DISTORTION OF WREATH PRODUCTS IN SOME FINITELY PRESENTED GROUPS

SEAN CLEARY

ABSTRACT. Wreath products such as $\mathbb{Z} \wr \mathbb{Z}$ are not finitely-presentable yet can occur as subgroups of finitely presented groups. Here we compute the distortion of $\mathbb{Z} \wr \mathbb{Z}$ as a subgroup of Thompson's group F and as a subgroup of Baumslag's metabelian group G. We find that $\mathbb{Z} \wr \mathbb{Z}$ is undistorted in F but is at least exponentially distorted in G.

1. INTRODUCTION

Here we consider aspects of the question of the distortion of infinitely-related groups as subgroups of finitely-presented groups. Higman [7] showed that every recursively-presentable group occurs as a subgroup of a finitely-presented group, but it is not clear in general what happens to the geometry of the group since this embedding uses complicated algebraic methods and methods from recursive function theory which may affect the geometry of the group severely. Ol′shanskiĭ [8] constructs isometric embeddings of recursively presentable groups into finitely presented groups using difficult methods that do not lead to easily constructed examples. In the particular concrete cases here, we consider concrete embeddings of one of the simplest finitely-generated but not finitely-presentable groups, $\mathbb{Z} \wr \mathbb{Z}$. We consider two embeddings of $\mathbb{Z} \wr \mathbb{Z}$ into finitely presented groups. The first is as a subgroup of Thompson's group F and the second is as subgroup of Baumslag's remarkable finitely presented metabelian group which contains $\mathbb{Z} \wr \mathbb{Z}$ and thus a free abelian subgroup of infinite rank. The distortion of the metric of $\mathbb{Z} \wr \mathbb{Z}$ is linear in Thompson's group F but is exponential in Baumslag's group.

2. BACKGROUND

2.1. **Metrics of wreath products.** We construct the wreath product in the standard manner, as special case of a semi-direct product. Given two

groups G and H, we form the wreath product $G \wr H$ by taking the direct product of $|H|$ copies of G with copy of G indexed by an element of H. The generators of G act on the conjugate copies of G, while generators of H act on the coordinates to determine to which of these conjugate copies of G the generators of G will be applied.

Two of the simplest infinite wreath products are the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$ and $\mathbb{Z} \wr \mathbb{Z}$. Cleary and Taback [5] analyzed aspects of the metric geometry of those groups and other wreath products. There are natural normal forms for elements in these groups which lead to geodesic words for elements in these groups with respect to their standard generating sets.

For $\mathbb{Z} \wr \mathbb{Z}$, we consider the standard presentation:

$$< a, t | [a^{t^i}, a^{t^j}], \text{ for } i, j \in \mathbb{Z} >$$

where we denote the conjugate $b^{-1}ab$ of a by b as a^b and the commutator $aba^{-1}b^{-1}$ by $[a, b]$.

Geometrically, we can think about this wreath product as a set of parallel copies of $\mathbb{Z}$ strung together along their respective origins. We can think of this as a string of counters, arranged from left to right with one counter distinguished as the origin. As in the lamplighter group, we imagine a cursor which moves along the string of counters and will point to a particular one of these counters as being of current interest. The generator a acts as a generator of $\mathbb{Z}$ in the factor to which the cursor currently points and increases the counter in that factor, and the generator t moves the cursor to the right to the next counter. A typical such word is illustrated in Figure1.

The starting configuration of these counters, corresponding to the identity element in $\mathbb{Z} \wr \mathbb{Z}$, is with all of the counters at zero and the cursor resting at the counter designated at the origin. We consider a word in these generators as a sequence of instructions to move the cursor and change the counter in the current factor. After application of a long string of the generators, we will be in a state where a finite number of counters are non-zero and the cursor points at a particular counter, called the "final position" of the cursor for that word.

We define $a_n = a^{t^n}$ and note that a_n is a generator of the conjugate copy of $\mathbb{Z}$ indexed by n. These a_n commute and we can put any word in the generators into one of two normal forms, 'right-first' and 'left-first', as described by Cleary and Taback in [5]:

$$rf(w) = a_{i_1}^{e_1} a_{i_2}^{e_2} \ldots a_{i_k}^{e_k} a_{-j_1}^{f_1} a_{-j_2}^{f_2} \ldots a_{-j_l}^{f_l} t^m$$

or

$$lf(w) = a_{-j_1}^{f_1} a_{-j_2}^{f_2} \ldots a_{-j_l}^{f_l} a_{i_1}^{e_i} a_{i_2}^{e_i} \ldots a_{i_k}^{e_k} t^m$$

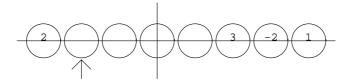with $i_k > \ldots i_2 > i_1 \geq 0$ and $j_l > \ldots j_2 > j_1 > 0$ and $e_i, f_j \neq 0$.

FIGURE 1. Diagram for $w = a_2^3 a_3^{-2} a_4 a_{-3}^2 t^{-2}$. The origin in the wreath product direction is indicated by a vertical line, empty circles denote counters which are zero, and the final cursor position is indicated by the arrow.

The final resting position of the cursor is easily seen to be m from either of these normal forms, and we can see that the leftmost non-zero counter is in position $-j_l$ and the rightmost non-zero counter is in position i_k.

In the 'right-first' form, $rf(w)$, the cursor moves first to the right from the origin, changing the counters in the appropriate factors as the cursor moves to the right. Then the cursor moves back to the origin not affecting any of the counters until passing the origin. Past the origin, the cursor continues to works leftwards, again changing the counters in the appropriate factors. Finally, the cursor moves to its ending location from the leftmost nonzero counter to the left of the origin.

The 'left-first' form is similar, but instead of initially moving to the right, the cursor begins by moving toward the left.

At least one of these normal forms will lead to minimal-length representation for w, depending upon the final location of the cursor. If m is non-negative, then the left-first normal form will lead to a geodesic representative, and if m is non-positive, the right-first normal form will lead to a geodesic representative, as described in [5], which gives the following measurement of length:

Proposition 2.1 (Prop 3.8 of [5]). *If a word $w \in \mathbb{Z} \wr \mathbb{Z}$ is in either normal form given above, we measure the word length of w with respect to $\{a, t\}$ and have*

$$|w| = \sum_{n=1}^{k} |e_{i_n}| + \sum_{n=1}^{l} |f_{j_n}| + min\{2j_l + i_k + |m - i_k|, 2i_k + j_l + |m + j_l|\}.$$

The first two terms are the minimum number of applications of $a^{\pm 1}$ needed to put all of the counters into their desired states and the last term is the minimum possible amount of movement required to visit the left- and right-most non-zero counters and then the final position of the cursor, counting the required applications of $t^{\pm 1}$.

The word $w = a_2^3 a_3^{-2} a_4 a_{-3}^2 t^{-2}$ pictured in Figure 1 has geodesic representatives in right-first normal form since the final position of the cursor is to the left of the origin. One such minimal representative is $t^2 a^3 t a^{-2} t a t^{-7} a^2 t$ of length 20.

3. $\mathbb{Z} \wr \mathbb{Z}$ as a subgroup of Thompson's group F

Thompson's group F is a remarkable, finitely-generated, finitely-presented group which can be understood via a wide range of perspectives. Cannon, Floyd, and Parry [3] give an excellent overview of the properties of F. The standard infinite presentation of F is given by:

$$< x_0, x_1, \ldots | x_n^{x_i} = x_{n+1} \text{ for } i < n > .$$

Since $x_2 = x_1^{x_0}$ and so on, F is generated by the first two generators and we can define $x_{n+1} = x_n^{x_0}$ to express all generators and thus all group elements in terms of x_0 and x_1. Furthermore, all of these infinitely many relations are consequences of the first two non-trivial relations, so we have the standard finite presentation:

$$< x_0, x_1 | x_2^{x_1} = x_3, x_3^{x_1} = x_4 > .$$

Thompson's group F can be described in terms of rooted tree pair diagrams, and there is a natural method of converting between words in a normal form with respect to the infinite generating set and tree pair diagrams, via the method of leaf exponents, as described in Cannon, Floyd and Parry [3]. There is a natural notion of a reduced tree pair diagram described there and there are efficient means to convert between the unique normal form for an element of F and the unique reduced tree pair diagram for that word.

We consider a rooted binary tree with n leaves as being constructed of $n - 1$ "carets," which are interior nodes of the tree together with the two downward directed edges from that node. The "left side" of a tree consists of nodes and edges which are connected to the root by a path consisting only of left edges, and similarly the "right side" of a tree consists of nodes and edges which are connected to the root by a path consisting only of right edges. A tree pair diagram (S, T) is made up of a 'positive' tree T and a 'negative' tree S.

The reduced tree pair diagrams for x_0 and x_1 and a typical word in normal form are pictured in Figures 2 and 3.

To understand the metric properties of F, we consider expressing words with respect to the finite generating set. Burillo, Cleary and Stein [2] estimated the word length in terms of the number of carets and showed that the
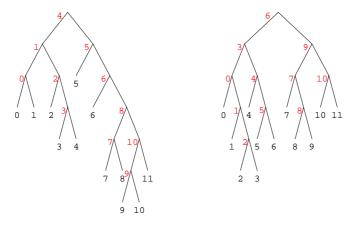
FIGURE 2. Tree pair diagram with carets and leaves numbered for $x_0^2 x_1 x_2 x_4 x_5 x_7 x_8 x_9^{-1} x_7^{-1} x_3^{-1} x_2^{-1} x_0^{-2}$.
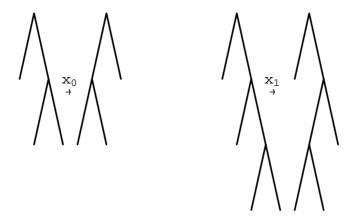

FIGURE 3. The tree pair diagrams representing the generators x_0 and x_1 of F.

number of carets is quasi-isometric to the word length. Fordham [6] developed a remarkable method using tree pair diagrams to efficiently compute exact word length and find minimal length representatives of words.

We can understand word length of elements represented as tree pair diagrams by understanding the how the generators change the tree pair diagram for w to that for wg for the generators, as described in Fordham [6] and in
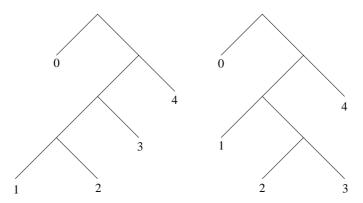
FIGURE 4. Tree pair diagram for $x_1 x_2 x_1^{-2}$, the image of a under ϕ.

Cleary and Taback [4]. The right actions of the generators can be described as 'rotations' which change the tree.

The wreath product $\mathbb{Z} \wr \mathbb{Z}$ is a subgroup of F and can be realized in many different ways. Perhaps the simplest is as the subgroup H generated by x_0 and $h = x_1 x_2 x_1^{-2}$, pictured in Figure 4. The isomorphism between this subgroup and $\mathbb{Z} \wr \mathbb{Z}$ is given by the homomorphism $\phi(w) : L \to F$ where $\phi(a) = x_1 x_2 x_1^{-2}$ and $\phi(t) = x_0$. The isomorphism is readily established after we see that x_0 conjugates $\phi(t)$ and its powers to elements that commute and that there are no other relations.

To understand the distortion of the subgroup H in F, we compare the word length of an element $w = a_{i_1}^{e_1} a_{i_2}^{e_2} \ldots a_{i_k}^{e_k} a_{-j_1}^{f_1} a_{-j_2}^{f_2} \ldots a_{-j_l}^{f_l} t^m$ with its image in F.

Theorem 3.1. *The subgroup H isomorphic to $\mathbb{Z} \wr \mathbb{Z}$ in F generated by $x_0 = \phi(t)$ and $h = x_1^2 x_2^{-1} x_1^{-1} = \phi(a)$ is undistorted.*

Proof: We count the number of carets of the image of a word w. First, we consider the case when $m = 0$ and then the cases where m is nonzero.

Case $m = 0$: Here, the image of the word as a tree pair diagram has a characteristic form where the root of the positive tree is paired with the root of the negative tree, such as that shown in Figure 5. In the general case where both k and l are positive, we have the following carets:

- A single root caret
- $i_k + 1$ right carets
- $\sum_{n=1}^{k} (|e_n| + 1)$ interior carets below the right arm of the tree
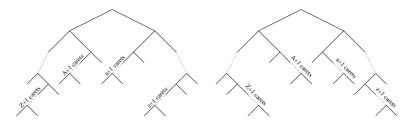
FIGURE 5. The tree pair diagram for the image $\phi(w)$ of a word $w = a_0^a a_1^b \ldots a_n^z a_{-1}^A a_{-2}^B \ldots a_{-m}^Z$ with t exponent sum 0 and all positive exponents for a_i.

- j_l left carets
- $\sum_{n=1}^{l}(|f_n| + 1)$ interior carets below the left arm of the tree

This gives a total $N(\phi(w)) = i_k + j_l + 2 + \sum_{n=1}^{k}(|e_n|+1) + \sum_{n=1}^{l}(|f_n|+1)$ carets in the image of w. By Burillo, Cleary and Stein [2], the number of carets is quasi-isometric to the word length in F with respect to $\{x_0, x_1\}$ and since the length of w in $\mathbb{Z} \wr \mathbb{Z}$ is $2j_l + 2i_k + \sum_{n=1}^{k}|e_n| + \sum_{n=1}^{l}|f_n|$, we see that these lengths are quasi-isometric.

The image of a typical word with all e_n and f_n positive is shown in Figure 5, corresponding to a series of rightward rotations at nodes distance one from the sides of the tree.

Case $m > 0$:

In this case, we start with the same tree pair diagram for the $m = 0$ case and apply x_0 on the right m times. Each application of x_0 will change the negative tree by moving the root caret to a right caret and the topmost left caret to the root, if there is a left caret. If there is no left caret, a new caret will need to be added for each such application. For each application of x_0 which requires a new caret, in the negative tree, that new caret will become the root caret and in the positive tree, the new caret will be added as the left child of the leftmost caret. Since there are j_l left carets, if $m \leq j_l$, we do not need to add any carets and the number of carets is $i_k + j_l + 2 + \sum_{n=1}^{k}(|e_n| + 1) + \sum_{n=1}^{l}(|f_n| + 1)$ as before. If $m > j_l$, we will need to add $m - j_l$ new carets and will have $i_k + j_l + 2 + \sum_{n=1}^{k}(|e_n| + 1) + \sum_{n=1}^{l}(|f_n| + 1) + m - j_l$ carets. Again, these quantities give lengths which are comparable to word length in $\mathbb{Z} \wr \mathbb{Z}$.

Case $m < 0$:

Again, we start with the same tree pair diagram for the $m = 0$ case and apply x_0^{-1} on the right m times. Each application of x_0^{-1} will change

the negative tree by moving the root caret to become a left caret and the topmost right caret to the root, if there is a right caret. If there is no right caret, a new caret will need to be added for each such application. Since there are i_k left carets, if $-m \leq i_k$, we have that the number of carets is $i_k + j_l + 2 + \sum_{n=1}^{k}(|e_n| + 1) + \sum_{n=1}^{l}(|f_n| + 1)$ as before. If $-m > i_k$, we have $i_k + j_l + 2 + \sum_{n=1}^{k}(|e_n| + 1) + \sum_{n=1}^{l}(|f_n| + 1) - m + i_k$ carets.

Thus in all cases ϕ does not distort distances more than linearly, so the subgroup H isomorphic to $\mathbb{Z} \wr \mathbb{Z}$ is undistorted in F.

$\square$.

We can obtain more precise estimates of the quasi-isometry constants using Fordham's method [6] for computing exact lengths in F. We can keep track of the particular caret pairings and their weights and we find that the caret pairings that occur are easily computed. Caret pairing types are described in [6] and [4]. For example, in the case where $m = 0$ and both l and k are positive, we note that we have the following caret pairs:

- One caret pair of type (L_0, L_0) from the leftmost carets, contributing no weight.
- j_l caret pairs of type (L_L, L_L) from the left side and root, contributing weight $2j_l$.
- $i_k - 1$ caret pairs of types (R_*, R_*) not of type (R_0, R_0), contributing weight $2(i_k - 1)$.
- One caret pair of type (R_0, R_0) from the rightmost carets, contributing no weight.
- For each $e_n > 0$, there will be a single pairing of type (I_0, I_0) contributing weight 2 and $e_n - 1$ pairings of type (I_0, I_R), contributing weight $4(e_n - 1)$.
- For each $e_n < 0$, there will be a single pairing of type (I_0, I_0) contributing weight 2 and $|e_n| - 1$ pairings of type (I_R, I_0), contributing weight $4(|e_n| - 1)$.
- Similarly, for the interior carets from the left side of the tree, we have for each f_n, there will be a single pairing of type (I_0, I_0) contributing weight 2 and $|f_n| - 1$ pairings of type (I_0, I_R) or (I_R, I_0), contributing weight $4(|f_n| - 1)$.

These will give a total weight of $2j_l + 2(i_k - 1) + 2k + 4\sum|e_n| + 2l + 4\sum|f_n| = 2j_l + 2i_k + 2k + 2l + 4\sum|e_n| + 4\sum|f_n| - 2$ in the case when $m = 0$, which compares to the corresponding length in $\mathbb{Z} \wr \mathbb{Z}$ of $2j_l + 2i_k + \sum|e_n| + \sum|f_n|$.

Again, these give lengths which are comparable to word length in $\mathbb{Z} \wr \mathbb{Z}$. After similar analysis for other cases, we have that for a word w in $\mathbb{Z} \wr \mathbb{Z}$, we have:

$$|w|_{\mathbb{Z} \wr \mathbb{Z}} - 2 \leq |\phi(w)|_F \leq 4|w|_{\mathbb{Z} \wr \mathbb{Z}}.$$

4. $\mathbb{Z} \wr \mathbb{Z}$ as a subgroup of Baumslag's metabelian group

Baumslag [1] introduced the group $G = <a, s, t | [s, t], [a^t, a], a^s = aa^t>$ to show that a finitely presented metabelian group can contain free abelian subgroups of infinite rank. This group in fact contains $\mathbb{Z} \wr \mathbb{Z}$ – all relators of the form $[a^{t^i}, a^{t^j}]$ are consequences of these three, so the subgroup H generated by a and t is isomorphic to $\mathbb{Z} \wr \mathbb{Z}$.

Here we examine the distortion of this subgroup in G

Theorem 4.1. *The subgroup H has at least exponential distortion in G.*

Proof:

First, we note that s conjugates elements in H to other elements in H in a manner illustrated here:

$$a^{(s^2)} = (a^s)^s = (aa^t)^s = a^s(a^s)^t = aa^t(aa^t)^t = aa^t a^t a^{t^2} = a_0 a_1^2 a_2$$

In terms of the notation described above, we have $a_n^s = a_n a_{n+1}$. Further conjugation by s leads to increasingly long words:

$$a^{(s^3)} = (a_0 a_1^2 a_2)^s = a_0 a_1 a_1^2 a_2^2 a_2 a_3 = a_0 a_1^3 a_2^3 a_3$$

and we notice the occurrence of the binomial coefficients with repeated iteration.

In general, we have:

$$a^{s^n} = a_0^{\binom{n}{0}} a_1^{\binom{n}{1}} \ldots a_n^{\binom{n}{n}}.$$

As an element of G, this has length $2n + 1$ and it lies in the subgroup H, as there is a representative with no occurrences of s.

To compute the length of this element in the subgroup H with respect to its generators a and t, we use the method described in Section 2 and get

$$|a^{s^n}|_H = 2n + \sum_{i=0}^{n} \binom{n}{i} = 2n + 2^n.$$

Thus we have $|a^{s^n}|_H = 2n + 2^n$ while $|a^{s^n}|_G = 2n + 1$, so the wreath product $\mathbb{Z} \wr \mathbb{Z}$ is exponentially distorted in G.

□

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Sean Cleary
Department of Mathematics
The City College of New York
City University of New York
New York, NY 10031
E-mail: cleary@sci.ccny.cuny.edu