THE ZARISKI-LEFSCHETZ PRINCIPLE FOR HIGHER HOMOTOPY GROUPS OF NONGENERIC PENCILS

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ABSTRACT. We prove a general Zariski-van Kampen-Lefschetz type theorem for higher homotopy groups of generic and nongeneric pencils on singular open complex spaces.

1. Introduction

We find natural generators for the "vanishing homotopy" which occurs in a pencil of hypersurfaces on an analytic variety X. This result is the analogue of the Zariski-Lefschetz theorem on "vanishing cycles" for homology groups. In the same time this generalises to higher homotopy groups the Zariski-van Kampen theorem on the fundamental group.

It turns out that our geometric method allows to prove such a result in a general setting, which includes for instance the use of nongeneric pencils, as we show in the following. Let Y be a compact complex analytic space and let $V \subset Y$ be a closed analytic subspace such that $X := Y \setminus V$ is connected, of pure complex dimension $n \geq 3$. A *pencil* on X is the restriction to X of a pencil on Y. The latter is by definition the ratio of two sections, f and g, of some holomorphic line bundle over Y. The meromorphic function $h := f/g : Y \dashrightarrow \mathbb{P}^1(\mathbb{C})$ is holomorphic over the complement $Y \setminus A$ of the axis $A := \{f = g = 0\}$. To define *singularities* of pencils in a generalised sense, we first consider a Nash blowing-up $\mathbb{Y}$ along the axis (see §2), with projections $p\colon \mathbb{Y} \to \mathbb{P}^1$ and $\sigma\colon \mathbb{Y} \to Y$. Then there is a well-defined singular locus of p with respect to a certain natural Whitney stratification $\mathcal{S}$, denoted by $\mathrm{Sing}_{\mathcal{S}} p$.

We shall assume throughout this paper that $\dim \mathrm{Sing}_{\mathcal{S}} p \leq 0$, i.e. the singularities are *isolated*. It should be pointed out that singularities may lie on the axis, i.e. that one may have $A \cap \sigma(\mathrm{Sing}_{\mathcal{S}} p) \neq \emptyset$. This occurs for

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instance if the transversality of the axis to strata of Y fails at finitely many points: see Proposition 2.2 and [Ti2, §2], [Ti3, §2] for more details.

Such nongeneric pencils were introduced in our preprint [Ti1] and this viewpoint allowed us to prove connectivity theorems of Lefschetz type [Ti2], and a far reaching extension of the Second Lefschetz Hyperplane Theorem [Ti3]. In the literature, examples of nongeneric pencils occured sporadically; more recently they got into light, e.g. [KPS, Ok2], precisely because one can use nongeneric pencils towards more efficient computations. For instance, M. Oka uses special pencils tangent to flex points of projective curves in order to compute the fundamental group of the complement, see e.g. [Ok1, Ok2]. Nongenericity also proved to be a useful concept for treating the topology of polynomial functions (see [NN, Ti3]) and of complements of arrangements (e.g. [LT]). A highly nongeneric situation occurs when V contains a member of the pencil, see §5.4.

In order to express how the "homotopy vanishing cycles" are generated we need to introduce a *homotopy variation map*. Roughly, this goes as follows (see §3). Let $a \in \Lambda = p(\mathrm{Sing}_{\mathcal{S}} p)$ be some critical value of the pencil and denote by $X_a := X \cap p^{-1}(a)$ the corresponding critical fibre. For a general fibre of the pencil X_c near to X_a, we denote by X_a^* the space $X_c \setminus \cup_j B_j$, where B_j is a Milnor ball at some point of the set $X_a \cap \sigma(\mathrm{Sing}_{\mathcal{S}} p)$. We construct in §3 a global variation map for higher homotopy groups:

$$\mathrm{hvar}_a : \pi_q(X_c, X_a^*, \centerdot) \to \pi_q(X_c, \centerdot), \tag{1}$$

for any $q \geq 3$. Under the condition that the involved spaces are path-connected and that $\pi_1(X_c, X_a^*, \centerdot)$ is trivial, this map commutes with the actions of $\pi_1(X_a^*, \centerdot)$ and of $\pi_1(X_c, \centerdot)$ via the surjection $\pi_1(X_a^*, \centerdot) \twoheadrightarrow \pi_1(X_c, \centerdot)$, respectively.

In case of homology groups, one encounters variation maps in the literature ever since Picard and Lefschetz studied the monodromy around an isolated singularity of a 2-variables holomorphic function [PS, Lef]. Generalised variation maps and Picard-Lefschetz formulas play a key role in studying the topology of pencils of hypersurfaces (e.g. [Ph, Mi, La, Lo, Ga, NN], [Va1, Va2]). We have introduced in [Ti3] a global variation map in homology which generalises the local variation map to pencils with isolated singularities on arbitrarily singular (open) underlying spaces.

Let us now state our general Zariski-van Kampen-Lefschetz type result which uses the homotopy variation map (1). We refer to §2 and §3 for the precise definitions of all the ingredients. The homotopy depth condition "hd" is recalled in Definition 4.5 and the notation "X_a^*" is explained in 3.4.

Theorem 1.1. *Let $h\colon Y \dashrightarrow \mathbb{P}^1$ define a pencil with isolated singularities such that the axis A of the pencil is not included in V and that the general* *fibre X_c on $X = Y \setminus V$ is path connected. Let us consider the following three conditions:*

(i) $(X_c, X_c \cap A)$ *is k-connected;*
(ii) (X_c, X_a^*) *is k-connected, for any critical value a of the pencil;*
(iii) $\mathrm{hd}_{\mathbb{X}\cap \mathrm{Sing}_{\mathcal{S}} P}\mathbb{X} \geq k+2$.

Then we have the following two conclusions:

(a) *If for some $k \geq 0$ the conditions* (i), (ii) *and* (iii) *hold then* $\pi_q(X, X_c, \centerdot) = 0$ *for all $q \leq k+1$.*
(b) *If for some $k \geq 2$ the conditions* (i) *and* (ii) *hold, and if condition* (iii) *is replaced by the following one:*
(iii') $\mathrm{hd}_{\mathbb{X}\cap \mathrm{Sing}_{\mathcal{S}} P}\mathbb{X} \geq k+3$,
then the kernel of the surjective map $\pi_{k+1}(X_c, \centerdot) \twoheadrightarrow \pi_{k+1}(X, \centerdot)$ is generated by the images of the variation maps hvar_a*.*

The conditions in our theorem are satisfied by a large class of spaces X. For instance, the conditions (ii) and (iii), resp. (iii'), are both fulfilled as soon as X is a singular space which is a locally complete intersection of dimension $n \geq k+2$, resp. $n \geq k+3$ (see §5.1). We send to §5 for many other comments on these conditions and on what they become in more particular situations.

In what concerns Part (a) of Theorem 1.1, this is a general "first Lefschetz hyperplane theorem" for homotopy groups. It has been proved in some particular cases in the past and it recovers, with significantly weakened hypotheses, our recent result [Ti2].

The focus of this paper is actually on Part (b) of Theorem 1.1, which treats the "homotopy vanishing cycles". Our approach is in the spirit of the Lefschetz method [Lef], as presented by Andreotti and Frankel in [AF]. Since the homology conterpart of our result (which is easier) is presented in [Ti3], we concentrate here on the more delicate constructions and ingredients of the proof which are needed in case of homotopy groups. We use homotopy excision (the Blackers-Massey theorem), more precisely the general version proved by Gray [Gr, Cor; 16.27], and the comparison between homotopy and homology groups via the Hurewicz map. Our result does not cover the case $k = 1$, hence the homotopy group π_2, because of the technical restriction in the definition of the variation map (1).

The cornerstone of the proof is perhaps Proposition 4.9 in which we identify the homotopy variation map to the boundary map of the long exact sequence of the pair $(\mathbb{X}_D^*, X_c)$, by expanding Milnor's explanation of the Wang exact sequence [Mi, pag. 67]. All those techniques are coupled with the general theory of stratified singular spaces and of Milnor fibrations of stratifies singularities of holomorphic functions (see e.g. [Lê, GM]). Notably

Proposition 4.8 is a key interplay between the use of the singular Milnor fibration and the homotopy excision.

Theorem 1.1 represents a synthetic viewpoint on Zariski-van Kampen type results for higher homotopy groups, having in the background the pioneering work of Lefschetz [Lef], Zariski and van Kampen [vK]. It recovers several cases considered before in the literature and in the same time it extends the range of applicability of the Zariski-van Kampen-Lefschetz principle. For instance we show in §5.3 how to get a version of Theorem 1.1 in the case of complements of singular projective spaces, which seems to be a new result. Some other possible applications of Theorem 1.1 parallel the ones for homology groups which have been discussed in [Ti2, Applications] and [LT].

Such a progress would not have been possible without the contribution of a long list of articles around the Lefschetz slicing principle, from which we took a part of the inspiration, such as [AF, Ch1, Ch2, Fu, FL, GM, HL1, HL2, HL3, La, Lef, Li, Lo, Mi]. Results of Zariski-van Kampen type have been proved by Libgober for generic Lefschetz pencils of hyperplanes and in the particular case when X is the complement in $\mathbb{P}^n$ or $\mathbb{C}^n$ of a hypersurface with isolated singularities [Li]. D. Chéniot, A. Libgober and C. Eyral authored in 2002 two preprints [CL, CE] treating, with different background, interesting particular cases of the problem we are concerned with here.

This paper completes in detail a certain part of the program "Lefschetz principle for nongeneric pencils" presented in our 2001 preprint [Ti1]. We own warm thanks to several institutions from which we have got hospitality in connection to this project: Newton Institute (2000), IAS Princeton (2002), CRM Barcelona (2004).

2. Pencils with singularities in the axis

As denoted before, let $X := Y \setminus V$ be the difference of two compact complex analytic spaces. We introduce the basic notations and recall from [Ti2] what are the singularities of a pencil, in a stratified sense.

Consider the compactification of the graph of h, namely the space:

$$\mathbb{Y} := \text{closure}\{y \in Y,\ [s;t] \in \mathbb{P}^1 \mid sf(y) - tg(y) = 0\} \subset Y \times \mathbb{P}^1.$$

Let us denote $\mathbb{X} := \mathbb{Y} \cap (X \times \mathbb{P}^1)$. Consider the projection $p : \mathbb{Y} \to \mathbb{P}^1$, its restriction $p_{|\mathbb{X}} : \mathbb{X} \to \mathbb{P}^1$ and the projection to the first factor $\sigma : \mathbb{Y} \to Y$. Notice that the restriction of p to $\mathbb{Y} \setminus (A \times \mathbb{P}^1)$ can be identified with h.

Let $\mathcal{W}$ be an analytic Whitney stratification of Y such that V is union of strata. Its restriction to the open set $Y \setminus A$ induces a Whitney stratification on $\mathbb{Y}\setminus(A\times\mathbb{P}^1)$, via the mentioned identification. We denote by $\mathcal{S}$ the coarsest analytic Whitney stratification on $\mathbb{Y}$ which coincides over $\mathbb{Y}\setminus(A\times\mathbb{P}^1)$ with the one induced by $\mathcal{W}$ on $Y\setminus A$. This stratification exists within a neighbourhood of $A\times\mathbb{P}^1$, by usual arguments (see e.g. [GLPW]), hence such stratification is well defined on $\mathbb{Y}$. We call it the *canonical stratification* of $\mathbb{Y}$ generated by the stratification $\mathcal{W}$ of Y.

Definition 2.1. [Ti2] One calls *singular locus* of p with respect to $\mathcal{S}$ the following subset of $\mathbb{Y}$:

$$\mathrm{Sing}_{\mathcal{S}}p := \bigcup_{\mathcal{S}_\beta\in\mathcal{S}} \mathrm{Sing}p_{|\mathcal{S}_\beta}.$$

The set $\Lambda := p(\mathrm{Sing}_{\mathcal{S}}p)$ is called the set of *critical values* of p with respect to $\mathcal{S}$. We say that the pencil defined by $h := f/g : Y \dashrightarrow \mathbb{P}^1$ is a *pencil with isolated singularities* if $\dim \mathrm{Sing}_{\mathcal{S}}p \leq 0$.

Remark that $\mathrm{Sing}_{\mathcal{S}}p$ is closed analytic. Remark also that p is proper and analytic in a neighbourhood of $A\times\mathbb{P}^1$ and that $\mathcal{S}$ has finitely many strata. Then, by a Bertini-Sard result which is a consequence of the First Isotopy Lemma [Th], see also [Ve], it follows that $\Lambda\subset\mathbb{P}^1$ is a finite set and that the maps $p : \mathbb{Y}\setminus p^{-1}(\Lambda)\to\mathbb{P}^1\setminus\Lambda$ and $p_{|\mathbb{X}} : \mathbb{X}\setminus p^{-1}(\Lambda)\to\mathbb{P}^1\setminus\Lambda$ are stratified locally trivial fibrations. In particular, $h : Y\setminus(A\cup h^{-1}(\Lambda))\to\mathbb{P}^1\setminus\Lambda$ is a locally trivial fibration.

Such nongeneric pencils may occur whenever the axis A is not in general position at finitely many points. More precisely, we have the following example of situation:

Proposition 2.2. [Ti2, Prop. 2.4]
Let $X = Y\setminus V\subset\mathbb{P}^N$ be a quasi-projective (singular) variety and let $\hat{h} = \hat{f}/\hat{g}$ define a pencil of hypersurfaces in $\mathbb{P}^N$ with axis $\hat{A}$. Let C denote the set of points on $\hat{A}\cap Y$ where some member of the pencil is singular or where $\hat{A}$ is not stratified transversal to Y. If $\dim C\leq 0$ and if the stratified critical points of the restriction $\hat{h}_| : Y\setminus A\to\mathbb{P}^1$ are isolated, then $\dim\mathrm{Sing}_{\mathcal{S}}p\leq 0$. □

We shall assume throughout the paper that our pencil on X has isolated singularities. Then, by the compactness of Y, the set $\mathrm{Sing}_{\mathcal{S}}p$ is a finite set. Let us emphasize that the points of $\mathrm{Sing}_{\mathcal{S}}p$ are not necessarily contained in $\mathbb{X}$ and that some of them may be on the blown-up axis $A\times\mathbb{P}^1$.

3. Homotopy variation maps

Inspired by the homology constructions throughout the literature and by a particular Zariski-van-Kampen type theorem for higher homotopy groups proved in the beginning of the 1990's by Ligbober [Li], we have given in the preprint [Ti1] (unpublished) the main lines of the construction of *global homotopy variation maps* for pencils with isolated singularities. We complete here in full detail this construction[1].

For any $M \subset \mathbb{P}^1$, we denote $\mathbb{Y}_M := p^{-1}(M)$ and $\mathbb{X}_M := \mathbb{X} \cap \mathbb{Y}_M$ and $X_M := \sigma(\mathbb{X}_M)$. Let $a_i \in \Lambda$ and let $\mathrm{Sing}_{\mathcal{S}} p = \{a_{ij}\}_{i,j} \subset \mathbb{Y}$, where $a_{ij} \in \mathbb{Y}_{a_i}$. For $c \in \mathbb{P}^1 \setminus \Lambda$ we say that $\mathbb{Y}_c$, respectively $\mathbb{X}_c$, respectively X_c, is a general fibre.

3.1. **Milnor fibration at isolated stratified critical points of holomorphic functions.** At some singularity a_{ij}, in local coordinates, we take a small ball B_{ij} centered at a_{ij}. For small enough radius of B_{ij}, this is a *Milnor ball*[2] of the local holomorphic germ of the function p at a_{ij}. Next we may take a small enough disc $D_i \subset \mathbb{P}^1$ at $a_i \in \mathbb{P}^1$, so that (B_{ij}, D_i) is *Milnor data* for p at a_{ij}. Moreover, we may do this for all (finitely many) singularities in the fibre $\mathbb{Y}_{a_i}$, keeping the same disc D_i, provided that it is small enough.

Since the function $p : \mathbb{Y} \to \mathbb{P}^1$ has isolated stratified singularities, the fibres of p are endowed with the stratification induced by $\mathcal{S}$ except that we have to introduce the point-strata $\{a_{ij}\}$. Every fibre has therefore a natural induced Whitney stratification.

By the general theory of holomorphic function germs [GM, Lê] we have that the restriction $p_| : \mathbb{Y}_{\partial \bar{D}_i} \to \partial \bar{D}_i$ is a *locally trivial* stratified fibration over the circle $\partial \bar{D}_i$. Since $p : \mathbb{Y} \to \mathbb{P}^1$ has a stratified isolated singularity at a_{ij}, the sphere $\partial \bar{B}_{ij}$ is stratified transversal to all the fibres of p above the points of D_i. It follows that the restriction of p to the pair of spaces $(\mathbb{Y}_{D_i} \setminus \cup_j B_{ij}, \mathbb{Y}_{D_i} \cap \cup_j \partial \bar{B}_{ij})$ is a *trivial stratified fibration* over D_i.

3.2. **Geometric monodromy.** Let's choose $c_i \in \partial \bar{D}_i$. One constructs in the usual way (see Looijenga's similar discussion in [Lo, 2.C, page 31]), by using a stratified vector field, a characteristic morphism $h_i : X_{c_i} \to X_{c_i}$ of the fibration over the circle $p_| : \mathbb{X}_{\partial \bar{D}_i} \to \partial \bar{D}_i$, which is a stratified homeomorphism and it is the identity on $X_{c_i} \setminus \cup_j B_{ij}$. This is called the *geometric monodromy* of the locally trivial fibration $p_| : \mathbb{X}_{\partial \bar{D}_i} \to \partial \bar{D}_i$ and corresponds to one counterclockwise loop around the circle $\partial \bar{D}_i$. Clearly h_i is not uniquely defined with these properties, but its relative isotopy class in the group $\mathrm{Iso}(X_{c_i}, X_{c_i} \setminus \cup_j B_{ij})$ of relative isotopy classes of stratified homeomorphisms which are the identity on $X_{c_i} \setminus \cup_j B_{ij}$ is unique.

[1]A similar construction in case of homology groups appears in [Ti3].

[2]One may consult [GM, Lê] for the definition and the terminology concerning the local Milnor fibration in case of a holomorphic function on a singular analytic space, at an isolated critical point with respect to a given stratification of the space.

3.3. **Construction of the homotopy variation map.** We assume in the rest of this section that $X_{c_i} \setminus \cup_j B_{ij}$ is path connected, for all i. This assumption is actually fulfilled under the hypotheses of our Theorem 1.1(b). More precisely, the fact that the general fibre X_c is path connected and that $(X_{c_i}, X_{c_i} \setminus \cup_j B_{ij})$ is 1-connected imply that $X_{c_i} \setminus \cup_j B_{ij}$ is path connected. Let $\gamma\colon (D^q, S^{q-1}, u) \to (X_{c_i}, X_{c_i} \setminus \cup_j B_{ij}, v)$, for $q \geq 3$, be some continuous map, and let $[\gamma]$ be its homotopy class in $\pi_q(X_{c_i}, X_{c_i} \setminus \cup_j B_{ij}, v)$. Note that $[h_i \circ \gamma]$ is a well-defined element of $\pi_q(X_{c_i}, X_{c_i} \setminus \cup_j B_{ij}, v)$ and does not depend on the representative h_i in its relative isotopy class. Consider the map $\rho\colon (D^q, S^{q-1}, u) \to (D^q, S^{q-1}, u)$ which is the reflection into some fixed generic hyperplane through the origin, which contains u. Then $[\gamma \circ \rho]$ is the inverse $-[\gamma]$ of $[\gamma]$. We use the additive notation since the relative homotopy groups π_q are abelian for $q \geq 3$.

Consider now the map:

$$\mu_i\colon \pi_q(X_{c_i}, X_{c_i} \setminus \cup_j B_{ij}, v) \to \pi_q(X_{c_i}, X_{c_i} \setminus \cup_j B_{ij}, v) \tag{2}$$

defined as follows: $\mu_i([\gamma]) = [(h_i \circ \gamma) * (\gamma \circ \rho)] = [h_i \circ \gamma] - [\gamma]$.

The notation "$*$" stands for the operation on maps which induces the group structure of the relative π_q. The map μ_i is well-defined and it is an automorphism of $\pi_q(X_{c_i}, X_{c_i} \setminus \cup_j B_{ij}, v)$, because of the abelianity of the relative homotopy groups π_q, for $q \geq 3$. For the time being, we cannot extend this construction to $q = 2$. In case $q = 1$ there are the monodromy relations which enter in the well-known Zariski-van Kampen theorem.

Claim: *the map $\nu_i := (h_i \circ \gamma) * (\gamma \circ \rho)\colon (D^q, S^{q-1}, u) \to (X_{c_i}, X_{c_i} \setminus \cup_j B_{ij}, v)$ is homotopic, relative to S^{q-1}, to a map $(S^q, u) \to (X_{c_i}, v)$.*

To prove the claim, we first observe that for the restriction of γ to S^{q-1}, which we shall call γ', we have $h_i \circ \gamma' = \gamma'$, since the geometric monodromy h_i is the identity on $\operatorname{im} \gamma'$. Without loss of generality, we may and shall suppose in the following that the geometric monodromy h_i is the identity on a small tubular neighbourhood $\mathcal{T}$ of $X_{c_i} \setminus \cup_j B_{ij}$ within X_{c_i}.

Let ν_i' denote the restriction of ν_i to S^{q-1}. We claim that one can shrink $\operatorname{im} \nu_i'$ to the point v through a homotopy within $\operatorname{im} \nu_i'$. The precise reason is that, by the definition of ρ, the map $\gamma' * (\gamma' \circ \rho)$ is homotopic to the constant map to the base-point v, through a deformation of the image of $\gamma' * (\gamma' \circ \rho)$ within itself. We may moreover extend this homotopy to a continuous deformation of the map ν_i in a small collar neighbourhood τ of S^{q-1} in D^q, which has its image in the tubular neighbourhood $\mathcal{T}$, such that it is the identity on the interior boundary of τ (which is a smaller sphere $S^{q-1}_{1-\varepsilon} \subset D^q$). Then we extend this deformation as the identity to the rest of D^q. In this way we have constructed a homotopy between ν_i and a map $(D^q, S^{q-1}, u) \to (X_{c_i}, v, v)$, which is nothing else than a map

$(D^q, u) \to (X_{c_i}, v)$. This represents an element of the group $\pi_q(X_{c_i}, v)$, so our claim is proved.

This construction is well-defined up to homotopy equivalences, hence it yields a map:

$$\text{hvar}_i : \pi_q(X_{c_i}, X_{c_i} \setminus \cup_j B_{ij}, v) \to \pi_q(X_{c_i}, v), \tag{3}$$

which we call the *(homotopy) variation map* of h_i.

Let us point out that hvar_i is a morphism of groups. Indeed, due to the abelianity we have:

$$\text{hvar}_i([\gamma] + [\delta]) = [(h_i \circ \gamma) * (h_i \circ \delta)] - [\gamma * \delta] = [h_i \circ \gamma] + [h_i \circ \delta] - [\gamma] - [\delta]$$

$$= [h_i \circ \gamma] - [\gamma] + [h_i \circ \delta] - [\delta] = \text{hvar}_i([\gamma]) + \text{hvar}_i([\delta]).$$

By its geometric definition, this variation map enters in the following commuting diagram:

$$\begin{tikzcd} \pi_q(X_{c_i}, X_{c_i} \setminus \cup_j B_{ij}, v) \arrow[rr, "\mu_i"] \arrow[dr, "\text{hvar}_i"'] & & \pi_q(X_{c_i}, X_{c_i} \setminus \cup_j B_{ij}, v) \\ & \pi_q(X_{c_i}, v) \arrow[ur, "j_\sharp"'] & \end{tikzcd} \tag{4}$$

where $j\colon (X_{c_i}, v) \to (X_{c_i}, X_{c_i} \setminus \cup_j B_{ij}, v)$ is the inclusion.

3.4. **Convention of notations.** In local coordinates at a_{ij}, $\mathbb{Y}_{a_i}$ is a germ of a stratified complex analytic space; hence, for a small enough ball B_{ij}, the set $\bar{B}_{ij} \cap X_{a_i} \setminus \cup_j a_{ij}$ retracts to $\partial \bar{B}_{ij} \cap X_{a_i}$, by the local conical structure of stratified analytic sets [BV]. It follows that $X_{a_i} \setminus \cup_j a_{ij}$ is homotopy equivalent, by retraction, to $X_{a_i} \setminus \cup_j B_{ij}$. We have seen in §3.1 that the restriction $p_| : \mathbb{X}_{D_i} \setminus \cup_j B_{ij} \to D_i$ is a trivial stratified fibration. This implies that the restriction to $\mathbb{X}$, i.e. $p_| : \mathbb{X}_{D_i} \setminus \cup_j B_{ij} \to D_i$ is a trivial stratified fibration too. Since $X_{a_i} \setminus \cup_j B_{ij}$ and $X_{c_i} \setminus \cup_j B_{ij}$ are both fibres of this fibration, we shall identify them via a fixed trivialisation along a chosen path in D_i.

In the remainder of this paper, for the sake of simplicity, we shall denote the spaces $X_{c_i} \setminus \cup_j B_{ij}$ and $X_{a_i} \setminus \cup_j B_{ij}$ by the same symbol $X^*_{a_i}$. In particular, $X_{a_i} \setminus \cup_j B_{ij}$ becomes in this way a subspace of X_{c_i}.

We shall use in the remainder the following notation for the variation map, only as a symbolic replacement for (3), but not having an intrinsic meaning:

$$\text{hvar}_i : \pi_q(X_{c_i}, X^*_{a_i}, \centerdot) \to \pi_q(X_{c_i}, \centerdot), \tag{5}$$

for any $q \geq 3$. This is actually the notation used in the statement of Theorem 1.1.

3.5. **The action of π_1.** We have the standard action of $\pi_1(X_c, \centerdot)$ on the higher homotopy groups $\pi_q(X_c, \centerdot)$ and on any relative homotopy group where X_c appears at the second position. Moreover, the hypotheses of Theorem 1.1(b) imply that $X^*_{a_i}$ is path connected for all i, as explained in §3.3, and that $\pi_1(X^*_{a_i}, \centerdot) \xrightarrow{i_*} \pi_1(X_{c_i}, \centerdot)$ is an isomorphism. We shall therefore identify the two actions modulo the isomorphism i_*.

So one has the action of $\pi_1(X^*_{a_i}, \centerdot)$ on those higher relative homotopy groups of pairs where $X^*_{a_i}$ is the second space of the pair, and one also has the action of $\pi_1(X^*_{a_i}, \centerdot)$, via the isomorphism i_*, on $\pi_q(X_{c_i}, \centerdot)$ and on the higher relative homotopy groups of pairs where X_{c_i} is the second space of the pair.

Let's then prove that the image of hvar_i is invariant under the action of $\pi_1(X_{c_i}, \centerdot)$. For some $\beta \in \pi_1(X^*_{a_i}, \centerdot)$, we have $\mathrm{hvar}_i(\beta[\gamma]) = \beta[h_i \circ \gamma] - \beta[\gamma] = \beta \mathrm{hvar}_i([\gamma])$ since, by the functoriality, the action of β commutes with the morphism induced by the geometric monodromy map $h_i : X_{c_i} \to X_{c_i}$.

We shall denote in the following by $\hat{\pi}_q$ the quotient of the corresponding higher homotopy group by the action of $\pi_1(X_c, \centerdot)$ or of $\pi_1(X^*_{a_i}, \centerdot)$, eventually through the isomorphism i_*. We shall use the same notation $\hat{\nu}$ for the passage to the quotients of some morphism of groups ν which commutes with the action of π_1.

4. Proof of Theorem 1.1

We shall prove in parallel (a) and (b). The only proof of (a) would be much simpler and can be easily extracted.

Relative to the claim (b), let us first remark that the kernel of $\pi_{k+1}(X_c, \centerdot) \twoheadrightarrow \pi_{k+1}(X, \centerdot)$ is invariant under the the $\pi_1(X_c, \centerdot)$-action on $\pi_{k+1}(X_c, \centerdot)$, by the functoriality of this action. So in order to prove (b) it suffices to show that the image of $\hat{\partial} : \hat{\pi}_{k+2}(X, X_c, \centerdot) \to \hat{\pi}_{k+1}(X_c, \centerdot)$ is generated by the images $\mathrm{im}(\widehat{\mathrm{hvar}_i})$, since $\mathrm{im}(\mathrm{hvar}_i)$ is also invariant under the π_1-action, see §3.5. This is what we shall do in the last part of the proof below.

When dealing with homotopy groups, we shall need to apply the homotopy excision theorem of Blackers and Massey in the form given by Gray [Gr, Cor; 16.27]. For the connectivity claim (a), we follow in the beginning the lines of [Ti2].

Let $A' := A \cap X_c$ and assume that $A' \neq \emptyset$. Let $K \subset \mathbb{P}^1$ be a closed disc with $K \cap \Lambda = \emptyset$ and let $\mathcal{D}$ denote the closure of its complement in $\mathbb{P}^1$. We denote by $S := K \cap \mathcal{D}$ the common boundary, which is a circle, and take a point $c \in S$. Then take standard paths ψ_i (non self-intersecting, non mutually intersecting) from c to c_i, such that $\psi_i \subset \mathcal{D} \setminus \cup_i D_i$. The configuration $\cup_i(\bar{D}_i \cup \psi_i)$ is a deformation retract of $\mathcal{D}$. We shall identify the fibre X_{c_i} to the fibre X_c, by parallel transport along the path ψ_i.

We have the natural commutative triangle:

$$
\begin{array}{ccc}
\pi_{k+2}(X_{\mathcal{D}}, X_c, \centerdot) & \xrightarrow{\iota_\sharp} & \pi_{k+2}(X, X_c, \centerdot) \\
{\scriptstyle \partial_0} \searrow & & \swarrow {\scriptstyle \partial} \\
& \pi_{k+1}(X_c, \centerdot) &
\end{array} \tag{6}
$$

where ∂ and ∂_0 are boundary morphisms and ι is the inclusion of pairs $(X_{\mathcal{D}}, X_c) \hookrightarrow (X, X_c)$.

One sais that an inclusion of pairs of topological spaces $(N, N') \hookrightarrow (M, M')$ is a **q-equivalence** if this inclusion induces an isomorphism of relative homotopy groups for $j < q$ and a surjection for $j = q$.

Lemma 4.1. *Let (X_c, A') be k-connected for $k \geq 0$. Then the inclusion of pairs $\iota : (X_{\mathcal{D}}, X_c) \hookrightarrow (X, X_c)$ is a $(k+2)$-equivalence, for $k \geq 0$. In particular, after taking $\pi_1(X_c, \centerdot)$-quotients in diagram (6), we have:* $\operatorname{im} \hat{\partial} = \operatorname{im} \hat{\partial}_0$.

Proof. Under the assumption (X_c, A') is k-connected, $k \geq 0$, it follows from [Ti2, Proposition 3.2(i)] that (X_S, X_c) is $(k+1)$-connected. From the exact sequence of the triple $(X_{\mathcal{D}}, X_S, X_c)$, we deduce that the inclusion $(X_{\mathcal{D}}, X_c) \hookrightarrow (X_{\mathcal{D}}, X_S)$ is a $(k+2)$-equivalence. Next, [Ti2, Proposition 3.2(ii)] shows that, by homotopy excision, the inclusion of pairs $(X_{\mathcal{D}}, X_S) \hookrightarrow (X, X_K)$ is a $(k+2)$-equivalence. Since X_K retracts in a fiberwise way to X_c, the pair (X, X_K) is homotopy equivalent to (X, X_c). □

We further go to the blow-up $\mathbb{X}$ and get:

Lemma 4.2. *The inclusion $(\mathbb{X}_{\mathcal{D}}, X_c) \to (X_{\mathcal{D}}, X_c)$ is a homotopy equivalence.*

Proof. It is a simple fact that $X_{\mathcal{D}}$ is homotopy equivalent to $\mathbb{X}_{\mathcal{D}}$ to which one attaches along $A' \times \mathcal{D}$ the product $A' \times \operatorname{Cone}(\mathcal{D})$ (see e.g. [Ti2, Lemma 3.1]). Since $\mathcal{D}$ is contractible, our claim follows. □

We also need the following result from [Ti2], which uses [Ti2, Lemma 3.1] and Switzer's result [Sw, 6.13]:

Lemma 4.3. [Ti2, Prop. 3.2(b)]
If $(\mathbb{X}_{D_i}, X_{c_i})$ is $(k+1)$-connected, $k \geq 0$, for all i, then $(\mathbb{X}_{\mathcal{D}}, X_c)$ is $(k+1)$-connected.

□

For all i, let $\iota_i : (\mathbb{X}_{D_i}, X_{c_i}) \to (\mathbb{X}_{\mathcal{D}}, X_c)$ denote the inclusion of pairs and let $\partial_i : \pi_{k+2}(\mathbb{X}_{D_i}, X_{c_i}, \centerdot) \to \pi_{k+1}(X_{c_i}, \centerdot)$ denote the boundary morphism in the pair $(\mathbb{X}_{D_i}, X_{c_i})$. Consider now the following commutative diagram, for $k \geq 1$:

$$
\begin{array}{ccc}
\oplus_i \pi_{k+2}(\mathbb{X}_{D_i}, X_{c_i}, \centerdot) & & \\
{\scriptstyle \sum \iota_{i\sharp}} \downarrow & & \searrow {\scriptstyle \sum \partial_i} \\
\pi_{k+2}(\mathbb{X}_{\mathcal{D}}, X_c, \centerdot) & \xrightarrow{\partial_0} & \pi_{k+1}(X_c, \centerdot),
\end{array}
\tag{7}
$$

where, for all i, we identify $(X_{c_i}, \centerdot)$ to $(X_c, \centerdot)$ together with the base points $\centerdot$ by using the paths ψ_i, as explained in the beginning of this section. We use the additive notations $\sum \partial_i$ and $\sum \iota_{i\sharp}$ since for $k \geq 1$ the groups $\pi_{k+1}(X_c, \centerdot)$ and $\pi_{k+2}(\mathbb{X}_{\mathcal{D}}, X_c, \centerdot)$ are abelian.

By the functoriality of the action of $\pi_1(X_c, \centerdot)$, we get the induced "hat" diagram:

$$
\begin{array}{ccc}
\oplus_i \hat{\pi}_{k+2}(\mathbb{X}_{D_i}, X_{c_i}, \centerdot) & & \\
{\scriptstyle \sum \hat{\iota}_{i\sharp}} \downarrow & & \searrow {\scriptstyle \sum \hat{\partial}_i} \\
\hat{\pi}_{k+2}(\mathbb{X}_{\mathcal{D}}, X_c, \centerdot) & \xrightarrow{\hat{\partial}_0} & \hat{\pi}_{k+1}(X_c, \centerdot),
\end{array}
\tag{8}
$$

With these notations we have the following:

Proposition 4.4. *Let $(\mathbb{X}_{D_i}, X_{c_i})$ be $(k+1)$-connected, $k \geq 1$, for all i. Then* $\operatorname{im} \hat{\partial}_0 = \operatorname{im}(\sum_i \hat{\partial}_i)$ *in diagram* (8).

Proof. We use Hurewicz maps between the relative homotopy and homology groups (denoted $\mathcal{H}_i$ and $\mathcal{H}$ below). We have the following commutative diagram:

$$
\begin{array}{ccc}
\oplus_i \hat{\pi}_{k+2}(\mathbb{X}_{D_i}, X_{c_i}, \centerdot) & \xrightarrow{\sum \hat{\iota}_{i\sharp}} & \hat{\pi}_{k+2}(\mathbb{X}_{\mathcal{D}}, X_c, \centerdot) \\
{\scriptstyle \oplus_i \mathcal{H}_i} \downarrow & & \downarrow {\scriptstyle \mathcal{H}} \\
\oplus_i H_{k+2}(\mathbb{X}_{D_i}, X_{c_i}) & \xrightarrow{\sum \iota_{i*}} & H_{k+2}(\mathbb{X}_{\mathcal{D}}, X_c).
\end{array}
$$

The additive notations are used because the relative homotopy groups are abelian for $k \geq 1$.

Under our hypothesis and by the relative Hurewicz isomorphism theorem (see e.g. [Sp, p. 397]), we get that the Hurewicz map $\mathcal{H}_i : \hat{\pi}_{k+2}(\mathbb{X}_{D_i}, X_{c_i}, \centerdot) \to H_{k+2}(\mathbb{X}_{D_i}, X_{c_i})$ is an isomorphism, for any i. The same is $\mathcal{H}$, since $(\mathbb{X}_{\mathcal{D}}, X_c)$ is $(k+1)$-connected, by Lemma 4.3.

Next, $\sum \iota_{i*}$ is a homology excision, hence an isomorphism too. It follows that $\sum \hat{\iota}_{i\sharp}$ is an isomorphism too, which proves our claim. $\square$

To complete the proof of our main theorem, we need the following:

- to prove that $(\mathbb{X}_{D_i}, X_{c_i})$ is $(k+1)$-connected, $k \geq 0$, for all i.
- to find the image of the map $\hat{\partial}_i : \hat{\pi}_{k+2}(\mathbb{X}_{D_i}, X_{c_i}, \centerdot) \to \hat{\pi}_{k+1}(X_{c_i}, \centerdot)$, $k \geq 1$, for all i.

First thing we do is to show that we may replace $\mathbb{X}_{D_i}$ by $\mathbb{X}^*_{D_i} := \mathbb{X}_{D_i} \setminus \mathrm{Sing}_{\mathcal{S}} p$, using the homotopy depth assumption, the definition of which we recall here.

Definition 4.5. For a discrete subset $\Phi \subset \mathbb{X}$, one sais that the **homotopy depth** of $\mathbb{X}$ at Φ, denoted by $\mathrm{hd}_\Phi \mathbb{X}$, is greater or equal to $q+1$ if, at any point $\alpha \in \Phi$, there is an arbitrarily small neighbourhood $\mathcal{N}$ of α such that the pair $(\mathcal{N}, \mathcal{N} \setminus \{\alpha\})$ is q-connected.

Lemma 4.6. *If* $\mathrm{hd}_{\mathbb{X}\cap \mathrm{Sing}_{\mathcal{S}} p}\mathbb{X} \geq q+1$*, for* $q \geq 1$*, then the inclusion of pairs* $(\mathbb{X}^*_{D_i}, X_{c_i}) \overset{j}{\hookrightarrow} (\mathbb{X}_{D_i}, X_{c_i})$ *is a* q*-equivalence, for all* i*.*

Proof. In the exact sequence of the triple $(\mathbb{X}_{D_i}, \mathbb{X}^*_{D_i}, X_{c_i})$, it is sufficient to prove that $(\mathbb{X}_{D_i}, \mathbb{X}^*_{D_i})$ is q-connected, for all i. By the homotopy depth assumption, for all j, the pair $(\mathbb{X}_{D_i} \cap B_{ij}, \mathbb{X}_{D_i} \cap B_{ij} \setminus \{a_{ij})$ is q-connected. By Switzer's result for CW-complexes [Sw, 6.13], the first space is obtained from the second by attaching cells of dimensions $\geq q+1$. It follows that $\mathbb{X}_{D_i}$ is obtained from $\mathbb{X}^*_{D_i}$ by replacing each $B_{ij} \setminus \{a_{ij}$ with B_{ij}, which amounts, as we have seen, to attaching cells of dimensions $\geq q+1$. So our claim is proved. □

Corollary 4.7. *If* $\mathrm{hd}_{\mathbb{X}\cap \mathrm{Sing}_{\mathcal{S}} p}\mathbb{X} \geq k+3$*,* $k \geq 0$*, then, for all* i*:*

$$\begin{aligned} &\mathrm{im}(\hat{\partial}_i : \hat{\pi}_{k+2}(\mathbb{X}_{D_i}, X_{c_i}, \centerdot) \to \hat{\pi}_{k+1}(X_{c_i}, \centerdot)) = \\ &\quad = \mathrm{im}(\hat{\partial}'_i : \hat{\pi}_{k+2}(\mathbb{X}^*_{D_i}, X_{c_i}, \centerdot) \to \hat{\pi}_{k+1}(X_{c_i}, \centerdot)). \end{aligned}$$

Proof. By Lemma 4.6, $j_\sharp : \pi_{k+2}(\mathbb{X}^*_{D_i}, X_{c_i}, \centerdot) \to \pi_{k+2}(\mathbb{X}_{D_i}, X_{c_i}, \centerdot)$ is surjective. Since $\partial'_i = \partial_i \circ j_\sharp$, we get our claim after taking the quotient by the $\pi_1(X_{c_i}, \centerdot)$. □

The last step in the proof of Theorem 1.1 consists of the following two key results.

Proposition 4.8. *If* $(X_{c_i}, X^*_{a_i})$ *is* k*-connected,* $k \geq 0$*, then* $(\mathbb{X}^*_{D_i}, X_{c_i})$ *is* $(k+1)$*-connected.*

Proof. We shall suppres the lower indices i in the following.
Let $D^* := D \setminus \{a\}$. We retract D^* to the circle ∂D and cover this circle by two arcs $I \cup J$. We have the homotopy equivalences $\mathbb{X}_{D^*} \overset{\mathrm{ht}}{\simeq} \mathbb{X}_I \cup \mathbb{X}_J$ and $X_c \overset{\mathrm{ht}}{\simeq} \mathbb{X}_J$. We remind that $\mathbb{X}_D \setminus \cup_j B_{ij}$ is the total space of the trivial fibration $p_| : \mathbb{X}_D \setminus \cup_j B_{ij} \to D$, its fibre being homotopy equivalent to $X_c \setminus \cup_j B_{ij}$ (which we have also denoted by X^*_a). Then the following inclusions are homotopy equivalences of pairs and they induce isomorphisms in all relative homotopy groups:

$$(\mathbb{X}_D^*, X_c) \hookleftarrow (\mathbb{X}_{D^*} \cup (\mathbb{X}_D \setminus \cup_j B_{ij}), X_c) \hookrightarrow (\mathbb{X}_{D^*} \cup (\mathbb{X}_D \setminus \cup_j B_{ij}), \mathbb{X}_J \cup (\mathbb{X}_D \setminus \cup_j B_{ij})) \hookleftarrow$$
$$\hookleftarrow (\mathbb{X}_I \cup \mathbb{X}_J \cup (\mathbb{X}_D \setminus \cup_j B_{ij}), \mathbb{X}_J \cup (\mathbb{X}_D \setminus \cup_j B_{ij})).$$

Let us consider the following homotopy excision:

$$(9) \quad (\mathbb{X}_I \cup \mathbb{X}_J \cup (\mathbb{X}_D \setminus \cup_j B_{ij}), \mathbb{X}_J \cup (\mathbb{X}_D \setminus \cup_j B_{ij})) \hookleftarrow (\mathbb{X}_I, \mathbb{X}_{\partial I} \cup (\mathbb{X}_I \setminus \cup_j B_{ij})).$$

The right hand side pair is homotopy equivalent to $(\mathbb{X}_I, X_c \times \partial I \cup X_a^* \times I)$, which is the product of pairs $(X_c, X_a^*) \times (I, \partial I)$. Since (X_c, X_a^*) is k-connected by our hypothesis, this product is $(k+1)$-connected[3].

By the Blakers-Massey theorem (cf Gray [Gr, Cor; 16.27]), it will follow that the excision (9) is a $(k+1+q)$-equivalence, provided that the pair $(\mathbb{X}_J \cup (\mathbb{X}_D \setminus \cup_j B_{ij}), \mathbb{X}_{\partial I} \cup (\mathbb{X}_I \setminus \cup_j B_{ij}))$ is q-connected. If $q \geq 0$, then this proves our claim that the original pair $(\mathbb{X}_D^*, X_c)$ is $(k+1)$-connected.

It therefore remains to evaluate the level q. Since the inclusions $\mathbb{X}_I \setminus \cup_j B_{ij} \hookrightarrow \mathbb{X}_D \setminus \cup_j B_{ij}$ and $X_c \hookrightarrow \mathbb{X}_J \setminus \cup_j B_{ij} \hookrightarrow \mathbb{X}_D \setminus \cup_j B_{ij}$ are homotopy equivalences, it follows that the following inclusions of pairs are homotopy equivalences and therefore induce isomorphisms in all relative homotopy groups:

$$(\mathbb{X}_J \cup (\mathbb{X}_D \setminus \cup_j B_{ij}), \mathbb{X}_{\partial I} \cup (\mathbb{X}_I \setminus \cup_j B_{ij})) \hookrightarrow (\mathbb{X}_J \cup (\mathbb{X}_D \setminus \cup_j B_{ij}), \mathbb{X}_{\partial I} \cup (\mathbb{X}_D \setminus \cup_j B_{ij})) \hookleftarrow$$
$$(\mathbb{X}_J \cup (\mathbb{X}_J \setminus \cup_j B_{ij}), \mathbb{X}_{\partial J} \cup (\mathbb{X}_J \setminus \cup_j B_{ij})) \hookleftarrow (\mathbb{X}_J, \mathbb{X}_{\partial J} \cup (\mathbb{X}_J \setminus \cup_j B_{ij})).$$

We also have:

$$(\mathbb{X}_J, \mathbb{X}_{\partial J} \cup (\mathbb{X}_J \setminus \cup_j B_{ij})) \overset{\text{ht}}{\simeq} (\mathbb{X}_c \times J, X_c \times \partial J \cup X_a^* \times J) = (X_c, X_a^*) \times (J, \partial J).$$

Since (X_c, X_a^*) is supposed k-connected, this implies that q is at least $k+1$. □

Proposition 4.9. *If $(X_{c_i}, X_{a_i}^*)$ is k-connected, $k \geq 2$, then* $\operatorname{im} \hat{\partial}_i' = \operatorname{im}(\widehat{\operatorname{hvar}}_i : \hat{\pi}_{k+1}(X_{c_i}, X_{a_i}^*, \centerdot) \to \hat{\pi}_{k+1}(X_{c_i}, \centerdot))$.

Proof. As in the above proof, we suppres the lower indices i. The final purpose is to show the commutativity of the diagram (13) below. The idea is to compare it with the analogous diagram for homology groups, via the Hurewicz maps. However, we cannot do it directly, since we do not have the k-connectivity of X_c from the upper right corner of (13). We only have that the pair (X_c, X_a^*) is k-connected. So we give a proof in two steps.

---

[3]This follows from the fact that the spaces are CW-complexes and by using Switzer's [Sw, Proposition 6.13].

Let us consider the following diagram:

$$
(10)\qquad \begin{array}{ccc}
\pi_{k+2}(\mathbb{X}_D^*, X_c \times I, \centerdot) \simeq \pi_{k+2}(\mathbb{X}_D^*, X_c, \centerdot) & \stackrel{\partial''}{\longrightarrow} & \pi_{k+1}(X_c, X_a^*, \centerdot) \\
{\scriptstyle \mathrm{exc}} \uparrow & & \uparrow {\scriptstyle \mu} \\
\pi_{k+2}(\mathbb{X}_I, X_c \times \partial I \cup X_a^* \times I, \centerdot) & \stackrel{\simeq}{\longleftarrow} & \pi_{k+1}(X_c, X_a^*, \centerdot),
\end{array}
$$

where μ is the morphism defined in (2) and ∂'' denotes the boundary morphism of the homotopy exact sequence of the triple $(\mathbb{X}_D^*, X_c, X_a^*)$. The fundamental group $\pi_1(X_a^*, \centerdot)$ acts on the groups at the right hand side of this diagram and via isomorphisms on the groups at the left hand side. Indeed we have the isomorphisms $\pi_1(X_a^*, \centerdot) \simeq \pi_1(X_c, \centerdot) \simeq \pi_1(X_c \times \partial I \cup X_a^* \times I, \centerdot)$ which follow from the connectivity hypothesis on (X_c, X_a^*) and from the inclusion of spaces $X_a^* \times I \subset X_c \times \partial I \cup X_a^* \times I \subset X_c \times I$. Therefore the natural diagram (10) passes to the quotient by the π_1-action and yields:

$$
(11)\qquad \begin{array}{ccc}
\hat{\pi}_{k+2}(\mathbb{X}_D^*, X_c \times I, \centerdot) \simeq \hat{\pi}_{k+2}(\mathbb{X}_D^*, X_c, \centerdot) & \stackrel{\hat{\partial}''}{\longrightarrow} & \hat{\pi}_{k+1}(X_c, X_a^*, \centerdot) \\
{\scriptstyle \widehat{\mathrm{exc}}} \uparrow & & \uparrow {\scriptstyle \hat{\mu}} \\
\hat{\pi}_{k+2}(\mathbb{X}_I, X_c \times \partial I \cup X_a^* \times I, \centerdot) & \stackrel{\simeq}{\longleftarrow} & \hat{\pi}_{k+1}(X_c, X_a^*, \centerdot),
\end{array}
$$

We claim that (11) is a commutative diagram. In homology, there is the following diagram:

$$
(12)\qquad \begin{array}{ccc}
H_{k+2}(\mathbb{X}_D^*, X_c) & \stackrel{\partial''}{\longrightarrow} & H_{k+1}(X_c, X_a^*) \\
{\scriptstyle \mathrm{exc}} \uparrow & & \uparrow {\scriptstyle h_* - \mathrm{id}} \\
H_{k+2}(\mathbb{X}_I, X_c \times \partial I \cup X_a^* \times I) & \stackrel{\simeq}{\longleftarrow} & H_{k+1}(X_c, X_a^*),
\end{array}
$$

which is a Wang type diagram, and therefore it is *commutative*. Milnor explained a simpler version of it in [Mi, pag. 67] and his explanation holds true in the relative homology. We have used in [Ti3, §3] an equivalent commutative diagram, which differs from (12) by the upper right corner.

Let's now prove the claim. We parallel the homotopy groups diagram (11) by the homology groups diagram (12) and connect them by the Hurewicz homomorphism. We get in this way a "cubic" diagram. Notice that the homology version of the homotopy map μ is just $h_* - \mathrm{id}$, by the very definition of μ given in §3.

By the naturality of the Hurewicz morphism, all the maps between homotopy groups are in commuting diagrams with their homology versions (as "faces" of the cubic diagram). Moreover, the Hurewicz theorem may be applied each time. The connectivity conditions that the 4 Hurewicz morphisms become isomorphisms are fullfilled: (X_c, X_a^*) is k-connected by hypothesis and $(\mathbb{X}_D^*, X_c)$ is $(k+1)$-connected by Proposition 4.8. This allows one to identify the homotopy diagram (11) to the homology one (12) and since the later commutes, it is the same with the former. Our claim is therefore proved.

We now claim that the following diagram, which differs from (11) by the upper right corner only, is commutative too:

$$
\begin{tikzcd}
\hat{\pi}_{k+2}(\mathbb{X}_D^*, X_c, \bullet) \arrow[r, "\hat{\partial}'"] & \hat{\pi}_{k+1}(X_c, \bullet) \\
\hat{\pi}_{k+2}(\mathbb{X}_I, X_c \times \partial I \cup X_a^* \times I, \bullet) \arrow[u, "\widehat{\mathrm{exc}}"] & \hat{\pi}_{k+1}(X_c, X_a^*, \bullet), \arrow[l, "\simeq"'] \arrow[u, "\widehat{\mathrm{hvar}}"']
\end{tikzcd}
\tag{13}
$$

Let us justify why we may replace the morphisms μ and ∂'' in (11) by the morphisms hvar and ∂' respectively. We have the following diagram:

$$
\begin{tikzcd}
 & & \hat{\pi}_{k+1}(X_c, X_a^*, \bullet) \\
\hat{\pi}_{k+2}(\mathbb{X}_D^*, X_c, \bullet) \arrow[r, "\hat{\partial}'"] \arrow[urr, bend left, "\hat{\partial}''"] & \hat{\pi}_{k+1}(X_c, \bullet) \arrow[ur, "\hat{\jmath}_\sharp"] & \\
 & \hat{\pi}_{k+1}(X_c, X_a^*, \bullet) \arrow[u, "\widehat{\mathrm{hvar}}"'] \arrow[uur, bend right, "\hat{\mu}"'] &
\end{tikzcd}
\tag{14}
$$

The two triangles of this diagram are commutative since: 1). the right hand side triangle coincides with diagram (4) after having taken quotients by the action of $\pi_1(X_c, \bullet)$, and 2). the upper-left triangle is commutative by the naturality of the boundary morphism.

Let now $\alpha \in \pi_{k+1}(X_c, X_a^*, \bullet)$. As explained in §3, an element of the form $\mu(\alpha)$ is homotopic, relative to X_a^*, to a certain element of the absolute group $\pi_{k+1}(X_c, \bullet)$, which we have denoted by $\mathrm{hvar}(\alpha)$. Furthermore, let us remark that, by the commutativity of diagram (11), $\hat{\mu}(\alpha) = \hat{\partial}''(\beta) \in \hat{\pi}_{k+1}(X_c, X_a^*, \bullet)$ for some $\beta \in \hat{\pi}_{k+2}(\mathbb{X}_D^*, X_c, \bullet)$. We get $\widehat{\mathrm{hvar}}(\alpha) = \hat{\partial}'(\beta)$.

We have thus shown that one can "pull-back" (as in the figure (14)) the upper right corner of the diagram (11).

The proof of our statement follows now from the commutativity of the diagram (13), and the fact that the botom row and the excision from the left are both isomorphisms.

□

We may now conclude the proof of Theorem 1.1 as follows:

- The claim (a) follows by chaining together Lemmas 4.1, 4.2, 4.3, 4.6 (for $q = k + 1$), and finally Proposition 4.8.
- The claim (b) follows by chaining together Lemma 4.1, Proposition 4.4, Corollary 4.7 and Proposition 4.9.

5. Some special cases of the main theorem

The Lefschetz method for generic pencils of hyperplanes and the comparison with the axis of the pencil have been used before by several authors in order to prove connectivity results of increasing generality. Our Theorem 1.1(a) improves [Ti2] by weakening the hypotheses. As we have explained in *loc.cit.*, our result [Ti2] recovered in its turn other previous connectivity results [La, Ch1, Ch2, Ey].

In the following we show what becomes Theorem 1.1(b) in some particular cases.

5.1. **Aspects of the homotopy depth condition.** The condition (iii) of Theorem 1.1 is satisfied as soon as X is a *locally complete intersection* of dimension $\geq k+2$. This follows from the fact that the *rectified homotopical depth*[4] is $\geq k+2$ for such an X (cf [HL3]). Indeed, one can easily show that, more generally, the condition $\mathrm{rhd}\, \mathbb{X} \geq k+2$ implies the condition (iii), i.e. that $\mathrm{hd}_{\mathbb{X}\cap \mathrm{Sing}_{\mathcal{S}} p} \mathbb{X} \geq k+2$.[5] Indeed, $\mathrm{rhd}\, X \geq k+2$ implies $\mathrm{rhd}\, \mathbb{X} \geq k+2$, since we may apply [HL3, Theorem 3.2.1] to the hypersurface $\mathbb{X}$ of $X \times \mathbb{P}^1$. This in turn implies $\mathrm{hd}_v \mathbb{X} \geq k+2$, for any point $v \in \mathbb{X}$, by definition.

Moreover, $\mathrm{rhd}\, \mathbb{X} \geq k+2$ implies that the pair $(\mathbb{X}_{D_i}, X_{c_i})$ is $(k+1)$-connected, by [Ti2, Proposition 4.1]. This provides a shortcut to the proof of Part (a) of Theorem 1.1, in the sense that the condition (ii) is not needed, and therefore Lemma 4.6 and Proposition 4.8 are useless.

In what concerns Part (b) of Theorem 1.1, the condition $\mathrm{rhd}\, X \geq k+3$ implies not only (iii'), as shown above, but also the condition (ii), which is important in the proof of Part (b). Indeed, the k-connectivity of the pair $(X_{c_i}, X^*_{a_i})$ follows from a sequence of homotopy excisions, which reduces the problem to proving the k-connectivity of each pair $(X_{c_i} \cap B_{ij}, X_{c_i} \cap \partial B_{ij})$. This is true since $X_{c_i} \cap B_{ij}$ is $(k+1)$-connected and $X_{c_i} \cap \partial B_{ij}$ is k-connected. The former follows from [Ti2, Proposition 3.4] which is based on [HL3, Corollary 4.2.2]. The proof goes as follows: $\mathrm{rhd}\, X \geq k+3$ implies $\mathrm{rhd}\, X_{a_i} \geq k+2$, by [HL3, Theorem 3.2.1]. This implies $\mathrm{hd}_{a_{ij}} X_{a_i} \geq k+2$, which shows that $X_{a_i} \cap \partial B_{ij}$ is k-connected. Finally, we already know that $X_{a_i} \cap \partial B_{ij}$ is homeomorphic to $X_{c_i} \cap \partial B_{ij}$, so this ends the proof of our claim.

5.2. **Pencils on X having no singularities in the axis.** Let us consider the situation $(A \times \mathbb{P}^1) \cap \mathbb{X} \cap \mathrm{Sing}_{\mathcal{S}} p = \emptyset$. We say that "$X$ does not contain singularities along the axis of the pencil". Even so, there might still be singularities in the axis on Y, and they certainly influence the topology of

---

[4] see [Ti2] and the source [HL3] for the definition and properties of this notion which goes back to Grothendieck; in particular, rhd does not depend on the stratification.

[5] see also [Ti3] for a similar remark in homology.

the pencil on X. However, this situation be further investigated as follows, see also §5.4 for an interesting case. We claim that the homotopy depth condition (iii), resp. (iii'), may be replaced by the following more general, global condition:

$$(X, X \setminus \Sigma) \text{ is } (k+2)\text{-connected, resp. } (k+3)\text{-connected,} \tag{15}$$

where $\Sigma := \sigma(\mathrm{Sing}_{\mathcal{S}} p) \subset Y$. Then our Theorem 1.1(a) specializes to a connectivity statement which recovers Eyral's main result [Ey], where such a condition was used for generic pencils. Indeed, we notice that the homotopy depth condition was used only when comparing $\mathbb{X}_{D_i}$ to $\mathbb{X}^*_{D_i}$. We may therefore get rid of this comparison by replacing everywhere in the proof of Theorem 1.1 the space X by the space $X \setminus \Sigma$. In particular, this reduces to tautologies Lemma 4.6 and Corollary 4.7. We then get the conclusion of Theorem 1.1 for the inclusion $X_c \hookrightarrow X \setminus \Sigma$ instead of the inclusion $X_c \hookrightarrow X$. But, at this stage, we may just plug in X in the place of $X \setminus \Sigma$ since the condition (15) tells that $X \setminus \Sigma$ and X have isomorphic homotopy groups up to π_{k+1}, and this is all we need.

Still in case $(A \times \mathbb{P}^1) \cap \mathbb{X} \cap \mathrm{Sing}_{\mathcal{S}} p = \emptyset$, let us observe that if the condition (i) of Theorem 1.1 is fulfilled, then the condition (ii) becomes equivalent to:

$$(X^*_{a_i}, A \cap X^*_{a_i}) \text{ is } (k-1)\text{-connected.} \tag{16}$$

Indeed, when there are no singularities in the axis, we have $A \cap X^*_{a_i} = A \cap X_c$, for any i and our claim follows from the exact sequence of the triple $(X_c, X^*_{a_i}, A \cap X^*_{a_i})$.

We have to point out that even if there are "no singularities

5.3. **The classical case: generic pencils on the projective space.** From the preceding observations on pencils without singularities in the axis, one may derive the following statement for *complements of arbitrarily singular subspaces* in $\mathbb{P}^n$. Its proof goes by induction, based on the abundance of generic pencils, i.e. pencils with no singularities along the axis and such that the axis A is transversal to all strata and is not included into the subspace V. The proof actually follows the pattern of the homology proof discussed in [Ti3, §4.3], supplemented by our homotopy considerations in §4 above.

Corollary 5.1. *Let $\mathcal{Y} = \mathbb{P}^n$. Let $V \subset \mathcal{Y}$ be a singular complex algebraic subspace, not necessarily irreducible. For any hyperplane $H \subset \mathcal{Y}$ transversal to all strata of V, we have:*

(a) *$H \cap (\mathcal{Y} \setminus V) \hookrightarrow \mathcal{Y} \setminus V$ is a $(n + \mathrm{codim} V - 2)$-equivalence.*
(b) *There exist generic pencils of hyperplanes having H as a generic member. If $\dim V \leq 2n - 5$ then the kernel of the surjection*

$\pi_{n+\mathrm{codim}V-2}(H\cap(\mathcal{Y}\setminus V),\centerdot) \twoheadrightarrow \pi_{n+\mathrm{codim}V-2}(\mathcal{Y}\setminus V,\centerdot)$ *is generated by the images of the variation maps* hvar_i *of such a generic pencil.* □

Proof. The part (a) of this Corollary is well-known, see e.g. [Ch1, Ey], and its proof goes by induction on $\dim V$, since by repeatedly slicing we arrive to the case $\dim V = 0$. The conditions of Theorem 1.1 are satisfied at each induction step, as follows. The generic slice X_c is path connected at each step. Condition (iii), resp. (iii'), is empty at every step, since a generic pencil has no singularities in $\mathbb{P}^{n-j}\setminus V$. Condition (ii) is implied by (i) since it is equivalent to (16), as shown above. The condition (i) in case $\dim V = 0$, is the level of connectivity of the pair $(\mathbb{P}^{n-\dim V-1}, \mathbb{P}^{n-\dim V-2})$, which is known to be equal to $k = 2(n-\dim V-1)-1$. Then Theorem 1.1(a) gives the level of connectivity $k+1$ of the higher pair $(\mathbb{P}^{n-\dim V}\setminus V\cap\mathbb{P}^{n-\dim V}, \mathbb{P}^{n-\dim V-1})$, which becomes the condition (i) for the next induction step. This proves part (a). Part (b) also follows from Theorem 1.1(b), since we have seen that the conditions are inductively fulfilled. In addition, we have to insure that the initial step verifies the requirement $k\geq 2$. This amounts to $\dim V\leq 2n-5$. □

The part (b) of this Corollary seems to be a new result since we consider the general situation of complements of arbitrarily singular subspaces. There is a single case which is not covered by the condition $\dim V\leq 2n-5$, namely the case $n=3$, $\dim V=2$ (since hvar is not well defined for π_2). Of course, the case $n=2$ does not matter here since it is covered by the classical Zariski-van Kampen theorem.

In the particular case $V\subset\mathbb{C}^n$ is a hypersurface with at most isolated singularities and transversal to the hyperplane at infinity, a similar type of result, but with different background, has been proved by Libgober [Li], including at the π_2 level.

5.4. **A complementary case.** It appears that the case when V contains a member of the pencil, which was excluded in our main theorem (since $A\subset V$ in this case), can also be treated. For homology groups, this was explained in [Ti3, §4-5]. Our main result can be reformulated, with much less restrictive conditions.

Theorem 5.2. *Let* $h: Y \dashrightarrow \mathbb{P}^1$ *define a pencil with isolated singularities, such that* V *contains a member of the pencil. Le* X_c *be path connected and let* $(X_c, X^*_{a_i})$ *be* k*-connected, for a general member* X_c *and any atypical one* $X^*_{a_i}$*. Then:*

(a) *If* $k\geq 0$ *and* $(X, X\setminus\Sigma)$ *is* $(k+1)$*-connected, then the inclusion* $X_c\hookrightarrow X$ *is a* $(k+1)$*-equivalence.*

(b) *If $k \geq 2$ and if $(X, X \setminus \Sigma)$ is $(k+2)$-connected, then the kernel of the surjection $\pi_{k+1}(X_c) \twoheadrightarrow \pi_{k+1}(X)$ is generated by the images of the variation maps* hvar_i.

□

Proof. The proof follows by revisiting the one of Theorem 1.1. We then observe that in our case $X_{\mathcal{D}}$ is just homotopy equivalent to X, according to the definition of $\mathcal{D}$. We have used the condition (i) only to compare these two spaces, in Lemma 4.1, so condition (i) does not occur any more in our situation. Let us further remark that the condition $(A \times \mathbb{P}^1) \cap \mathbb{X} \cap \mathrm{Sing}_{\mathcal{S}} p = \emptyset$ considered in §5.2 is satisfied here, and so we may use condition (15) istead of (iii) and (iii'), respectively. □

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