ON BOAS-TYPE PROBLEM

SERGEY TIKHONOV

ABSTRACT. R.P. Boas has found necessary and sufficient conditions of belonging of function to Lipschitz class. From his findings it turned out, that the conditions on sine and cosine coefficients for belonging of function to Lip $\alpha\,(0<\alpha<1)$ are the same, but for Lip 1 are different. Later his results were generalized by many authors in the viewpoint of generalization of condition on the majorant of modulus of continuity. The aim of this paper is to obtain Boas-type theorems for generalized Lipschitz classes. To define generalized Lipschitz classes we use the concept of modulus of smoothness of fractional order.

1. INTRODUCTION

It is a well known fact that the local behavior of the sum of a Fourier series with nonnegative coefficients determines the asymptotic behavior of the coefficients, and vice versa. As a main reference, we would like to mention the remarkable book [5] by R.P.Boas Jr., where one can find many results on the subject, which we will discuss below.

As it was shown by R.P.Boas [5, p. 45] (see also [4]), the conditions on sine and cosine coefficients for Lip $\alpha\,(0<\alpha<1)$ from this point of view are the same:
If $\lambda_n \geq 0$ are the Fourier sine or cosine coefficients of $\psi(x)$, then

$$\psi \in \mathrm{Lip}\ \alpha \qquad \textit{if and only if} \qquad \sum_{k=n}^{\infty} \lambda_k = O\left(n^{-\alpha}\right) \tag{1}$$

$$\textit{or, equivalently,} \qquad \sum_{k=1}^{n} k\lambda_k = O\left(n^{1-\alpha}\right).$$

This result does not hold for $\alpha = 1$. Firstly,

$$\psi \in \mathrm{Z} \qquad \textit{if and only if} \qquad \sum_{k=n}^{\infty} \lambda_k = O\left(n^{-1}\right), \tag{2}$$

where Z is the Zygmund class, i.e., $\Big\{\psi \in C : \|\psi(x+h) - 2\psi(x) + \psi(x-h)\| = O(h)\Big\}$. On the other hand (see [5, p. 46-47]) the conditions of belonging to Lip 1 for sine series and for cosine series are different:

$$f \in \text{Lip } 1 \quad \textit{if and only if} \quad \begin{array}{l} (1).\ \sum_{k=1}^{n} k^2 a_k = O(n) \\ (2).\ \sum_{k=1}^{n} k a_k \sin kx = O(1) \ \textit{uniformly in } x. \end{array} \tag{3}$$

$$g \in \text{Lip } 1 \quad \textit{if and only if} \quad \sum_{k=1}^{n} k b_k = O(1). \tag{4}$$

Here and further, we assume that the series

$$\sum_{k=1}^{\infty} a_k \cos kx, \qquad a_k \geq 0 \tag{5}$$

and

$$\sum_{k=1}^{\infty} b_k \sin kx, \qquad b_k \geq 0 \tag{6}$$

converge uniformly to functions $f(x)$ and $g(x)$, respectively and (5), (6) are Fourier series of $f(x)$ and $g(x)$.

The aim of this paper is to present Boas-type results for generalized Lipschitz classes.
The paper is organized in the following way. Section 2 contains some definitions and preliminaries. We start with the discussion of the well-known results and conclude this section describing the problem, which we will consider. In section 3, we present our main results. Section 4 contains lemmas that we will need and proofs of theorems. We finish by Section 5, where we provide a few remarks.

2. Definitions, Results, Comments

2.1. **Basic Notations and Definitions.** Let $\psi(x)$ be a continuous function and $\|\psi(\cdot)\| = \max_{x \in [0,2\pi]} |\psi(x)|$. The key concept in the paper will be the modulus of smoothness of fractional order introduced by Butzer, Dyckhoff, Göerlich and Stens [6].

The modulus of smoothness of order β $(\beta > 0)$ of function $\psi \in C$ is given by

$$\omega_\beta(\psi, t) = \sup_{|h| \leq t} \left\| \triangle_h^\beta \psi(\cdot) \right\|,$$

where

$$\triangle_h^{\beta}\psi(x) = \sum_{\nu=0}^{\infty}(-1)^{\nu}\binom{\beta}{\nu}\psi(x+(\beta-\nu)h),$$

and $\binom{\beta}{\nu} = \frac{\beta(\beta-1)\cdots(\beta-\nu+1)}{\nu!}$ for $\nu \geq 1$, and $\binom{\beta}{\nu} = 1$ for $\nu = 0$. Let $\beta > 0$ and $\{\gamma_n\}$ be a positive sequence. By $\mathrm{Lip}_\beta\ \gamma$ we denote generalized Lipschitz classes, i.e.,

$$\mathrm{Lip}_\beta\ \gamma = \left\{ f \in C : \omega_\beta\Big(\psi, \frac{1}{n}\Big) = O(\gamma_n)\right\}.$$

By this definition, $\mathrm{Lip}\ \alpha \equiv \mathrm{Lip}_1\ \gamma$ with $\{\gamma_n = n^{-\alpha}\}$ and $\mathrm{Z} \equiv \mathrm{Lip}_2\ \gamma$ with $\{\gamma_n = n^{-1}\}$.

We will denote by $E_n(\psi)$ the error of best trigonometric approximation of order n of a function ψ.

Now we define several conditions on positive sequences and functions. A sequence $\gamma := \{\gamma_n\}$ of positive terms will be called almost increasing (almost decreasing), if there exists a constant $K := K(\gamma) \geq 1$ such that

$$K\gamma_n \geq \gamma_m \quad (\gamma_n \leq K\gamma_m)$$

holds for any $n \geq m$. This concept was introduced by Bernstein[1].

We will say that a sequence $\gamma := \{\gamma_n\}$ of positive terms satisfies the condition (SQ), if there exists $\varepsilon \in (0,1)$, such that $\{n^{\varepsilon}\gamma_n\}$ is almost decreasing. And a sequence $\gamma := \{\gamma_n\}$ of positive terms satisfies the condition (SQ_β), if there exists $\varepsilon \in (0,1)$, such that $\{n^{\beta-\varepsilon}\gamma_n\}$ is almost increasing.

Also, we will give the following definition by Chan [9]. Let $Y[a,b]$, $a \leq b$ be the collection of all positive functions $\gamma(u)$ defined on $[X,\infty)$, $X > 0$ such that $u^{-a}\gamma(u)$ is nondecreasing and $u^{-b}\gamma(u)$ is nonincreasing.

Finally, let Φ_σ $(\sigma \in \mathbf{R})$ be the set of nonnegative, bounded functions $\gamma(u)$ on $(0,\infty)$, such that

(a): $\gamma(u) \to 0$ as $u \to 0$,
(b): $\gamma(u)$ is nondecreasing,
(c): $\gamma(u)\,u^{-\sigma}$ is nonincreasing.

2.2. **Results and Comments.** After the Boas's book [5] was published, there has been done a lot of work on generalization of his results (1)-(4) to more general classes of functions than Lip α. Different ways of generalization were considered by M. and S. Izumi ([10]), J. Nemeth ([13]), L.-Y. Chan ([9]), L. Leindler ([11]). It is natural to consider different conditions on a majorant of modulus of continuity $\omega_1(f,\frac{1}{n}) = O(\gamma_n)$ or modulus of smoothness $\omega_2(f,\frac{1}{n}) = O(\gamma_n)$. Recently, we have obtained ([15]) the interrelation between several well-known conditions (in particular, from [9]-[11]

[1]See reference to S.N. Bernstein in [2].

and [13]-[14])) on a majorant. Also, we have proved theorem which generalizes all previous results for the case, when the conditions on Fourier sine or cosine coefficients are the same (i.e., the analogue of Boas's results for Lip α, $0<\alpha<1$, and Z).

Theorem 2.1. ([15]) *Let $\gamma=\{\gamma_n\}$ be a positive sequence and $\beta>0$. Let $\lambda_n\ge 0$ be the Fourier sine or cosine coefficients of $\psi(x)$.*
(A). *If $\gamma\in SQ\cap SQ_\beta$, then for any function $\psi\in C$ conditions*

$$\sum_{k=n}^{\infty}\lambda_k=O\left(\gamma_n\right), \tag{7}$$

$$\sum_{k=1}^{n}k^\beta\lambda_k=O\left(n^\beta\gamma_n\right), \tag{8}$$

$$\omega_\beta\Big(\psi,\frac{1}{n}\Big)=O(\gamma_n) \tag{9}$$

are equivalent.
(B). *Let γ be a non-increasing sequence. If for any function $\psi\in C$ conditions (7), (8) and (9) are equivalent, then $\gamma\in SQ\cap SQ_\beta$.*

Now we recall the following equivalence result on order of decay of the modulus of smoothness.

Theorem 2.2. ([16]) (A). *If $\psi(\cdot)\in C$ and $\beta>0$, then there exists a function $\gamma(\cdot)\in\Phi_\beta$, such that*

$$\gamma(t)\le\omega_\beta(\psi,t)\le C(\beta)\gamma(t)\ \ (0<t<\infty),$$

where $C(\beta)$ is a positive constant depending only on β.
(B). *If $\gamma(\cdot)\in\Phi_\beta$, $\beta>0$, then there exist a function $\psi(\cdot)\in C$ and a constant $t_1>0$, such that*

$$C_1(\beta)\omega_\beta(\psi,t)\le\gamma(t)\le C_2(\beta)\omega_\beta(\psi,t)\ \ (0<t<t_1),$$

where $C_1(\beta),C_2(\beta)$ are positive constants depending only on β.

It is clear, that the class $\big\{\gamma(\cdot)\in\Phi_\beta\big\}$ is wider then $\big\{\{\gamma_n=\gamma\left(\frac{1}{n}\right)\}\in SQ\cap SQ_\beta\big\}$. We give as examples two following majorants

$$(\mathbf{i})\quad \gamma=\big\{\gamma_n=n^{-\beta}\xi(n)\big\}\notin SQ_\beta \text{ and } (\mathbf{ii})\quad \gamma=\Big\{\gamma_n=(\xi(n))^{-1}\Big\}\notin SQ,$$

where $\xi(\cdot)$ is positive nondecreasing slowly varying function[2].

[2]By definition, a positive measurable function $\xi(\cdot)$ on $[A,\infty)$, $A>0$ is slowly varying if for all $\lambda>0$ we have $\lim\limits_{t\to\infty}\frac{\xi(\lambda t)}{\xi(t)}=1$. Note (see [3, p. 54]), that, by our assumptions on function $\xi(\cdot)$, it is sufficient to have $\lim\limits_{t\to\infty}\frac{\xi(\lambda^* t)}{\xi(t)}=1$ only for a positive $\lambda^*\ne 1$. By Theorem 2.2, the natural restriction in the case **(ii)** is $\lim\limits_{t\to\infty}\xi(t)=\infty$.

So, Theorem 2.1 does not provide Boas-type criterion of belonging of function to $\mathrm{Lip}_\beta\,\gamma$ for such cases as **(i)** and **(ii)**. These cases are of particular interest since the conditions for cosine and sine series are different (see, for instance, (3), (4)).

In the case $\beta=1$, the Boas-type result for a majorant as in the case **(i)** was obtained by Chan [9]. We recall his result.

Theorem 2.3. (Cosine) *Let* $\gamma(\cdot)\in Y[c_1,c_2]$, $0<c_1\le c_2<\infty$. *Then*

$$\omega_1(f,\delta)=O\left[\gamma(\delta)\right]\quad \textit{iff}\quad (1).\ \sum_{k=1}^{n}k^2a_k=O\left[n^2\gamma\left(\frac{1}{n}\right)\right]$$

$$(2).\ \sum_{k=1}^{n}ka_k\sin kx=O\left[n\gamma\left(\frac{1}{n}\right)\right]\ \textit{uniformly in } x.$$

Theorem 2.4. (Sine) *Let* $\gamma(\cdot)\in Y[c_1,c_2]$, $0<c_1\le c_2<\infty$. *Then*

$$\omega_1(g,\delta)=O\left[\gamma(\delta)\right]\qquad \textit{if and only if}\qquad \sum_{k=1}^{n}kb_k=O\left[n\gamma\left(\frac{1}{n}\right)\right].$$

It is clear that $\gamma(\cdot)\in Y[c_1,c_2]$ implies $\left\{\gamma\left(\frac{1}{n}\right)\right\}\in SQ\cap SQ_{c_2+\varepsilon}\subset SQ$ for any $\varepsilon>0$ but the converse is not true.

The following result was obtained in [14].

Theorem 2.5. (Cosine) *Let* $\gamma=\{\gamma_n\}$ *be the positive sequence and* $\gamma\in SQ_1$. *Then*

$$\omega_1\Big(f,\frac{1}{n}\Big)=O(\gamma_n)\qquad \textit{if and only if}\qquad \sum_{k=n}^{\infty}a_k=O\left(\gamma_n\right).$$

Naturally, for a majorant γ as in the case **(ii)** we have $\gamma\in SQ_1$.

Our main idea in this paper is to prove Boas-type criterion for generalized Lipschitz classes $\mathrm{Lip}_\beta\,\gamma$ where a majorant γ satisfies either SQ or SQ_β condition. So, if $\gamma\in SQ$, then, as an example, we can consider the case **(i)** $\gamma=\left\{\gamma_n=n^{-\beta}\xi(n)\right\}$ and we will have the generalization of Theorems 2.3 and 2.4, and classical Boas's results (3),(4) for Lip 1 in particular. If $\gamma\in SQ_\beta$, then, as an example, we can consider the case **(ii)** $\gamma=\left\{\gamma_n=\left(\xi(n)\right)^{-1}\right\}$ we will have Theorem 2.5 for $\beta=1$.

Finally, if $\gamma\in SQ\cap SQ_\beta$ then the results below coincide Theorem 2.1 and we have Boas's criterion (1) for $\mathrm{Lip}\,\alpha$ $(0<\alpha<1)$ and (2) for Z in particular.

3. Main Results

Theorem 3.1. (Cosine) *Let* $\gamma=\{\gamma_n\}$ *be the positive sequence and* $\beta>0$.
(A). *If* $\beta\neq 2l-1\,(l\in\mathbf{N})$ *and* $\gamma\in SQ$, *then for any function* $f\in C$ *with*

Fourier series (5) conditions

$$\omega_\beta\Big(f,\frac{1}{n}\Big)=O(\gamma_n) \tag{10}$$

and

$$\sum_{k=1}^{n}k^\beta a_k=O\left(n^\beta\gamma_n\right) \tag{11}$$

are equivalent.
(B). *If $\beta=2l-1\,(l\in\mathbf{N})$ and $\gamma\in SQ$, then for any function $f\in C$ with Fourier series (5) conditions*

$$\omega_\beta\Big(f,\frac{1}{n}\Big)=O(\gamma_n) \tag{12}$$

and

$$\sum_{k=1}^{n}k^{\beta+1} a_k=O\left(n^{\beta+1}\gamma_n\right), \tag{13}$$

$$\sum_{k=1}^{n}k^{\beta} a_k\sin kx=O\left(n^{\beta}\gamma_n\right) \quad \text{uniformly in} \quad x \tag{14}$$

are equivalent.
(C). *If $\gamma\in SQ_\beta$, then for any function $f\in C$ with Fourier series (5) conditions*

$$\omega_\beta\Big(f,\frac{1}{n}\Big)=O(\gamma_n) \tag{15}$$

and

$$\sum_{k=n}^{\infty}a_k=O\left(\gamma_n\right) \tag{16}$$

are equivalent.

Here every equivalence result (items (A), (B), (C)) should be understood in the following sense: if for any function $f\in C$ with Fourier series (5) condition $\omega_\beta\left(f,\frac{1}{n}\right)=O(\gamma_n)$ holds, then suitable condition on coefficients $\{a_k\}$ are true. Conversely, if for any $\{a_k\geq 0\}_{k=1}^\infty$ given condition holds, then the series (5) converges uniformly to the continuous function $f(x)$ and (5) is Fourier series of $f(x)$ and $\omega_\beta\left(f,\frac{1}{n}\right)=O(\gamma_n)$. We will use the same assumptions in the Theorem 3.2 too.

Theorem 3.2. (Sine) *Let $\gamma=\{\gamma_n\}$ be the positive sequence and $\beta>0$.*
(A). *If $\beta\neq 2l\,(l\in\mathbf{N})$ and $\gamma\in SQ$, then for any function $g\in C$ with Fourier series (6) conditions*

$$\omega_\beta\Big(g,\frac{1}{n}\Big)=O(\gamma_n) \tag{17}$$

and

$$\sum_{k=1}^{n}k^\beta b_k=O\left(n^\beta\gamma_n\right) \tag{18}$$

are equivalent.
(B). *If $\beta=2l\,(l\in\mathbf{N})$ and $\gamma\in SQ$, then for any function $g\in C$ with Fourier series (6) conditions*

$$\omega_\beta\Big(g,\frac{1}{n}\Big)=O(\gamma_n) \tag{19}$$

and

$$\sum_{k=1}^{n}k^{\beta+1} b_k=O\left(n^{\beta+1}\gamma_n\right), \tag{20}$$

$$\sum_{k=1}^{n}k^\beta b_k\sin kx=O\left(n^\beta\gamma_n\right)\quad \textit{uniformly in}\quad x \tag{21}$$

are equivalent.
(C). *If $\gamma\in SQ_\beta$, then for any function $g\in C$ with Fourier series (6) condition*

$$\sum_{k=n}^{\infty}b_k=O\left(\gamma_n\right) \tag{22}$$

implies

$$\omega_\beta\Big(g,\frac{1}{n}\Big)=O(\gamma_n). \tag{23}$$

4. Proofs.

The following lemmas[3] play the central role in the proofs of main theorems.

[3]See more about results of this type in [13], [17] and references there.

Lemma 4.1. (Cosine) *If $f(x) \in C$ has a Fourier series (5), then*

$$n^{-\beta}\sum_{k=1}^{n} k^{\beta} a_k \le C(\beta)\omega_\beta\Big(f, \frac{1}{n}\Big), \quad \textit{for} \quad \beta \ne 2l-1, l = 1, 2, \cdots$$

Proof of Lemma 4.1. If $f(x) \sim \sum_{n=1}^{\infty} a_n \cos nx$, then

$$\triangle_h^\beta f\Big(x - \frac{\beta h}{2}\Big) \sim \sum_{n=1}^{\infty} a_n \left(2\sin\frac{kh}{2}\right)^{\beta} \cos\Big(kx + \frac{\pi\beta}{2}\Big) =: J. \qquad (24)$$

Note, that for any $\xi(x) \in L$ such that $\|\xi\|_1 \ne 0$ we have

$$\|\triangle_h^\beta f\Big(\cdot - \frac{\beta h}{2}\Big)\| \ge \left(\int_0^{2\pi} |\xi(x)|\, dx\right)^{-1} \left|\int_0^{2\pi} \xi(x)\triangle_h^\beta f\Big(x - \frac{\beta h}{2}\Big)\, dx\right|.$$

We take as $\xi(x)$ the Fejer kernel ([19]), i.e.,

$$\xi(x) = K_{2n}(x) = \frac{1}{2} + \sum_{m=1}^{2n}\left(1 - \frac{m}{2n+1}\right)\cos mx,\ n \in \mathbf{N},\ \int_0^{2\pi} |K_{2n}(x)|\, dx = \pi.$$

Then the Parseval equation implies

$$\begin{aligned}
\|\triangle_h^\beta f(\cdot)\| &\ge \frac{1}{\pi}\left|\int_0^{2\pi} K_{2n}(x)\, J\, dx\right| \\
&= 2^{\beta}\left|\sum_{m=1}^{2n}\left(1 - \frac{m}{2n+1}\right)\left(\sin\frac{mh}{2}\right)^{\beta}\cos\frac{\pi\beta}{2} a_m\right|.
\end{aligned}$$

Therefore, for $h \in (0, \frac{1}{n}]$ we get

$$\begin{aligned}
\|\triangle_h^\beta f(\cdot)\| &\ge C(\beta)h^{\beta}\sum_{m=1}^{2n}\left(1 - \frac{m}{2n+1}\right) m^{\beta} a_m \\
&\ge C(\beta)h^{\beta}\sum_{m=1}^{n} m^{\beta} a_m.
\end{aligned}$$

The proof is complete.

Lemma 4.2. (Sine) *If $g(x) \in C$ has a Fourier series (6), then*

$$n^{-\beta}\sum_{k=1}^{n} k^{\beta} b_k \le C(\beta)\omega_\beta\Big(g, \frac{1}{n}\Big), \quad \textit{for} \quad \beta \ne 2l, l = 1, 2, \cdots$$

The proof of Lemma 4.2 is analogous to the previous lemma.

Also we will need two auxiliary results for sequences.

Lemma 4.3. ([12]). *Let $\gamma = \{\gamma_n\}$ be a positive sequence such that the inequalities*

$$\alpha_1 \min(\gamma_{2^k}, \gamma_{2^{k+1}}) \le \gamma_n \le \alpha_2 \max(\gamma_{2^k}, \gamma_{2^{k+1}}), \quad 0 < \alpha_1 \le \alpha_2 < \infty$$

hold for any $2^k \le n \le 2^{k+1}$, $k = 1, 2, \cdots$. Then the inequalities

$$\sum_{k=1}^{n} \gamma_{2^k} \le C\gamma_{2^n} \ (n \in \mathbf{N}, C \ge 1) \quad \left(or \ \sum_{k=n}^{\infty} \gamma_{2^k} \le C\gamma_{2^n} \quad (n \in \mathbf{N}, C \ge 1) \right)$$

hold if and only if the sequence $\{n^{\varepsilon}\gamma_n\}$ is almost increasing (decreasing) with a certain negative (positive) number ε, respectively.

Lemma 4.4. *Let $\gamma = \{\gamma_n > 0\}$, $\mu = \{\mu_n \ge 0\}$ and $\beta > 0$.*

(i). *If $\gamma \in SQ_{\beta}$, then*

$$\sum_{k=n}^{\infty} \mu_k = O\left(\gamma_n\right) \tag{25}$$

implies

$$\sum_{k=1}^{n} k^{\beta}\mu_k = O\left(\gamma_n n^{\beta}\right). \tag{26}$$

(ii). *If $\gamma \in SQ$, then (26) implies (25).*

Proof of Lemma 4.4. (i). It is clear that the condition (25) implies the condition $\sum_{k=n}^{2n} k^{\beta}\mu_k = O\left(n^{\beta}\gamma_n\right)$. We choose s such that $2^{s-1} \le n < 2^s$. By Lemma 4.3, we have

$$\begin{aligned} \sum_{k=1}^{n} k^{\beta}\mu_k &\le \sum_{k=2^{s-1}}^{n} k^{\beta}\mu_k + \sum_{k=0}^{s-1} \sum_{\xi=2^k}^{2^{k+1}-1} \xi^{\beta}\mu_{\xi} \\ &= O\left(2^{\beta(s-1)}\gamma_{2^{s-1}} + \sum_{k=0}^{s-1} 2^{\beta k}\gamma_{2^k}\right) \\ &= O\left(2^{\beta(s-1)}\gamma_{2^{s-1}}\right). \end{aligned}$$

Since $\gamma \in SQ_{\beta}$, $\{n^{\beta}\gamma_n\}$ is almost increasing and (26) is true. The proof of (ii) is analogous.

Proof of Theorem 3.1. We will use several times the following trivial consequence of Theorem 2.2

$$\omega_{\beta}(f, \delta) \le \omega_{\beta}(f, s\delta) \le C(\beta)s^{\beta}\omega_{\beta}(f, \delta) \quad (s \ge 1). \tag{27}$$

(A). First, suppose that (10) holds. By Lemma 4.1, we have (10) implies (11). On the other hand, by Lemma 4.4 (ii), (11) gives the inequality $\sum\limits_{k=n}^{\infty} a_k = O(\gamma_n)$. Thus, the series (5) converges uniformly to the function $f(x)$, $f(x) = \sum\limits_{k=1}^{\infty} a_k \cos kx$.

We will use representation (24). For given h one can choose n: $\frac{1}{n+1} < h \le \frac{1}{n}$. Then

$$\begin{aligned} \left|\triangle_h^{\beta} f\Big(x - \frac{\beta h}{2}\Big)\right| &\le \sum_{k=1}^{\infty} \left|2\sin\frac{kh}{2}\right|^{\beta} a_k \\ &= O\left(h^{\beta}\sum_{k=1}^{n} k^{\beta} a_k + \sum_{k=n+1}^{\infty} a_k\right) \\ &= O(\gamma_n). \end{aligned} \tag{28}$$

Therefore, we have $\left|\triangle_h^{\beta} f\left(x - \frac{\beta h}{2}\right)\right| = O(\gamma_n)$ with $0 < h \le \frac{1}{n}$, and so $\omega_{\beta}(f, \frac{1}{n}) = O(\gamma_n)$.

(B). Let $\beta = 2l-1$. We will follow the proving line of [9]. First, we note that for $|k\beta h| \le 1$ we have $1 - \cos k\beta h = O\left(k^2h^2\right)$ and $\sin kh = kh + O\left(k^3h^3\right)$. Also, $\sin(kx + k\beta h) = \sin kx - (1 - \cos k\beta h)\sin kx + \sin k\beta h \cos kx$. Let $[\alpha] = \max\{\alpha^* \in \mathbf{Z} : \alpha^* \le \alpha\}$. We define $s := \left[(\beta h)^{-1}\right]$. Therefore, from (24) we have[4]

$$\begin{aligned} &\triangle_{2h}^{\beta} f(x) = \\ &= (-1)^l 2^{\beta}\left(\sum_{k=1}^{s} + \sum_{k=s+1}^{\infty}\right)(\sin kh)^{\beta} a_k \sin(kx + k\beta h) \\ &= (-1)^l 2^{\beta}\sum_{k=1}^{s} a_k (kh)^{\beta}\sin kx + O\left(\sum_{k=1}^{s} a_k (kh)^{\beta+1}\right) + O\left(\sum_{k=s+1}^{\infty} a_k\right) \\ &=: I_1 + I_2 + I_3. \end{aligned} \tag{29}$$

Let (12) be true. Using (27) and the following property of the modulus of smoothness (see [6]) $\omega_{\alpha+\beta}(f,\delta)_p \le C(\alpha)\omega_{\beta}(f,\delta)_p$ for $\alpha \ge 0$ we can write

$$\omega_{\beta}(f, 2h) = O(\gamma_s) \qquad \text{and} \qquad \omega_{\beta+1}(f, h) = O(\gamma_s). \tag{30}$$

---

[4] For $s = 0$ we put $\sum\limits_{k=1}^{s} := 0$

Since $\beta+1$ is even, then, from Lemma 4.1, (30) implies $I_2 = O\left(\omega_{\beta+1}\left(f,h\right)\right)$ $= O(\gamma_s)$. Then Lemma 4.4 implies $I_3 = O(\gamma_s)$. Hence, one has (14) from (29).

Conversely, if (13) and (14) hold, then, by Lemma 4.4, $I_2 = O(\gamma_s)$ implies $I_3 = O(\gamma_s)$ and $f(x) = \sum_{k=1}^{\infty} a_k \cos kx$. Further, (13) and (14) give $\omega_\beta\left(f, 2h\right) = O(\gamma_s)$. Finally, we have (12) from (27).

(C). Let (15) be satisfied. Then, using the inequality (see [1]) $\sum_{k=2n}^{\infty} a_k \leq 4E_n(f)$, and the Jackson inequality (see [6]) $E_n(f) \leq C(\beta)\omega_\beta(f, \frac{\pi}{n})$, and (27), we get

$$\begin{aligned}\sum_{k=n}^{\infty} a_k &\leq 4E_{\left[\frac{n}{2}\right]}(f)\\ &\leq C(\beta)\omega_\beta\Big(f, \frac{1}{n}\Big)\\ &= O\left(\gamma_n\right). \end{aligned} \tag{31}$$

Let (16) be true. Then, by Lemma 4.4 (i), we have $\sum_{k=1}^{n} k^\beta a_k = O\left(n^\beta \gamma_n\right)$ and, using (28), we have (15). The proof of Theorem 3.1 is complete.

Proof of Theorem 3.2. (A). Let (17) be true. By Lemma 4.2, (17) implies (18). Conversely, if we have (18), then by Lemma 4.4 (ii) we can write $\sum_{k=n}^{\infty} b_k = O\left(\gamma_n\right)$. Further, we will use

$$\triangle_h^\beta g\Big(x - \frac{\beta h}{2}\Big) \sim \sum_{k=1}^{\infty} b_k \left(2\sin\frac{kh}{2}\right)^\beta \sin\left(kx + \frac{\pi\beta}{2}\right). \tag{32}$$

Then, for $\frac{1}{n+1} < h \leq \frac{1}{n}$ we get

$$\begin{aligned}\left|\triangle_h^\beta g\Big(x - \frac{\beta h}{2}\Big)\right| &= O\left(h^\beta \sum_{k=1}^{n} k^\beta b_k + \sum_{k=n+1}^{\infty} b_k\right)\\ &= O\left(\gamma_n\right)\end{aligned} \tag{33}$$

and $\omega_\beta\left(g, \frac{1}{n}\right) = O\left(\gamma_n\right)$ holds.

(B). Let $\beta = 2l\, (l \in \mathbf{N})$, $s := \left[(\beta h)^{-1}\right]$. Similar to the proof of the part (B) of Theorem 3.1, we will have

$\triangle_{2h}^\beta g(x) =$

$$\begin{aligned} &= (-1)^l 2^\beta \left(\sum_{k=1}^{s} + \sum_{k=s+1}^{\infty}\right) (\sin kh)^\beta\, b_k \sin(kx + k\beta h) \\ &= (-1)^l 2^\beta \sum_{k=1}^{s} b_k\,(kh)^\beta \sin kx + O\left(\sum_{k=1}^{s} b_k\,(kh)^{\beta+1}\right) + O\left(\sum_{k=s+1}^{\infty} b_k\right) \\ &=: J_1 + J_2 + J_3. \end{aligned}$$

Using the properties of modulus of smoothness, Lemmas 4.2 and 4.4, we have that (19) implies $J_2, J_3 = O(\gamma_n)$ and therefore, (20) and (21) hold. On the other hand, (20) and (21) imply (19).

(C). From (22), by Lemma 4.4 (i), we have $\sum\limits_{k=1}^{n} k^\beta b_k = O\left(n^\beta \gamma_n\right)$ and from (33) we get (23). The proof of Theorem 3.2 is complete.

5. Concluding remarks

1. We note (see also [8]) that Theorem 3.1 (A) for $\beta = 2j$ and $\gamma_n = n^{-\beta}$ and Theorem 3.2 (A) for $\beta = 2j-1$ and $\gamma_n = n^{-\beta}$ answer to the following question by Boas [5, p. 25, Question 4.25]: What conditions are necessary and sufficient for $\sum n^{2j} a_n$ or $\sum n^{2j-1} b_n$ to converge?

2. We mention that for $\beta = 1$ Theorem 2.5 coincides with Theorem 3.1 (C) and [11, Theorem 3.5]) with Theorem 3.2 (A) . Also, we note, that for $\beta = 1$ and $\beta = 2$ some particular cases of Theorem 3.1 and Theorem 3.2 were considered in [13], [14].

3. If only $\gamma \in SQ_\beta$, then we can not have the equivalence between (22) and (23) in the part (C) of Theorem 3.2 for all $g \in C$. Indeed, for any $\beta > 0$ there exist a majorant γ with $\gamma \in SQ_\beta$ and a function g such that (23) holds but (22) does not hold.

As an example, we consider $\gamma_n = \ln^{-1-\varepsilon} 2n$ and the following continuous function

$$g(x) = \sum_{k=1}^{\infty} b_k \sin kx, \quad \text{where} \quad b_k = \frac{1}{k}\,\frac{1}{\ln^{1+\varepsilon} 2k} \quad \text{and} \quad \varepsilon > 0.$$

It is clear, that for any $\beta > 0$ we have $\gamma \in SQ_\beta$ and (22) is not true. To prove (23) for $\beta = 1$ we can use the ideas of the proof of [18, Lemma 4]. Then, using $\omega_{\alpha+1}(g,\delta)_p \le C(\alpha)\omega_1(g,\delta)_p$ $(\alpha > 0)$, we have (23) for $\beta > 1$. For $0 < \beta < 1$ the Marshaud inequality (see [6]) implies

$$\omega_\beta\Big(g, \frac{1}{n}\Big) \le C n^{-\beta} \sum_{\nu=1}^{n} \nu^{\beta-1} \omega_1\Big(g, \frac{1}{n}\Big)$$

$$\begin{aligned} &\le Cn^{-\beta}\sum_{\nu=1}^{n}\nu^{\beta-1}\ln^{-1-\varepsilon}2\nu \\ &\le C\ln^{-1-\varepsilon}2n. \end{aligned}$$

So, (23) holds for all $\beta > 0$.

4. One can assume that the condition

$$\sum_{k=1}^{n} k^{\beta} b_k = O\left(n^{\beta}\gamma_n\right) \tag{34}$$

is necessary and sufficient for the condition (23) to hold in the case $\gamma \in SQ_\beta$. The functions

$$g(x) = \sum_{k=1}^{\infty}\frac{1}{4k^2}\sin 3^{4k^2}x \qquad \text{and} \qquad \gamma_n = \log_3^{-1} 2n$$

give the negative answer to this conjecture for any $\beta > 0$. In fact, we have $\gamma \in SQ_\beta$, the condition (34) holds, but the condition (23) is not true. It follows from the Jackson inequality and the estimate of $E_n(g)$ for lacunary series

$$\begin{aligned} \omega_\beta\Big(g,\frac{1}{n}\Big) &\ge CE_n(g) \\ &\ge C\sum_{k:\,3^{4k^2}>n}\frac{1}{4k^2} \\ &\ge C\log_3^{-\frac{1}{2}} 2n. \end{aligned}$$

5. We note that if $\gamma \in SQ_\beta$ and, additionally, $\gamma \in SQ$, then (22) and (23) (and (34)) are equivalent (see Theorem 2.1).

6. One can write different equivalent conditions for conditions $\gamma \in SQ$ or $\gamma \in SQ_\beta$. Several of them were written in [2], [15]. We mention two more conditions.

First, we recall the following definition by Matuszewska (see [3, p. 68]). Let $f(\cdot)$ be positive defined on $[X,\infty)$, $X > 0$. Its upper Matuszewska index $\alpha(f)$ is the infimum of those α for which there exists a constant $C = C(\alpha)$ such that for each $\Lambda > 1$,

$$\frac{f(\lambda x)}{f(x)} \le C\lambda^{\alpha}\{1+o(1)\} \qquad (x\to\infty) \quad \text{uniformly in} \quad \lambda\in[1,\Lambda];$$

its lower Matuszewska index $\beta(f)$ is the supremum of those β for which, for some $D = D(\beta) > 0$ and for all $\Lambda > 1$,

$$\frac{f(\lambda x)}{f(x)} \geq D\lambda^{\beta}\{1 + o(1)\} \qquad (x \to \infty) \quad \text{uniformly in} \quad \lambda \in [1, \Lambda].$$

Lemma 5.1. *Let $\gamma = \{\gamma_n\}$ be a positive sequence and $\beta > 0$. If we define a function $\gamma(t)$, $t \in [1, \infty)$ as $\gamma(n) := \gamma_n$, $n \in \mathbf{N}$ and monotonic for $n \leq t \leq n+1$, then*

(i).: *$\{\gamma_n\}$ satisfies (SQ_β)-condition if and only if $\beta(\gamma) > -\beta$,*
(ii).: *$\{\gamma_n\}$ satisfies (SQ)-condition if and only if $\alpha(\gamma) < 0$.*

The proof of Lemma 5.1 follows from [3, Theorem 2.2.2, p.72].

Also, we write the following condition from [18, p. 152]. Let φ be a gauge function, i.e., continuous and nondecreasing function on $[0, 1]$. Let φ_0 be the infimum of those α, for which

$$\sum_{k=0}^{n} 2^{k\alpha}\varphi\left(\frac{1}{2^k}\right) \leq C_\alpha 2^{n\alpha}\varphi\left(\frac{1}{2^n}\right)$$

is true with a constant C_α. If $\gamma_n = \varphi\left(\frac{1}{n}\right)$, then $\gamma \in SQ_\beta$ iff $\beta > \varphi_0$.

7. The elementary corollary of the Jackson inequality is as follows:

$$\mathrm{Lip}_\beta\, \gamma \subset \mathrm{Lip}_{\beta_1}\, \gamma \subset E(\gamma) := \{f \in C : E_n(f) = O(\gamma_n)\}, \qquad \beta_1 > \beta.$$

Note, if $\gamma \in SQ_\beta$, then the inverse theorem of trigonometric approximation implies that $\mathrm{Lip}_\beta\, \gamma$ is independent of β for all $\beta_1 > \beta$, i.e.,

$$\mathrm{Lip}_\beta\, \gamma = \mathrm{Lip}_{\beta_1}\, \gamma = E(\gamma), \qquad \beta_1 > \beta.$$

Finally, we rewrite the equivalence between modulus of smoothness and fractional K-functional (see [6], [7]):

$$C_1(\beta)\omega_\beta(f, t)_p \leq K(f, t^\beta, L_p, W_p^\beta) \leq C_2(\beta)\omega_\beta(f, t)_p,$$

where K-functional is given by $K(f, t, L_p, W_p^\beta) := \inf_{g \in W_p^\beta}\left(\|f-g\|_p + t\left\|D^\beta g\right\|_p\right)$.

6. Acknowledgements

The author acknowledges financial support from the European Commission (Contract MIF1-CT-2004-509465). The paper was partially written while he was staying at the Centre de Recerca Matematica (CRM) in Barcelona, Spain. He is grateful to the staff of CRM for their hospitality.

References

[1] N.K. Bari, On best approximation of two conjugate functions by trigonometric polynomials // *Izv. Akad. Nauk SSSR, Ser. Mat.* 19, No. 5, 285-302 (1955).

[2] N. K. Bari; S. B. Stečkin, Best approximations and differential properties of two conjugate functions // *Trudy Moskov. Mat. Obšč.* 5 (1956), 483–522

[3] N.H. Bingham, C.M. Goldie and J.L. Teugels, *Regular variation.* Cambridge Univ. Press, 1987.

[4] R. P. Boas Jr., Fourier series with positive coefficients // *J. Math. Anal. Appl.*,**17** (1967), 463–483.

[5] R. P. Boas Jr., *Integrability theorems for trigonometric transforms.* Springer-Verlag, New York, 1967.

[6] P.L. Butzer, H. Dyckhoff, E. Göerlich, R.L. Stens, Best trigonometric approximation, fractional order derivatives and Lipschitz classes // *Can. J. Math.* 29, 781–793, 1977.

[7] P.L. Butzer, U. Westphal, *An access to fractional differentiation via fractional difference quotients.* Fractional calculus and its applications. Lecture Notes in Math., Vol. 457, Springer, 1975.

[8] Ling-Yau Chan, On Fourier series with nonnegative coefficients and two problems of R. P. Boas // *J. Math. Anal. Appl.* 110, N 1, 116–129, 1985.

[9] Ling-Yau Chan, Generalized Lipschitz classes and asymptotic behavior of Fourier series // *J. Math. Anal. Appl.* 155, N 2, 371–377, 1991.

[10] M. Izumi and S. Izumi, Lipschitz classes and Fourier coefficients // *J. Math. Mech.*, **18**, 857–870, 1969.

[11] L. Leindler, Power-monotone sequences and Fourier series with positive coefficients // *J. Inequal. Pure Appl. Math.*, **1**(1), Article No.1, 2000. Available online at `http://jipam.vu.edu.au/`.

[12] L. Leindler; J. Nemeth, On the connection between quasi power-monotone and quasi geometrical sequences with application to integrability theorems for power series // *Acta Math. Hung.*, **68**, N 1-2, 7–19, 1995.

[13] J. Nemeth, Fourier series with positive coefficients and generalized Lipschitz classes // *Acta Sci. Math.* (Szeged) **54**, N 3-4, 291–304, 1990.

[14] J. Nemeth, Power-monotone sequences and Fourier series with positive coefficients // *J. Inequal. Pure Appl. Math.*, **2**(2), Article No.14 2001. Available online at `http://jipam.vu.edu.au/`.

[15] S. Tikhonov, Generalized Lipschitz classes and Fourier coefficients //*Mat. Zametki*, **75**, no. 6, 947–951, 2004; translation in *Math. Notes*, **75**, no. 6, 885–889, 2004.

[16] S. Tikhonov, On moduli of smoothness of fractional order //*CRM Preprints*, in press, 2005.

[17] S. Tikhonov, On two theorem of Lorentz // Izv. Ross. Akad. Nauk, Ser. Mat. Vol. 69, N 1, 165–178, 2005, translation in Russian Acad. Sci. Izv. Math, 2005.

[18] V. Totik, On the strong approximation of Fourier series // *Acta Math. Acad. Sci. Hung.* 35, 151–172, 1980.

[19] A. Zygmund, *Trigonometric series.* Cambridge University Press, 1959.

Centre de Recerca Matemàtica(CRM)
Bellaterra (Barcelona) E-08193, Spain
e-mail: `stikhonov@crm.es`