GROWTH OF POSITIVE WORDS AND LOWER BOUNDS OF THE GROWTH RATE FOR THOMPSON'S GROUPS
$F(p)$

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ABSTRACT. Let $F(p)$, $p \geq 2$ be the family of generalized Thompson's groups. Here $F(2)$ is the famous Richard Thompson's group usually denoted by F. We find the growth rate of the monoid of positive words in $F(p)$ and show that it does not exceed $p + 1/2$. Also we describe new normal forms for elements of $F(p)$ and, using these forms, we find a lower bound for the growth rate of $F(p)$ in its natural generators. This lower bound asymptotically equals $(p-1/2)\log_2 e + 1/2$ for large values of p.

INTRODUCTION

The family of generalized Thompson's groups $F(p)$ was introduced by K. S. Brown in [6]. Additional facts about these groups can be found in [8, 22]. The case $p = 2$ corresponds to the famous Richard Thompson's group F. See the survey [9] for details about this group.

The groups $F(p)$ have many common features. All of them are embeddable into each other [4]. None of them has free non-abelian subgroups. None of these groups satisfy any nontrivial group law. The derived subgroups of each of the $F(p)$ is simple (infinitely generated). Every proper homomorphic image of $F(p)$ is abelian (so these groups are not residually finite). Each $F(p)$ is finitely presented and has quadratic Dehn function [16].

Each of these groups has a faithful representation by piecewise linear functions. The word problem has an easy solution in each of these groups. Also all these groups are diagram groups in the sense of [17]. Namely, $F(p)$ is a diagram group over a very simple semigroup presentation $\langle\, x \mid x = x^p \,\rangle$. It

follows then from [17, Section 15] that $F(p)$ has solvable conjugacy problem. Each group $F(p)$ satisfies homological finiteness condition $\mathcal{F}_\infty$. All integer homology groups $H_n(F(p), \mathbb{Z})$ are free abelian of finite rank and the Poincaré series are rational [19].

However, there is some difference between the groups of this family. Brin [3] described the group $\mathrm{Aut}\, F$ for $F = F(2)$. Some information about automorphisms of $F(p)$, where $p > 2$, can be found in [4], where it is shown that already for $p = 3$ there are "wild" automorphisms of $F(p)$.

The goal of this article is to obtain analogs of some results for the group F. The first author found the growth function of the monoid of positive elements of F. This function is rational, namely, it equals

$$\frac{1 - x^2}{1 - 2x - x^2 + x^3}.$$

Notice that the elements x_0, x_1, ..., x_{p-1} generate a free submonoid of rank p in $F(p)$. Thus the growth rate of positive elements in $F(p)$ is at least p. In this paper we show that for any p, the exact value of the growth rate of positive elements is only slightly higher than p — it never exceeds $p + 1/2$.

Guba and Sapir [18] found two new normal forms for elements of F. One of them is a normal form in the infinite set of generators. This normal form is locally testable (unlike the standard normal form). It has the same feature as the normal form in the free group: a word is in a normal form if and only if all its subwords of length 2 are in the normal form. In this paper, we find such a form for every $F(p)$. Another normal form constructed in [18] for F allows one to construct a regular set of normal forms in F. We find an analogous construction for each $F(p)$.

Using the above regular normal form, the second author proved in [15] that the growth rate of the group F in generators x_0, x_1 is at least $(3+\sqrt{5})/2$. Notice that neither the growth function, nor the growth rate for F is known at the present. In this paper we find a lower bound of the growth rate for each of the groups $F(p)$, where the generating set consists of x_0, x_1, ..., x_{p-1}. We show that the lower bound is a root of a certain algebraic equation and find the asymptotic behaviour of this root. For large values of p, this is $(p - 1/2) \log_2 e + 1/2$, where $\log_2 e = 1.442695\ldots$.

The plan of the paper is as follows. In Section 1 we recall the definition of the family $F(p)$ of generalized Thompson's groups and some basic facts about growth functions and growth rates. This Section also contains a description of (positive) elements in $F(p)$ in terms of rooted p-trees.

In Section 2 we describe Fordham's method to calculate the word length in $F(p)$. We restrict ourselves to the case of positive words only (the description for this case is much simpler). Recall that for the case $p = 2$, a

fast algoritm to find the word length metric was described in [11, 12]. This algorithm is very effective but it has quite a complicated description. A simplification of the method due to Belk and Brown can be found in [1]. One of the easiest algorithms to find the word length in F (the so-called Length Formula) is contained in [15, Section 5]. Notice that for $p > 2$, none of the simplified versions exists so we use Fordham's approach from [13].

In Section 3, using Fordham's method, we find equations for generating functions describing the growth of $F_+(p)$. We solve these equations in Section 4 and show that the generating function for positive words in $F(p)$ is irrational provided $p > 2$ (unlike the case $p = 2$). Then we find the growth rate of positive words in $F(p)$ as a root of an algebraic equation. We prove that this growth rate never exceeds $p + 1/2$ approaching this value as p approaches infinity. Thus the set $F_+(p)$ of all positive words is not much higher than the free submonoid generated by x_0, x_1, ..., x_{p-1}.

Section 5 describes two new normal forms of elements in $F(p)$. The first of these forms is locally testable (one needs to test only subwords of length 2, similar to a free group). The second of the normal forms leads to a regular language that represents each element of $F(p)$ exactly once. Based on that regular language, we construct the corresponding automaton and find a lower bound for the growth rate of $F(p)$ in Section 6. This lower bound is given as a root of an algebraic equation. We also describe its asymptotic behaviour showing that it approaches $(p - 1/2) \log_2 e + 1/2$ for large values of p.

1. Preliminaries

The family of generalized Thompson's group can be defined as follows. The group $F(p)$ is the group of all piecewise linear self homeomorphisms of the unit interval $[0, 1]$ that are orientation preserving (that is, send 0 to zero and 1 to 1) with all slopes integer powers of p and such that their singularities (breakpoints of the derivative) belong to $\mathbb{Z}[\frac{1}{p}]$. The group $F(p)$ admits a presentation given by

$$(1) \qquad \langle\, x_i \ (i \geq 0) \mid x_j x_i = x_i x_{j+p-1} \ (i < j) \,\rangle.$$

This presentation is infinite, but a close examination shows that the group is actually finitely generated, since x_0, x_1, ..., x_{p-1} are sufficient to generate it. In fact, the group is finitely presented. The finite presentation is awkward and it is not used much. The symmetric and simple nature of the infinite presentation makes it much more adequate for almost all purposes.

One such example where the infinite presentation is particularly appropriate is in the construction of the normal form. A word given in the generators

x_i and their inverses, can have its generators moved around according to the relators, and the result is the following well-known statement:

Theorem 1.1. *An element in $F(p)$ always admits an expression of the form*

$$x_{i_1} x_{i_2} \cdots x_{i_m} x_{j_n}^{-1} \cdots x_{j_2}^{-1} x_{j_1}^{-1},$$

where

$$i_1 \leq i_2 \leq \cdots \leq i_m, \ j_1 \leq j_2 \leq \cdots \leq j_n.$$

In general, this expression is not unique, but for every element there is a unique word of this type which satisfies certain technical condition (see [9] for details). This unique word is called the *standard normal form* for the element of $F(p)$.

Observe that the infinite presentation for $F(p)$ is actually a monoid presentation. Hence $F(p)$ admits a submonoid, the submonoid $F_+(p)$ given by the same presentation, whose elements are called *positive words*. Theorem 1.1 shows that $F(p)$ is the group of right fractions of this monoid.

An element of $F(p)$ can be represented by two subdivisions of the interval $[0, 1]$, namely, the subdivision into intervals which get mapped linearly to each other. A subdivision of this type, where the dividing points are all in $\mathbb{Z}[\frac{1}{p}]$, can always be obtained by subsequent subdivisions of the interval into p equal pieces. Hence, a subdivision of the interval is equivalent to a rooted tree where each vertex has valence $p + 1$ except the root, which has valence p (or 1 in case when the tree consists of the root only), and the *leaves*, which have valence 1. A node (except the root and the leaves) is pictured to have one edge going up and p edges going down to its p *children*. These trees will be called *rooted p-trees*. An element of $F(p)$ is then represented by a pair of rooted p-trees called the *source* tree and the *target* tree. This representation has been extensively studied in the case $p = 2$. Note that positive words can be represented by a single p-tree, because the other tree is always the same: the tree which consists of all right carets.

A piece of these p-trees consisting of a node and its p edges going down to its children is called a *caret*. Carets are the building blocks of the trees and they give rise to the algorithm for finding the word metric in $F(p)$, see Section 2.

As stated in the introduction, the exact growth function for the groups $F(p)$ is not known. In this paper we will give lower bounds for growth rates of these groups, computing lower bounds for the number of elements in each length.

To be precise, given a finitely generated group G with finite generating set X, denote its sphere of radius n by

$$\mathbf{S}(n) = \{ g \in G \mid \ell(g) = n \},$$

where $\ell(g)$ is the length of $g \in G$ in the set of generators X. We also have the ball of radius n

$$\mathbf{B}(n) = \bigcup_{k=0}^{n} \mathbf{S}(k).$$

If $\gamma_n = \#\mathbf{B}(n)$, the series

$$\Gamma(x) = \sum_{n=0}^{\infty} \gamma_n x^n$$

is called the (general) *growth function* for G with respect to X, and the number

$$\gamma = \lim_{n \to \infty} \gamma_n$$

is the *growth rate* of G with respect to X. The limit always exists due to the submultiplicative property of γ_n, that is, $\gamma_{m+n} \le \gamma_m \gamma_n$ for all $m, n \ge 0$. Also, the *spherical growth function* is given by $\sigma_n = \#\mathbf{S}(n)$ and

$$\Sigma(x) = \sum_{n=0}^{\infty} \sigma_n x^n,$$

which has the same growth rate as the general growth function (for all infinite groups). For details about growth functions, see, for instance, [14].

If $P \subseteq G$ is a subset of a group, not necessarily a subgroup, we can define the growth functions of the set P by the same formulas as above but where the coefficients are actually the cardinals of the sets $P \cap \mathbf{B}(n)$ or $P \cap \mathbf{S}(n)$. The goal for one of the next sections is to compute the growth series of the subset $F_+(p)$ in $F(p)$. In order to do that, we need to describe the algorithm for calculating the word metric in $F(p)$.

2. Positive words in Thompson's groups $F(p)$ and Fordham's method

In 1995, S. Blake Fordham [11] constructed an algorithm which, for any given element in $F = F(2)$, finds its distance to the identity in the word metric given by generators x_0, x_1. This algorithm consists in defining different types of carets, then having each caret of the source tree paired to its corresponding caret in the target tree, and assigning a weight to each type of pairs of carets. A table is given for all possible pairs of types, with the assignment of the weight. The sum of all the weights of all the pairs is the exact distance from the element to the identity. In a set of unpublished notes [13], Fordham extends his method to the groups $F(p)$. This method will be the starting block of the computation.

The method used to compute this growth will be an extension to $F(p)$ of the method developed in [7] for the case of $F = F(2)$. Consider a positive

element of $F(p)$. As we know, the element can be represented by a rooted p-tree. We are going to define different types of carets and their weights, following Fordham [13].

A caret will be called *left* or *right* if it is situated in the leftmost edge of the tree or in the rightmost edge, and *middle* or *interior* if it is situated in the middle, i.e. if it is not right or left. For instance, a caret is left if it represents a subinterval of $[0, 1]$ which has left endpoint equal to zero. Middle carets will be subdivided into $p - 1$ types, denoted by $\mathcal{M}^1$, $\mathcal{M}^2$, ..., $\mathcal{M}^{p-1}$ according to which caret they are children of, and its position as child.

The children of a caret are subdivided in two types, the predecessors and the successors. This subdivision will give a total order to the set of carets, with a caret being always after its predecessor children and before its successors. The definitions of the caret types are as follows:

- The root caret is special. Its children are:
    - Its left child is a left caret and it is the only predecessor.
    - Its middle children are successors, and have types $\mathcal{M}^1$, $\mathcal{M}^2$, ..., $\mathcal{M}^{p-2}$, in order-preserving way.
    - Its right child is obviously a successor and a right caret.
- A left caret has the following children:
    - Its only predecessor is the left child, a left caret.
    - All the other children are successors, all middle carets, and of types $\mathcal{M}^1$, $\mathcal{M}^2$, ..., $\mathcal{M}^{p-1}$, in order.
- A right caret has the following children:
    - One single predecessor of type $\mathcal{M}^{p-1}$.
    - It has $p - 1$ successors, which in order are of types $\mathcal{M}^1$, $\mathcal{M}^2$, ..., $\mathcal{M}^{p-2}$ and the last one of type $\mathcal{R}$.
- A caret of type $\mathcal{M}^i$ ($1 \leq i \leq p - 1$) has the following children:
    - The first $p - i$ children are predecessors, and their types are $\mathcal{M}^i$, ..., $\mathcal{M}^{p-1}$.
    - The other i children are successors, and they are of types $\mathcal{M}^1$, $\mathcal{M}^2$, ..., $\mathcal{M}^i$.

For the purposes of computing the length of an element, these caret types are subdivided in further types depending on the existence of predecessor and successor types. This classification is actually more complicated in Fordham's paper but we do not need the total strength of the method since we are dealing only with positive words. We will indicate also which is the weight of each caret for the purposes of the computation of the length of a positive word.

The caret types are as follows:

- The root, which has always weight zero.
- Left carets, which have always weight one.
- Carets of type $\mathcal{R}_\varnothing$ are right carets whose all successors are right carets, i.e., it has no middle successors. Its only successors hang from its rightmost leaf. These carets carry weight zero.
- Carets of type $\mathcal{R}_M$ are right carets which are not $\mathcal{R}_\varnothing$, that is, which have middle successors. Observe that the middle successors do not have to be immediate successors, they can be successors of successors. Carets of type $\mathcal{R}_M$ have weight two.
- Carets of type $\mathcal{M}^i_\varnothing$ are middle carets which do not have any successor children. They carry weight one.
- Carets of type $\mathcal{M}^i_M$ are middle carets which have at least a successor child. These carets have weight three.

Observe that the index on the middle carets is only necessary to identify its successors, but it has no role in the weight assignment beyond that one.

Now, the main theorem giving the length is as follows:

Theorem 2.1. (S. B. Fordham) [13] *Given a positive word in $F(p)$ represented by a rooted p-tree, the distance from this element to the identity (in the word metric for $F(p)$ with generators x_0, x_1, ..., x_{p-1}) is equal to the total sum of the weights of its carets.*

3. Generating functions for the growth of positive words

Once the theorem for the length has been established, now the computation of the growth function is reduced to a combinatorial problem, namely, finding how many trees have a given weight, according to the rules above. The method for finding the number of trees with a given weight is to split the trees in several ones in such a way that recurrences can be found. The reader can see details about generating functions in [23], and can see this method used already in [7].

We will make use of several sequences:

- The sequence $s_n = \#(F_+(p) \cap \mathbf{S}(n))$. This is the number of trees which have weight n.
- The sequence l_n. This sequence gives the number of subtrees which can be left subtrees of a rooted p-tree and such that its total weight is n. The subtrees are required to be strict, that is, the main tree does not qualify as a left subtree.
- Analogously the sequence r_n is the sequence of possible right subtrees of weight n.
- The sequence $m_n^{(i)}$ for $i = 1, \ldots, p-1$, gives the number of interior subtrees which start with a caret of type $\mathcal{M}^i$. Observe that this

subtree is completely composed of middle carets, and also with total weight n.

Observe that the subtrees are always considered as subtrees of the main tree, which means that, for instance, a left subtree never has carets of type $\mathcal{R}$ because that would mean it is the total tree. A subtree which starts in an $\mathcal{M}^i$ caret has all interior carets.

Each one of these sequences will have its generating function:

$$S(x) = \sum_{n=0}^{\infty} s_n x^n \qquad L(x) = \sum_{n=0}^{\infty} l_n x^n$$

$$R(x) = \sum_{n=0}^{\infty} r_n x^n \qquad M_i(x) = \sum_{n=0}^{\infty} m_n^{(i)} x^n.$$

Now we will establish relations between the sequences which will give functional equations for their generating functions, which then will allow us to find the growth of the submonoid of positive words. For instance, if one considers the tree representing a word, and assumes the tree has total weight n, since the root has weight zero, the weight has to be distributed among all the p children subtrees. Hence, a tree of total weight n will be obtained every time that we take a family of subtrees such that the sum of their separate weights as subtrees is n.

This fact gives the first formula satisfied by the sequences, and also by the generating functions:

$$(2) \qquad\qquad s_n = \sum_{j_0 + \cdots + j_{p-1} = n} l_{j_0} m_{j_1}^{(1)} \cdots m_{j_{p-2}}^{(p-2)} r_{j_{p-1}}$$

$$(3) \qquad\qquad S = LM_1 \cdots M_{p-2} R.$$

To find a formula for the function $L(x)$ of left subtrees, one needs to consider that left carets have weight 1. Hence the different subtrees only have to add up to $n-1$. The formula is

$$l_n = \sum_{j_0 + \cdots + j_{p-1} = n-1} l_{j_0} m_{j_1}^{(1)} \cdots m_{j_{p-2}}^{(p-2)} m_{j_{p-1}}^{(p-1)}$$

$$(4) \qquad\qquad L - 1 = xLM_1 M_2 \cdots M_{p-1}.$$

The formula for the generating functions is obtained by multiplying each side of the formula for sequences by x^n. The right hand side has an x multiplying because the indices are shifted by one.

For the function for right trees, one has to take into account the fact that a right caret can be of type $\mathcal{R}_\varnothing$ or $\mathcal{R}_M$, with weights zero and two respectively. For the first possibility, the caret is of type $\mathcal{R}_\varnothing$, and all its

successors have no weight. Observe that in a positive word there can be one and only one caret of type $\mathcal{R}_\varnothing$, because any others would be reducible. Hence, if the caret is of type $\mathcal{R}_\varnothing$, all the weight is concentrated in its only predecessor. So there are as many right subtrees of this type as trees of the type $\mathcal{M}^{p-1}$ with the same weight, which gives the first part of the recurrence equal to $m_n^{(p-1)}$.

If the right caret is of type $\mathcal{R}_M$, it carries weight 2 and one the successors is necessarily nonempty with a middle caret somewhere. Hence if one of the successors is necessarily nonempty, the term in the recurrence has all possible weights for these successors. The formula is

$$r_n = m_n^{(p-1)} + \sum_{\substack{j_0+\cdots+j_{p-1}=n-2 \\ j_1+\cdots+j_{p-1}\geq 1}} m_{j_0}^{(p-1)} m_{j_1}^{(1)} \cdots m_{j_{p-2}}^{(p-2)} r_{j_{p-1}}$$

(5) $$R = M_{p-1} + x^2(M_1 M_2 \cdots M_{p-1} R - M_{p-1}).$$

Finally, the middle subtrees are the ones whose children are also middle subtrees and hence facilitate the resolution of the equations. A middle caret of type $\mathcal{M}^i$ has either weight 1 if its successors are empty or weight 3 if one of the successor subtrees is nonempty. Both cases correspond to the two adding terms of the formula for the sequence:
$m_n^{(i)} =$

$$= \sum_{j_i+\cdots+j_{p-1}=n-1} m_{j_i}^{(i)} \cdots m_{j_{p-1}}^{(p-1)} + \sum_{\substack{j_0+j_1+\cdots+j_{p-1}=n-3 \\ j_{p-i}+\cdots+j_{p-1}\geq 1}} m_{j_0}^{(i)} \cdots m_{j_{p-i-1}}^{(p-1)} m_{j_{p-i}}^{(1)} \cdots m_{j_{p-1}}^{(i)}$$

which gives the following formula for the generating functions:

(6) $M_i - 1 = x M_i M_{i+1} \cdots M_{p-1} + x^3 M_i M_{i+1} \cdots M_{p-1}(M_1 \cdots M_{i-1} M_i - 1).$

Solving these equations will give us information on the function $S(x)$, which is the one we are interested in, and the growth of positive elements in the groups $F(p)$.

4. GROWTH FUNCTIONS AND GROWTH RATES OF $F_+(p)$

Now we collect formulas (3), (4), (5), (6) to find the equation on $S(x)$ and the radius of convergence of the corresponding series. First of all, we have to mention that $F_+(p)$ has a free submonoid generated by x_0, x_1, ..., x_{p-1} and so the growth rate of $F_+(p)$ is at least p. As we will see at the end of this Section, the exact value of the growth rate is only slightly larger than p. (In fact, it is always less than $p + 1/2$.)

Let

$$M(x) = M_1(x) M_2(x) \cdots M_{p-1}(x).$$

Lemma 4.1. *For all $0 \leq i \leq p-1$, we have*

$$M_1 M_2 \cdots M_i = \frac{x^{-2}}{(1 - x^3 M)^i} + 1 - x^2.$$

Proof. We proceed by induction on i. If $i = 0$, then the result is obvious. Let $1 \leq i \leq p-1$. Formula (6) can be written as

$$M_i = 1 + \frac{xM}{M_1 \cdots M_{i-1}} + x^3 \left(M_i M - \frac{M}{M_1 \cdots M_{i-1}} \right).$$

Therefore,

$$M_1 \cdots M_i = M_1 \cdots M_{i-1} + xM + x^3 M \cdot M_1 \cdots M_i - x^3 M$$

and so

$$M_1 \cdots M_i (1 - x^3 M) = (x - x^3) M + M_1 \cdots M_{i-1}.$$

Using the inductive assumption, we have

$$
\begin{aligned}
M_1 \cdots M_i (1 - x^3 M) &= (x - x^3) M + \frac{x^{-2}}{(1 - x^3 M)^{i-1}} + 1 - x^2 = \\
&= \frac{x^{-2}}{(1 - x^3 M)^{i-1}} + (1 - x^2)(1 - x^3 M).
\end{aligned}
$$

Now the only thing left to do is to divide by $1 - x^3 M$. $\qquad\square$

Taking $i = p - 1$ gives us

Corollary 4.2. *The function $M = M(x)$ satisfies*

$$x^2 M = \frac{1}{(1 - x^3 M)^{p-1}} + x^2 - 1.$$

Now we express $S(x)$ in terms of $M(x)$. It follows from (4) and (5) that

$$L = \frac{1}{1 - xM} \qquad R = \frac{(1 - x^2) M_{p-1}}{1 - x^2 M}.$$

Now, using (3), we have

(7) $$S = \frac{LMR}{M_{p-1}} = \frac{(1 - x^2) M}{(1 - xM)(1 - x^2 M)}.$$

The first author proved in [7] that the growth function $S(x)$ of positive elements of $F = F(2)$ is rational (although $M(x)$ is irrational). Now we have the following

Theorem 4.3. *The growth function $S(x)$ of positive elements in $F(p)$ is irrational provided $p \geq 3$.*

Proof. Let $N = (1 - x^3 M)^{-1}$. From Corollary 4.2 we have

$$1 - N^{-1} = x^3 M = xN^{p-1} + x^3 - x.$$

Hence $N = N(x)$ satisfies the equation

(8) $$xN^p + (x^3 - x - 1)N + 1 = 0.$$

Suppose that $S(x)$ is rational. Then it follows from (7) that $M(x)$ satisfies a quadratic equation with coefficients in the field $\mathbb{Q}(x)$ of rational functions. Since $M = x^{-3}(1 - N^{-1})$, the function $N(x)$ also satisfies an equation of degree at most 2 over $\mathbb{Q}(x)$. This implies that the polynomial $f(t) = xt^p + (x^3 - x - 1)t + 1$ from $\mathbb{Q}(x)[t]$ is divisible by a polynomial of degree at most 2. Since $p \geq 3$, the polynomial $f(t)$ is reducible over $\mathbb{Q}(x)$. A standard algebraic trick (using Gauss' lemma) implies that $f(t)$ is a product of two polynomials from $\mathbb{Z}[x][t]$ of degree less than p. Taking $x = 1$, we obtain that the polynomial $t^p - t + 1$ is reducible over $\mathbb{Q}$. However, this contradicts a result from [20]. $\qquad\square$

Now we will find the growth rate of $F_+(p)$. To do that, we need to take the radius of convergence of the series for $S(x)$ and take the reciprocal. Observe that from (2) we deduce $m_n \leq s_n$ for all $n \geq 0$. This implies that

$$(\limsup_{n\to\infty} m_n)^{-1} \geq (\limsup_{n\to\infty} s_n)^{-1},$$

that is, the radius of convergence of the series $S(x)$ does not exceed the one for the series $M(x)$. Let $x > 0$ be a real number such that $S(x)$ converges. Then $M(x)$ also converges and formula (7) holds.

To find the radius of convergence of $S(x)$, we need to find the smallest positive real number such that the denominator of the right hand side of (7) is zero. Since $M(x)$ is increasing and $0 < x < 1$, the smallest positive solution of the equation $M(x) = x^{-1}$ will not exceed the smallest positive solution of the equation $M(x) = x^{-2}$. Therefore, we need to solve the equation $M(x) = x^{-1}$. Notice that $M(x)$ increases and x^{-1} decreases so we can just speak about a positive root of this equation. Using (4.2), we get $x = (1 - x^2)^{-(p-1)} + x^2 - 1$, that is, we need to find the positive root of

(9) $$(1 - x^2)^{p-1}(1 + x - x^2) = 1.$$

The growth rate of $F_+(p)$ will thus be equal to x^{-1}. We already know that the growth rate of $F_+(p)$ is at least p, as it was mentioned in the beginning of this Section. Hence $x \leq 1/p$.

Let us rewrite this equation in the following form:

$$p - 1 = \frac{\ln(1 + x - x^2)}{-\ln(1 - x^2)}.$$

From the Taylor formula for $\ln(1 + y)$, we deduce the inequality

$$y - y^2/2 < \ln(1 + y) < y - y^2/2 + y^3/3,$$

where $y > 0$, and then we get $\ln(1 + x - x^2) < x - 3x^2/2 + 4x^3/3 - 3x^4/2 + x^5 - x^6/3 < x - 3x^2/2 + 4x^3/3$. Since $-\ln(1 - x^2) > x^2$, we have $p - 1 < x^{-1} - 3/2 + 4x/3 \leq x^{-1} - 3/2 + 4/3p$. So $x^{-1} > p + 1/2 - 4/3p = p + 1/2 + o(1)$ as $p \to \infty$.

Now we want to show that $x^{-1} < p + 1/2$. We have $\ln(1 + x - x^2) > x - x^2 - (x - x^2)^2/2 = x - 3x^2/2 + x^3 - x^4/2$ and $-\ln(1 - x^2) = x^2 + x^4/2 + x^6/3 + \cdots < x^2 + x^4(1 + x^2 + x^4 + \cdots)/2 = x^2 + x^4/(2 - 2x^2) \leq x^2 + 2x^4/3$ because $x \leq 1/p \leq 1/2$. This gives $p - 1 > (x - 3x^2/2 + x^3 - x^4/2)/(x^2 + 2x^4/3) = (1 - 3x/2 + x^2 - x^3/2)/(x + 2x^3/3)$. Finally,

$$p - 1 > \frac{1 - 3x/2 + x^2 - x^3/2}{x + 2x^3/3} = \frac{1}{x} - \frac{9 - 2x + 3x^3}{2(3 + 2x^2)} > 1/x - 3/2$$

since $3x^2 - 6x - 2 < 0$ on $[0; 1]$. This gives $x^{-1} < p + 1/2$, as desired. So we get the following result.

Theorem 4.4. *The growth rate of the monoid $F_+(p)$ of positive elements in the group $F(p)$ generated by x_0, x_1, ..., x_{p-1} is a number ζ_p, which is the root of equation*

$$(y^2 - 1)^{p-1}(y^2 + y - 1) = y^{2p}.$$

This number has the form $\zeta_p = p + \lambda_p$, where $0 < \lambda_p < 1/2$ for all p and $\lambda_p \to 1/2$ as $p \to \infty$.

Indeed, we proved inequalities $p + 1/2 - 4/3p < x^{-1} < p + 1/2$, where x is the solution of (9). The inequality $x^{-1} > p$ obviously follows for $p \geq 3$; if $p = 2$, then it is known from [7] that $\zeta_2 > 2.24$.

The equation in the statement of Theorem 4.4 is equivalent to (9) via the substitution $y = 1/x$. Notice that x and y are roots of polynomials of degree $2p - 1$ with integer coefficients. Also let us mention without proof that λ_p is strictly increasing with respect to p.

The number ζ_p gives a lower bound for the growth rate of the group $F(p)$. However, this estimate can be essentially improved.

5. New normal forms for elements of $F(p)$

We are going to find two new normal forms for elements of $F(p)$. They will be analogs of the normal forms constructed in [18] for the case $F = F(2)$.

The first of these normal forms will involve the infinite set of generators $\Sigma = \{ x_i \ (i \geq 0) \}$. Consider the following rewriting system $\Gamma = \Gamma(p)$ over the alphabet $\Sigma^{\pm 1} = \Sigma \cup \Sigma^{-1}$ (basic facts about rewriting systems can be found in [2, 10]):

(1) $x_i^{\varepsilon} x_i^{-\varepsilon} \to 1$ $(i \geq 0, \, \varepsilon = \pm 1)$

(2) $x_j^{\varepsilon} x_i \to x_i x_{j+p-1}^{\varepsilon}$ $(j > i, \, \varepsilon = \pm 1)$

(3) $x_{j+p-1}^{\varepsilon} x_i^{-1} \to x_i^{-1} x_j^{\varepsilon}$ $(j > i, \, \varepsilon = \pm 1)$

Notice that for every rewriting rule of Γ, the left hand side and the right hand side are equal in $F(p)$.

It is easy to see that Γ is *terminating*, that is, for every word w, the process of applying rewriting rules to w always terminates. Indeed, Γ either decreases the length of a word or it preserves the length. In the second case, if we make a vector that consists of subscripts of a word, the rewriting rules will decrease this vector lexicographically.

Since Γ is terminating, applying the rewriting rules to a word w gives us a word v that cannot be reduced (that is, no more rewriting rules can be applied to v). We say that v is an *irreducible form* of w. Now we are going to check that Γ is also *confluent*, that is, every word has a unique irreducible form. To do that, we apply the Diamond Lemma. In our case, this means that if we have rewriting rules of the form $ab \to u$, $bc \to v$, where a, b, c, d are letters and u, v are words, then uc and av have a common descendant. There are only finitely many cases to check, and all of them are easy. We will show one of these cases, the rest is left to the reader.

Let us take the rewriting rules $x_{k+p-1}^{\varepsilon} x_j^{-1} \to x_j^{-1} x_k^{\varepsilon}$ and $x_j^{-1} x_i \to x_i x_{j+p-1}^{-1}$, where $k > j > i$, $\varepsilon = \pm 1$. We have:

$$x_j^{-1} x_k^{\varepsilon} x_i \to x_j^{-1} x_i x_{k+p-1}^{\varepsilon} \to x_i x_{j+p-1}^{-1} x_{k+p-1}^{\varepsilon}$$

and

$$x_{k+p-1}^{\varepsilon} x_i x_{j+p-1}^{-1} \to x_i x_{k+2p-2}^{\varepsilon} x_{j-p-1}^{-1} \to x_i x_{j+p-1}^{-1} x_{k+p-1}^{\varepsilon}.$$

So the words have a common descendant.

Now we know that Γ is *complete*, that is, terminating and confluent. Therefore, each element of $F(p)$ can be uniquely represented by an irreducible word. So we have proved the following

Theorem 5.1. *Each element $g \in F(p)$ can be uniquely represented as a word of the form*

$$N(g) = x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \cdots x_{i_m}^{\varepsilon_m},$$

where $m \geq 0$, $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m = \pm 1$, and for every $1 \leq k < m$ one of the following conditions holds:

- $i_k < i_{k+1}$
- $i_k = i_{k+1}$ *and* $\varepsilon_k = \varepsilon_{k+1}$
- $0 < i_k - i_{k+1} < p$ *and* $\varepsilon_{k+1} = -1$.

Indeed, the conditions listed in the statement exactly mean that the word $N(g)$ is irreducible, that is, it has no subwords that are left hand sides of

the rewriting rules of Γ. The set of these irreducible words over $\Sigma^{\pm 1}$ will be denoted by $\mathcal{N}_{\text{inf}}$.

Notice that the set $\mathcal{N}_{\text{inf}}$ has the following property: a word belongs to $\mathcal{N}_{\text{inf}}$ if and only if all its subwords of length 2 belong to $\mathcal{N}_{\text{inf}}$. That is, the normal form of Theorem 5.1 is locally testable.

Now we will construct another normal form for elements of $F(p)$. Now all words will involve only the finite set of generators $x_0^{\pm 1}$, $x_1^{\pm 1}$, ..., $x_{p-1}^{\pm 1}$. Moreover, these normal forms will give a regular language closed under taking subwords. Notice that this gives a regular spanning tree in the Cayley graph of $F(p)$ in the above generators. As in [18] for the case $p = 2$, this tree is not geodesic.

It is possible to write down a new rewriting system in order to get the normal form we wish to construct. However, it will take too much effort to prove that the rewriting system ijs complete. We choose an approach that differs from [18].

Let $j \geq 1$. Then j can be uniquely expressed in the form $j = r + d(p-1)$, where $1 \leq r \leq p - 1$, $d \geq 0$. In this case x_j equals in $F(p)$ to the word $x_0^{-d} x_r x_0^d$. For any word w over $\Sigma^{\pm 1}$, replace each letter of the form x_j^ε ($j \geq 1$, $\varepsilon = \pm 1$) by $x_0^{-d} x_r x_0^d$, where $j = r + d(p-1)$, $1 \leq r \leq p - 1$, $d \geq 0$ and then freely reduce all subwords of the form $x_0^\varepsilon x_0^{-\varepsilon}$ ($\varepsilon = \pm 1$). We obtain a word in generators $x_0^{\pm 1}$, $x_1^{\pm 1}$, ..., $x_{p-1}^{\pm 1}$ denoted by $\bar{w}$.

Lemma 5.2. *If $w \in \mathcal{N}_{\text{inf}}$, then $\bar{w}$ has no subwords of the following form:*

(1) $x_i^\varepsilon x_i^{-\varepsilon}$ $(0 \leq i \leq r - 1)$
(2) $x_\alpha^\varepsilon x_0^k x_\beta$ $(k \geq 0,\ 1 \leq \beta < \alpha \leq r - 1)$
(3) $x_\alpha^\varepsilon x_0^{k+1} x_\beta^{-1}$ $(k \geq 0,\ 1 \leq \beta < \alpha \leq r - 1)$
(4) $x_\alpha^\varepsilon x_0^{k+1} x_\beta$ $(k \geq 0,\ 1 \leq \alpha \leq \beta \leq r - 1)$
(5) $x_\alpha^\varepsilon x_0^{k+2} x_\beta^{-1}$ $(k \geq 0,\ 1 \leq \alpha \leq \beta \leq r - 1)$

The words of the form 1) – 5) are called *forbidden subwords*. The set of words in $\{ x_0^{\pm 1}, x_1^{\pm 1}, \ldots, x_{p-1}^{\pm 1} \}$ without forbidden subwords will be denoted by $\mathcal{N}_{\text{fin}}$.

Proof. Let $w \in \mathcal{N}_{\text{inf}}$ have the form

$$(10) \qquad w = x_0^{k_0} x_{\alpha_1}^{l_1} x_0^{k_1} x_{\alpha_2}^{l_2} \cdots x_0^{k_{h-1}} x_{\alpha_h}^{l_h} x_0^{k_h},$$

where $h \geq 0$, $\alpha_i = r_i + d_i(p-1)$, $1 \leq r_i \leq p - 1$, $d_i \geq 0$, $l_i \neq 0$ for all $1 \leq i \leq h$. By definition,

$$(11) \qquad \bar{w} = x_0^{k_0 - d_1} x_{r_1}^{l_1} x_0^{d_1 + k_1 - d_2} x_{r_2}^{l_2} \cdots x_0^{d_{h-1} + k_{h-1} - d_h} x_{r_h}^{l_h} x_0^{d_h + k_h}.$$

Suppose that $\bar{w}$ is not freely irreducible. Then there exist an i from 1 to $h - 1$ such that $d_i + k_i - d_{i+1} = 0$, $r_i = r_{i+1}$, and and $l_i l_{i+1} < 0$. By

definition, words from $\mathcal{N}_{\mathrm{inf}}$ have no subwords of the form $x_j^{\pm 1} x_0$ for $j > 0$ and also have no subwords of the form $x_j^{\pm 1} x_0^{-1}$ for $j \geq p$. This implies $k_i \leq 0$.

Suppose that $k_i < 0$. Then $d_i = 0$ (otherwise $\alpha_i \geq p$). Since $d_{i+1} \geq 0$, we obtain $d_i + k_i - d_{i+1} = k_i - d_{i+1} < 0$. This is a contradiction. Therefore, $k_i = 0$ and so $d_i = d_{i+1}$. This implies $\alpha_i = r_i + d_i(p-1) = r_{i+1} + d_{i+1}(p-1) = \alpha_{i+1}$. Thus the word w is not freely irreducible since $l_i l_{i+1} < 0$. We have a contradiction. This proves that $\bar{w}$ has no subwords of the form 1).

Suppose that $\bar{w}$ has a subword of one of the forms 2) – 5). Let $x_{r_i}^{\pm 1} x_0^{d_i + k_i - d_{i+1}} x_{r_{i+1}}^{\pm 1}$ be such a subword, where $1 \leq i < h$. As above, $k_i \leq 0$. Suppose that $k_i \neq 0$. This implies $d_i = 0$ and $d_i + k_i - d_{i+1} < 0$. But none of the words 2) – 5) can contain x_0^{-1}. This allows us to conclude that $k_i = 0$ and $\bar{w}$ contains $v = x_{r_i}^{\pm 1} x_0^{d_i - d_{i+1}} x_{r_{i+1}}^{\pm 1}$ as a subword.

Suppose that v satisfies condition 2). This means that $d_i \geq d_{i+1}$, $r_i > r_{i+1}$, $l_{i+1} > 0$. Hence w contains $x_{\alpha_i}^{\pm 1} x_{\alpha_{i+1}}$, where $\alpha_i = r_i + d_i(p-1) > r_{i+1} + d_{i+1}(p-1)$. So w does not belong to $\mathcal{N}_{\mathrm{inf}}$, which is impossible.

Suppose that v satisfies condition 3). Now $d_i - d_{i+1} \geq 1$, $r_i > r_{i+1}$, $l_{i+1} < 0$. This leads to $\alpha_i - \alpha_{i+1} = (r_i - r_{i+1}) + (d_i - d_{i+1})(p-1) \geq p$, which also contradicts $w \in \mathcal{N}_{\mathrm{inf}}$.

Suppose that v satisfies condition 4). Then $d_i - d_{i+1} \geq 1$, $r_i \leq r_{i+1}$, $l_{i+1} > 0$. Now $r_i - r_{i+1} \geq 1 - (p-1) = 2 - p$ and so $\alpha_i - \alpha_{i+1} = (r_i - r_{i+1}) + (d_i - d_{i+1})(p-1) \geq (p-1) + (2-p) > 0$. Thus w contains $x_{\alpha_i}^{\pm 1} x_{\alpha_{i+1}}$ with $\alpha_i > \alpha_{i+1}$. This cannot happen by definition of $\mathcal{N}_{\mathrm{inf}}$.

Finally, suppose that v satisfies condition 5). Now we have $d_i - d_{i+1} \geq 2$ and so $\alpha_i - \alpha_{i+1} = (r_i - r_{i+1}) + (d_i - d_{i+1})(p-1) \geq 2(p-1) + (2-p) = p$. However, it should be $\alpha_i - \alpha_{i+1} < p$ because $w \in \mathcal{N}_{\mathrm{inf}}$.

The proof is complete. $\square$

For every $g \in F(p)$, we have the word $\overline{N(g)} \in \mathcal{N}_{\mathrm{fin}}$ that represents g. We will prove that g is represented uniquely by a word from $\mathcal{N}_{\mathrm{fin}}$. This will follow from

Lemma 5.3. *The mapping* $w \mapsto \bar{w}$ *from* $\mathcal{N}_{\mathrm{inf}}$ *to* $\mathcal{N}_{\mathrm{fin}}$ *is a bijection.*

Proof. We prove first that the mapping $w \mapsto \bar{w}$ from $\mathcal{N}_{\mathrm{inf}}$ to $\mathcal{N}_{\mathrm{fin}}$ is injective. As above, let $w \in \mathcal{N}_{\mathrm{inf}}$ have the form (10). Thus $\bar{w}$ equals (11). Suppose that we know the word $\bar{w}$, that is, we know the numbers $m_0 = k_0 - d_1$, $m_1 = d_1 + k_1 - d_2, \ldots, m_{h-1} = d_{h-1} + k_{h-1} - d_h$, $m_h = k_h + d_h$. Our aim is to recover the numbers $k_0, d_1, k_1, \ldots, d_{h-1}, k_{h-1}, d_h, k_h$.

Let $h \geq 1$. It follows from the definition of $\mathcal{N}_{\mathrm{inf}}$ that $k_h \leq 0$. Moreover, either $k_h < 0$ and $d_h = 0$, or $k_h = 0$. In the first case $m_h = k_h + d_h = k_h < 0$, in the second case $m_h = k_h + d_h = d_h \geq 0$. Since we know m_h, we can

distinguish between these two cases. Namely, if $m_h < 0$, then $d_h = 0$, $k_h = m_h$. If $m_h \geq 0$, then $k_h = 0$, $d_h = m_h$. Now we know d_h and k_h.

If $h \geq 2$, then $k_{h-1} \leq 0$. As above, we have one of the two cases: $k_{h-1} < 0$, $d_{h-1} = 0$, or $k_{h-1} = 0$. The number $k_{h-1} + d_{h-1}$ is negative in the first case and nonnegative in the second case. But this number equals $m_{h-1} + d_h$, so we know it and thus we are able to distinguish these cases. In the first case we have $d_{h-1} = 0$, $k_{h-1} = m_{h-1} + d_h$; in the second case — $k_{h-1} = 0$, $d_{h-1} = m_{h-1} + d_h$. Therefore, we know d_{h-1} and k_{h-1}.

Continuing in this way, we get the values of d_{h-2}, k_{h-2}, ..., d_1, k_1. At the final step we get $k_0 = m_0 + d_1$.

Now we show that the mapping is surjective. We start with a word from $\mathcal{N}_{\mathrm{fin}}$. This word has the form

$$(12) \qquad x_0^{m_0} x_{r_1}^{l_1} x_0^{m_1} x_{r_2}^{l_2} \cdots x_0^{m_{h-1}} x_{r_h}^{l_h} x_0^{m_h}.$$

Using the rules described in the first part of the proof, we define the numbers k_0, d_1, k_1, ..., d_h, k_h. It follows that $d_i \geq 0$ for all i from 1 to h. So we can form a word w as in (10), where $\alpha_i = r_i + d_i(p-1)$ $(1 \leq i \leq h)$. It is obvious that $\bar{w}$ equals the word (12). It remains to prove that w belongs to $\mathcal{N}_{\mathrm{inf}}$.

Let us assume the contrary. Since w has no subwords of the form $x_0^\varepsilon x_0^{-\varepsilon}$, it should contain one of the following subwords:

a) $x_i^\varepsilon x_i^{-\varepsilon}$ $(i \geq 1, \varepsilon = \pm 1)$;

b) $x_j^{\pm 1} x_i$ $(j > i)$;

c) $x_{j+p-1}^{\pm 1} x_i^{-1}$ $(j > i)$.

In case a), $\bar{w}$ will contain a forbidden subword of the form $x_r^\varepsilon x_r^{-\varepsilon}$. Notice that $k_i \leq 0$ for all $1 \leq i \leq h$; if $k_i < 0$, then $d_i = 0$. This means that in cases b) and c) one has $i \geq 1$. Let $j = \alpha + d(p-1)$, $i = \beta + d'(p-1)$, where $1 \leq \alpha, \beta \leq r-1$, $d, d' \geq 0$. Applying the "bar" mapping to b) and c), we see that the word $\bar{w}$ contains $u = x_\alpha^{\pm 1} x_0^{d-d'} x_\beta$ in case b) and $v = x_\alpha^{\pm 1} x_0^{d-d'+1} x_\beta^{-1}$ in case c). Since $j > i$, we have $(d - d')(p-1) > \beta - \alpha > -(p-1)$. Hence $d - d' \geq 0$. If u is not forbidden, then $\alpha \leq \beta$. But in this case $d - d' > 0$ so u has to be forbidden anyway. If v is not forbidden, then we also have $\alpha \leq \beta$, which implies $d - d' + 1 \geq 2$. We have a final contradiction.

The proof is complete. □

From Lemmas 5.2 and 5.3 we obtain

Theorem 5.4. *Each element $g \in F(p)$ can be uniquely represented by a word $w \in \mathcal{N}_{\mathrm{fin}}$. This means that for every $g \in F(p)$ there is exactly one word over $\{ x_0^{\pm 1}, x_1^{\pm 1}, \ldots, x_{p-1}^{-1} \}$ that represents g and has no forbidden subwords. This gives a regular set of normal forms for the group $F(p)$.*

Indeed, the set of forbidden subwords is a regular language. So the set $\mathcal{N}_{\text{fin}}$ of words that do not contain forbidden subwords will be also regular. Throughout the rest of the paper, we will denote this language by $\mathcal{L}_p$. (For basic properties of regular languages see [21].)

6. Lower bounds for the growth rates of $F(p)$

A lower bound of $(3 + \sqrt{5})/2 = 2.618\ldots$ for the growth rate of $F = F(2)$ was obtained by the second author in [15]. Now we will find a similar lower bound for each $F(p)$. In the previous section, we constructed a regular language $\mathcal{L}_p$ of normal forms for $F(p)$. Each word of length n in $\mathcal{L}_p$ is at a distance at most n from the identity in the Cayley graph of $F(p)$. So the growth function of $\mathcal{L}_p$ does not exceed the number of elements in the ball of radius n for $F(p)$. Then, finding the growth function and the growth rate of $\mathcal{L}_p$, we find a lower bound for the growth rate of the group $F(p)$.

An automaton to recognize the language $\mathcal{L}_p$ has $3p + 1$ states. However, it is easier to construct a directed graph with only $2p + 1$ vertices (states). This graph will be denoted by $\mathcal{A}_p$ and we will also call it an automaton although its edges have no labels. The description of $\mathcal{A}_p$ is as follows.

The vertices (states) of $\mathcal{A}_p$ are denoted by $q, q_0, q_1, \ldots, q_{p-1}, q_{1,0}, q_{2,0}, \ldots, q_{p-1,0}, \bar{q}$. They will correspond to the following partition of $\mathcal{L}_p$ into disjoint subsets:

- The set $\{1\}$ that consists of the empty word (state q).
- The set of words that end with $x_0^{\pm 1}$ and do not have a terminal segment of the form $x_i^{\pm 1} x_0^k$, where $1 \leq i \leq p - 1$, $k \geq 1$ (state q_0).
- The set of words that end with $x_i^{\pm 1}$ (state q_i for each $1 \leq i \leq p-1$).
- The set of words that end with $x_i^{\pm 1} x_0$ (state $q_{i,0}$ for each $1 \leq i \leq p - 1$).
- The set of words that end with $x_i^{\pm 1} x_0^k$ for some $1 \leq i \leq p - 1$ and $k \geq 2$ (state $\bar{q}$).

Let $w \in \mathcal{L}_p$. If w is empty, then $wx_i^{\pm 1}$ will be in $\mathcal{L}_p$ for all $0 \leq i \leq p-1$. We draw two arrows from q to q_i for each $0 \leq i \leq p - 1$.

Let w correspond to the state q_0. Then $w = vx_0^\varepsilon$ for some word v and for some $\varepsilon = \pm 1$. The word wx_0^ε will be in $\mathcal{L}_p$; for each $1 \leq i \leq p - 1$ the word $wx_i^{\pm 1}$ will be also in $\mathcal{L}_p$ since w has no terminal segments of the form $x_i^{\pm 1} x_0^k$ ($1 \leq i \leq p - 1$, $k \geq 1$. Thus we draw an arrow from q_0 to itself and two arrows from q_0 to each q_i ($1 \leq i \leq p - 1$).

Let w correspond to q_i ($1 \leq i \leq p-1$). The words wx_0^{-1} and wx_0 belong to $\mathcal{L}_p$; we draw an arrow from q_i to q_0 and an arrow from q_i to $q_{i,0}$. The words wx_j belong to $\mathcal{L}_p$ whenever $i < j \leq p - 1$; the words wx_j^{-1} belong to

$\mathcal{L}_p$ for all $1 \le j \le p-1$. So we draw one arrow from q_i to each $q_1, \ldots, q_i$ and two arrows from q_i to each $q_{i+1}, \ldots, q_{p-1}$.

Let w correspond to $q_{i,0}$ $(1 \le i \le p-1)$. The word wx_0 is in $\mathcal{L}_p$ and we draw an arrow from $q_{i,0}$ to $\bar{q}$. Also $wx_j^{-1} \in \mathcal{L}_p$ whenever $i \le j \le p-1$. So one arrow goes from $q_{i,0}$ to each $q_i, \ldots, q_{p-1}$. No other arrows can appear.

Finally, let w correspond to $\bar{q}$. Now only wx_0 leads to a word in $\mathcal{L}_p$; it corresponds to an arrow from $\bar{q}$ to itself.

The description of $\mathcal{A}_p$ is complete. Notice that the number of words in $\mathcal{L}_p$ of length n is exactly the number of (directed) paths of length n in $\mathcal{A}_p$ starting at q. We would like to compute the number of paths in $\mathcal{A}_p$ of length n starting at q and ending at a given state. For each state we consider the corresponding generating function. Namely, to each vertex v we assign a series of the form $\sum_{n=0}^{\infty} a_n t^n$, where a_n is the number of paths in $\mathcal{A}_p$ starting at q and ending at v. These generating functions will be denoted by f, f_i $(0 \le i \le p-1)$, $f_{i,0}$ $(1 \le i \le p-1)$, $\bar{f}$ for each of the states, respectively. We will write down a system of equations for these functions.

First of all, it is clear that $f(t) = 1$. To find f_0, we mention that two arrows go from q into q_0 and one arrow goes into q_0 from each of the states $q_0, q_1, \ldots, q_{p-1}$. Hence

$$(13) \qquad\qquad f_0 = t(2f + f_0 + f_1 + \cdots + f_{p-1}).$$

Given a vertex q_i $(1 \le i \le p-1)$, we observe that two arrows go into q_i from $q, q_0, \ldots, q_{i-1}$ and one arrow from $q_i, \ldots, q_{p-1}$. Also one arrow goes into q_i from each $q_{1,0}, \ldots, q_{i,0}$. Thus

$$(14) \quad f_i = t(2f + 2f_0 + \cdots + 2f_{i-1} + f_i + \cdots + f_{p-1}) + t(f_{1,0} + \cdots + f_{i,0}).$$

Notice that $f_{i,0} = tf_i$ for each $1 \le i \le p-1$ because only one arrow goes into $q_{i,0}$ (from the state q_i). Thus we can rewrite (14) as follows:

$$(15) \quad f_i = t(2f + 2f_0 + \cdots + 2f_{i-1} + f_i + \cdots + f_{p-1}) + t^2(f_1 + \cdots + f_i).$$

Finally, there is one arrow that goes into $\bar{q}$ from each of the states $q_{1,0}, \ldots, q_{p-1,0}, \bar{q}$. So

$$(16) \qquad \bar{f} = t(f_{1,0} + \cdots + f_{p-1,0} + \bar{f}) = t^2(f_1 + \cdots + f_{p-1}) + t\bar{f}.$$

In order to solve the system, let us consider the difference of equation (15) with $i = 1$ and (13). This gives $f_1 - f_0 = tf_0 + t^2 f_1$, that is, $(1 - t^2)f_1 = (1 + t)f_0$. So $f_0 = (1 - t)f_1$.

Now suppose that $1 \le i < p-1$. If we take the difference between f_{i+1} and f_i using (15), we obtain $f_{i+1} - f_i = tf_i + t^2 f_{i+1}$, which implies $f_i = (1 - t)f_{i+1}$.

Now for all $0 \leq i \leq p-1$ one has $f_i = (1-t)^{p-1-i} f_{p-1}$. The equation (13) becomes

$$(1-t)^{p-1} f_{p-1} = 2t + t f_{p-1}((1-t)^{p-1} + \cdots + (1-t) + 1) = 2t + f_{p-1}(1 - (1-t)^p).$$

This gives

(17) $$f_{p-1}(t) = \frac{2t}{(1-t)^p + (1-t)^{p-1} - 1}.$$

In order to find the number of words in $\mathcal{L}_p$ having length n, we need to add all the generating functions for all states. The result will be

$$\Phi_p(t) = 1 + f_0 + f_1 + \cdots + f_{p-1} + t(f_1 + \cdots + f_{p-1}) + \frac{t^2}{1-t} \cdot (f_1 + \cdots + f_{p-1})$$

(here we used (16) to express $\bar{f}$). Taking into account that

$$f_1 + \cdots + f_{p-1} = \frac{1-t}{t} \cdot f_0 - 2$$

from (13), we have

$$\Phi_p(t) = 1 + f_0 + \left(1 + t + \frac{t^2}{1-t}\right)(f_1 + \cdots + f_{p-1}) = 1 + \left(1 + \frac{1}{t}\right) f_0 - \frac{2}{1-t}.$$

Now, using $f_0 = (1-t)^{p-1} f_{p-1}$ and (17), we finally have

(18) $$\Phi_p(t) = \frac{1+t}{1-t} \cdot \frac{1 - t(1-t)^{p-1}}{(1-t)^p + (1-t)^{p-1} - 1}.$$

This is the generating function for $\mathcal{L}_p$. Thus the growth rate of $\mathcal{L}_p$ will be the reciprocal of t, where t is the smallest positive root of the denominator of the right hand side of (18).

The number $y = (1-t)^{-1}$ is the root of $y^p = y + 1$. It is clear that $y > 1$. Let $y = 1 + x$, where $x > 0$. We would like to solve the equation $(1+x)^p = 2 + x$. Notice that $0 < x < 1$. Since $(1+x)^p < 3$, the root x approaches 0 as p goes to infinity.

The equation $(1+x)^p = 2 + x$ can be written as

$$p = \frac{\ln(2+x)}{\ln(1+x)} = \frac{\ln 2 + \ln(1+x/2)}{\ln(1+x)} = \frac{\ln 2 + x/2 + o(x)}{x - x^2/2 + o(x^2)}.$$

Therefore,

(19) $$\frac{p - \frac{1}{2}}{\ln 2} = \frac{1 + o(x)}{x - x^2 + ox^2} = x^{-1}(1 + x/2 + o(x)) = x^{-1} + \frac{1}{2} + o(1)$$

as $p \to \infty$.

We are interested in the number $\xi_p = t^{-1}$, where t is the root of $(1 - t)^p + (1 - t)^{p-1} - 1 = 0$. Here $(1 - t)^{-1} = y$, where $y = 1 + x$ is the root of $y^p = y + 1$. It is easy to see that $\xi_p = 1 + x^{-1}$. So we deduce from (19) that

$$\xi_p = \frac{p - \frac{1}{2}}{\ln 2} + \frac{1}{2} + o(1), \quad p \to \infty.$$

It is also easy to see that $\xi_p = t^{-1}$ satisfies the following equation: $(2\xi - 1)(\xi - 1)^{p-1} = \xi^p$. So we proved

Theorem 6.1. *The growth rate of the group $F(p)$, $p \geq 2$ has a lower bound of ξ_p, where ξ_p satisfies the equation*

$$(2\xi - 1)(\xi - 1)^{p-1} = \xi^p.$$

The following asymptotic formula holds:

$$\xi_p = \frac{p - \frac{1}{2}}{\ln 2} + \frac{1}{2} + o(1), \quad p \to \infty.$$

Here are several numerical values of ξ_p:

$$\xi_2 = 2.618033989$$
$$\xi_3 = 4.079595623$$
$$\xi_4 = 5.530132718$$
$$\xi_5 = 6.977144180$$

and so on. For large values of p, the growth rate of $F(p)$ is at least $0.72(2p - 1)$ (recall that $2p - 1$ is the maximum value of the growth rate of a p-generated group; this happens if and only if the group is free of rank p).

It would be interesting to find nontrivial upper bounds for the growth rates of $F(p)$. This means to find a constant $c < 1$ such that the growth rate of $F(p)$ in its natural generators does not exceed $c(2p - 1)$.

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