ON AN UPPER BOUND FOR THE ARITHMETIC SELF-INTERSECTION NUMBER OF THE DUALIZING SHEAF ON ARITHMETIC SURFACES

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ABSTRACT. We give an upper bound for the arithmetic self-intersection number of the dualizing sheaf on arithmetic surfaces that arise from Belyi morphisms of a particular kind. In particular our result provides an upper bound for the arithmetic self-intersection number of the dualizing sheaf on minimal regular models of many elliptic modular curves associated with congruence subgroups $\Gamma_0(N)$ and $\Gamma(N)$ of $\mathrm{SL}_2(\mathbb{Z})$, as well as for the Fermat curves with prime exponent.

INTRODUCTION

0.1. Arakelov theory is motivated by the analogy between the intersection theory on an algebraic surface fibered over a complete curve $S \to B$ and arithmetic intersection theory on a regular model $f : \mathcal{X} \to \operatorname{Spec} \mathcal{O}_K$ of a curve X over the ring of integers of a number field K. A major result in the theory of algebraic surfaces is the Bogomolov-Miyaoke-Yau inequality $\mathrm{c}_1(S)^2 \leq 3\,\mathrm{c}_2(S)$. This inequality may for certain fibered surfaces also be reformulated as

$$\omega_{S/B}^2 \leq (2g-2)(2q-2) + 3\sum_{b\in B} \delta_b,$$

here g is the genus of the generic fiber of $S \to B$, q is the genus of the base curve and δ_b equals the number of singular points in the fiber above $b \in B$. The question whether in Arakelov theory an analogous inequality $\widehat{\mathrm{c}}_1(\mathcal{X})^2 \leq a\,\widehat{\mathrm{c}}_2(\mathcal{X})$ of suitable defined arithmetic Chern numbers with some $a > 0$ holds is still open.

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From now on we assume that the genus g of X is larger than 1. Then, it is natural to define $\widehat{\mathrm{c}}_1(\mathcal{X})^2$ by the arithmetic intersection number

$$\widehat{\mathrm{c}}_1(\mathcal{X})^2 = \overline{\omega}^2_{\mathrm{Ar}},$$

where $\overline{\omega}^2_{\mathrm{Ar}}$ is the dualizing sheaf $\omega_{\mathcal{X}} = \omega_{\mathcal{X}/\mathcal{O}_K} \otimes f^*\omega_{\mathcal{O}_K/\mathbb{Z}}$ equipped with the Arakelov metric (see [Ar], p.1177, [MB1] p.75). It is a delicate problem to define the other quantity $\widehat{\mathrm{c}}_2(\mathcal{X})$. A possible candidate is as follows

$$\begin{aligned}\widehat{\mathrm{c}}_2(\mathcal{X}) =& a_1(2g-2)\log|\Delta_{K|\mathbb{Q}}| + \\ & a_2\left(\sum_{x\in\mathcal{X}^{\mathrm{sing}}}\log(\#k(x)) + \sum_{\sigma:K\to\mathbb{C}}\delta_{\mathrm{Fal}}(\mathcal{X}_\sigma(\mathbb{C}))\right) + a_3;\end{aligned}$$

here a_1, a_2, $a_3 \in \mathbb{R}$, $\Delta_{K|\mathbb{Q}}$ is the absolute discriminant of the number field K and δ_{Fal} is a function on the moduli space of compact Riemann surfaces . One should view δ_{Fal} as the logarithm of a certain distance of being a singular curve. If with the above choices an inequality $\widehat{\mathrm{c}}_1(\mathcal{X})^2 \leq a\,\widehat{\mathrm{c}}_2(\mathcal{X})$ holds, then via the Kodaira-Parshin construction an effective version of Mordell's conjecture is implied (cf. [Pa], [MB2]). The latter conjecture is equivalent to the uniform *abc*-conjecture for number fields [Fr], which in turn has lots of different applications and consequences in number theory.

Here we consider inequalities for the arithmetic self-intersection number of the dualizing sheaf of the minimal regular model $\mathcal{X}$ equipped with its natural Arakelov metric of type

$$(*) \qquad \overline{\omega}^2_{\mathrm{Ar}} \overset{?}{\leq} a_1(2g-2)\log|\Delta_{K|\mathbb{Q}}| + a_2;$$

here a_1 is assumed to be an absolute constant and a_2 should be a real valued function depending on g, $[K:\mathbb{Q}]$ and the bad reduction of $\mathcal{X}$ (see e.g. [La], p. 166, or [Sz], p.244). We emphasize that if (*) holds for all curves X that vary in a single family provided by the Kodaira-Parshin construction, then δ_{Fal} is bounded and (*) would also imply the desired effective Mordell.

0.2. In this article we approach (*) by means of a Belyi uniformization $X(\Gamma)$ of an algebraic curve X. Recall that a complex curve X is defined over a number field, if and only if there exist a morphism $\boldsymbol{\beta}: X \to \mathbb{P}^1$ with only three ramification points, if and only $X(\mathbb{C})$ is isomorphic to a modular curve $X(\Gamma)$ associated with a finite index subgroup of $\Gamma(1) = \mathrm{Sl}_2(\mathbb{Z})$. The later isomorphism is called a *Belyi uniformization* of degree $[\Gamma(1):\Gamma]$.

A regular model $\mathcal{X}$ of X associated with a Belyi uniformization $\boldsymbol{\beta}$ is an arithmetic surface together with an morphism $\boldsymbol{\beta}: \mathcal{X} \to \mathbb{P}^1_{\mathcal{O}_K}$ which extends $\boldsymbol{\beta}: X \to \mathbb{P}^1$. Since the regular model $\mathcal{X}(1)$ of the modular curve $X(1)$

equals $\mathbb{P}^1_{\mathbb{Z}}$, we could obtain such regular models $\mathcal{X}$ of X by taking a desingularisation of the normalization of $\mathcal{X}(1)$ in X with respect to the morphism $X(\Gamma) \to X(1)$. In many cases the arithmetic surface $\mathcal{X} \to \mathcal{X}(1)$ can be chosen to be the minimal regular model, but we have to stress the fact that in general the arithmetic surface $\mathcal{X}$ is not the minimal regular model.

Theorem I. *Let $\boldsymbol{\beta} : \mathcal{X} \to \mathcal{X}(1)$ be an arithmetic surface associated with a Belyi uniformization $X(\Gamma)$ of a curve X defined over a number field K. Assume that all cusps are K-rational points and that all cuspidal divisors (= divisors on X with support in the cusps of degree zero) are torsion. Then there exists an absolute constant $\kappa \in \mathbb{R}$ independent of X such that the arithmetic self-intersection number of the dualizing sheaf on $\mathcal{X}$ satisfies the inequality*

$$\overline{\omega}^2_{Ar} \leq (4g-4)\left(\log|\Delta_{K|\mathbb{Q}}| + [K:\mathbb{Q}]\kappa\right) + \sum_{\mathfrak{p}\text{ bad}} a_{\mathfrak{p}} \log \mathrm{Nm}(\mathfrak{p}), \tag{0.2.1}$$

where a prime $\mathfrak{p}$ is said to be bad if the fiber of $\mathcal{X}(\Gamma)$ above $\mathfrak{p}$ is reducible. The coefficients $a_{\mathfrak{p}}$ are rational numbers, which can be calculated explicitly. In particular, if the fiber of $\mathcal{X}$ above $\mathfrak{p}$ has $r_{\mathfrak{p}}$ irreducible components $C_j^{(\mathfrak{p})}$, then

$$a_{\mathfrak{p}} \leq 4g(g-1)(r_{\mathfrak{p}}-1)^2 \max_{j,k} |C_j^{(\mathfrak{p})}.C_k^{(\mathfrak{p})}|.$$

Moreover if $\boldsymbol{\beta} : \mathcal{X} \to \mathcal{X}(1)$ is a Galois cover, then $a_{\mathfrak{p}} \leq 0$.

This result is still far from giving an affirmative answer of the conjectured inequality (*), since it is not clear whether a general curve possesses such a particular Belyi uniformization. If $\mathcal{X}$ is not the minimal model $\mathcal{X}_{\min}$ of X_K, then our formula (0.2.1) will become additional contributions coming from those primes of $\mathcal{O}_K$ that give rise to fibers of $\mathcal{X}$ which contain a (-1)-curve. Indeed let $\pi : \mathcal{X} \to \mathcal{X}_{\min}$ be the morphism that exists by the minimality of $\mathcal{X}$, then there exist a vertical divisor $\mathcal{W}$ with support in those fibers which contain a (-1)-curve such that $\pi^*\omega_{\mathcal{X}_{\min}} = \omega_{\mathcal{X}} \otimes \mathcal{O}(\mathcal{W})$. Therefore we have

$$\begin{aligned}\overline{\omega}^2_{\mathcal{X}_{\min},\mathrm{Ar}} = \pi^*\overline{\omega}^2_{\mathcal{X}_{\min},\mathrm{Ar}} &= \overline{\omega}^2_{\mathcal{X},\mathrm{Ar}} + 2\omega_{\mathcal{X}}.\mathcal{O}(\mathcal{W}) + \mathcal{O}(\mathcal{W})^2 \\ &= \overline{\omega}^2_{\mathcal{X},\mathrm{Ar}} + \sum_{\mathfrak{p}\text{ bad}} b_{\mathfrak{p}} \log \mathrm{Nm}(\mathfrak{p}).\end{aligned}$$

Notice that, the coefficients $a_{\mathfrak{p}} + b_{\mathfrak{p}}$, which could be seen as a measure of how complicated the minimal regular model $\mathcal{X}_{\min}$ is, may be arbitrarily large compared to the number of singular points.

We can apply our result whenever Γ is a congruence subgroup, this is because of the Manin-Drinfeld theorem (see e.g. [El]). In particular if Γ is of certain kind, then the coefficients $a_{\mathfrak{p}}$ in (0.2.1) could be calculated explicitly by using the descriptions of models for $X(\Gamma)$ (see e.g. [KM], [DR]). We illustrate this with the following theorems.

Theorem II. *Let $\mathcal{X}_0(N)$ be the minimal regular model of the modular curve $X_0(N)$, where N is a square free integer having at least two different prime factors and $(N, 6) = 1$. Then the arithmetic self-intersection number of its dualizing sheaf equipped with the Arakelov metric is bounded from above by*

$$\overline{\omega}_{Ar}^2 \leq (4g-4)\kappa + (3g+1)\sum_{p|N} \frac{p+1}{p-1} \log p,$$

where $\kappa \in \mathbb{R}$ is an absolute constant independent of N.

The modular curves $X_0(N)$, with square free N and $(6, N) = 1$, are defined over $\mathbb{Q}$. We point to the fact that the completely different methods in [AU],[MU],[JK], which depend strongly on the specific arithmetic of $\Gamma_0(N)$, give the slightly better estimate

$$\overline{\omega}_{\mathcal{X}_0(N),\mathrm{Ar}}^2 = 3g \log(N)(1 + O(\log\log(N)/\log(N)),$$

which is the best possible one.

Theorem III. *Let $\mathcal{X}(N)$ be the minimal regular model of the modular curve $X(N)$, where N has at least two different prime divisors. Then the arithmetic self-intersection number of its dualizing sheaf equipped with the Arakelov metric is bounded from above by*

$$\overline{\omega}_{Ar}^2 \leq (4g-4)\left(\log|\Delta_{\mathbb{Q}(\zeta_N)|\mathbb{Q}}| + [\mathbb{Q}(\zeta_N):\mathbb{Q}]\kappa\right)$$

where $\kappa \in \mathbb{R}$ is an absolute constant independent of N.

Other examples of curves where our result could be applied are the Fermat curves $x^n + y^n = z^n$. Here we consider just the Fermat curves with prime exponents.

Theorem IV. *Let $\mathcal{X}$ be the desingularisation of the closure in $\mathbb{P}^2_{\mathbb{Z}[\zeta_p]}$ of the Fermat curve $x^p + y^p = z^p$ with prime exponent p (see* [Mc]*). Then the arithmetic self-intersection number of its dualizing sheaf equipped with the Arakelov metric is bounded from above by*

$$\begin{aligned}\overline{\omega}_{Ar}^2 \leq (4g-4)\Big(&\log|\Delta_{\mathbb{Q}(\zeta_N)|\mathbb{Q}}| \\ &+ [\mathbb{Q}(\zeta_N):\mathbb{Q}]\kappa\Big) + 4g(g-1)(r_p-1)^2(p-1)\log(p).\end{aligned}$$

where $\kappa \in \mathbb{R}$ is independent of N and $r_p \leq 4 + p(p-1)/2$.

0.3. The starting point in the proof of the main result Theorem I is the following inequality of arithmetic intersection numbers

$$\overline{\omega}_{\mathrm{Ar}}^2 \leq (4g-4)\log|\Delta_{K|\mathbb{Q}}| - 4g(g-1)\overline{\mathcal{O}}(P)_{\mathrm{Ar}}^2,$$

that holds for all sections P of $\mathcal{X}(\Gamma)$. Under our assumptions this holds in particular for all cusps S_j of $\mathcal{X}$. Furthermore, our assumptions allow then to express the weighted sum $\sum b_j \overline{\mathcal{O}}(S_j)^2_{\mathrm{Ar}}$, here b_j denotes the width of the cusp S_j, in terms of the arithmetic self-intersection number of $\mathcal{O}(1)$ on $\mathcal{X}(1)$ and an integral depending on a hermitian metric on $\mathcal{O}(1)$ (see Theorem 3.5); for our convenience we have chosen the Fubini-Study metric on $\mathcal{O}(1)$.

The second step is to bound the integral of the first step, i.e. to control the contribution that measures the difference of the Fubini-Study metric and the canonical metric on $X(\Gamma)$. We obtain this upper bound of the integral in question (see Theorem 4.4) by means of techniques that were first used by J. Jorgenson and J. Kramer in their study of sup-norm bounds of automorphic forms in [JK2].

0.4. Plan of paper. In the first and second sections of this paper we present the necessary background material on Arakelov theory on arithmetic surfaces. After this preparatory work we study in the third section the behavior of arithmetic intersection numbers with respect to a finite morphism to $\mathbb{P}^1$. The fourth section is devoted to the analytical aspects needed in our bound (0.2.1). In section five we study the geometric aspects used in the bound of the quantities $a_{\mathfrak{p}}$ of (0.2.1). In the last section we finally prove the previously stated theorems II to IV.

1. Intersection numbers of hermitian line bundles

In this section we give an overview of Arakelov theory for arithmetic surfaces. For more details we refer to the articles [Ga], [So] and [Kü]; we use the notation, if not explained in the following, as given in these references.

1.1. Notation. Let K be a number field, $\mathcal{O}_K$ its ring of integers. Let Σ be the set of infinite places of K, i.e., Σ is the set of complex embeddings of K in $\mathbb{C}$. Let X_K be a smooth, projective curve of genus g bigger than two defined over K. An arithmetic surface $\mathcal{X}$ is a regular scheme of dimension 2 together with a projective flat morphism $f : \mathcal{X} \longrightarrow \operatorname{Spec} \mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of integers of K. Moreover we assume that the generic fiber X_K of f is geometrically irreducible, i.e., $\mathcal{X}$ is a regular model for X_K over $\operatorname{Spec} \mathcal{O}_K$. We let $\mathcal{X}_\infty$ be the set of complex points of $\mathcal{X}$ of the scheme defined over $\operatorname{Spec} \mathbb{Z}$ defined by $\mathcal{X}$, i.e.

$$\mathcal{X}_\infty = \coprod_{\sigma \in \Sigma} X_\sigma(\mathbb{C}).$$

Note the complex conjugation F_∞ acts on $\mathcal{X}_\infty$. Finally, by abuse of notation we set

$$\int_{\mathcal{X}_\infty} := \sum_{\sigma\in\Sigma} \int_{X_\sigma(\mathbb{C})}$$

1.2. Remark. Note the assumption $g \geq 2$ implies that, after a possible replacement of K by a finite extension, there exist a unique minimal regular model $\mathcal{X}$ for X_K. It has the property, that for any regular model $\mathcal{X}'$ the induced birational morphism $\mathcal{X}' \to \mathcal{X}$ is in fact a morphism. Moreover $\mathcal{X}$ is a semi-stable arithmetic surface associated to X, i.e., $\mathcal{X}$ is a regular scheme of dimension 2, together with a projective flat morphism $\pi : \mathcal{X} \longrightarrow \operatorname{Spec} \mathcal{O}_K$, which has generic fiber X_K and such that the special fibers of π do not contain a copy of $\mathbb{P}^1$ with self-intersection (-1).

1.3. We call a line bundle $\mathcal{L}$ on $\mathcal{X}$ equipped with a hermitian metric h a *hermitian line bundle* and denote it by $\overline{\mathcal{L}} = (\mathcal{L}, h)$. The analytic degree of a hermitian line bundle $\overline{\mathcal{M}}$ is given by

$$\deg(\overline{\mathcal{M}}) = \frac{1}{[K:\mathbb{Q}]} \int_{\mathcal{X}_\infty} \mathrm{c}_1(\overline{\mathcal{M}}),$$

where the first Chern form $\mathrm{c}_1(\overline{\mathcal{M}})$ is outside the zeros of a section m of $\mathcal{M}$ given by the $(1,1)$-form

$$\mathrm{c}_1(\overline{\mathcal{M}})\big|_{\mathcal{X}_\sigma} = \mathrm{dd}^c - \log \|m^\sigma\|^2.$$

It equals the geometric degree of $\mathcal{M}$ restricted to the generic fiber X_K of $\mathcal{X}$.

Two hermitian line bundles $\overline{\mathcal{L}}, \overline{\mathcal{M}}$ on $\mathcal{X}$ are *isomorphic*, if

$$\overline{\mathcal{L}} \otimes \overline{\mathcal{M}}^{-1} \cong (\mathcal{O}_{\mathcal{X}}, |\cdot|).$$

The *arithmetic Picard group*, denoted by $\widehat{\operatorname{Pic}}(\mathcal{X})$, is the group of isomorphy classes of hermitian line bundles $\overline{\mathcal{L}}$ on $\mathcal{X}$ the group structure being given by the tensor product.

1.4. Definition. Let $\overline{\mathcal{L}}, \overline{\mathcal{M}}$ be two hermitian line bundles on $\mathcal{X}$ and l, m (resp.) be non-trivial, global sections, whose induced divisors on $\mathcal{X}_\infty$ have no points in common. Then, the *arithmetic intersection number* $\overline{\mathcal{L}}.\overline{\mathcal{M}}$ *of* $\overline{\mathcal{L}}$ *and* $\overline{\mathcal{M}}$ is given by

$$\overline{\mathcal{L}}.\overline{\mathcal{M}} := (l.m)_{\mathrm{fin}} + (l.m)_\infty; \tag{1.4.1}$$

here $(l.m)_{\rm fin}$ is defined by Serre's Tor-formula, which for l, m having proper intersection specializes to

$$(l.m)_{\rm fin} = \sum_{x \in \mathcal{X}} \log \sharp \left(\mathcal{O}_{\mathcal{X},x}/(l_x, m_x) \right),$$

where l_x, m_x are the local equations of l, m respectively at the point $x \in \mathcal{X}$ and

$$(l.m)_\infty = -(\log \|m\|)\big[\mathrm{div}(l)\big] + \int_{\mathcal{X}_\infty} \log \|l\| \cdot c_1(\overline{\mathcal{M}}). \tag{1.4.2}$$

Note that by convention $-(\log \|m\|)\big[\sum n_p P\big]$ equals $\sum -n_P(\log \|m\|)(P)$.

1.5. Proposition. (Arakelov, Deligne) *The formula* (1.4.1) *induces a bilinear, symmetric pairing*

$$\widehat{\mathrm{Pic}}(\mathcal{X}) \times \widehat{\mathrm{Pic}}(\mathcal{X}) \quad \longrightarrow \quad \mathbb{R}.$$

$\square$

1.6. Arithmetic Chow groups. Instead of the arithmetic Picard groups $\widehat{\mathrm{Pic}}(\mathcal{X})$ we could also consider the arithmetic Chow groups $\widehat{\mathrm{CH}}(\mathcal{X})$. In the sequel it will be more convenient to use these groups to check some identities on arithmetic intersection numbers, this is why we briefly discuss them here. The objects of these arithmetic Chow groups are equivalence classes of arithmetic cycles represented by pairs (D, g_D), where D is a divisor on $\mathcal{X}$ and g_D is Green function for $D_\infty = \prod_{\sigma \in \Sigma} D_\sigma(\mathbb{C})$. If $\overline{\mathcal{L}}$ is a hermitian line bundle, then the first arithmetic Chern class $\widehat{\mathrm{c}}_1(\overline{\mathcal{L}}) \in \widehat{\mathrm{CH}}^1(\mathcal{X}, \mathcal{D}_{\rm pre})$ can be represented by any of the pairs $(\mathrm{div}(s), -\log \|s\|^2)$ where s is a section for $\mathcal{L}$. The assignment $\overline{\mathcal{L}} \mapsto \widehat{\mathrm{c}}_1(\overline{\mathcal{L}})$ induces a morphism $\widehat{\mathrm{Pic}}(\mathcal{X}) \to \widehat{\mathrm{CH}}^1(\mathcal{X})$ that is compatible with the arithmetic intersection pairings in both groups. In particular we have the formulas

$$(\mathrm{div}(s), -\log \|s\|^2).(0, g) = \frac{1}{2} \int_{\mathcal{X}_\infty} g \, \mathrm{c}_1(\overline{\mathcal{L}}), \tag{1.6.1}$$

$$(\mathrm{div}(s), -\log \|s\|^2).(D, 0) = (\mathrm{div}(s), D)_{\rm fin}, \tag{1.6.2}$$

which are just reformulations of particular cases in (1.4.1).

1.7. We put $\widehat{\mathrm{Pic}}(\mathcal{X})_\mathbb{Q} = \widehat{\mathrm{Pic}}(\mathcal{X}) \otimes_\mathbb{Z} \mathbb{Q}$. Let $\widehat{\mathrm{Pic}}^0(\mathcal{X})_\mathbb{Q} \subset \widehat{\mathrm{Pic}}(\mathcal{X})_\mathbb{Q}$ denote the subgroup generated by the hermitian line bundles $\overline{\mathcal{L}} = (\mathcal{L}, \| \cdot \|)$ satisfying $\deg(\mathcal{L}|_{\mathcal{C}_l^{(\mathfrak{p})}}) = 0$ for all irreducible components $\mathcal{C}_l^{(\mathfrak{p})}$ of the fiber $f^{-1}(\mathfrak{p})$ above $\mathfrak{p} \in \mathrm{Spec}\, \mathcal{O}_K$, and $\mathrm{c}_1(\overline{\mathcal{L}}) = 0$.

We recall that if D is a divisor on X_K with $\deg D = 0$, then there exist a divisor $\mathcal{D}$, which may have rational coefficients, on $\mathcal{X}$ satisfying $\mathcal{D}_K = D$ and a hermitian metric $\|\cdot\|$ on $\mathcal{O}(\mathcal{D})_\infty$ such that $\overline{\mathcal{O}}(\mathcal{D}) = (\mathcal{O}(\mathcal{D}), \|\cdot\|) \in \widehat{\mathrm{Pic}}^0(\mathcal{X})_{\mathbb{Q}}$. The divisor $\mathcal{D}$ is unique up to multiples of the fibers of f and the metric $\|\cdot\|$ is unique up to multiplication by scalars.

1.8. Theorem. (Faltings-Hriljac) *Let $\mathcal{X} \to \operatorname{Spec} \mathcal{O}_K$ be an arithmetic surface. Let D be a divisor on X_K with* $\deg D = 0$*. Then, for all extensions $\overline{\mathcal{O}(\mathcal{D})}$ of $\mathcal{O}(D)$ to $\widehat{\mathrm{Pic}}^0(\mathcal{X})_{\mathbb{Q}}$, there is an equality*

$$\overline{\mathcal{O}}(\mathcal{D}).\overline{\mathcal{O}}(\mathcal{D}) = -[K:\mathbb{Q}]\langle D, D\rangle_{NT} \leq 0,$$

where $\langle D, D\rangle_{NT}$ is the Néron-Tate height pairing of the induced class in the Picard group of X_K and $\overline{\mathcal{O}(\mathcal{D})}.\overline{\mathcal{O}(\mathcal{D})}$ denotes the arithmetic intersection number.

Proof. The statement follows immediately from [Hr] theorem 3.1 and [Hr] proposition 3.3. □

1.9. Definition. Let $\mathcal{X}$ be an arithmetic surface. Then, $\mathcal{X}_\infty = \coprod_{\sigma\in\Sigma} X_\sigma(\mathbb{C})$ is a finite union of compact Riemann surfaces. By abuse of notation we call a $(1,1)$-form ν on $\mathcal{X}_\infty$ such that $\nu = \coprod_{\sigma\in\Sigma}\nu_\sigma$, where ν_σ is a Kähler form on $X_\sigma(\mathbb{C})$, also a Kähler form on $\mathcal{X}_\infty$. A hermitian line bundle $\overline{\mathcal{L}}$ is called *ν-admissible* if $\mathrm{c}_1(\overline{\mathcal{L}}) = \nu$.

1.10. If the genus of X is greater than 1, then for each σ we have on $X_\sigma(\mathbb{C})$ the *canonical Kähler form*

$$\nu^\sigma_{Ar}(z) = \frac{i}{2g}\sum_j |f^\sigma_j|^2 \mathrm{d}z \wedge \mathrm{d}\overline{z},$$

where $f^\sigma_1(z)\mathrm{d}z$, ... $f^\sigma_g(z)\mathrm{d}z$ is an orthonormal basis of $H^0(X_\sigma(\mathbb{C}), \Omega^1)$ equipped with the natural scalar product. We write ν_{Ar} for the induced Kähler form on $\mathcal{X}_\infty$.

1.11. Definition. Let ν be a fixed Kähler form on $\mathcal{X}_\infty$. Given a divisor $D = \sum n_P P$ on $\mathcal{X}$ we define

$$\overline{\mathcal{O}}(D)_\nu = (\mathcal{O}(D), \|\cdot\|_\nu),$$

where the metric on the line bundle $\mathcal{O}(D_\sigma)$ on $X_\sigma(\mathbb{C})$ is such that

$$-\log\|1_{D_\sigma}\|^2_\nu = \sum n_p g_{\nu_\sigma}(z, P),$$

where $g_{\nu_\sigma}(z,P)$ is the unique *normalized Green function* for $P_\sigma(\mathbb{C})$ associated with ν_σ (see definition 2.1).

Observe $\overline{\mathcal{O}}(D)_\nu$ is a ν-admissible line bundle. We note that if g is an algebraic function on X, then in general

$$\overline{\mathcal{O}}_X \neq \overline{\mathcal{O}}(\mathrm{div}(g))_\nu,$$

although both sides have vanishing Chern forms.

1.12. Theorem.(Arakelov) *There is a unique metric $\|\cdot\|_{Ar}$ on $\omega_{\mathcal{X}}$ such that for all sections P of $\mathcal{X}$ it holds the equality*

$$\overline{\omega}_{Ar}.\overline{\mathcal{O}}(P)_{Ar} + \overline{\mathcal{O}}(P)^2_{Ar} = \log|\Delta_{K|\mathbb{Q}}|.$$

Moreover $\overline{\omega}_{Ar} = (\omega_{\mathcal{X}}, \|\cdot\|_{Ar})$ is a ν_{Ar}-admissible line bundle. □

For the convenience of the reader we recall the convention

$$\omega_{\mathcal{X}} = \omega_{\mathcal{X}/\mathcal{O}_K} \otimes f^*\omega_{\mathcal{O}_K/\mathbb{Z}},$$

where $\omega_{\mathcal{O}_K/\mathbb{Z}} = \partial^{-1}_{K|\mathbb{Q}}$. The above result is equivalent to the adjunction formula

$$\overline{\omega}_{\mathcal{X}/\mathcal{O}_K,\mathrm{Ar}}.\overline{\mathcal{O}}(P)_{\mathrm{Ar}} + \overline{\mathcal{O}}(P)^2_{\mathrm{Ar}} = 0,$$

which holds for arithmetic surfaces in the sense of 1.1 (see e.g. [La], p. 101), since by construction $\overline{\omega}_{\mathrm{Ar}} = \overline{\omega}_{\mathcal{X}/\mathcal{O}_K,\mathrm{Ar}} \otimes f^*\overline{\omega}_{\mathcal{O}_K/\mathbb{Z}}$, where the later metric is the natural one.

1.13. Lemma. *Let P be a section of $\mathcal{X} \to \operatorname{Spec}\mathcal{O}_K$, then there exists a fibral divisor $\mathcal{F}_P$ such that*

$$\overline{\omega}^2_{Ar} \leq (4g-4)\log|\Delta_{K|\mathbb{Q}}| - 4g(g-1)\,\overline{\mathcal{O}}(P)^2_{Ar} + \mathcal{O}(\mathcal{F}_P)^2,$$

in particular

$$\overline{\omega}^2_{Ar} \leq (4g-4)\log|\Delta_{K|\mathbb{Q}}| - 4g(g-1)\,\overline{\mathcal{O}}(P)^2_{Ar}.$$

Proof. These formulae are well-known for semi-stable arithmetic surfaces (see e.g. [Fa], p. 410), however the essential ingredients namely theorem 1.8 and theorem 1.12 hold also for arithmetic surfaces in the sense of 1.1. For convenience we recall the proof:

Let $\mathcal{F}_p$ be a fibral divisor such that

$$\overline{\omega}_{\mathrm{Ar}} \otimes \overline{\mathcal{O}}(P)^{2-2g}_{\mathrm{Ar}} \otimes \mathcal{O}(\mathcal{F}_P) \in \widehat{\mathrm{Pic}}^0(\mathcal{X})_{\mathbb{Q}}. \tag{1.13.1}$$

Note $\mathcal{F}_P$ has support in the fibers of bad reduction of $\mathcal{X}$. Then because of the Faltings Hriljac theorem, the definition of $\widehat{\mathrm{Pic}}^0(\mathcal{X})_{\mathbb{Q}}$ and the adjunction formula we have

$$
\begin{aligned}
0 &\geq \Big(\overline{\omega}_{\mathrm{Ar}} \otimes \overline{\mathcal{O}}(P)_{\mathrm{Ar}}^{2-2g} \otimes \mathcal{O}(\mathcal{F}_P)\Big) \cdot \Big(\overline{\omega}_{\mathrm{Ar}} \otimes \overline{\mathcal{O}}(P)_{\mathrm{Ar}}^{2-2g} \otimes \mathcal{O}(\mathcal{F}_P)\Big)\\
&= \overline{\omega}_{\mathrm{Ar}}^2 + (4-4g)\overline{\omega}_{\mathrm{Ar}}.\overline{\mathcal{O}}(P)_{\mathrm{Ar}} + (2g-2)^2\overline{\mathcal{O}}(P)_{\mathrm{Ar}}^2 - \mathcal{O}(\mathcal{F}_P)^2\\
&= \overline{\omega}_{\mathrm{Ar}}^2 + \big((4g-4)+(2g-2)^2\big)\,\overline{\mathcal{O}}(P)_{\mathrm{Ar}}^2 + (4-4g)\log|\Delta_{K|\mathbb{Q}}| - \mathcal{O}(\mathcal{F}_P)^2.
\end{aligned}
$$

Hence

$$
\overline{\omega}_{\mathrm{Ar}}^2 \leq (4g-4)\log|\Delta_{K|\mathbb{Q}}| - 4g(g-1)\,\overline{\mathcal{O}}(P)_{\mathrm{Ar}}^2 + \mathcal{O}(\mathcal{F}_P)^2,
$$

and with the fact $\mathcal{O}(\mathcal{F}_P)^2 \leq 0$ we derive the desired inequality. $\square$

2. Basic properties of Green functions.

We recall here some basic facts of Green functions on a compact Riemann surface X.

2.1. Green functions. Let ν be a Kähler form, i.e. a smooth, symmetric $(1,1)$-form with $\int_X \nu(z) = 1$. A Green function associated with ν is a real valued function on $X \times X$ which is smooth outside the diagonal and has near the diagonal an expansion $g(z,w) = -\log|z-w|^2 + h(z,w)$ with a smooth function h. As a current it satisfies

$$
\mathrm{dd}^c\,[g(w,z)] + \delta_w = [\nu(z)] \tag{2.1.1}
$$

and it is called normalized if in addition for all $w \in X$

$$
\int_X g(z,w)\nu(z) = 0. \tag{2.1.2}
$$

One can show that there is a unique normalized Green function associated to ν.

2.2. Remark. We note that there are also other normalizations for Green functions used in the literature. Our normalization is as in [SABK], it is twice of the green function advocated in [BKK1] and it is -2 times the normalization used in [Sz] and [Fa].

2.3. Resolvent Kernel. Normalized Green functions are also referred to as the resolvent kernel. In order to describe this we consider the space $\mathcal{C}^\infty(X,\mathbb{C})$ of complex valued $\mathcal{C}^\infty$-functions on X. Then the Laplace operator $\Delta = \Delta_\nu$ associated with a Kähler form ν is defined by

$$
\Delta f \cdot \nu = \frac{1}{\pi i}\partial\bar{\partial} f = -2\,\mathrm{dd}^c f, \tag{2.3.1}
$$

here $f \in \mathcal{C}^\infty(X,\mathbb{C})$. The eigenvalues of Δ are positive reel numbers

$$
0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_m \leq \ldots
$$

and $\lim_{m\to\infty}\lambda_m = \infty$. We let $\phi_0 = 1$, ϕ_1, ... denoted the corresponding normalized eigenfunctions, i.e.,

$$\int_X \phi_n \overline{\phi}_m \nu = \delta_{n,m}.$$

If in addition $f \in L^2(X,\nu) \cap \mathcal{C}^\infty(X,\mathbb{C})$, then f has an expansion $f(z) = \sum_{m\geq 0} a_m\phi_m(z)$ with $a_m = \int_X f(u)\overline{\phi}_m(u)\nu(u)$.

2.4. Theorem. *For $f(z) = \sum_{m\geq 0} a_m\phi_m(z) \in L^2(X,\nu)\cap\mathcal{C}^\infty(X,\mathbb{C})$ we set*

$$\tilde{G}_\nu(f)(z) = \sum_{m>0} \frac{2a_m}{\lambda_m}\phi_m(z).$$

Then we have

$$\tilde{G}_\nu(f)(z) = \int_X g(z,w)f(w)\nu(w).$$

Proof. It suffices to show that for all $f \in L^2(X,\nu) \cap \mathcal{C}^\infty(X,\mathbb{C})$ with $\int_X f(z)\nu(z) = 0$ we have the equality

$$2f(P) = \int_X g(P,z)\Delta f(z)\nu(z),$$

since the kernel of Δ is spanned by the constant function 1. Using the identity (2.3.1) and the Green's equation (2.1.1) we derive

$$\begin{aligned}\int_X g(P,z)\Delta f(z)\nu(z) &= -2\int_X g(P,z)\,\mathrm{dd}^c f(z) = -2\,\mathrm{dd}^c[g(P,z)](f)\\ &= 2\delta_P(f) - 2[\nu](f) = 2f(P).\end{aligned}$$

□

2.5. Corollary. *If in addition $\int_X f(z)\nu(z) = 0$, then*

$$\mathrm{dd}^c - \int_X g(z,w)f(w)\nu(w) = f(z)\nu(z).$$

Proof. By theorem 2.4 we have $\Delta\tilde{G}_\nu(f)(z) = 2f(z)$. Now using the identity (2.3.1) we deduce

$$\mathrm{dd}^c\left(-\int_X g(z,w)f(w)\nu(w)\right) = \frac{1}{2}\Delta\tilde{G}_\nu(f)(z)\cdot\nu(z) = f(z)\nu(z).$$

□

2.6. Corollary. *Let $g_\nu(z,w)$, $g_\mu(z,w)$ be the normalized Green functions associated with the Kähler forms ν, μ respectively. Then*

$$g_\nu(z,w) = g_\mu(z,w) + a_{\mu,\nu}(z) + a_{\mu,\nu}(w) + c_{\mu,\nu}, \tag{2.6.1}$$

where for $z \in X$ we set

$$a_{\mu,\nu}(z) = -\int_X g_\mu(z,u)\nu(u), \tag{2.6.2}$$

$$c_{\mu,\nu} = -\int_X a_{\mu,\nu}(w)\nu(w) = \iint_{X\times X} g_\mu(w,u)\nu(u)\nu(w). \tag{2.6.3}$$

Proof. The right hand is except of a logarithmic singularity along the diagonal in $X \times X$ smooth. It is also orthogonal to ν in both variables. We write now $\nu(u) = (f(u)+1)\cdot\mu(u)$. Observe that $\int_X f(u)\mu(u) = 0$ and since g_μ is normalized we have $a_{\mu,\nu} = -\int_X g_\mu(z,u)f(u)\mu(u)$. From corollary 2.5 we deduce $\mathrm{dd}^c\, a_{\mu,\nu} = f(z)\mu(z) = \nu(z) - \mu(z)$, hence the right hand side equals by unicity g_ν. $\square$

2.7. Lemma. *Let ν, μ be Kähler forms on X, then for any $P \in X$*

$$\int_X (g_\nu(z,P) - g_\mu(z,P))\cdot(\mu(z)+\nu(z)) = 2\, a_{\mu,\nu}(P) + c_{\mu,\nu}.$$

Proof. By means of corollary 2.6 we have

$$\begin{aligned} g_\nu(z,P) - g_\mu(z,P) &= g_\mu(z,P) + a_{\mu,\nu}(z) + a_{\mu,\nu}(P) + c_{\mu,\nu} - g_\mu(z,P) \\ &= a_{\mu,\nu}(z) + a_{\mu,\nu}(P) + c_{\mu,\nu}. \end{aligned}$$

By changing the order of integration we derive the identity

$$\int a_{\mu,\nu}(z)\,(\mu(z)+\nu(z)) = \int \left(\int -g_\mu(z,w)\nu(w)\right)(\mu(z)+\nu(z)) = -c_{\mu,\nu},$$

thus, since $\int_X (\mu(z)+\nu(z)) = 2$, we obtain the claim. $\square$

2.8. Lemma. *Write $\nu(z) = (f(z)+1)\cdot\mu(z)$, then we have*

$$0 \le c_{\mu,\nu} \le \frac{2}{\lambda_1}||f||^2,$$

here λ_1 is the first non zero eigenvalue of Δ_μ and $||f||^2 = \int_X f(z)\overline{f(z)}\mu(z)$.

Proof. By theorem 2.4 we have since $f(z) = \overline{f(z)}$

$$0 \le c_{\mu,\nu} = \int_X \tilde{G}_\mu(f)(z)\overline{f(z)}\mu(z) = \sum_{m>0} 2\frac{|a_m|^2}{\lambda_m}$$
$$\le \frac{2}{\lambda_1}\sum_{m>0}|a_m|^2 = \frac{2}{\lambda_1}\int_X f(z)\overline{f(z)}\mu(z).$$

□

3. Upper bounds through morphisms

3.1. Models. Let X be a curve defined over a number field K and let $\boldsymbol{\beta} : X \to \mathbb{P}^1_K$ be a morphism of degree d; in this section we don't require that $\boldsymbol{\beta}$ is a Belyi morphism.

We denote by $X_{\mathcal{O}_K}$ the normalization of $\mathbb{P}^1_{\mathcal{O}_K}$ in $k(X)$; this is a normal, integral, projective scheme (see [EGAII], chapter 6.3), which is flat over $\operatorname{Spec}\mathcal{O}_K$ (see [Ha], chapter III, 9.1.3). The singularities of $X_{\mathcal{O}_K}$ are isolated since it is a normal scheme. By the theorem of Lipman these can be resolved (see [Li], p. 151). Let $\mathcal{X} \longrightarrow X_{\mathcal{O}_K}$ be a desingularisation of $X_{\mathcal{O}_K}$. We call $\mathcal{X}$ *an arithmetic surface associated to* $\boldsymbol{\beta}$. We denote the induced morphism from $\mathcal{X}$ to $\mathbb{P}^1_{\mathcal{O}_K}$ also by $\boldsymbol{\beta}$; its degree is $\deg(\boldsymbol{\beta}) = d$. Finally, we note that every dominant morphism from $\mathcal{X}$ to $\mathbb{P}^1$ factorizes through $X_{\mathcal{O}_K}$ (see [EGAII], chapter 6.3), i.e., we have the following commutative diagram

(3.1.1)
$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[dr, "\boldsymbol{\beta}"'] & X_{\mathcal{O}_K} \arrow[d] \\ & \mathbb{P}^1_{\mathcal{O}_K} \end{tikzcd}$$

On $\mathcal{X}$ we consider the line bundle $\mathcal{M}$ defined by $\mathcal{M} = \boldsymbol{\beta}^*\mathcal{O}(1)$, where $\mathcal{O}(1) = \mathcal{O}_{\mathbb{P}^1_{\mathcal{O}_K}}(1)$. Note by construction every function $h \in k(\mathcal{X})^*$ on $\mathcal{X}$ can be written as $h = f/g$ with $f, g \in H^0(\mathcal{X}, \mathcal{M}^{\otimes n})$ for some $n \in \mathbb{N}$.

3.2. Lemma. *Let $\boldsymbol{\beta} : X \to \mathbb{P}^1_K$ be a morphism of degree d. Assume there exist K-rational points S_K, resp. R_K, of X, resp. $\mathbb{P}^1_K$, such that $dS_K - (\boldsymbol{\beta}^* R_K) \in \operatorname{Div}(X)$ is a torsion divisor. We denote by $\boldsymbol{\beta} : \mathcal{X} \to \mathbb{P}^1$ an arithmetic surface associated with $\boldsymbol{\beta}$ and by S, resp. R the sections induced by S_K, resp. R_K.*

(i) There exists a global section r of $\mathcal{M}$ with $\operatorname{div}(r) = \boldsymbol{\beta}^* R$ and for large enough $e \in \mathbb{N}$ there exists a global section g of $\mathcal{M}^{\otimes e}$ with $\operatorname{div}(g)_K = ed\, S_K$.

(ii) With the notation as in (i), we have for any hermitian metric $\|\cdot\|$ on $\mathcal{M}_\infty$ that

$$d^2\,\overline{\mathcal{O}}(S)^2_{Ar} = \overline{\mathcal{O}}(\boldsymbol{\beta}^*R)^2_{Ar} + d\int_{\mathcal{X}_\infty}\log\|g^{1/e}\|^2\nu_{Ar} - d\int_{\mathcal{X}_\infty}\log\|r\|^2\nu_{Ar} + 2\mathcal{O}(\boldsymbol{\beta}^*R).\mathcal{O}(\mathcal{G}_S) + \mathcal{O}(\mathcal{G}_S)^2,$$

where $\mathcal{G}_S$ denotes the divisor

$$\mathcal{G}_S = dS - \frac{1}{e}\operatorname{div}(g) \in \operatorname{Div}(\mathcal{X})_{\mathbb{Q}}. \tag{3.2.1}$$

Proof. (i) The existence of the global section r is obvious. Since $d\,S_K - \boldsymbol{\beta}^*R_K$ is a torsion divisor, there exists a natural number e such that for some rational function h_K on X

$$ed\,S_K = e\,\boldsymbol{\beta}^*R_K + \operatorname{div}(h)_K = e\operatorname{div}(r)_K + \operatorname{div}(h)_K = \operatorname{div}(r^e\,h)_K;$$

indeed since $k(X) = k(\mathcal{X})$ we have have that $r^e h$ is a section of $\mathcal{M}^{\otimes e}$. We denote this section by g.

(ii) By the linearity of the arithmetic intersection product, we are allowed to work with divisors that have rational coefficients. Thus from now on we consider $g^{1/e}$ as a section of $\mathcal{M}$. The claim might be checked by using solely the above definition for the arithmetic intersection number. But it is more conceptually presented in the language of arithmetic Chow rings, which we will use for a moment. It follows in the first arithmetic Chow group the identities

$$\begin{aligned} d\widehat{\mathrm{c}}_1(\overline{\mathcal{O}}(S)_{\mathrm{Ar}}) &= \widehat{\mathrm{c}}_1(\overline{\mathcal{O}}(\boldsymbol{\beta}^*R)_{\mathrm{Ar}}) + (d\,S - \operatorname{div}(g^{1/e}), 0) \\ &\quad + (\operatorname{div}(g^{1/e}/r), g_{\mathrm{Ar}}(dS, z) - g_{\mathrm{Ar}}(\boldsymbol{\beta}^*R, z)) \\ &= \widehat{\mathrm{c}}_1(\overline{\mathcal{O}}(\boldsymbol{\beta}^*R)_{\mathrm{Ar}}) + (\mathcal{G}_S, 0) + (\operatorname{div}(g^{1/e}/r), -\log|g^{1/e}/r|^2) \\ &\quad + \left(0, \left(\int_{X_\sigma}\log|(g^\sigma)^{1/e}/r^\sigma|^2\nu^\sigma_{\mathrm{Ar}}\right)_{\sigma\in\Sigma}\right) \\ &= \widehat{\mathrm{c}}_1(\overline{\mathcal{O}}(\boldsymbol{\beta}^*R)_{\mathrm{Ar}}) + (\mathcal{G}_S, 0) \\ &\quad + \left(0, \left(\int_{X_\sigma}\log\left|(g^\sigma)^{1/e}/r^\sigma\right|^2\nu^\sigma_{\mathrm{Ar}}\right)_{\sigma\in\Sigma}\right). \end{aligned}$$

Taking the square we obtain with help of (1.6.1) and (1.6.2) the claim. □

3.3. Lemma. *Let $\boldsymbol{\beta}:\mathcal{X}\to\mathbb{P}^1$ be an arithmetic surface associated with a morphism $\boldsymbol{\beta}: X\to\mathbb{P}^1_K$ of degree d. Let R be a section of $\mathbb{P}^1$ and let r be a* *section of* $\mathcal{M}$ *with* $\operatorname{div}(r) = f^*R$. *Let* $\overline{\mathcal{O}}(1) = (\mathcal{O}(1), \|\cdot\|) \in \widehat{\operatorname{Pic}}(\mathbb{P}^1)$*, then with* $\overline{\mathcal{M}} = \boldsymbol{\beta}^*\overline{\mathcal{O}(1)}$ *and* $\mu = \boldsymbol{\beta}^* \operatorname{c}_1(\overline{\mathcal{O}}(1))/d$ *it holds the equality*

$$\overline{\mathcal{O}}(\boldsymbol{\beta}^* R)^2_{Ar} = d \cdot \overline{\mathcal{O}}(1)^2 + d \int_{\mathcal{X}_\infty} \log \|r\|^2 \nu_{Ar} + \frac{d}{2} c_{\mu,\nu_{Ar}}.$$

Proof. Again the proof is most easily deduced in the language of arithmetic Chow rings, which we will use for a moment. We write

$$\begin{aligned}
\widehat{\operatorname{c}}_1(\overline{\mathcal{O}}(\boldsymbol{\beta}^* R)_{\mathrm{Ar}}) &= (\operatorname{div}(r), g_{\nu_{\mathrm{Ar}}}(\boldsymbol{\beta}^* R, z)) \\
&= (\operatorname{div}(r), -\log\|r\|^2) + (0, g_{\nu_{\mathrm{Ar}}}(\boldsymbol{\beta}^* R, z) + \log\|r\|^2)) \\
&= \widehat{\operatorname{c}}_1(\overline{\mathcal{M}}) + (0, g_{\nu_{\mathrm{Ar}}}(\boldsymbol{\beta}^* R, z) - g_\mu(\boldsymbol{\beta}^* R, z)) \\
&\quad + (0, g_\mu(\boldsymbol{\beta}^* R, z) + \log\|r\|^2);
\end{aligned}$$

observe the last term is in fact a vector of constants. Then we obtain with the help of lemma 2.7 the equalities

$$\begin{aligned}
\overline{\mathcal{O}}(\boldsymbol{\beta}^* R)^2_{\mathrm{Ar}} &= \overline{\mathcal{M}}^2 + \frac{d}{2}\int_{\mathcal{X}_\infty} \big(g_{\nu_{\mathrm{Ar}}}(\boldsymbol{\beta}^* R, z) - g_\mu(\boldsymbol{\beta}^* R, z)\big) \wedge (\mu(z) + \nu(z)) \\
&\quad + d\left(g_\mu(\boldsymbol{\beta}^* R, z) + \log\|r\|^2\right) \\
&= \boldsymbol{\beta}^*\overline{\mathcal{O}}(1)^2 + d\, a_{\mu,\nu}(\boldsymbol{\beta}^* R) + \frac{d}{2} c_{\mu,\nu_{\mathrm{Ar}}} + d(g_\mu(\boldsymbol{\beta}^* R, z) + \log\|r\|^2) \\
&= d \cdot \overline{\mathcal{O}}(1)^2 + d\int_{\mathcal{X}_\infty} \log\|r\|^2 \nu_{\mathrm{Ar}} + \frac{d}{2} c_{\mu,\nu_{\mathrm{Ar}}}.
\end{aligned}$$

□

3.4. Lemma. *Let* $\boldsymbol{\beta} : X \to \mathbb{P}^1_K$ *be a morphism of degree* d*. Let* P_K *be* K*-rational point on* $\mathbb{P}^1_K$ *and write* $\boldsymbol{\beta}^* P_K = \sum_j b_j S_j$*. Assume that all* S_i *are* K*-rational points and that all divisors* $\sum n_i S_i$ *of degree zero are torsion divisors. We denote by* $\boldsymbol{\beta} : \mathcal{X} \to \mathbb{P}^1_{\mathcal{O}_K}$ *an arithmetic surface associated with* $\boldsymbol{\beta}$ *and by* $\boldsymbol{\beta}^* P$ *the horizontal divisor induced by the divisor* $\boldsymbol{\beta}^* P_K$*. Let* $\overline{\mathcal{O}}(1) = (\mathcal{O}(1), \|\cdot\|) \in \widehat{\operatorname{Pic}}(\mathbb{P}^1)$*, then with* $\overline{\mathcal{M}} = \boldsymbol{\beta}^*\overline{\mathcal{O}(1)}$ *it holds the inequality*

$$\int_{\mathcal{X}_\infty} \log\|p\|^2 \nu_{Ar} = \frac{1}{d}\sum_j b_j \left(\int_{\mathcal{X}_\infty} \log\|g_j^{1/e}\|^2 \nu_{Ar} + \frac{2}{d} \boldsymbol{\beta}^* \mathcal{O}(1).\mathcal{O}(\mathcal{G}_j)\right)$$

where p *is the unique section of* $\mathcal{M}$ *whose divisor equals* $\boldsymbol{\beta}^* P$ *and for each cusp* $\mathcal{G}_j$ *is a vertical divisor as in* (3.2.1).

Proof. We first note that the assumptions of Lemma 3.2 are satisfied with $R_K = P_K$ and $S_K = S_j$, thus there exist sections g_j for all S_j. The difference $\sum b_j \log\|g_j^{1/e_j}\| - d\log\|p\|$ is a constant function on $\mathcal{X}_\infty$, which we study now. From the equalities

$$\sum b_j \left(\operatorname{div}(g_j^{1/e_j}), -\log \|g_j^{1/e_j}\|^2 \right) = d\, \widehat{c}_1(\overline{\mathcal{M}}) = d \left(\operatorname{div}(p), -\log \|p\|^2 \right)$$

of classes in $\widehat{\mathrm{CH}}^1(\mathcal{X})_{\mathbb{Q}}$, we deduce that there exists an element $a \in \mathcal{O}_K$ with

$$\begin{aligned}
&(\operatorname{div}(a), -\log |a|^2) \\
&= \left(d\boldsymbol{\beta}^* P - \sum b_j \operatorname{div}(g_j^{1/e_j}), -d \log \|p\|^2 + \sum b_j \log \|g_j^{1/e_j}\|^2 \right) \\
&= \left(\sum b_j \mathcal{G}_j, -d \log \|p\|^2 + \sum b_j \log \|g_j^{1/e_j}\|^2 \right).
\end{aligned}$$

So we get the identities

$$\sum b_j \int_{\mathcal{X}_\infty} \log \|g_j^{1/e}\| \nu_{\mathrm{Ar}} = d \int_{\mathcal{X}_\infty} \log \|p\| \nu_{\mathrm{Ar}} - \log \mathrm{Nm}(a)$$

$$\begin{aligned}
\frac{1}{d} \sum b_j \boldsymbol{\beta}^* \mathcal{O}(1).\mathcal{O}(\mathcal{G}_j) &= \frac{1}{d} \boldsymbol{\beta}^* \mathcal{O}(1).\operatorname{div}(a) \\
&= \frac{\deg(\boldsymbol{\beta}^* \mathcal{O}(1))}{d} \log \mathrm{Nm}(a) = \log \mathrm{Nm}(a).
\end{aligned}$$

□

3.5. Theorem. *Let $\boldsymbol{\beta} : X \to \mathbb{P}^1_K$ be a morphism of degree d. Let P_K be K-rational point on $\mathbb{P}^1_K$ and write $\boldsymbol{\beta}^* P_K = \sum_j b_j S_j$. Assume that all S_i are K-rational points and that all divisors $\sum n_i S_i$ of degree zero are torsion divisors. We denote by $\boldsymbol{\beta} : \mathcal{X} \to \mathbb{P}^1_{\mathcal{O}_K}$ an arithmetic surface associated with $\boldsymbol{\beta}$ and by $\boldsymbol{\beta}^* P$ the horizontal divisor induced by the divisor $\boldsymbol{\beta}^* P_K$. Let $\overline{\mathcal{O}}(1) = (\mathcal{O}(1), \|\cdot\|) \in \widehat{\mathrm{Pic}}(\mathbb{P}^1)$ and put $\overline{\mathcal{M}} = \boldsymbol{\beta}^* \overline{\mathcal{O}(1)}$. Then it holds the inequality*

$$\begin{aligned}
\overline{\omega}^2_{Ar} \leq & (4g-4) \log |\Delta_{K|\mathbb{Q}}| - \frac{4g(g-1)}{d} \left(\overline{\mathcal{O}}(1)^2 + \int_{\mathcal{X}_\infty} \log \|p\|^2 \nu_{Ar} \right) \\
& - \frac{4g(g-1)}{d^3} \sum_j b_j\, \mathcal{O}(\mathcal{G}_j)^2 + \frac{1}{d} \sum_j b_j\, \mathcal{O}(\mathcal{F}_j)^2; \qquad (3.5.1)
\end{aligned}$$

where p is the unique section of $\mathcal{M}$ whose divisor equals $\boldsymbol{\beta}^* P$ and for each cusp $\mathcal{F}_j$ and $\mathcal{G}_j$ are vertical divisors as in (1.13.1) *and* (3.2.1).

Proof. Let $S_j \in \{\boldsymbol{\beta}^{-1}(P_K)\}$, then for any K-rational point R of $\mathbb{P}^1_K$ the divisor $dS_j - \boldsymbol{\beta}^* R$ is a torsion divisor since by assumption $d\, S_j - \boldsymbol{\beta}^* P$ is one. We let g_j be a section of $\mathcal{M}^{\otimes e}$ and $-\mathcal{G}_j$ its vertical divisor as in lemma 3.2. By abuse of notation we write S_j also for the section induced by S_j. We write $\mathcal{F}_j$ for the vertical divisor determined by S_j by means of formula

(1.13.1). Then we have because of the previous lemmata

$$
\begin{aligned}
&\Big(\overline{\omega}^2_{\mathrm{Ar}} - (4g-4)\log|\Delta_{K|\mathbb{Q}}| - \mathcal{O}(\mathcal{F}_j)^2\Big)/(4g(g-1)) \\
&\overset{1.13}{\le} -\overline{\mathcal{O}}(S_j)^2_{\mathrm{Ar}} \\
&\overset{3.2}{=} -\frac{1}{d^2}\Bigg(\overline{\mathcal{O}}(\boldsymbol{\beta}^*R)^2_{\mathrm{Ar}} + d\int_{\mathcal{X}(\Gamma)_\infty}\Big(\log\|g_j^{1/e}\|^2 - \log\|r\|^2\Big)\ \nu_{\mathrm{Ar}} \\
&\qquad + 2\mathcal{O}(\boldsymbol{\beta}^*R).\mathcal{O}(\mathcal{G}_j) + \mathcal{O}(\mathcal{G}_j)^2\Bigg) \\
&\overset{3.3}{=} -\frac{1}{d}\Bigg(\overline{\mathcal{O}}(1)^2 + 2\int_{\mathcal{X}_\infty}\log\|g_j^{1/e}\|\nu_{\mathrm{Ar}} + \frac{c_{\mu,\nu_{\mathrm{Ar}}}}{2} \\
&\qquad + \frac{2}{d}\boldsymbol{\beta}^*\mathcal{O}(1).\mathcal{O}(\mathcal{G}_j) + \frac{1}{d}\mathcal{O}(\mathcal{G}_j)^2\Bigg) \\
&\overset{2.8}{\le} -\frac{1}{d}\left(\overline{\mathcal{O}}(1)^2 + 2\int_{\mathcal{X}_\infty}\log\|g_j^{1/e}\|\nu_{\mathrm{Ar}} + \frac{2}{d}\boldsymbol{\beta}^*\mathcal{O}(1).\mathcal{O}(\mathcal{G}_j) + \frac{1}{d}\mathcal{O}(\mathcal{G}_j)^2\right).
\end{aligned}
$$

We now add these inequalities for all $S_j \in \{\boldsymbol{\beta}^{-1}(P)\}$ weighted with the factor b_j/d, which is determined by $\boldsymbol{\beta}^*P = \sum b_jS_j$ and obtain by means of lemma 3.4 the claim. $\square$

4. Upper bounds for the analytic contributions

If in theorem 3.5 the morphism $\boldsymbol{\beta}$ is a Belyi morphism, P equals the unique cusp ∞ of $X(1)$ and the metric on $\mathcal{O}(1)$ is the Fubini Study metric, then the contributions

$$-\frac{4g(g-1)}{d}\left(\overline{\mathcal{O}}(1)^2 + \int_{\mathcal{X}_\infty}\log\|p\|^2\nu_{\mathrm{Ar}}\right)$$

which contain analytical data can be bounded from above by means of formula (4.3.1) and theorem 4.4 below.

4.1. Modular curves over $\mathbb{C}$. Let $\mathbb{H} := \{\tau = x+iy \in \mathbb{C} \,|\, \mathrm{Im}\,\tau = y > 0\}$ denote the upper half plane. The full modular group $\mathrm{SL}_2(\mathbb{Z})$ acts properly discontinuously on $\mathbb{H}$ via fractional linear transformations:

$$\tau \mapsto \frac{a\tau+b}{c\tau+d}; \quad \tau\in\mathbb{H}, \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

Define $\Gamma(1) := \mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{\pm1\}$. For any subgroup Γ of finite index in $\Gamma(1)$ the quotient $\Gamma\setminus\mathbb{H}$ has the structure of a Riemann surface, which can be compactified by adding finitely many cusps. We denote this compact Riemann surface by $X(\Gamma)$; in fact, we have $X(\Gamma)=\Gamma\backslash(\mathbb{H}\cup\mathbb{P}^1(\mathbb{Q}))$. Following [Se], p. 71, we call $X(\Gamma)$ a (general) modular curve. On $X(\Gamma)$ there may be also finitely many elliptic points, which correspond to $\tau\in\mathbb{H}$ with a non-trivial isotropy subgroup in Γ.

The compact Riemann surface $X(1):=X(\Gamma(1))$ is known to be isomorphic to $\mathbb{P}^1(\mathbb{C})$, the isomorphism being induced by the classical j-invariant. For a subgroup Γ of finite index in $\Gamma(1)$ we finally have a finite, holomorphic map

$$\pi_\Gamma : X(\Gamma)\longrightarrow X(1)$$

of degree $\deg\pi_\Gamma=[\Gamma(1):\Gamma]$. This morphism of compact Riemann surfaces is ramified only above the two elliptic points P_i, P_ρ and the cusp S_∞ of $X(1)$. For more details, cf. [Sh].

4.2. Hyperbolic metric on $X(\Gamma)$. The volume form μ associated to the hyperbolic metric on $\mathbb{H}$ is given by the $(1,1)$-form

$$\mu=\frac{dx\wedge dy}{y^2}=\frac{i}{2}\cdot\frac{d\tau\wedge d\bar\tau}{(\operatorname{Im}\tau)^2}.$$

By abuse of notation we denote the $(1,1)$-form induced by μ on $X(\Gamma)$ also by μ. It is singular and its singularities occur at the elliptic fixed points and at the cusps of $X(\Gamma)$ (see e.g. [Kü], p. 222). We write

$$\mu_\Gamma=\frac{1}{[\Gamma(1):\Gamma]}\frac{3}{\pi}\mu$$

for the log-log singular Kähler form on $X(\Gamma)$ determined by μ; i.e. we have the formula $\int_{X(\Gamma)}\mu_\Gamma=1$.

4.3. Fubini-Study metric. A modular form f of weight k is a holomorphic function $f:\mathbb{H}\to\mathbb{C}$, such that $f(\gamma(\tau))=(c\tau+d)^kf(\tau)$ for all $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma$ and such that f is holomorphic in the cusps. Important examples are the modular forms Δ and $E_4^3=j\Delta$ of weight 12.

The vector space of modular forms of weight k with $12|k$ is naturally identified with the vector space $H^0(X(\Gamma),\mathcal{M}^{\otimes e})$, where $\mathcal{M}^{\otimes e}=\pi_\Gamma^*\mathcal{O}(\infty)^{\otimes e}$ and $e=k/12$. The Fubini-Study metric on $\mathcal{M}^{\otimes e}$ is for the modular form f corresponding to a section f given by

$$\|f\|^2=\frac{|f|^2}{(|E_4|^6+|\Delta|^2)^e}.$$

The metrized line bundle $\overline{\mathcal{M}}=(\mathcal{M},\|\cdot\|)$ is a hermitian line bundle. It is well known, that on $\mathcal{X}(1)=\mathbb{P}^1_{\mathbb{Z}}$

$$\overline{\mathcal{M}}^2=\overline{\mathcal{O}}(1)^2=\frac{1}{2}. \tag{4.3.1}$$

4.4. Theorem. *Let $f : X(\Gamma) \to X(1)$ be a Belyi uniformization of degree d. There exists a constant κ such that*

$$\int_{X(\Gamma)} -\log \|\Delta\| \, \nu_{Ar} \leq \frac{d}{g}\, \kappa,$$

here $\log\|\Delta\|$ denotes the Fubini-Study metric of the weight 12 cusp form Δ for $\Gamma(1)$ and ν_{Ar} is the canonical Kähler form.

Proof. It is shown by Jorgenson and Kramer in [JK2] that there exists a constant κ only depending on $X(1)$ such that for all $X(\Gamma)$ of genus $g \geq 2$ it holds the estimate

$$\nu_{\mathrm{Ar}} \leq \kappa \frac{[\Gamma(1):\Gamma]}{g} \mu_\Gamma. \tag{4.4.1}$$

However, the logarithmic singularities of $\log|\Delta|^2$ and the singularities of μ_Γ near the cusps prevent us to apply the inequality (4.4.1) directly. For a small ε we set

$$X(\Gamma)_\varepsilon = X(\Gamma) \setminus f^* B_\varepsilon(\infty).$$

Since $0 \leq \nu_{\mathrm{Ar}} \leq \kappa_1 \dfrac{d}{g}\, \mu_\Gamma$ we have

$$\begin{aligned}
\int_{X(\Gamma)_\varepsilon} -\log\|\Delta\|^2 \, \nu_{\mathrm{Ar}} &\leq \int_{X(\Gamma)_\varepsilon} \left[-\log\|\Delta\|^2\right]^+ \nu_{\mathrm{Ar}} \\
&\leq \kappa_1 \frac{d}{g} \int_{X(\Gamma)_\varepsilon} \left[-\log\|\Delta\|^2\right]^+ \mu_\Gamma \\
&\leq \kappa_1 \frac{d}{g} \int_{X(1)_\varepsilon} \left[-\log\|\Delta\|^2\right]^+ \mu_{\Gamma(1)} \\
&\leq \frac{d}{g} \kappa_2(\varepsilon),
\end{aligned}$$

here $[-\log\|\Delta\|^2]^+ = (|-\log\|\Delta\|^2| - \log\|\Delta\|^2)/2$ is the positive part of $-\log\|\Delta\|^2$.

It remains to bound the integral over the closure of $f^*B_\varepsilon(\infty)$. We write $f^*B_\varepsilon(\infty) = \cup_{S_j} B_\varepsilon(S_j)$. Observe that near the cusp S_∞ we have $-\log\|\Delta\|^2 = 4\pi y + O(1)$.

We have

$$\begin{aligned}
\int_{\overline{B_\varepsilon(S_j)}} O(1)\nu_{\mathrm{Ar}} &\leq \kappa_1 \frac{d}{g} \int_{\overline{B_\varepsilon(S_j)}} |O(1)|\mu_\Gamma \\
&\leq \kappa_1 \frac{d}{g} \frac{b_j}{d} \int_{\overline{B_\varepsilon(\infty)}} |O(1)|\mu_{\Gamma(1)} \\
&\leq \frac{b_j}{g} \kappa_3(\varepsilon)
\end{aligned}$$

and with the same considerations as in [JK2], p. 1274, we get

$$\int_{\overline{B_\varepsilon(S_j)}} 4\pi f^*(y)\, \nu_{\mathrm{Ar}} \leq \kappa_1 \frac{b_j}{g} \kappa_4(\varepsilon)$$

Taking into account that the sum of the width of cusps equal d we derive

$$\int_{\overline{B_\varepsilon(S_j)}} -\log\|\Delta\|^2 \nu_{\mathrm{Ar}} \leq \kappa_1 \frac{d}{g} \kappa_5(\varepsilon)$$

□

5. Upper bounds for the geometric contribution

Here we give general bounds for the geometric contributions

$$\sum_{\mathfrak{p}\ \mathrm{bad}} a_{\mathfrak{p}} \log \mathrm{Nm}(\mathfrak{p}) := -\frac{4g(g-1)}{d^3} \sum_j b_j\, \mathcal{O}(\mathcal{G}_j)^2 + \frac{1}{d} \sum_j b_j \mathcal{O}(\mathcal{F}_j)^2$$

in the inequality provided by theorem 3.5, which in turn completes the proof of theorem I.

5.1. Proposition. *Keep the notation as in 3.5. For each prime $\mathfrak{p}$ of bad reduction, we let $\mathcal{X}(\overline{\mathbb{F}}_{\mathfrak{p}}) = \sum_{j=1}^{r_{\mathfrak{p}}} m_j C_j^{(\mathfrak{p})}$ and set $s_{\mathfrak{p}} = \max_{i,j} |C_i^{(\mathfrak{p})}.C_j^{(\mathfrak{p})}|$. Then we have*

$$\sum_{\mathfrak{p}\ \mathrm{bad}} a_{\mathfrak{p}} \log \mathrm{Nm}(\mathfrak{p}) \leq 4g(g-1) \sum_{\mathfrak{p}\ \mathrm{bad}} (r_{\mathfrak{p}}-1)^2 s_{\mathfrak{p}} \log \mathrm{Nm}(\mathfrak{p}).$$

Proof. Since $\mathcal{O}(\mathcal{F}_j)^2 \leq 0$ for all j, it suffices to bound the terms involving $\mathcal{O}(\mathcal{G}_j)^2$. After possibly renumbering the irreducible components and adding full fibers, we may assume $0 \neq \mathcal{G}_j^{(\mathfrak{p})} = \sum_{k=2}^{r} n_k C_k^{(\mathfrak{p})}$ with all $n_k \geq 0$ and $n_1 = 0$. Then we have the upper bound

$$\deg\left(\mathcal{M}|_{C_1^{(\mathfrak{p})}}\right) = \left(dS_j + \mathcal{G}_j^{(\mathfrak{p})}\right).C_1^{(\mathfrak{p})} = \sum_{k=2}^{r_{\mathfrak{p}}} n_k C_k^{(\mathfrak{p})}.C_1^{(\mathfrak{p})} \leq d.$$

Since the natural numbers $C_k^{(\mathfrak{p})}.C_1^{(\mathfrak{p})}$ are semi-positive whenever $k \neq 1$ and since at least one of it doesn't vanish, we derive that all $n_k \leq d$. So we obtain

$$\begin{aligned} -\mathcal{O}\left(\mathcal{G}_j^{(\mathfrak{p})}\right)^2 &= -\left(\sum_{j,k=2}^{r_{\mathfrak{p}}} n_k n_j C_k^{(\mathfrak{p})}.C_j^{(\mathfrak{p})}\right) \log \mathrm{Nm}(\mathfrak{p}) \\ &\leq \left((r_{\mathfrak{p}}-1)^2 d^2 \max_{j,k} \left|C_k^{(\mathfrak{p})}.C_j^{(\mathfrak{p})}\right|\right) \log \mathrm{Nm}(\mathfrak{p}). \end{aligned}$$

Now summing up over all S_j and over all bad primes yield the claim. □

5.2. Proposition. *Keep the notation as in theorem 3.5. If* $\omega_{Ar}^{\otimes d} \otimes \mathcal{M}^{\otimes(2-2g)} \in \widehat{\mathrm{Pic}}^0(\mathcal{X})$*, then we have*

$$\sum_{\mathfrak{p} \text{ bad}} a_{\mathfrak{p}} \log \mathrm{Nm}(\mathfrak{p}) \leq 0$$

.

Proof. Because of lemma 3.2 we have $\mathcal{M} \cong \mathcal{O}(dS_j) \otimes \mathcal{O}(-\mathcal{G}_j)$. We deduce using the assumption that

$$\overline{\omega}_{\mathrm{Ar}} \otimes \overline{\mathcal{O}}(S_j)_{\mathrm{Ar}}^{2-2g} \otimes \mathcal{O}(-\frac{2-2g}{d}\mathcal{G}_j) \in \widehat{\mathrm{Pic}}^0(\mathcal{X})_{\mathbb{Q}},$$

which by means of formula (1.13.1) implies that $\mathcal{F}_j^2 = (2g-2)^2/d^2\, \mathcal{O}(\mathcal{G}_j)^2$. Therefore

$$-\frac{4g(g-1)}{d^3} \sum_j b_j\, \mathcal{O}(\mathcal{G}_j)^2 + \frac{1}{d} \sum_j b_j \mathcal{F}_j^2 = \frac{4}{d^3} \sum_j b_j\, \mathcal{O}(\mathcal{G}_j)^2 \leq 0,$$

which we wanted to to show. □

As the following lemma shows, the assumptions of the above proposition are always satisfied if $\boldsymbol{\beta} : \mathcal{X} \to \mathbb{P}^1$ is a Galois cover.

5.3. Lemma. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a Galois cover of arithmetic surfaces, then there exists a line bundle $\mathcal{L}$ on $\mathcal{Y}$ with:*

$$\omega_{\mathcal{X}}^{\deg(f)} \cong f^*\mathcal{L}.$$

Proof. By assumption $\mathcal{Y} = \mathcal{X}/G$ for a finite group G. Let $s \in H^0(\mathcal{X}, \omega_{\mathcal{X}})$, then also $s^\sigma \in H^0(\mathcal{X}, \omega_{\mathcal{X}})$ for all $\sigma \in G$ and therefore

$$g = \prod_{\sigma \in G} s^\sigma \in H^0(\mathcal{X}, \omega_{\mathcal{X}}^{\otimes |G|}).$$

However, since $g^\sigma = g$ for all $\sigma \in G$, we have $\mathcal{O}(\mathrm{div}(g)) = f^*\mathcal{L}$ for some line bundle $\mathcal{L}$ on $\mathcal{Y}$. Hence

$$\omega_{\mathcal{X}}^{\otimes |G|} \cong \mathcal{O}(\mathrm{div}(g)) \cong f^*\mathcal{L},$$

from which we deduce the first claim by using the fact $|G| = \deg(f)$. □

5.4. Proof of Theorem I. Applying the upper bounds for the analytical given by means of formula (4.3.1) and theorem 4.4 and those for the geometric contribution given above in the inequality of theorem 3.5 yield our main result. □

6. Explicit calculations

A first application of Theorem I is the following general result for modular curves.

6.1. Theorem. *Let $X(\Gamma)$ be the modular curve associated with a congruence subgroup of $\Gamma(1)$. Assume the level of Γ equals N. Then there exists a constant κ only depending on $\mathcal{X}(1)$ such that the arithmetic self-intersection number of the dualizing sheaf on the minimal regular model $\mathcal{X}$ of $X(\Gamma)$ satisfies the inequality*

$$\overline{\omega}^2_{Ar} \leq (4g-4)\big(\log|\Delta_{K|\mathbb{Q}}| + [K:\mathbb{Q}]\kappa\big) + \sum_{p|N} a_p \log p, \tag{6.1.1}$$

where the coefficients a_p are rational numbers.

Proof. Recall that a modular curve of level N and its cusps are defined over a certain subfield K of $\mathbb{Q}(\zeta_N)$. We also remind to the theorem of Manin and Drinfeld that on modular curves $X(\Gamma)$ associated with congruence subgroups of $\Gamma(1)$ all divisors with support in the cusps of degree zero are torsion divisors in $\operatorname{Pic}^0(X(\Gamma))$; hence the assumptions of theorem 3.5 are satisfied with the set of cusps of $X(\Gamma)$. Two of the quantities in (3.5.1) are determined in (4.3.1) and theorem 4.4, which in turn provides that on regular models $\mathcal{X}(\Gamma)$ as in 3.1 we have the formula

$$\overline{\omega}^2_{\mathrm{Ar}} \leq (4g-4)\big(\log|\Delta_{K|\mathbb{Q}}| + [K:\mathbb{Q}]\kappa\big) \tag{1}$$

$$+\frac{1}{d}\sum_j b_j \mathcal{O}(\mathcal{F}_j)^2 - \frac{4g(g-1)}{d^3}\sum_j b_j\,\mathcal{O}(\mathcal{G}_j)^2; \tag{6.1.2}$$

where $\kappa \in \mathbb{R}$ is a positive constant independent of Γ and for each cusp S_j of width b_j the vertical divisors $\mathcal{F}_j$ and $\mathcal{G}_j$ are as in (1.13.1) and (3.2.1). As we have seen, the claim now follows immediately from the fact that the fibers above the primes $p \nmid N$ are smooth (see e.g. [KM]). □

6.2. Modular curves $X_0(N)$. We now aim to give explicit formulas for the coefficients a_p in (6.1.1) for the modular curves $X_0(N)$. These curves have a model over $\mathbb{Q}$ and its complex valued points correspond to the compact Riemann surfaces $\Gamma_0(N)\setminus\big(\mathbb{H}\cup\mathbb{P}^1(\mathbb{Q})\big)$, where $\Gamma_0(N) = \big\{\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma(1) \mid c \equiv 0 \mod N\big\}$. The index in $\Gamma(1)$ is given by the formula

$$d = [\Gamma(1):\Gamma_0(N)] = \prod_{p|N}(p+1).$$

From now on N will be a square free integer with $(6,N) = 1$ having at least two different prime factors, because then the cusps are also rational points [Og]. The minimal model $\mathcal{X}_0(N)$ over $\operatorname{Spec}\mathbb{Z}$ has been determined by Deligne and Rapoport to be as follows: The curve $X_0(N)$ is smooth over $\mathbb{Z}[1/N]$. If $p|N$, then the scheme $\mathcal{X}_0(N)\otimes\mathbb{Z}/p\mathbb{Z}$ is reduced and singular over $\mathbb{Z}/p\mathbb{Z}$. We write $N = pM = pq_1 \cdot \ldots \cdot q_\nu$ and set $Q = \prod_{i=1}^{\nu}(q_i + 1)$ and define

$$u = \begin{cases} 1 & \text{if } p \equiv 7 \text{ or } 11 \mod 12 \text{ and all } q_i \equiv 1 \mod 4, \;\; i = 1 \ldots \nu \\ 0 & \text{else} \end{cases}$$
$$v = \begin{cases} 1 & \text{if } p \equiv 5 \text{ or } 11 \mod 12 \text{ and all } q_i \equiv 1 \mod 3, \;\; i = 1 \ldots \nu \\ 0 & \text{else} \end{cases}$$

The fiber $\mathcal{X}_0(N)\otimes\mathbb{Z}/p\mathbb{Z}$ is the union of two copies C_0, C_∞ of $\mathcal{X}_0(M)\otimes\mathbb{Z}/p\mathbb{Z}$ crossing transversely in certain super singular points and some chains of projective lines connecting the remaining supersingular points with those of the other copy. More precisely, if $u = 1$ then there are 2^ν projective lines F_i, if $v = 1$ then there are 2^ν pairs G_i, H_i of projective lines and if $u = v = 0$ there are no such projective lines. All theses projective lines have self-intersection number -2. The intersection behavior between these irreducible components is given by the figure 1 below.

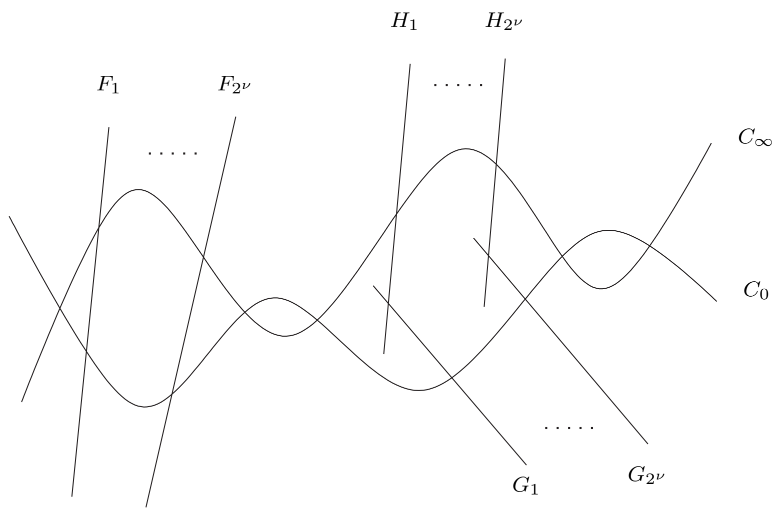

FIGURE 1.

In figure 1 all intersections are transversal, i.e., all intersection multiplicities are equal to 1. Finally we mention the formula

$$C_0.C_\infty = d\frac{p-1}{12(p+1)} - 2^\nu\left(\frac{u}{2}+\frac{v}{3}\right).$$

6.3. Lemma. *With the notation as in theorem 3.5 we have on $\mathcal{X}_0(N)$ the formulae*

$$-\frac{4g(g-1)}{d^3}\sum_j b_j \mathcal{O}(\mathcal{G}_j)^2 + \frac{1}{d}\sum_{S_j} b_j \mathcal{O}(\mathcal{F}_j)^2 \leq (3g+1)\sum_{p|N}\frac{p+1}{p-1}\log p,$$

Proof. We can consider the above inequality prime by prime. Let S_j be any cusp, then we set

$$\operatorname{div}(g_j^{1/e})\big|_{\mathcal{X}_0(N)\otimes\mathbb{Z}/p\mathbb{Z}} = d\,S_j + \alpha\left(C_0 + \sum_\nu \beta_\nu F_\nu + \sum_\mu \gamma_\mu G_\mu + \sum_\rho \delta_\rho H_\rho\right).$$

The relations $\operatorname{div}(g_j^{1/e})\big|_{F_\nu} = \operatorname{div}(g_j^{1/e})\big|_{G_\mu} = \operatorname{div}(g_j^{1/e})\big|_{H_\rho} = 0$ imply $\beta_\mu = 1/2$, $\gamma_\nu = 2/3$ and $\delta_\rho = 1/3$, for all possible μ, ν and ρ. We put

$$\mathcal{F}_0 = C_0 + \frac{1}{2}\sum_\nu F_\nu + \frac{2}{3}\sum_\mu G_\mu + \frac{1}{3}\sum_\rho H_\rho$$

$$\mathcal{F}_\infty = \mathcal{X}_0(N)\otimes\mathbb{Z}/p\mathbb{Z} - \mathcal{F}_0 = C_\infty + \frac{1}{2}\sum_\nu F_\nu + \frac{1}{3}\sum_\mu G_\mu + \frac{2}{3}\sum_\rho H_\rho;$$

where if some of the components F_ν, G_μ, H_ρ do not exist, then these are set to be the zero divisors. The other relation to be satisfied is

$$f = \operatorname{div}(g_j^{1/e})\big|_{C_\infty} = (d\,S_j + \alpha\mathcal{F}_0)\,.C_\infty = d\,S_j.C_\infty + \alpha d\frac{p-1}{12(p+1)}.$$

Hence, since $|f - d(S_j.C_\infty)| \leq d$, we derive

$$|\alpha| = \frac{|f - d(S_j.C_\infty)|}{d\frac{p-1}{p+1}\frac{1}{12}} \leq 12\frac{p+1}{p-1}.$$

From $(\mathcal{F}_0 + \mathcal{F}_\infty)^2 = 0$, we deduce

$$\mathcal{F}_0^2 = -\mathcal{F}_0.\mathcal{F}_\infty = -\mathcal{F}_0.C_\infty = -d\frac{p-1}{12(p+1)},$$

which in turn implies that

$$-(\mathcal{G}_j^{(p)})^2 = -\alpha^2\,\mathcal{F}_0^2 \leq 12d\frac{p+1}{p-1}.$$

Now summing up over all cusps S_j yields the bound

$$-\sum_{S_j} b_j \mathcal{O}(\mathcal{G}_j^{(p)})^2 \le \sum_{S_j} b_j \cdot 12d^2 \frac{p+1}{p-1} \log(p) \le 12d^2 \frac{p+1}{p-1} \log(p)$$

For the formula

$$-\sum_{S_j} b_j \mathcal{O}(\mathcal{F}_j^{(p)})^2 = 12(g-1)^2 \frac{p+1}{p-1} \log p,$$

which can be proven in a similar way, we refer to [AU], Proposition 4.2.1, p.63. Now we conclude prime by prime and obtain

$$\begin{aligned}
&-\frac{4g(g-1)}{d^3} \sum_j b_j \mathcal{O}(\mathcal{G}_j)^2 + \frac{1}{d} \sum_{S_j} b_j \mathcal{O}(\mathcal{F}_j)^2 \\
&\le \frac{48g(g-1) - 12(g-1)^2}{d} \sum_{p|N} \frac{p+1}{p-1} \log p \\
&\le (3g+1) \sum_{p|N} \frac{p+1}{p-1} \log p,
\end{aligned}$$

where in the last inequality we used $g - 1 \le \frac{1}{12} d$. $\square$

We are now in position to state

6.4. Theorem. *Let $\mathcal{X}_0(N)$ be the minimal regular model of the modular curve $X_0(N)$, where N is a square free integer with $(N, 6) = 1$. Then the arithmetic self-intersection number of its dualizing sheaf equipped with the Arakelov metric is bounded from above by*

$$\overline{\omega}^2_{Ar} \le (4g-4)\kappa + (3g+1) \sum_{p|N} \frac{p+1}{p-1} \log\, p,$$

where $\kappa \in \mathbb{R}$ is an absolute constant independent of N.

6.5. Modular curves $X(N)$. We now consider the modular curves $X(N)$, where N has at least two different prime divisors. These curves have a model over $\mathbb{Q}(\zeta_N)$, where ζ_N is a primitive N-th root of unity and its complex valued points correspond to the compact Riemann surfaces $\Gamma(N)\backslash\big(\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})\big)$, where $\Gamma(N) = \big\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(1) \,\big|\, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \mod N \big\}$.

6.6. Theorem. *Let $\mathcal{X}(N)$ be the minimal regular model of the modular curve $X(N)$, where N has at least two different prime divisors. Then the arithmetic self-intersection number of its dualizing sheaf equipped with the Arakelov metric is bounded from above by*

$$\overline{\omega}^2_{Ar} \le (4g-4) \log |\Delta_{\mathbb{Q}(\zeta_N)|\mathbb{Q}}| + (g-1)[\mathbb{Q}(\zeta_N) : \mathbb{Q}]\kappa$$

where $\kappa \in \mathbb{R}$ is an absolute constant independent of N.

Proof. The normalization $\mathcal{X}(N)$ of $\mathcal{X}(1)$ in the function field of $X(\Gamma(N))$ equals the minimal regular model (see e.g. [DR]). Moreover the natural morphism $\boldsymbol{\beta} : \mathcal{X}(N) \to \mathcal{X}(1)$ is a Galois cover (see e.g. [DR]). Therefore by the previous lemma we get $\omega^{\otimes \deg \boldsymbol{\beta}} \otimes \mathcal{M}^{\otimes(2-2g)} \in \widehat{\mathrm{Pic}}^0(\mathcal{X}(N))$, hence by the theorem of Manin and Drinfeld we derive the claim. □

6.7. Fermat curves. We now consider the Fermat curves $F_n : x^n + y^n = z^n$. The morphism $\boldsymbol{\beta} : F_N \to \mathbb{P}^1$ given by $(x : y : z) \mapsto (x : z)$ determines a Galois covering with group $(\mathbb{Z}/n\mathbb{Z})^2$. Since $\boldsymbol{\beta}$ has the three ramification points $0, 1, \infty$, it is a Belyi morphism. In contrast to the previous examples, we proceed with the identification $\mathbb{P}^1 \setminus \{0, 1, \infty\} = \Gamma(2) \setminus \mathbb{H}$. The principal congruence subgroup $\Gamma(2)$ is a free group on two generators A, B. Let $\Gamma_n = \left\{\gamma = A^{e_1} B^{f_1} \cdots A^{e_r} B^{f_r} \in \Gamma(2) \mid \sum e_i = \sum f_j = n\right\}$, then $\Gamma(2)/\Gamma_n \cong (\mathbb{Z}/n\mathbb{Z})^2$. Therefore $F_n = X(\Gamma_n)$ and $\boldsymbol{\beta}$ is induced by the natural morphism $\Gamma_n \setminus \mathbb{H} \to \Gamma(2) \setminus \mathbb{H}$. The cusps are defined over $\mathbb{Q}(\zeta_n)$ and the group of cuspidal divisors on F_n with respect to the uniformization given by Γ_n modulo rational equivalence is a torsion subgroup, in fact its structure is determined in [Ro].

Let $\mathcal{X}$ be the desingularisation of the closure in $\mathbb{P}^2_{\mathbb{Z}[\zeta_p]}$ of the Fermat curve $x^p + y^p = z^p$ with prime exponent p (see [Mc]). Let $\mathfrak{p}$ be a prime above p, then $\mathcal{X}(\overline{\mathbb{F}}_{\mathfrak{p}})$ has at most

$$r_{\mathfrak{p}} := 3 + p(p-3)/2 \tag{6.7.1}$$

irreducible components $C_l^{(\mathfrak{p})}$, which are all isomorphic to $\mathbb{P}^1$, and

$$s_{\mathfrak{p}} := \max_{l,m} \left| C_l^{(\mathfrak{p})} . C_m^{(\mathfrak{p})} \right| \leq p. \tag{6.7.2}$$

6.8. Proof of theorem IV. Because of the previous discussion, we are allowed to apply theorem 3.5 with $P = \infty$. It is easy to check that the analytic contribution can be bounded by the method of Theorem 4.4, but now applied to covers of $X(2)$. The geometric contributions are bounded by means of proposition 5.1 applied with (6.7.1) and (6.7.2). □

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