ON MODULI OF SMOOTHNESS OF FRACTIONAL ORDER

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Abstract. In this paper we consider the properties of moduli of smoothness of fractional order. The main result of the paper describes the equivalence of the modulus of smoothness and a function from some class.

1. Introduction

In 1977 P.L. Butzer, H. Dyckhoff, E. Goerlich, R.L. Stens (see [2]) and R.Tabersky (see [14]) introduced the modulus of smoothness of fractional order. This notion could be considered as a direct generalization of the classical modulus of smoothness, and it is more natural to use it for a number of problems of harmonic analysis (see, for example, [2], [5], [7], [10]).

The important problem of approximation theory and theory of Fourier series is the problem of description of moduli of smoothness (see [1], [4], [8], [11]). One can consider this problem from the viewpoint of description of majorant of smoothness moduli. Recently, A. Medvedev (see [6]) has proved that for any modulus of continuity on $[0,\infty)$ there exists a concave majorant that is infinitely differentiable. In this paper, we obtain the description of the modulus of smoothness of fractional order from the viewpoint of the order of decreasing to zero of the modulus of smoothness.

Let us introduce several definitions. If $1 \le p < \infty$, let L_p be the space of 2π-periodic, measurable functions $f(x)$ such that $\|f\|_p = \left(\int_0^{2\pi} |f(x)|^p \, dx\right)^{\frac{1}{p}} < \infty$. Similarly, let L_∞ be the space of 2π-periodic, continuous functions $f(x)$ with $\|f\|_\infty = \max_{x\in[0,2\pi]} |f(x)|$. We will define the difference of fractional order β $(\beta > 0)$ of function $f(x)$ at the point x $(x \in \mathbf{R})$ with increment h $(h \in \mathbf{R})$

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by

$$\triangle_h^\beta f(x) = \sum_{\nu=0}^{\infty} (-1)^\nu \binom{\beta}{\nu} f(x + (\beta - \nu)h),$$

where $\binom{\beta}{\nu} = \frac{\beta(\beta-1)\cdots(\beta-\nu+1)}{\nu!}$ for $\nu > 1$, $\binom{\beta}{\nu} = \beta$ for $\nu = 1$, $\binom{\beta}{\nu} = 1$ for $\nu = 0$.

The modulus of smoothness of order β $(\beta > 0)$ of function $f \in L_p$, $1 \le p \le \infty$, is given by $\omega_\beta(f,t)_p = \sup_{|h|\le t} \left\| \triangle_h^\beta f(\cdot) \right\|_p$ (see [2],[14]).

Let Φ_γ $(\gamma \in \mathbf{R})$ be the set of nonnegative, bounded functions $\varphi(\delta)$ on $(0,\infty)$ such that

a): $\varphi(\delta) \to 0$ as $\delta \to 0$,
b): $\varphi(\delta)$ is nondecreasing,
c): $\varphi(\delta)\delta^{-\gamma}$ is nonincreasing.

If for $f \in L_p$ there exists $g \in L_p$ such that $\lim_{h\to 0+} \left\| h^{-\beta} \triangle_h^\beta f(\cdot) - g(\cdot) \right\|_p = 0$ then g is called the Liouville-Grunwald-Letnikov derivative of order $\beta > 0$ of a function f in the L_p-norm, denoted by $g = D^\beta f$ (see [2], [12]). Set $W_p^\beta := \left\{ f \in L_p : D^\beta f \text{ exists as element in } L_p \right\}$. The K-functional is given by $K(f,t,L_p,W_p^\beta) := \inf_{g\in W_p^\beta} \left(\|f-g\|_p + t \left\| D^\beta g \right\|_p \right)$.

2. Results

Let $f(x) \in L_p$, $p \in [1,\infty]$ and $\beta > 0$. It is clear that (see [12])

$$\left| \binom{\beta}{\nu} \right| = \left| \frac{\beta(\beta-1)\cdots(\beta-\nu+1)}{\nu!} \right| \le \frac{C(\beta)}{\nu^{\beta+1}}, \quad \nu \in \mathbf{N}$$

implies $C^*(\beta) := \sum_{\nu=0}^{\infty} \left| \binom{\beta}{\nu} \right| < \infty$ and the fractional difference $\triangle_h^\beta f(x)$ is defined almost everywhere and belongs to L_p:

$$\|\triangle_h^\beta f(\cdot)\|_p \le C^*(\beta) \, \|f(\cdot)\|_p. \tag{1}$$

It is easy to write the following representation for $C^*(\beta)$ (see [14]):

$$C^*(\beta) = \begin{cases} 2\sum_{\nu=0}^{k} \binom{\beta}{2\nu}, & \text{if} \quad 2k < \beta \le 2k+1 \ (k = 0,1,2,\cdots), \\ 2\sum_{\nu=0}^{k} \binom{\beta}{2\nu+1}, & \text{if} \quad 2k+1 < \beta \le 2k+2 \ (k = 0,1,2,\cdots). \end{cases} \tag{2}$$

The fractional differences and moduli of smoothness have some useful properties and we shall establish some of them in the following lemmas.

Lemma 2.1. ([2], [14]) *Let $f \in L_p$, $p \in [1,\infty]$, $\alpha,\beta > 0$; $h \in \mathbf{R}$. Then*

(a): $\triangle_h^\alpha(\triangle_h^\beta f(x)) = \triangle_h^{\alpha+\beta} f(x)$ *for almost every* x;

(b): $\|\triangle_h^{\alpha+\beta} f(\cdot)\|_p \leq C^*(\alpha)\|\triangle_h^{\beta} f(\cdot)\|_p$;
(c): $\lim\limits_{h\to 0+} \|\triangle_h^{\alpha} f(\cdot)\|_p = 0$.

Lemma 2.2. *Let $f, f_1, f_2 \in L_p$, $p \in [1,\infty]$, $\alpha,\beta > 0$; $x, h \in \mathbf{R}$. Then*

(a): *$\omega_\beta(f,\delta)_p$ is nondecreasing nonnegative function of δ on $(0,\infty)$ with $\lim\limits_{\delta\to 0+} \omega_\beta(f,\delta)_p = 0$;*
(b): $\omega_\beta(f_1+f_2,\delta)_p \leq \omega_\beta(f_1,\delta)_p + \omega_\beta(f_2,\delta)_p$;
(c): $\omega_{\alpha+\beta}(f,\delta)_p \leq C^*(\alpha)\omega_\beta(f,\delta)_p$;
(d): *if $\lambda \geq 1$, then* $\omega_\beta(f,\lambda\delta)_p \leq C(\beta)\lambda^\beta \omega_\beta(f,\delta)_p$;
(e): *if $0 < t \leq \delta$, then* $\omega_\beta(f,\delta)_p\, \delta^{-\beta} \leq C(\beta)\omega_\beta(f,t)_p\, t^{-\beta}$.

Indeed, we immediately have $(a)-(c)$ from Lemma 2.1, (d) was proved in [2], and (d) implies (e):

$$\omega_\beta(f,\delta)_p = \omega_\beta\left(f, \frac{\delta}{t}t\right)_p \leq C(\beta)\left(\frac{\delta}{t}\right)^\beta \omega_\beta(f,t)_p.$$

Lemma 2.3. *Let $f \in L_p$, $p \in [1,\infty]$, $\beta > 0$.*

(a): *If $\beta \in \mathbf{N}$, then* $\left\|\triangle_\pi^\beta f(\cdot)\right\|_p \leq 2^{[\frac{\beta+1}{2}]}\left\|\triangle_{\frac{\pi}{2}}^\beta f(\cdot)\right\|_p$.
(b): *If $\beta \notin \mathbf{N}$, then* $\left\|\triangle_\pi^\beta f(\cdot)\right\|_p \leq 2^{[\frac{\beta+1}{2}]+1}\left\|\triangle_{\frac{\pi}{2}}^\beta f(\cdot)\right\|_p$.

Corollary 2.4. *For a function $\varphi(t) = t^\alpha$ $(0 \leq t \leq \pi)$ to be a modulus of smoothness of order $\beta\,(\beta > 0)$ of a function $f(\cdot) \in L_p$, $1 \leq p \leq \infty$ it is necessary to have $\alpha \leq \left[\frac{\beta+1}{2}\right] + 1$.*

Theorem 2.5. *Let $p \in [1,\infty]$, $\beta > 0$.*

(A): *If $f(\cdot) \in L_p$, then there exists a function $\varphi(\cdot) \in \Phi_\beta$ such that*

$$\varphi(t) \leq \omega_\beta(f,t)_p \leq C(\beta)\varphi(t) \ \ (0 < t < \infty),$$

where $C(\beta)$ is a positive constant depending only on β.

(B): *If $\varphi(\cdot) \in \Phi_\beta$, then there exist a function $f(\cdot) \in L_p$ and a constant $t_1 > 0$ such that*

$$C_1(\beta)\omega_\beta(f,t)_p \leq \varphi(t) \leq C_2(\beta)\omega_\beta(f,t)_p \ \ (0 < t < t_1),$$

where $C_1(\beta), C_2(\beta)$ are positive constants depending only on β.

Corollary 2.6. *Let $p \in [1,\infty]$, $\beta > 0$.*

(A): *If $f(\cdot) \in L_p$, then there exists a function $\varphi(\cdot) \in \Phi_\beta$ such that*

$$C_1(\beta)\varphi(t) \leq K(f, t^\beta, L_p, W_p^\beta) \leq C_2(\beta)\varphi(t). \tag{3}$$

is true.

(B): *If $\varphi(\cdot) \in \Phi_\beta$, then there exists a function $f(\cdot) \in L_p$ such that (3) is true.*

Remark 2.7. 1). *We can replace the condition $f \in L_p$ by condition $f \in L_\infty$ in the part* (B) *of Theorem 2.5.*
2). *Note that theorem 2.5 for $\beta \in \mathbf{N}$ was proved in* [11]. *Also, for H^p-spaces the analogue of Corollary 2.6 for $\beta \in \mathbf{R}_+$ and the analogue of theorem 2.5 for $\beta \in \mathbf{N}$ were proved in* [5].

3. Proofs

Proof of Lemma 2.3. The first inequality was proved in [3]. Let $\beta > 1, \notin \mathbf{N}$. We shall use the following representation (see [14])

$$\triangle_{2h}^{\beta} f(x-2\beta h) = \sum_{\nu=0}^{\infty} \binom{\beta}{\nu} \triangle_h^{\beta} f(x-\beta h-\nu h) \qquad \text{for almost every} \quad x \quad (4)$$

By Lemma 2.1(a) and part (a) of this Lemma, it follows that

$$\begin{aligned} \left\| \triangle_{\pi}^{\beta} f(\cdot)\right\|_p &= \left\| \left(\triangle_{\pi}^{[\beta]} (\triangle_{\pi}^{\beta-[\beta]} f)\right)(\cdot) \right\|_p \\ &\le 2^{\left[\frac{[\beta]+1}{2}\right]} \left\| \left(\triangle_{\frac{\pi}{2}}^{[\beta]} (\triangle_{\pi}^{\beta-[\beta]} f)\right)(\cdot) \right\|_p . \end{aligned}$$

Here we use (4) for $h = \frac{\pi}{2}$. We have

$$\begin{aligned} \left\| \triangle_{\pi}^{\beta} f(\cdot)\right\|_p &\le 2^{\left[\frac{[\beta]+1}{2}\right]} \left\| \triangle_{\frac{\pi}{2}}^{[\beta]} \left\{ \sum_{\nu=0}^{\infty} \binom{\beta-[\beta]}{\nu} \triangle_{\frac{\pi}{2}}^{\beta-[\beta]} f \right\} (\cdot - \frac{\beta\pi}{2} - \frac{\nu\pi}{2}) \right\|_p \\ &= 2^{\left[\frac{[\beta]+1}{2}\right]} \left\| \sum_{\nu=0}^{\infty} \binom{\beta-[\beta]}{\nu} \left(\triangle_{\frac{\pi}{2}}^{[\beta]} (\triangle_{\frac{\pi}{2}}^{\beta-[\beta]} f)\right)(\cdot) \right\|_p . \end{aligned}$$

Thus, by Lemma 2.1(a) and inequality (1), we get

$$\left\| \triangle_{\pi}^{\beta} f(\cdot)\right\|_p \le C^*(\beta-[\beta])\, 2^{\left[\frac{[\beta]+1}{2}\right]} \left\| \triangle_{\frac{\pi}{2}}^{\beta} f(\cdot) \right\|_p .$$

If we combine this result with $C^*(\beta-[\beta]) = 2$ (see (2)) and $2^{\left[\frac{[\beta]+1}{2}\right]} = 2^{\left[\frac{\beta+1}{2}\right]}$, we obtain the required inequality. If $0 < \beta < 1$, then we use (1) and (4). This completes the proof of Lemma.

We will need the following lemma.

Lemma 3.1. *Let $\beta > 0$, $n \in \mathbf{N}$, $\delta > 0$.*

(a): *If $f(x) = \sin x$ and $p \in [1, \infty]$, then there exist $t_1 > 0$ and $C_1(\beta), C_2(\beta) > 0$ such that for any $\delta \in (0, t_1)$ we have*

$$C_1(\beta)\delta^\beta \le \omega_\beta(f,\delta)_p \le C_2(\beta)\delta^\beta. \qquad (5)$$

(b): *If* $f(x)=\sin nx$ *and* $p\in[1,\infty]$, *then for any* $\delta\in\left(0,\frac{\pi}{2}\right]$ *we have*[1] $\|\triangle_\delta^\beta f(\cdot)\|_p\le(2\pi)^{\frac{1}{p}}\,(n\delta)^\beta$.

(c): *If* $f(x)=\sin nx$, *then* $\|\triangle_{\pi/n}^\beta f(\cdot)\|_1=2^{\beta+2}$.

(d): *If* $f(x)=\sin nx$, *then for any* $\delta\in\left(0,\frac{\pi}{n}\right]$ *we have* $\|\triangle_\delta^\beta f(\cdot)\|_1\ge 4\left(\frac{2}{\pi}\right)^\beta(\delta n)^\beta$.

Proof of Lemma 3.1 Let $T_n(x)=\sum\limits_{\nu=-n}^{n}c_\nu e^{i\nu x}$, then

$$\triangle_\delta^\beta T_n(x-\frac{\beta\delta}{2})=\sum_{\nu=-n}^{n}\left(2i\sin\frac{\nu\delta}{2}\right)^\beta c_\nu e^{i\nu x}.$$

Thus, for $f(x)=\sin nx$, $n\in\mathbf{N}$, we get

$$\triangle_\delta^\beta f(x-\frac{\beta\delta}{2})=\left(2\sin\frac{n\delta}{2}\right)^\beta\sin\left(nx+\frac{\beta\pi}{2}\right). \tag{6}$$

For $n=1$ we obviously have $C_1(\beta)\left(2\left|\sin\frac{\delta}{2}\right|\right)^\beta\le\|\triangle_\delta^\beta\sin(\cdot)\|_p\le C_2(\beta)\left(2\left|\sin\frac{\delta}{2}\right|\right)^\beta$. If we combine this inequality with $\sin t\le t$ $(t\ge0)$ and $\sin t\ge\frac{2t}{\pi}$ $(0\le t\le\frac{\pi}{2})$, then we obtain (5). In the same way, by (6), we shall have the proofs of $(b)-(d)$. This completes the proof of Lemma.

Proof of Theorem 2.5. (A). Let us define $\varphi(t):=t^\beta\inf\limits_{0<\xi\le t}\{\xi^{-\beta}\omega_\beta(f,\xi)_p\}$. We immediately have $\varphi(t)\in\Phi_\beta$ from [13, §2]. It is trivial, that $\varphi(t)\le\omega_\beta(f,t)_p$. By Lemma 2.2(e), we have $\omega_\beta(f,t)_p\le C(\beta)\varphi(t)$:

$$\begin{aligned}\omega_\beta(f,t)_p &= t^\beta t^{-\beta}\omega_\beta(f,t)_p\\ &\le C(\beta)t^\beta\inf_{0<\xi\le t}\left\{\xi^{-\beta}\omega_\beta(f,\xi)_p\right\}\\ &= C(\beta)\varphi(t).\end{aligned}$$

Therefore, for any $t>0$ the following inequality $\varphi(t)\le\omega_\beta(f,t)_p\le C(\beta)\varphi(t)$ holds and (A) follows.

(B). *1 case.* Let $\lim\limits_{t\to0}\frac{\varphi(t)}{t^\beta}=C\,(0\le C<\infty)$. Then, by virtue of monotonicity of $\frac{\varphi(t)}{t^\beta}$, we write

$(*)$ $\varphi(t)\le Ct^\beta$ for $0<t\le\pi$;

$(**)$ there exists $t_1>0$ such that $\varphi(t)\ge\frac{Ct^\beta}{2}$ for $0<t\le t_1$.

Indeed, $(*)$ is trivial like $(**)$ for $C=0$. If $C>0$ and $\lim\limits_{t\to0}\frac{\varphi(t)}{t^\beta}=C$, then for any $\varepsilon>0$ there exists $t_1>0$ such that $C-\frac{\varphi(t)}{t^\beta}\le\varepsilon$ for $0<t\le t_1$. Then $\frac{\varphi(t)}{t^\beta}\ge C-\varepsilon$, and choosing small ε we have $(**)$.

[1]Here $\frac{1}{\infty}=0$.

Define $f(x)=C\sin x$. By Lemma 3.1(a), we have

$$\omega_\beta(f,\delta)_p \geq CC_1(\beta)\delta^\beta \geq C_2(\beta)\varphi(\delta) \quad \text{for} \quad 0<\delta\leq \pi,$$

$$\omega_\beta(f,\delta)_p \leq CC_3(\beta)\delta^\beta \leq C_4(\beta)\varphi(\delta) \quad \text{for} \quad 0<\delta\leq t_1,$$

completing the proof in this case.

2 case. Let $\lim\limits_{t\to 0}\frac{\varphi(t)}{t^\beta}=+\infty$. Then $\lim\limits_{t\to 0}\varphi(t)=0$ and $\lim\limits_{t\to 0}\frac{t^\beta}{\varphi(t)}=0$. We fix $a\geq 2$. Then, following Oskolkov ([9]), we define the sequence $\{n_\nu\}_{\nu=1}^\infty$, where $n_\nu=2^{m_\nu}$ are the numbers m_ν such that

$$m_1=2,$$

$$m_{\nu+1}=\min\left\{m\in\mathbf{N}:\ \max\left(\frac{\varphi(2^{-m})}{\varphi(2^{-m_\nu})},\frac{2^{m_\nu\beta}\varphi(2^{-m_\nu})}{2^{m\beta}\varphi(2^{-m})}\right)\leq\frac{1}{a}\right\} \quad (\nu\in\mathbf{N}).$$

From the definition of $\{n_\nu\}_{\nu=1}^\infty$ it follows that $m_{\nu+1}>m_\nu$, $n_{\nu+1}\geq 2n_\nu$ and for any $\nu\in\mathbf{N}$ we have

$$\varphi\left(\frac{1}{n_{\nu+1}}\right)\leq\frac{1}{a}\varphi\left(\frac{1}{n_\nu}\right); \tag{7}$$

$$n_\nu^\beta\varphi\left(\frac{1}{n_\nu}\right)\leq\frac{1}{a}n_{\nu+1}^\beta\varphi\left(\frac{1}{n_{\nu+1}}\right). \tag{8}$$

Let us fix $\varkappa=2^d\,(d\in\mathbf{N})$ such that $\varkappa>2\pi$. Note that (7) implies $\sum\limits_{\nu=1}^\infty\varphi\left(\frac{1}{n_\nu}\right)\leq\varphi\left(\frac{1}{n_1}\right)\sum\limits_{\nu=1}^\infty a^{1-\nu}<\infty$, and, therefore, we can define the function $f(x)=\sum\limits_{\nu=1}^\infty\varphi\left(\frac{1}{n_\nu}\right)\sin(\varkappa n_\nu x)$.

First, we shall estimate $\omega_\beta(f,\delta)_p$ from above. By the inequality $\|f\|_p\leq(2\pi)^{\frac{1}{p}}\|f\|_\infty\leq 2\pi\|f\|_\infty$, $p\in[1,\infty)$, it is enough to prove $\omega_\beta(f,\delta)_\infty\leq C(\beta)\varphi(\delta)$. Let $\delta\in(0,\frac{1}{n_1}]$. For all $h\in(0,\frac{1}{n_1}]$ we can find the number $N\in\mathbf{N}$ such that $\frac{1}{n_{N+1}}<h\leq\frac{1}{n_N}$. Then

$$\begin{aligned}\left\|\triangle_h^\beta f(x)\right\|_\infty &\leq \left\|\sum_{\nu=1}^N\varphi\left(\frac{1}{n_\nu}\right)\triangle_h^\beta\sin(\varkappa n_\nu x)\right\|_\infty + \\ &+ \left\|\sum_{\nu=N+1}^\infty\varphi\left(\frac{1}{n_\nu}\right)\triangle_h^\beta\sin(\varkappa n_\nu x)\right\|_\infty \\ &=: I_1+I_2.\end{aligned}$$

Combining Lemma 3.1(b), inequality (8), and condition (c) in the definition of Φ_β, we get

$$\begin{aligned}
I_1 &\le \sum_{\nu=1}^{N} \varphi\left(\frac{1}{n_\nu}\right) \left\| \triangle_h^\beta \sin(\varkappa n_\nu x) \right\|_\infty \\
&\le C(\beta)\,(\varkappa h)^\beta\, \varphi\left(\frac{1}{n_N}\right) n_N^\beta \sum_{\nu=1}^{N} a^{-(N-\nu)} \\
&\le C(\beta)\,(n_N h)^\beta\, \varphi\left(\frac{1}{n_N}\right) \\
&\le C(\beta)\varphi\left(h\right).
\end{aligned}$$

Inequalities (1) and (7) yield that

$$\begin{aligned}
I_2 &\le \sum_{\nu=N+1}^{\infty} \varphi\left(\frac{1}{n_\nu}\right) \left\| \triangle_h^\beta \sin(\varkappa n_\nu x) \right\|_\infty \\
&\le C(\beta) \sum_{\nu=N+1}^{\infty} \varphi\left(\frac{1}{n_\nu}\right) \\
&\le C(\beta)\varphi\left(\frac{1}{n_{N+1}}\right) \sum_{\nu=N+1}^{\infty} a^{N+1-\nu} \\
&\le C(\beta)\varphi\left(\frac{1}{n_{N+1}}\right) \\
&\le C(\beta)\varphi\left(h\right).
\end{aligned}$$

Therefore, if $h \in (\frac{1}{n_{N+1}}, \frac{1}{n_N}],\ N \in \mathbf{N}$, then $\|\triangle_h^\beta f(x)\|_\infty \le C(\beta)\varphi\left(h\right)$, which implies $\omega_\beta(f,\delta)_\infty \le C(\beta)\varphi(\delta)$.

Now we shall obtain the inequality $\varphi(\delta) \le C(\beta)\omega_\beta(f,\delta)_p$. From the inequality $\|f\|_1 \le 2\pi\|f\|_p,\ p \in [1,\infty]$ it is sufficient to prove $\varphi(\delta) \le C(\beta)\omega_\beta(f,\delta)_1$. Also, we note that if the last inequality holds for $\delta = \frac{\pi}{2^k},\ k = N, N+1, N+2, \cdots$, where $N \in \mathbf{N}$, then it holds for $\delta \in \left(\frac{\pi}{2^k}, \frac{\pi}{2^{k+1}}\right)$. Indeed, from the monotonicity of $t^{-\beta}\varphi(t)$, we see that the estimate $\varphi\left(\delta\right) \le C(\beta)\varphi\left(\frac{\pi}{2^k}\right)$ is true. By Lemma 2.2(a), we get

$$\begin{aligned}
\varphi\left(\delta\right) &\le C(\beta)\varphi\left(\frac{\pi}{2^k}\right) \\
&\le C(\beta)\omega_\beta\left(f, \frac{\pi}{2^k}\right)_1 \\
&\le C(\beta)\omega_\beta(f,\delta)_1.
\end{aligned}$$

To go further, we suppose that $\delta = \frac{\pi}{2^k}$.

Let M be the integer, $M>1$, and, let $h_1=\frac{\pi}{\varkappa n_M}$. We shall show that

$$\left\|\triangle^{\beta}_{h_1} f(x)\right\|_1 \geq 4\varphi\left(\frac{1}{n_M}\right)\left(2^{\beta}-\frac{\pi^{\beta+1}}{a}\right). \tag{9}$$

For this purpose, we shall use the following representation of a function $f(x)$:

$$\begin{aligned} f(x) &= \sum_{\nu=1}^{M-1}\varphi\left(\frac{1}{n_\nu}\right)\sin(\varkappa n_\nu x)+\varphi\left(\frac{1}{n_M}\right)\sin(\varkappa n_M x)+\\ &\quad+\sum_{\nu=M+1}^{\infty}\varphi\left(\frac{1}{n_\nu}\right)\sin(\varkappa n_\nu x)\\ &=: f_1+f_2+f_3. \end{aligned}$$

Note, that $\sin(\varkappa n_\nu x+\frac{\pi n_\nu}{n_M})=\sin(\varkappa n_\nu x)$ for $\nu>M$, and $f_3(x)$ has the period $T=h_1=\frac{\pi}{\varkappa n_M}$. We therefore obtain

$$\triangle^{\beta}_{h_1} f_3(x)=f(x+\beta h_1)\sum_{\xi=0}^{\infty}(-1)^{\xi}\binom{\beta}{\xi}=0.$$

By Lemma 3.1(b) and (8), we have

$$\begin{aligned} \left\|\triangle^{\beta}_{h_1} f_1(x)\right\|_1 &\leq \sum_{\nu=1}^{M-1}\varphi\left(\frac{1}{n_\nu}\right)\left\|\triangle^{\beta}_{h_1}\sin(\varkappa n_\nu x)\right\|_1\\ &\leq \sum_{\nu=1}^{M-1}2\pi\left(\varkappa n_\nu h_1\right)^{\beta}\varphi\left(\frac{1}{n_\nu}\right)\\ &= 2\pi\left(\frac{\pi}{n_M}\right)^{\beta}\sum_{\nu=1}^{M-1}\varphi\left(\frac{1}{n_\nu}\right)n_\nu^{\beta}\\ &\leq 2\pi\left(\frac{\pi}{n_M}\right)^{\beta}\varphi\left(\frac{1}{n_{M-1}}\right)n_{M-1}^{\beta}\sum_{\nu=1}^{M-1}a^{-(M-1-\nu)}. \end{aligned}$$

Using $\sum\limits_{\nu=1}^{M-1}a^{-(M-1-\nu)}\leq 2$ and (8), we obtain $\left\|\triangle^{\beta}_{h_1} f_1(x)\right\|_1\leq\frac{4\pi^{\beta+1}}{a}\varphi\left(\frac{1}{n_M}\right)$.
By Lemma 3.1(c), $\left\|\triangle^{\beta}_{h_1} f_2(x)\right\|_1=\varphi\left(\frac{1}{n_M}\right)\left\|\triangle^{\beta}_{h_1}\sin(\varkappa n_M x)\right\|_1=2^{\beta+2}\varphi\left(\frac{1}{n_M}\right)$.

Therefore, for $h_1 = \frac{\pi}{\varkappa n_M}$, the inequality $|f| \geq |f_2| - |f_1| - |f_3|$ implies

$$\begin{aligned} \left\|\triangle_{h_1}^{\beta} f(x)\right\|_1 &\geq \left\|\triangle_{h_1}^{\beta} f_2(x)\right\|_1 - \left\|\triangle_{h_1}^{\beta} f_1(x)\right\|_1 - \left\|\triangle_{h_1}^{\beta} f_3(x)\right\|_1 \\ &= \left\|\triangle_{h_1}^{\beta} f_2(x)\right\|_1 - \left\|\triangle_{h_1}^{\beta} f_1(x)\right\|_1 \\ &\geq 4\varphi\left(\frac{1}{n_M}\right)\left(2^{\beta} - \frac{\pi^{\beta+1}}{a}\right), \end{aligned}$$

i.e. we obtain (9).
Further, we choose the integer i such that

$$\frac{1}{n_{i+1}} = \frac{1}{2^{m_{i+1}}} < \delta \leq \frac{1}{2^{m_i}} = \frac{1}{n_i}.$$

Note, that, by definition of m_i, at the least one of the following inequalities is true:

$$2^{\beta(m_{i+1}-1)}\varphi\left(\frac{1}{2^{m_{i+1}-1}}\right) < a\, 2^{\beta m_i}\varphi\left(\frac{1}{2^{m_i}}\right), \tag{10}$$

$$\varphi\left(\frac{1}{2^{m_{i+1}-1}}\right) > \frac{1}{a}\,\varphi\left(\frac{1}{2^{m_i}}\right) \tag{11}$$

Case 2(a). Let (10) be true. Using the monotonicity of $\varphi(t)$ and (10), we get

$$\begin{aligned} n_{i+1}^{\beta}\varphi\left(\frac{1}{n_{i+1}}\right) &\leq 2^{\beta}2^{\beta(m_{i+1}-1)}\varphi\left(\frac{1}{2^{m_{i+1}-1}}\right) \\ &< a\,2^{\beta}n_i^{\beta}\varphi\left(\frac{1}{n_i}\right). \end{aligned} \tag{12}$$

We write

$$\begin{aligned} f(x) &= \sum_{\nu=1}^{i-1}\varphi\left(\frac{1}{n_\nu}\right)\sin(\varkappa n_\nu x) + \varphi\left(\frac{1}{n_i}\right)\sin(\varkappa n_i x) + \\ &\quad + \sum_{\nu=i+1}^{\infty}\varphi\left(\frac{1}{n_\nu}\right)\sin(\varkappa n_\nu x) \\ &=: f_1 + f_2 + f_3. \end{aligned}$$

It is clear, that the function f_3 has a period $T = \frac{2\pi}{\varkappa n_{i+1}}$. Then, for $\varkappa = 2^d > 2\pi$ we have $\delta = \frac{\pi}{2^r} > \frac{1}{n_{i+1}} > T$, therefore, f_3 has a period δ and $\triangle_{\delta}^{\beta} f_3(x) = 0$.

For $0 < \delta \le \frac{\pi}{\varkappa n_i}$, by Lemma 3.1(d), we have

$$\begin{aligned}\left\|\triangle_\delta^\beta f_2(x)\right\|_1 &= \varphi\left(\frac{1}{n_i}\right)\left\|\triangle_\delta^\beta \sin(\varkappa n_i x)\right\|_1 \\ &\ge 4\left(\frac{2}{\pi}\right)^\beta \varphi\left(\frac{1}{n_i}\right)(\varkappa n_i\delta)^\beta .\end{aligned}$$

Using Lemma 3.1(b) and inequality (8), we estimate f_1:

$$\begin{aligned}\left\|\triangle_{h_1}^\beta f_1(x)\right\|_1 &\le \sum_{\nu=1}^{i-1}\varphi\left(\frac{1}{n_\nu}\right)\left\|\triangle_\delta^\beta \sin(\varkappa n_\nu x)\right\|_1 \\ &\le \sum_{\nu=1}^{i-1} 2\pi\,(\varkappa n_\nu\delta)^\beta \varphi\left(\frac{1}{n_\nu}\right) \\ &\le 4\pi\,(\varkappa n_{i-1}\delta)^\beta \varphi\left(\frac{1}{n_{i-1}}\right) \\ &\le 4\pi\,(\varkappa n_i\delta)^\beta \frac{1}{a}\varphi\left(\frac{1}{n_i}\right).\end{aligned}$$

For $\frac{1}{n_{i+1}} < \delta \le \frac{\pi}{\varkappa n_i}$ we obtain

$$\begin{aligned}\left\|\triangle_\delta^\beta f(x)\right\|_1 &\ge \left\|\triangle_\delta^\beta f_2(x)\right\|_1 - \left\|\triangle_\delta^\beta f_1(x)\right\|_1 \\ &\ge \varphi\left(\frac{1}{n_i}\right)(\varkappa n_i\delta)^\beta\left\{4\left(\frac{2}{\pi}\right)^\beta - \frac{4\pi}{a}\right\}.\end{aligned}$$

Now we choose a such that $2^\beta - \frac{\pi^{\beta+1}}{a} = \gamma_1 > 0$ $\left(\text{then } 4\left(\frac{2}{\pi}\right)^\beta - \frac{4\pi}{a} = \gamma_2 > 0\right)$. From (12) and the condition (c) in the definition of Φ_β, we have

$$\begin{aligned}(\delta n_i)^\beta \varphi\left(\frac{1}{n_i}\right) &\ge \left(\frac{\delta n_{i+1}}{2}\right)^\beta \frac{1}{a}\varphi\left(\frac{1}{n_{i+1}}\right) \\ &\ge 2^{-\beta}\frac{1}{a}\varphi(\delta).\end{aligned}$$

Thus, the inequality $\omega_\beta(f,\delta)_p \ge C(\beta)\varphi(\delta)$ holds for $\frac{1}{n_{i+1}} < \delta \le \frac{\pi}{\varkappa n_i}$. If $\frac{\pi}{\varkappa n_i} < \delta \le \frac{1}{n_i}$, then (9) implies

$$\begin{aligned}\omega_\beta(f,\delta)_p &\ge \omega_\beta\left(f, \frac{\pi}{\varkappa n_i}\right)_p \\ &\ge C(\beta)\varphi\left(\frac{1}{n_i}\right) \\ &\ge C(\beta)\varphi(\delta).\end{aligned}$$

The theorem has been proved in case $2(a)$.

Case 2(b). Let (11) be true. By virtue of monotonicity of $\frac{\varphi(t)}{t^\beta}$, we write $\varphi\left(\frac{1}{2^{m_{i+1}-1}}\right) \leq 2^\beta \varphi\left(\frac{1}{2^{m_{i+1}}}\right)$.
Hence,

$$\begin{aligned}
\varphi\left(\frac{1}{n_{i+1}}\right) &= \varphi\left(\frac{1}{2^{m_{i+1}}}\right) \\
&\geq 2^{-\beta}\varphi\left(\frac{1}{2^{m_{i+1}-1}}\right) \\
&> \frac{2^{-\beta}}{a}\varphi\left(\frac{1}{2^{m_i}}\right) \\
&= \frac{2^{-\beta}}{a}\varphi\left(\frac{1}{n_i}\right).
\end{aligned} \tag{13}$$

It follows from (9) and (13) that

$$\begin{aligned}
\omega_\beta\left(f,\delta\right)_1 &\geq \omega_\beta\left(f, \frac{1}{n_{i+1}}\right)_1 \\
&\geq \omega_\beta\left(f, \frac{\pi}{\varkappa n_{i+1}}\right)_1 \\
&\geq C(\beta)\varphi\left(\frac{1}{n_{i+1}}\right) \\
&\geq C(\beta)\varphi\left(\frac{1}{n_i}\right) \\
&\geq C(\beta)\varphi\left(\delta\right).
\end{aligned}$$

This completes the proof of case $2(b)$ and Theorem 2.5.

Proof of Corollary 2.6 follows from the following estimates (see [2]):

$$C_1(\beta)\omega_\beta(f,t)_p \leq K(f, t^\beta, L_p, W_p^\beta) \leq C_2(\beta)\omega_\beta(f,t)_p.$$

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