\mathrm{ab}} \cong \langle t \mid \rangle \oplus F_n^{\mathrm{ab}}/Im(\phi^{\mathrm{ab}} - Id)$. Then, $F_n^{\mathrm{ab}}/Im(\phi^{\mathrm{ab}} - Id)$ is finite if and only if $rank_{\mathbb{Z}}(Im(\phi^{\mathrm{ab}} - Id)) = rank_{\mathbb{Z}}(F_n^{\mathrm{ab}}) = n$. And this happens if and only if $rank_{\mathbb{Z}}(Ker(\phi^{\mathrm{ab}} - Id)) = 0$, which is the same as saying that ϕ^{ab} has no eigenvalue 1.

We shall now prove the remaining assertions under the assumption that these conditions hold. Consider the abelianisation map $M_\phi \to M_\phi^{\mathrm{ab}}$. Since $F_n^{\mathrm{ab}}/Im(\phi^{\mathrm{ab}} - Id)$ is the torsion subgroup of M_ϕ^{ab}, its full pre-image F_n is characteristic in M_ϕ. Hence, every automorphism of M_ϕ leaves F_n invariant. At this point, it is clear that $Aut^+(M_\phi)$ is a normal subgroup of $Aut(M_\phi)$ of index at most 2, and so is $Out^+(M_\phi)$ in $Out(M_\phi)$.

Assuming $n \geqslant 2$, it remains to prove that $Out^+(M_\phi) \cong C([\phi])/\langle[\phi]\rangle$. Define the map

$$\begin{array}{rcl} f\colon Aut^+(M_\phi) & \to & C([\phi])/\langle[\phi]\rangle \\ \Psi & \mapsto & [\Psi|_{F_n}]\langle[\phi]\rangle. \end{array}$$

Note that by Proposition 2.3, the image of this map lies in $C([\phi])/\langle[\phi]\rangle$. Clearly, f is a homomorphism.

First, we will prove that $Im\, f = C([\phi])/\langle[\phi]\rangle$. Let $\psi \in Aut(F_n)$ be such that $[\psi] \in C([\phi])$. Then, $\phi\psi = \psi\phi\gamma_w$ for some $w \in F_n$. In this situation, it is straightforward to verify that ψ extends to a well defined automorphism Ψ of M_ϕ by just sending t to tw. Clearly, $\Psi \in Aut^+(M_\phi)$ and its f-image is $[\psi]\langle[\phi]\rangle$.

Now we will prove that $Ker\, f = Inn(M_\phi)$. For every element $g = t^k w \in M_\phi$, we have that $\Gamma_g|_{F_n} = \phi^k\gamma_w$ and so, Γ_g maps under f to the identity element of $C([\phi])/\langle[\phi]\rangle$. This means that $Ker\, f \geqslant Inn(M_\phi)$. Conversely, let $\Psi \in Ker\, f$. Then, $\Psi|_{F_n} = \phi^k\gamma_w = \Gamma_{t^k w}|_{F_n}$ for some integer k and some $w \in F_n$. Also, applying Ψ to both sides of the equation $t^{-1}xt = x\phi$ and using the positivity of Ψ and the fact $n \geqslant 2$, it is straightforward to check that $t\Psi = t(w\phi)^{-1}w = t\Gamma_{t^k w}$. Thus, $\Psi = \Gamma_{t^k w}$. This completes the prove that $Ker\, f = Inn(M_\phi)$ and so, $Out^+(M_\phi) \cong C([\phi])/\langle[\phi]\rangle$. $\square$

The extreme opposite case to the one considered above is when ϕ is the identity automorphism (or, in fact, an inner automorphism). We also calculate the automorphism group in this case.

Theorem 2.5. *Let $F_n = \langle x_1, \ldots, x_n\rangle$ be a free group of rank $n \geqslant 2$, and let $M = M_{Id} = F_n \times \mathbb{Z}$. Consider the group $\mathbb{Z}^n \rtimes C_2$ where C_2 is the cyclic group of order 2 which acts by sending u to $-u$ for all $u \in \mathbb{Z}^n$ (think u as a column vector); also, consider the action of $Aut(F_n)$ (and also $Out(F_n)$) on it given by the trivial action on C_2, and the natural action after abelianisation on $\mathbb{Z}^n$. Then, $Aut(M) \cong (\mathbb{Z}^n \rtimes C_2) \rtimes Aut(F_n)$ and $Out(M) \cong (\mathbb{Z}^n \rtimes C_2) \rtimes Out(F_n)$.*

Proof. Clearly, distinct automorphisms of F_n extend to distinct positive automorphisms of M by sending t to t. In this sense, we shall think $Aut(F_n)$ as a subgroup of $Aut(M)$. On the other hand, consider $\mathbb{Z}^n \rtimes C_2 = \mathbb{Z}^n \rtimes \langle v\rangle$ so that $v^{-1}uv = -u$ for all $u \in \mathbb{Z}^n$. It is straightforward to verify that this group acts faithfully on $M = F_n \times \mathbb{Z}$, whereby an element (v^ϵ, u), $u = (u_1, \ldots, u_n)^T$, sends x_i to $t^{u_i}x_i$ and t to $t^{1-2\epsilon}$, where $\epsilon = 0, 1$. So, we shall think $\mathbb{Z}^n \rtimes C_2 \leqslant Aut(M)$.

Note that $Aut(F_n)$ and $\mathbb{Z}^n \rtimes C_2$ have trivial intersection as subgroups of $Aut(M)$. We shall now show that they generate $Aut(M)$. Let $\Psi \in Aut(M)$. It will be sufficient to show that we can multiply Ψ by elements in $Aut(F_n)$ and $\mathbb{Z}^n \rtimes C_2$ until we get the identity. Note that, since the centre of M is its infinite cyclic subgroup generated by t (use $n \geqslant 2$ and see Lemma 2.2), $\langle t\rangle$ is characteristic in M and so, $t\Psi = t^{\pm 1}$. Thus, after possibly composing with $v \in \mathbb{Z}^n \rtimes C_2$, we can assume that $t\Psi = t$. Then, write $x_i\Psi = t^{u_i}w_i$ for $i = 1, \ldots, n$, and $u = (u_1, \ldots, u_n)^T$. Since Ψ is an automorphism, $\{w_1, \ldots, w_n\}$ must generate (and so form a basis of) F_n. Composing Ψ with the automorphism which fixes t and sends w_i back to x_i, $i = 1, \ldots, n$, we obtain the element $(v^0, u) \in \mathbb{Z}^n \rtimes C_2$. This proves that $Aut(F_n)$ together with $\mathbb{Z}^n \rtimes C_2$ generate $Aut(M)$.

Consider now elements of $Aut(M)$, $\Theta \in Aut(F_n)$ and $(v^\epsilon, u) \in \mathbb{Z}^n \rtimes C_2$, $u = (u_1, \ldots, u_n)^T$. We shall calculate the conjugate $\Theta(v^\epsilon, u)\Theta^{-1}$ as an element of $Aut(M)$. Let $\theta = \Theta|_{F_n} \in Aut(F_n)$ and let $\theta^{\,\mathrm{ab}} = (b_{i,j}) \in GL_n(\mathbb{Z})$ be its abelianisation (an $n \times n$ integral matrix whose i-th row describes the total exponent sums of $x_i\theta$). Bearing this in mind, $\Theta(v^\epsilon, u)\Theta^{-1}$ acts as

$$\begin{array}{ccccccc} x_i & \mapsto & x_i\Theta & \mapsto & t^{c_i}(x_i\Theta) & \mapsto & t^{c_i}x_i \\ t & \mapsto & t & \mapsto & t^{1-2\epsilon} & \mapsto & t^{1-2\epsilon}, \end{array}$$

where $c_i = \sum_{j=1}^n b_{i,j}u_j$ is the i-th entry of the column vector $\theta^{\,\mathrm{ab}}u$. In other words,

$$\Theta(v^\epsilon, u)\Theta^{-1} = (v^\epsilon, \theta^{\,\mathrm{ab}}u)$$

for every $u = (u_1, \ldots, u_n)^T \in \mathbb{Z}^n$. This immediately shows that $\mathbb{Z}^n \rtimes C_2$ is normal in $Aut(M)$. Hence, $Aut(F_n \times \mathbb{Z}) \cong (\mathbb{Z}^n \rtimes C_2) \rtimes Aut(F_n)$, where the action of $Aut(F_n)$ in this last semi-direct product is the trivial one over the C_2 part, and the natural one after abelianisation over the $\mathbb{Z}^n$ part.

Lastly, to prove the final statement note that, since $\langle t \rangle$ is central, inner automorphisms of $F_n \times \mathbb{Z}$ are just inner automorphisms by elements of F_n, and all of them fix t. Thus, $Inn(M) = Inn(F_n) \leqslant Aut(F_n)$ and so $Out(M) \cong (\mathbb{Z}^n \rtimes C_2) \rtimes Out(F_n)$, where the actions are just like before but factorised by $Inn(F_n)$. □

3. $Aut(F_2)$ AND $GL_2(\mathbb{Z})$

For the rest of the paper we will be considering only the case $n = 2$. Hence, we shall avoid unnecessary subscripts by using the letters $\{a, b\}$ as free generators of $F_2 = \langle a, b \rangle$.

In this section we will briefly review some well known facts about $Aut(F_2)$, $Out(F_2)$ and $GL_2(\mathbb{Z})$. The abelianisation map $F_2 \to F_2^{\mathrm{ab}}$ induces naturally a surjective map $Aut(F_2) \twoheadrightarrow GL_2(\mathbb{Z})$ for which, abusing notation, we shall write $\phi \mapsto \phi^{\,\mathrm{ab}}$. More precisely, $\phi^{\,\mathrm{ab}}$ is the 2×2 integral matrix whose first (second) row counts the total a- and b-exponent sums of $a\phi$ (of $b\phi$). Clearly, this is well defined for any rank, but in rank 2 it has some special properties which make this case easier to study. The main specificity of the rank 2 case is the following well known result.

Theorem 3.1 (Nielsen, Prop. 4.5 in [4]). *The kernel of the map from $Aut(F_2)$ to $GL_2(\mathbb{Z})$ consists of precisely the inner automorphisms of F_2. That is, $Out(F_2) \cong GL_2(\mathbb{Z})$.*

This means that, for every automorphism $\phi \in Aut(F_2)$, the 2×2 integral matrix $\phi^{\,\mathrm{ab}}$ is enough to recover the automorphism ϕ up to conjugation.

Since the isomorphism type of M_ϕ (and so that of $Aut(M_\phi)$) only depends on ϕ up to conjugation, ϕ^{ab} contains all the algebraic information we may want about M_ϕ and $Aut(M_\phi)$.

Matrices in $GL_2(\mathbb{Z})$ can be classified according to their eigenvalues and dynamics, often leading to useful information for $Aut(F_2)$. This is often done in the following way.

Definition 3.2. Let A be a 2×2 integral invertible matrix, $A \in GL_2(\mathbb{Z})$. If A^2 is the identity matrix, $A^2 = I_2$, we say that A is *elliptic*. Otherwise, A is called *hyperbolic* if $|trace(A^2)| > 2$, *parabolic* if $|trace(A^2)| = 2$, and *elliptic* if $|trace(A^2)| < 2$.

Suppose $A \in GL_2(\mathbb{Z})$ is a hyperbolic matrix. Then, $A^2 \neq I_2$ has two real eigenvalues, $\alpha, 1/\alpha$, such that $|\alpha| > 1$. Since A preserves the one dimensional eigenspaces of A^2, A must also have two real eigenvalues, $\beta, \pm 1/\beta$, such that $|\beta| > 1$. In particular, A does not have 1 as an eigenvalue.

Suppose $A \in GL_2(\mathbb{Z})$ is a parabolic matrix. Then, $A^2 \neq I_2$ has characteristic polynomial equal to $(x \pm 1)^2$. This implies that A^2 is conjugate, in $GL_2(\mathbb{Z})$, to $\pm \left(\begin{smallmatrix} 1 & k' \\ 0 & 1 \end{smallmatrix}\right)$ for some $0 \neq k' \in \mathbb{Z}$ (take a rational eigenvector of eigenvalue ± 1, multiply it by a scalar to obtain an integral vector v with coprime entries, and then extend to a basis $\{u, v\}$ of $\mathbb{Z}^2$). A simple calculation shows now that A must then be conjugate to one of the following matrices

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} -1 & k \\ 0 & -1 \end{pmatrix},$$

for some integer $k \neq 0$. These are infinite order matrices and the first of these will turn out to be the most challenging case to consider.

Finally, suppose $A \in GL_2(\mathbb{Z})$ is an elliptic matrix. Then, either $A^2 = I_2$ or the characteristic polynomial of A^2 is equal to $x^2 + 1$, $x^2 + x + 1$ or $x^2 - x + 1$. In the first case, A is either $\pm I_2$ or is conjugate, in $GL_2(\mathbb{Z})$, to $\left(\begin{smallmatrix} 1 & k \\ 0 & -1 \end{smallmatrix}\right)$ for some $k \in \mathbb{Z}$ (by a similar reasoning as above). Otherwise, A have complex conjugate roots and, in particular, it does not have 1 as an eigenvalue.

4. The rank 2 case: proof of Theorem 1.1

Given $\phi \in Aut(F_2)$, we shall analyse $Aut(M_\phi)$ and prove Theorem 1.1 by following the classification of matrices in the previous section for ϕ^{ab}.

First, we state the following two lemmas for later use.

Lemma 4.1. *Every non-central element in $GL_2(\mathbb{Z})$ generates a finite index subgroup of its own centraliser. The centre consists precisely of the matrices $\pm I_2$.*

Proof. The first statement of the lemma follows easily from the presentation of $GL_2(\mathbb{Z})$ as an amalgamated product, $GL_2(\mathbb{Z}) \cong D_4 *_{D_2} D_6$, where D_n is the dihedral group of order $2n$. The statement about central elements is elementary. □

Lemma 4.2. *Consider the automorphism ϕ of $F_2 = \langle a, b\rangle$ given by $a\phi = ab^k$ and $b\phi = b$, where $k \neq 0$. Then, for every integer $r \neq 0$ and every $w \in F_2$,*

i) $Fix\,\phi = Fix\,\phi^r = \langle aba^{-1}, b\rangle$,
ii) *if $w\phi^r$ is conjugate to w, then w is conjugate to an element fixed by ϕ.*

Proof. Given an arbitrary reduced word $w \in F_2$, let us split it into *pieces* each of the form b^m, ab^m, $b^m a^{-1}$ or $ab^m a^{-1}$, where m is some integer. There can be many variants of such a splitting, but we shall use the special one defined by putting breaking points precisely before each occurrence of a and after each occurrence of a^{-1}. One can easily see that this splitting is invariant under the action of ϕ, and that the pieces do not interact under iterates of ϕ. Hence, if $w\phi^r = w$ then the corresponding pieces in the splitting of w must also be fixed by ϕ^r, which rules out the possibilities ab^m and $b^m a^{-1}$ (because $rk \neq 0$). This proves that $Fix\,\phi^r \leqslant \langle aba^{-1}, b\rangle$. The inclusions $\langle aba^{-1}, b\rangle \leqslant Fix\,\phi \leqslant Fix\,\phi^r$ are obvious. This proves (i).

In order to prove (ii), note that we can assume w is cyclically reduced. If w is a power of b there is nothing to prove. So, we may also assume that w contains $a^{\pm 1}$. Moreover, by inverting and cyclically permuting if necessary, we may assume that w begins with a. So, in this particular situation, assume that $w\phi^r$ is conjugate to w. The splitting of w must begin with a piece of the form ab^m or $ab^m a^{-1}$, and must end with a piece of the form b^m or ab^m. But this splitting is stable under iterates of ϕ hence, the first and last pieces in $w\phi^r$ will be of the corresponding same types. In particular, $w\phi^r$ is still cyclically reduced. Thus, $w\phi^r$ must be a cyclic permutation of w. Then, for a suitable s, $w\phi^{rs} = w$ which, by (i), means that w is fixed by ϕ. This completes the proof. □

Proof of Theorem 1.1. All along the prove, let us fix the following notation. Let $F_2 = \langle a, b\rangle$ be a free group of rank $n = 2$, let $\phi \in Aut(F_2)$ and let $M_\phi = F_2 \rtimes_\phi \mathbb{Z}$ be the mapping torus of ϕ. Let $\phi^{\,\mathrm{ab}} \in GL_2(\mathbb{Z})$ be the map induced by ϕ on $F_2^{\mathrm{ab}} \cong \mathbb{Z}^2$, i.e. the 2×2 integral matrix whose rows count the total a- and b-exponent sums of $\{a\phi, b\phi\}$.

First of all, note that the discussions in the previous section show that a generic matrix $\phi^{\,\mathrm{ab}} \in GL_2(\mathbb{Z})$ fits into one of the cases distinguished in Theorem 1.1. Namely, if A is hyperbolic then it satisfies (iii), if it is parabolic it satisfies either (iii) or (v), and if it is elliptic then it fits into either (i), (ii) or (iv). Uniqueness is a straightforward exercise in linear algebra.

Suppose $\phi^{\rm ab} = I_2$. Then, $\phi = \gamma_w$ for some $w \in F_2$. Hence, using Lemma 2.1, $M_\phi \cong M_{Id} = F_2 \times \mathbb{Z}$. Now, using Theorem 2.5, we have

$$Out(M_\phi) \cong Out(M_{Id}) \cong (\mathbb{Z}^2 \rtimes C_2) \rtimes GL_2(\mathbb{Z}),$$

where the actions are the natural ones described above. This proves Theorem 1.1 (i).

Suppose $\phi^{\rm ab} = -I_2$. Then it does not have 1 as an eigenvalue. Hence, by Theorem 2.4, $Out^+(M_\phi)$ is a normal subgroup of $Out(M_\phi)$ of index at most two which is isomorphic to $C(\phi^{\rm ab})/\langle \phi^{\rm ab}\rangle$, where $C(\phi^{\rm ab})$ is the centraliser of $\phi^{\rm ab}$ in $Out(F_2) = GL_2(\mathbb{Z})$. But $\phi^{\rm ab} = -I_2$, which is central in $GL_2(\mathbb{Z})$ so,

$$Out^+(M_\phi) \cong C(\phi^{\rm ab})/\langle \phi^{\rm ab}\rangle = GL_2(\mathbb{Z})/\{\pm I_2\} = PGL_2(\mathbb{Z}).$$

On the other hand, $a \mapsto a$, $b \mapsto b$, $t \mapsto t^{-1}$ determines a (well-defined) negative automorphism Υ of M_ϕ and so, $Aut^+(M_\phi) \trianglelefteq_2 Aut(M_\phi)$. Furthermore, note that Υ has order two and commutes with every $\Psi \in Aut^+(M_\phi)$ (which has the form $a \mapsto w_1$, $b \mapsto w_2$, $t \mapsto tw_3$ where w_3 is a *palindrome*, $w_3^R = w_3$). Hence, $Aut(M_\phi) \cong Aut^+(M_\phi) \times C_2$. Finally, since $Inn(M_\phi) \leqslant Aut^+(M_\phi)$, we have

$$Out(M_\phi) \cong Out^+(M_\phi) \times C_2 \cong PGL_2(\mathbb{Z}) \times C_2.$$

This proves Theorem 1.1 (ii).

Suppose that $\phi^{\rm ab} \neq -I_2$ does not have 1 as an eigenvalue. Then, applying Theorem 2.4 and Lemma 4.1, we deduce that $Out(M_\phi)$ is finite. This proves Theorem 1.1 (iii).

Now, suppose that $\phi^{\rm ab}$ is conjugate in $GL_2(\mathbb{Z})$ to $\left(\begin{smallmatrix} 1 & k \\ 0 & -1 \end{smallmatrix}\right)$ for some $k \in \mathbb{Z}$. Then, Proposition 4.3 shows that $Out(M_\phi)$ has an infinite cyclic subgroup of finite index. This completes Theorem 1.1 (iv).

Finally, suppose that $\phi^{\rm ab}$ is conjugate in $GL_2(\mathbb{Z})$ to $\left(\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right)$ for some $0 \neq k \in \mathbb{Z}$. Then, Proposition 4.4 will complete the proof of Theorem 1.1 (v) by showing that $Out(M_\phi)$ also has an infinite cyclic subgroup of finite index. □

Proposition 4.3. *With the notation above, assume $\phi^{\rm ab}$ is conjugate in $GL_2(\mathbb{Z})$ to $\left(\begin{smallmatrix} 1 & k \\ 0 & -1 \end{smallmatrix}\right)$, where $k \in \mathbb{Z}$. Then $Out(M_\phi)$ has an infinite cyclic subgroup of finite index.*

Proof. Using Lemma 2.1, we can assume $\phi^{\rm ab} = \left(\begin{smallmatrix} 1 & k \\ 0 & -1 \end{smallmatrix}\right)$. Furthermore, composing ϕ by a suitable inner automorphism of F_2, we can assume $a\phi = ab^k$ and $b\phi = b^{-1}$. So,

$$M_\phi = \langle a, b, t \mid t^{-1}at = ab^k,\ t^{-1}bt = b^{-1}\rangle.$$

It is straightforward to check that

$$\begin{array}{rcl} a & \mapsto & ta \\ b & \mapsto & b \\ t & \mapsto & t. \end{array}$$

defines an automorphism Ψ of M_ϕ such that $[\Psi] \in Out(M_\phi)$ is an infinite order outer automorphism (because inner automorphisms of M_ϕ leave F_2 invariant). Let us prove now that the infinite cyclic subgroup $\langle [\Psi] \rangle$ has finite index in $Out(M_\phi)$.

Clearly, $\phi^2 = Id$ and so, t^2 is in the centre of M_ϕ. Also, $\langle a, b, t^2 \rangle \cong F_2 \times \mathbb{Z}$ is an index 2 subgroup of M_ϕ. So, all those automorphisms of M_ϕ which restrict to an automorphism of $\langle a, b, t^2 \rangle$ form a finite index subgroup of $Aut(M_\phi)$ (since M_ϕ is finitely generated and so has finitely many index 2 subgroups). Moreover, the centre of $\langle a, b, t^2 \rangle$ is $\langle t^2 \rangle$ so, any such automorphism sends t^2 to $t^{\pm 2}$. Hence, all those automorphisms of M_ϕ which restrict to an automorphism of $\langle a, b, t^2 \rangle$ and fix t^2 still form a finite index subgroup of $Aut(M_\phi)$, containing Ψ^2. Thus, we can confine our attention to the subgroup $G \leqslant Aut(M_\phi)$ consisting on these automorphisms, and prove that $\langle [\Psi^2] \rangle$ has finite index in $[G] \leqslant Out(M_\phi)$.

Let $\Theta \in G$. Note that $[a, t] = a^{-1}t^{-1}at = b^k$ and $[b, t] = b^{-1}t^{-1}bt = b^{-2}$ and hence $M'_\phi = \langle b^2 \rangle F'_2$ if k is even, and $M'_\phi = \langle b \rangle F'_2$ if k is odd. Also, M_ϕ^{ab} has torsion subgroup generated by b if k is even, and is torsion-free if k is odd. In any case, the preimage in M_ϕ of the (possibly trivial) torsion in M_ϕ^{ab} is $\langle b \rangle M'_\phi = \langle b \rangle F'_2$. In particular, this subgroup is characteristic in M_ϕ and so Θ acts on $\langle a, b, t^2 \rangle$ in the following way:

$$\begin{array}{rcl} a & \mapsto & t^r u \\ b & \mapsto & v \\ t^2 & \mapsto & t^2, \end{array}$$

where $u \in F_2$, $v \in \langle b \rangle F'_2$, and r is even. Since $\langle a, b, t^2 \rangle = \langle t^r u, v, t^2 \rangle = \langle u, v, t^2 \rangle$ and t^2 lies in the center of M_ϕ, it follows that $\langle u, v \rangle = \langle a, b \rangle$. Then, from this and the form of v, we deduce that $u \in a^\epsilon \langle b \rangle F'_2$ for some $\epsilon = \pm 1$.

Consider now the automorphism $\Lambda = \Theta \Psi^{-\epsilon r}$ which acts like

$$\begin{array}{rcccl} a & \mapsto & t^r u & \mapsto & t^r (u \Psi^{-\epsilon r}) = x \in F_2 \\ b & \mapsto & v & \mapsto & v \Psi^{-\epsilon r} = y \in F_2 \\ t^2 & \mapsto & t^2 & \mapsto & t^2 \end{array}$$

(note that u has a-exponent sum equal to ϵ and so, the t-exponent sum of $u\Psi^{-\epsilon r}$ is $-r$, showing that $x \in F_2$; also, the a-exponent sum of v is 0 and so $y \in F_2$ too). Writing $t\Lambda = t^s w$ and imposing t^2 to be fixed, we deduce $t\Lambda = tz$ for some $z \in F_2$. Thus, Λ is a positive automorphism of M_ϕ and, by Proposition 2.3, $(\Lambda|_{F_2})^{\mathrm{ab}}$ lies in the centraliser of ϕ^{ab}. But a straightforward matrix calculation shows that $C(\phi^{\mathrm{ab}})$ is finite and so, (x, y) takes only finitely many values up to conjugacy in F_2. Since, by Proposition 2.3, z is uniquely determined by (x, y), Λ also takes only finitely many values up to conjugacy, while Θ runs over all G. In other words, $\langle[\Psi^2]\rangle$ has finite index in $[G]$. □

Proposition 4.4. *With the notation above, assume ϕ^{ab} is conjugate in $GL_2(\mathbb{Z})$ to $\left(\begin{smallmatrix}1 & k\\ 0 & 1\end{smallmatrix}\right)$, where $0 \neq k \in \mathbb{Z}$. Then $Out(M_\phi)$ has an infinite cyclic subgroup of finite index.*

Proof. Using Lemma 2.1, we can assume $\phi^{\mathrm{ab}} = \left(\begin{smallmatrix}1 & k\\ 0 & 1\end{smallmatrix}\right)$ and, furthermore, composing ϕ by a suitable inner automorphism of F_2, we can assume $a\phi = ab^k$ and $b\phi = b$, $k \neq 0$. So, we have to understand the automorphism group of the group

$$M_\phi = \langle a, b, t \mid t^{-1}at = ab^k,\ t^{-1}bt = b\rangle.$$

Before going into the analysis of $Aut(M_\phi)$, note that both relators have a- and t-exponent sums equal to zero. So, it makes sense to talk about a- and t-exponent sums of elements in M_ϕ. Strictly speaking, the maps from M_ϕ to $\mathbb{Z}$ killing b and t and sending a to the generator of $\mathbb{Z}$ (resp. killing a and b and sending t to the generator of $\mathbb{Z}$) are well defined surjective homomorphisms. They count the total a- and t-exponent sums, respectively (note that the notion of b-exponent sum makes no sense in M_ϕ).

It is straightforward to check that

$$\begin{array}{rcl} a & \mapsto & ta \\ b & \mapsto & b \\ t & \mapsto & t \end{array}$$

defines an automorphism Ψ of M_ϕ such that $[\Psi] \in Out(M_\phi)$ is an infinite order outer automorphism (because inner automorphisms of M_ϕ leave F_2 invariant). Let us prove now that the infinite cyclic subgroup $\langle[\Psi]\rangle$ has finite index in $Out(M_\phi)$.

Consider the three automorphisms of M_ϕ defined on the generators by

$$\begin{array}{ccc} \Omega & \Delta & \Xi \\ \begin{array}{rcl} a & \mapsto & a \\ b & \mapsto & b^{-1} \\ t & \mapsto & t^{-1} \end{array} & \begin{array}{rcl} a & \mapsto & a^{-1} \\ b & \mapsto & b^{-1} \\ t & \mapsto & tb^{-k} \end{array} & \begin{array}{rcl} a & \mapsto & ab \\ b & \mapsto & b \\ t & \mapsto & t \end{array} \end{array}$$

(as above, checking that they are well-defined is a straightforward exercise). ***Claim:*** for any given $\Theta \in Aut(M_\phi)$, there exists an integer m and an element $g \in M_\phi$ such that $\Theta\Psi^m\Gamma_g$ is equal to one of Ξ^i, $\Xi^i\Omega$, $\Xi^i\Delta$ or $\Xi^i\Delta\Omega$, for some $0 \leqslant i \leqslant |k| - 1$.

This automatically will imply that $\langle[\Psi]\rangle$ has finite index in $Out(M_\phi)$. In order to prove this claim note that, since $Inn(M_\phi)$ is a normal subgroup of

$Aut(M_\phi)$, we may apply inner automorphisms at any point in the product $\Theta\Psi^m$.

So, let Θ be an arbitrary automorphism of M_ϕ, and write normal forms for the images of generators, $a\Theta = t^p w_1$, $b\Theta = t^l w_2'$, $t\Theta = t^q w_3$, where $w_1, w_2', w_3 \in F_2$, and $p, l, q \in \mathbb{Z}$. Write also $w_2' = w_2^r$, where $r \geqslant 1$ and w_2 is either trivial or not a proper power. Applying Θ to the equality $t^{-1}at = ab^k$ we get

$$w_3^{-1}t^{-q}t^p w_1 t^q w_3 = t^p w_1 (t^l w_2')^k.$$

Comparing the t-exponent sums we immediately see that $kl = 0$ and hence $l = 0$ and $w_2 \neq 1$. Now, applying Θ to $t^{-1}bt = b$, we also get

$$w_3^{-1}(w_2\phi^q)^r w_3 = w_3^{-1}t^{-q}w_2' t^q w_3 = w_2' = w_2^r.$$

Thus, $w_2\phi^q$ is conjugate to w_2 in F_2. By applying Lemma 4.2 (ii), we obtain that w_2 is conjugate to an element fixed by ϕ, say $w_2 = xv_2x^{-1}$, where $x \in F_2$ and $1 \neq v_2 = v_2\phi$ is not a proper power. Now, $\Theta\Gamma_x$ is an automorphism of M_ϕ acting as

$$\begin{array}{ccccc} a & \mapsto & t^p w_1 & \mapsto & x^{-1}t^p w_1 x = t^p v_1 \\ b & \mapsto & w_2' & \mapsto & x^{-1}w_2^r x = v_2^r \\ t & \mapsto & t^q w_3 & \mapsto & x^{-1}t^q w_3 x = t^q v_3, \end{array}$$

where $v_1, v_3 \in F_2$. Since b commutes with t, v_2^r must commute with $t^q v_3$. But v_2 commutes with t since it is fixed by ϕ. Therefore v_2 commutes with v_3 and hence, $v_3 = v_2^s$ for some integer s. Finally, observe that $\{t^p v_1,\, v_2^r,\, t^q v_2^s\}$ must generate M_ϕ. Thus, since the a-exponent sum of v_2 is zero (by Lemma 4.2 (i)), the a-exponent sum of v_1 must be ± 1. So, without loss of generality, we may assume that Θ acts as

$$\begin{array}{ccc} a & \mapsto & t^p v_1 \\ b & \mapsto & v_2^r \\ t & \mapsto & t^q v_2^s, \end{array}$$

where $v_1, v_2 \in F_2$, $p, q, r, s \in \mathbb{Z}$, v_2 is fixed by ϕ and has a-exponent sum equal to zero, and v_1 has a-exponent sum equal to $\epsilon = \pm 1$.

Now let $m = -\epsilon p$. It is straightforward to verify that $\Theta\Psi^m$ acts in the following form,

$$\begin{array}{ccccc} a & \mapsto & t^p v_1 & \mapsto & t^p(v_1\Psi^m) = u_1 \in F_2 \\ b & \mapsto & v_2^r & \mapsto & u_2 \in F_2 \\ t & \mapsto & t^q v_2^s & \mapsto & t^q u_3, \end{array}$$

where $u_1, u_2, u_3 \in F_2$. By Proposition 2.3, it follows now that $\Theta\Psi^m$ restricts to an automorphism of F_2 with signum $q = \pm 1$.

Consider now the automorphisms $\Theta\Psi^m$, $\Theta\Psi^m\Omega$, $\Theta\Psi^m\Delta$ and $\Theta\Psi^m\Omega\Delta$. Each of these leaves F_2 invariant and have signum q, $-q$, q and $-q$, respectively. Also, the traces of the abelianisations of their restrictions to F_2 are d, e, $-d$ and $-e$, respectively, for some $d, e \in \mathbb{Z}$. So, one of these four automorphisms, say Υ, is positive and its restriction to F_2 abelianises to a matrix with non-negative trace.

We shall show that, up to an inner automorphism, Υ coincides with Ξ^i for some $0 \leqslant i \leqslant |k| - 1$. This will prove the claim since both Ω and Δ above have order two in $Aut(M_\phi)$.

Since Υ is a positive automorphism, Proposition 2.3 ensures us that the matrices $(\Upsilon|_{F_2})^{\text{ab}}$ and $\phi^{\text{ab}} = \left(\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right)$ do commute. But the centraliser of ϕ^{ab} in $GL_2(\mathbb{Z})$ is the set of matrices of the form $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} -1 & * \\ 0 & -1 \end{smallmatrix}\right)$. So, since $(\Upsilon|_{F_2})^{\text{ab}}$ has non-negative trace, we deduce that, for some $z \in F_2$, $(\Upsilon\Gamma_z)|_{F_2}$ acts as $a \mapsto ab^j$, $b \mapsto b$, for some integer j. Write $j = i + \lambda k$ with $0 \leqslant i \leqslant |k| - 1$, and notice that the inner automorphism Γ_t of M_ϕ acts as

$$\begin{array}{rcl} a & \mapsto & ab^k \\ b & \mapsto & b \\ t & \mapsto & t. \end{array}$$

Hence $\Upsilon\Gamma_z\Gamma_t^{-\lambda}$ agrees with Ξ^i on F_2. And they both are positive so, by Proposition 2.3 (ii), they must also agree on t. Hence, up to a conjugation, Υ coincides with Ξ^i. This completes the proof. □

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