

# PURE BRAID SUBGROUPS OF BRAIDED THOMPSON'S GROUPS

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ABSTRACT. We describe pure braided versions of Thompson's group  $F$ . These groups,  $BF$  and  $\widehat{BF}$ , are subgroups of the braided versions of Thompson's group  $V$ , introduced by Brin and Dehornoy. Unlike  $V$ , elements of  $F$  are order-preserving self-maps of the interval and we use pure braids together with elements of  $F$  thus preserving order. We define these groups and give normal forms for elements and describe infinite and finite presentations of these groups.

## 1. INTRODUCTION AND DEFINITIONS

Thompson's groups  $F$  and  $V$  have been studied from many perspectives. Both groups can be understood as groups of locally orientation-preserving piecewise-linear maps of the unit interval. In the case of  $F$ , these maps are homeomorphisms, and in the case of  $V$  the maps are right-continuous bijections. In both cases the breakpoints and discontinuities are restricted to be dyadic rational numbers, and the slopes, when defined, are powers of 2. Both groups can also be understood by means of rooted binary tree pair diagrams—order-preserving in the case of  $F$ . Cannon, Floyd and Parry [7] give an excellent introduction to these groups and several approaches to understanding their properties.

A rooted binary tree is a finite tree where every node has valence 3 except the root, which has valence two, and the leaves, which have valence one. We draw such trees with the root on top and the nodes descending from it to the leaves along the bottom. The two nodes immediately below a node are

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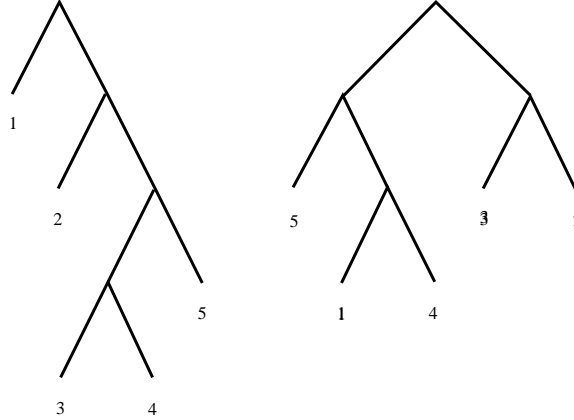
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FIGURE 1. An element of  $V$ 

its *children*. A node and its two children form a *caret*. A caret whose two children are leaves is called an *exposed caret*. We number the leaves of a rooted binary tree with  $n$  carets and  $n + 1$  leaves from 0 to  $n$  in any order, although we frequently choose to number the leaves of a rooted tree in order from left to right.

A *tree pair diagram* is a triple  $(T_-, \pi, T_+)$ , where  $T_-$  and  $T_+$  are two binary trees with the same number of leaves  $n$ , and  $\pi$  is a permutation in  $S_n$ . A reduction can be performed in a diagram if both leaf numbers of an exposed caret in  $T_-$  are mapped by  $\pi$  to the two leaf numbers of an exposed caret in  $T_+$ . In cases where a reduction is possible, we can replace the exposed caret with a leaf and renumber leaves to give an equivalent representative with a new permutation in  $S_{n-1}$ . A tree pair diagram is *reduced* if no reductions are possible. The set of binary tree pair diagrams thus admits an equivalence relation, whose classes consist of those diagrams which have a common reduced representative, with such reduced representatives being unique. In Figure 1 a reduced diagram is depicted, where the leaf numbering describes the permutation. The elements of  $V$  which are actually in  $F$  are precisely those elements for which the permutation is the identity.

Composition in  $V$  can be understood by means of these binary tree diagrams. If two elements in  $V$  are given by their representative diagrams  $(T_-, \pi, T_+)$  and  $(S_-, \sigma, S_+)$ , their composition can be found by finding two tree diagrams in the corresponding equivalence classes,  $(T'_-, \pi', T'_+)$  and  $(S'_-, \sigma', S'_+)$ , such that  $T'_+ = S'_-$ . When this is achieved, the product element is represented by the diagram  $(T'_-, \pi' \circ \sigma', S'_+)$ .

Brin [2, 3] and Dehornoy [8] describe braided Thompson's group  $BV$ , incorporating braids into tree pair diagrams. The permutation  $\pi$  in a triple for an element of  $V$  is replaced by a braid, giving the notion of a *braided tree diagram*. A braided tree diagram is then a triple  $(T_-, b, T_+)$ , where the two trees have the same number of leaves  $n$ , and  $b \in B_n$  is a braid in  $n$  strands. Reductions can still be performed if two exposed carets are joined by two parallel strands. In Figure 2 an element of  $BV$  is depicted. We now draw the rooted tree  $T_-$  with the root at the top, and the tree  $T_+$  below with the root at the bottom, and then draw the braid between the leaves of the two trees as indicated.

In [2], Brin also describes a "larger" group  $\widehat{BV}$  as a group of braided forest diagrams having all but finitely many of the forests trivial. Not only does  $\widehat{BV}$  naturally contain  $BV$  as the subgroup of forest diagrams where all trees are trivial except the first pair, but  $\widehat{BV}$  also sits inside  $BV$  as the subgroup of braided tree diagrams where the rightmost strand is always unbraided. We begin this paper by providing finite and infinite presentations of  $BV$  which contain the presentations provided by Brin in [2] as subpresentations. Next, we describe subgroups  $BF$  of  $BV$  and  $\widehat{BF}$  of  $\widehat{BV}$ . Just as  $F$  is the subgroup of  $V$  of order-preserving right continuous bijections of  $V$ , the groups  $BF$  and  $\widehat{BF}$  are the subgroups of order-preserving elements of the braided versions of  $V$ . The order is preserved by using generators which come from  $F$  and generators which involve pure braids. We describe normal forms for elements in these subgroups and obtain infinite and finite presentations for these groups. Dehornoy [8] calls this pure braid subgroup  $PB_\bullet$ , the group of pure parenthesized braids.

## 2. THE BRAIDED THOMPSON'S GROUP $BV$

In [2], an infinite presentation for  $\widehat{BV}$  is given. The generators in this presentation are the generators in the standard infinite presentation for Thompson's group  $F$ , as well as the generators for  $B_\infty$ . Here  $B_\infty$  is considered as a direct limit of the groups  $B_n$ , where  $B_n$  is included in  $B_{n+1}$  via adding one strand at the right. Now  $\widehat{BV}$  sits naturally as a subgroup of  $BV$ ; it is isomorphic to the subgroup of all elements represented by braided tree diagrams in which the rightmost strand is unbraided. Although presentations for  $BV$ , both finite and infinite, are given in [3], they are not related in a simple way to the presentation for  $\widehat{BV}$ . Instead, we give a presentation for  $BV$  which contains Brin's presentation for  $\widehat{BV}$  as a subpresentation. First, we define the set of generators. Any element of  $BV$  can be represented by a braided tree diagram  $(T_-, b, T_+)$  where both  $T_-$  and  $T_+$  have  $n$  leaves and  $b$  is a braid in  $B_n$ . A single tree can be thought

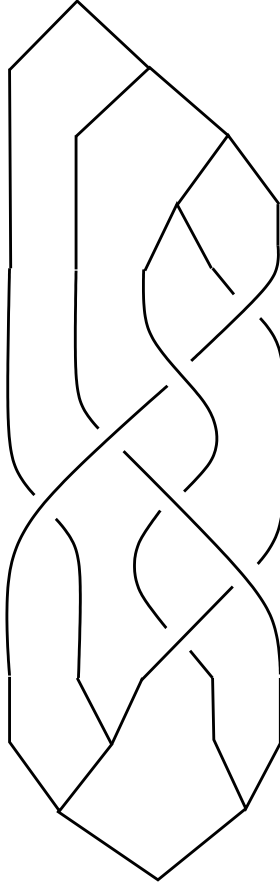


FIGURE 2. An element of  $BV$ , one of the preimages of the element drawn in Figure 1 under the map  $\tilde{\phi}$ .

of as a *positive* element of Thompson's group  $F$ , when we take it as being paired with an *all-right* tree, which is a tree whose caretts are all right children of their parent caretts. These positive elements correspond to elements which are positive words in  $F$  with respect to the infinite generating set  $\{x_0, x_1, \dots\}$ . The correspondence between tree pair diagrams and normal forms with respect to the infinite generating set is given by the process of exponents of leaves, as described by Cannon, Floyd and Parry [7] and Fordham [9]. All-right trees have all leaf exponents zero, and thus the normal forms for tree pair diagrams which involve one all-right tree will be purely

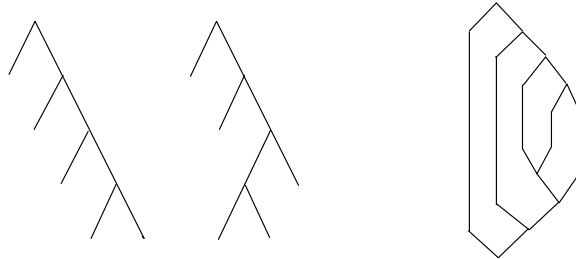


FIGURE 3. The generator  $x_2$  of  $F$ , in standard and in braided form.

negative or purely positive. We will denote by  $R_n$  the all-right tree which has  $n$  leaves.

We can factor an element  $(T_-, b, T_+)$  into three pieces, using all-right trees of the appropriate number of leaves, in a manner similar to that done for elements of Thompson's group  $T$  by Burillo, Cleary, Stein and Taback [6]. The resulting three elements in this factorization are

$$(T_-, id, R_n) \quad (R_n, b, R_n) \quad (R_n, id, T_+),$$

and the product of the elements represented by these three diagrams yields the original group element. In general, these three tree pair diagrams will not be reduced; in order for each of them to have the same number of carets, we may need to take unreduced representatives for as many as two of the three terms. By enlarging trees in this manner, it is clear that every element of  $BV$  can be factored this way.

Hence, we can always think of an element of  $BV$  as if it were composed of two elements of Thompson's group  $F$ , one positive and one negative, and one braid. It makes sense then to consider, as a set of generators of  $BV$ , the set of generators for  $F$  and the set of generators for the braid groups, interpreted as braided tree pair diagrams between all-right trees.

The infinite set of generators for  $F$  consists of the elements  $x_i$  with  $i \geq 0$ . Figure 3 shows  $x_2$  in both tree pair diagram form and in braided tree pair form. These generators from  $F$  are enough to produce the two elements  $(T_-, id, R_n)$  and  $(R_n, id, T_+)$  in  $BV$ .

We can consider the element  $(R_n, b, R_n)$  as an element in the appropriate braid group  $B_n$ . Now this copy of  $B_n$  is generated by  $n - 1$  transpositions, the  $i^{th}$  of which braids strand  $i$  over strand  $i + 1$ . We do not, however, need to include all of these as generators. The generators which do not involve braiding the last two strands can be obtained from the generators of  $B_{n-1}$  by splitting the last strand.

For this reason we must consider two sets of generating braids, one which leaves the rightmost strand unbraided and one which does not. We define  $\sigma_i$  to be the element represented by the braided tree diagram  $(R_{i+2}, a_i, R_{i+2})$ , where  $a_i$  is the braid on  $i+2$  strands which crosses strand  $i$  over strand  $i+1$ . Similarly,  $\tau_i$  is the element represented by the diagram  $(R_{i+1}, b_i, R_{i+1})$ , where  $b_i$  is the braid on  $i+1$  strands which crosses strand  $i$  over strand  $i+1$ . Then the set  $\{\sigma_1, \sigma_2, \dots, \sigma_{n-2}, \tau_{n-1}\}$  generates the copy of  $B_n$  containing all elements of  $BV$  represented by diagrams of the form  $(R_n, p, R_n)$ . Notice that the  $x_i$  together with the  $\sigma_i$  generate the copy of  $\widehat{BV}$  inside  $BV$ , and they correspond to Brin's generators. We have shown:

**Proposition 2.1.** *The elements  $x_i$ , for  $i \geq 0$ ,  $\sigma_i$  for  $i \geq 1$ , and  $\tau_i$  for  $i \geq 1$  form a set of generators for  $BV$ .*

There are three types of natural relations among these generators. First, there are the generators involving only the generators of  $F$ , namely  $x_j x_i = x_i x_{i+j}$  for  $j > i$ . These are the generators for the standard presentation for  $F$ . Next, we expect to need generators for each copy of  $B_n$ . These yield four types of relations:

- $\sigma_i \sigma_j = \sigma_j \sigma_i$ , for  $j \geq i+2$
- $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$
- $\sigma_i \tau_j = \tau_j \sigma_i$ , for  $j \geq i+2$
- $\sigma_i \tau_{i+1} \sigma_i = \tau_{i+1} \sigma_i \tau_{i+1}$ .

Finally, there are relations governing the interactions between the generators for  $F$  and the generators for the braid groups  $B_n$ .

- $\sigma_i x_j = x_j \sigma_i$ , for  $i < j$
- $\sigma_i x_i = x_{i-1} \sigma_{i+1} \sigma_i$
- $\sigma_i x_j = x_j \sigma_{i+1}$ , for  $i \geq j+2$
- $\sigma_{i+1} x_i = x_{i+1} \sigma_{i+1} \sigma_{i+2}$
- $\tau_i x_j = x_j \tau_{i+1}$ , for  $i \geq j+2$
- $\tau_i x_{i-1} = \sigma_i \tau_{i+1}$
- $\tau_i = x_{i-1} \tau_{i+1} \sigma_i$ .

In preparation for showing that the relations above give a presentation, we first introduce a special class of words in the generators. We would like to identify those words in the generators which could be identified easily with a triple of diagrams in  $BV$ . As noted earlier, any element of  $BV$  can be represented by a triple of braided tree diagrams of the form  $(T_-, id, R_n)$   $(R_n, b, R_n)$   $(R_n, id, T_+)$ . Such a triple leads easily to a word in the generators as follows. The group element represented by the first diagram is a positive element  $a$  of  $F \subset BV$ , and may be expressed uniquely as a word of the form  $x_{i_1}^{r_1} x_{i_2}^{r_2} \dots x_{i_k}^{r_k}$ , where  $i_1 < i_2 < \dots < i_k$  and  $r_m \geq 1$  for all  $m$ . Similarly, the group element  $c \in F \subset BV$  represented by the third

diagram can be uniquely expressed as a word of the form  $x_{j_1}^{-s_1} \dots x_{j_2}^{-s_2} x_{j_1}^{-s_1}$ , where  $j_1 < j_2 < \dots < j_l$  and  $s_m \geq 1$  for all  $m$ . Now the group element represented by the middle diagram may be represented as some word in the generators  $\sigma_1, \sigma_2, \dots, \sigma_{n-2}, \tau_{n-1}$ , and their inverses. For convenience, we will call any word in this set of generators and their inverses *a word in the  $B_n$  generators*. Note that if such a word contains no  $\tau$  generators, it can be considered a word in the  $B_n$  generators for many values of  $n$ . Notice that the minimum number of carets required in the trees for tree pair diagrams representing  $a$  and  $c$  respectively, is at most  $n - 1$ . The concatenation of the three words described above yield a word which cannot serve as a normal form, since we have not specified preferred arrangements of the  $\sigma$ 's and  $\tau$  and furthermore, there are many different triples of tree pair diagrams representing any element. However, these words are nice in that any word of the above special form can be easily translated into a triple of diagrams, and we find them to be useful tools.

Given a word  $w \in F$ , denote by  $N(w)$  the number of carets in the reduced binary tree diagram representing it. Here is the algebraic description of blocks, which are these words which come from a single triple of diagrams.

**Definition 2.2.** *A word in the generators  $x_i^{\pm 1}, \sigma_i^{\pm 1}, \tau_i^{\pm 1}$  is called a block if it is of the form  $w_1 w_2 w_3^{-1}$  where*

- (1)  $w_1$  is of the form  $x_{i_1}^{r_1} x_{i_2}^{r_2} \dots x_{i_k}^{r_k}$ , where  $i_1 < i_2 < \dots < i_k$  and  $r_m \geq 1$  for all  $m$ .
- (2)  $w_3$  is of the form  $x_{j_1}^{s_1} x_{j_2}^{s_2} \dots x_{j_l}^{s_l}$ , where  $j_1 < j_2 < \dots < j_l$  and  $s_m \geq 1$  for all  $m$ , and by  $w_3^{-1}$  we mean the word  $x_{j_1}^{-s_1} \dots x_{j_2}^{-s_2} x_{j_1}^{-s_1}$ .
- (3) Let  $N = \max(N(w_1), N(w_2))$ . Then there exists an integer  $n$ ,  $n \geq N + 1$ , such that  $w_2$  is a word in the  $B_n$  generators.

Then we have the following lemma:

**Lemma 2.3.** *A block  $w_1 w_2 w_3^{-1}$  is the identity in  $BV$  if and only if  $w_1$  and  $w_3$  are the same word, and  $w_2$  is the identity in the copy of the braid group  $B_n$  generated by  $\tau_{n-1}^{\pm 1}$  and  $\sigma_i^{\pm 1}$  where  $1 \leq i \leq n - 2$ .*

*Proof.* The lemma follows directly from the fact that any word which is a block  $w_1 w_2 w_3^{-1}$  can be represented by a braided tree diagram  $(T_-, b, T_+)$  where  $w_1$  is represented by  $(T_-, id, R_n)$ ,  $w_2$  is represented by  $(R_n, b, R_n)$ , and  $w_3^{-1}$  is represented by  $(R_n, id, T_+)$ , and from any such triple of diagrams a block can be read off, unique up to the choice of the word in  $B_n$  expressing  $b$ . Since the identity in  $BV$  can be represented by the diagram consisting of the tree with only one vertex, and the trivial braid on one strand, all other diagrams representing the identity result from splitting strands, and will

always have two identical trees with the trivial braid. But such diagrams translate into blocks of the form described in the lemma.  $\square$

We will use these blocks to prove:

**Theorem 2.4.** *The group  $BV$  admits a presentation with generators:*

- $x_i$ , for  $i \geq 0$ ,
- $\sigma_i$ , for  $i \geq 1$ ,
- $\tau_i$ , for  $i \geq 1$ .

and relators

- (A)  $x_j x_i = x_i x_{j+1}$  for  $j > i$
- (B1)  $\sigma_i \sigma_j = \sigma_j \sigma_i$  for  $j - i \geq 2$
- (B2)  $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$
- (B3)  $\sigma_i \tau_j = \tau_j \sigma_i$  for  $j - i \geq 2$
- (B4)  $\sigma_i \tau_{i+1} \sigma_i = \tau_{i+1} \sigma_i \tau_{i+1}$ .
- (C1)  $\sigma_i x_j = x_j \sigma_i$ , for  $i < j$
- (C2)  $\sigma_i x_i = x_{i-1} \sigma_{i+1} \sigma_i$
- (C3)  $\sigma_i x_j = x_j \sigma_{i+1}$ , for  $i \geq j + 2$
- (C4)  $\sigma_{i+1} x_i = x_{i+1} \sigma_{i+1} \sigma_{i+2}$
- (D1)  $\tau_i x_j = x_j \tau_{i+1}$ , for  $i - j \geq 2$
- (D2)  $\tau_i x_{i-1} = \sigma_i \tau_{i+1}$
- (D3)  $\tau_i = x_{i-1} \tau_{i+1} \sigma_i$ .

This presentation appears without proof in J. Belk's thesis [1].

*Proof.* Let  $G$  be the abstract group given by the presentation above. We map  $G$  to  $BV$  via  $\phi$  by sending each generator to the element of  $BV$  with the same name. All relations in the presentation hold in  $BV$ , so  $\phi$  is a well-defined homomorphism. Proposition 2.1 shows that the map is surjective, so it remains only to show that  $\phi$  is injective. To show this, we must show that any word in the generators which maps to the identity in  $BV$  is already the identity in  $G$ . Now by Lemma 2.3, this is true if the word in question happens to be a block. So we are done once we show that the relations in  $G$  are sufficient to transform any word into a block. But since any generator is itself a block, an arbitrary word of length  $k$  is trivially the product of  $k$  blocks. So to show  $\phi$  is injective it is sufficient to prove that a word in  $G$  which is the product of two blocks can be rewritten, using the relations in  $G$ , as a single block. We first prove a series of 3 preliminary lemmas, from which we will deduce this fact in Lemma 2.8, which will complete the proof of the theorem.  $\square$

Our first lemma permits us to push the  $x_i^{\pm 1}$  generators to the left or right of the braid generators, which helps move a word toward block form.



**Lemma 2.5.** *If  $w$  is a word in the  $B_n$  generators, and  $i \leq n - 2$ , then  $x_i^{-1}w$  is equivalent in  $G$  to either  $\bar{w}x_{i'}^{-1}$  or  $\bar{w}$ , where  $\bar{w}$  is a word in the  $B_{n+1}$  generators, and  $i' \leq n - 2$ . Similarly, under the same conditions on all indices, a word  $wx_i$  may be replaced by either  $x_{i'}\bar{w}$  or  $\bar{w}$ .*

*Proof.* We describe first how to push  $x_i^{-1}$  past the  $\sigma$  and  $\tau$  generators. Pushing past  $\sigma$  type generators is always possible, but in order to push past  $\tau$ 's we must carefully keep track of the index of the  $x^{-1}$  as it moves along. Using the relations of type C, we may replace  $x_k^{-1}\sigma_j^{\pm 1}$  by  $w(\sigma)x_{k'}^{-1}$ , where  $w(\sigma)$  is a word in the  $\sigma$  generators and their inverses of length one or two. Furthermore, the maximum index appearing in  $w(\sigma)$  is  $j + 1$ . Now the index  $k'$  can, in general, be either  $k, k - 1$ , or  $k + 1$ . However, it only increases to  $k + 1$  in the case where  $j = k + 1$  also. So since the initial index  $i$  satisfies  $i \leq n - 2$  and  $j \leq n - 2$ , even a series of such replacements results in the presence of  $x_{i'}^{-1}$  with  $i' \leq n - 2$ . This is important, since relations (D2) and (D3) allow replacement of  $x_{n-2}^{-1}\tau_{n-1}^{\pm 1}$  by  $\tau_n^{\pm 1}\sigma_{n-1}^{\pm 1}$ , and relations (D1) allow us to replace  $x_k^{-1}\tau_{n-1}^{\pm 1}$  by  $\tau_n^{\pm 1}x_k^{-1}$  if  $k \leq n - 3$ . Hence,  $x_i^{-1}w$  can be replaced by either  $\bar{w}x_{i'}^{-1}$  or simply  $\bar{w}$  as claimed. The argument for  $wx_i$  is similar.  $\square$

Next we prove a lemma showing that a word in the  $B_n$  generators can always be pumped up to a word in the  $B_{n+1}$  generators at the expense of tacking on an  $x$  generator.

**Lemma 2.6.** *Let  $w$  be a word in the  $B_n$  generators. Then using the relators in  $G$ ,  $w$  may be replaced by either  $\bar{w}$  or  $\bar{w}x_i^{-1}$  where  $i \leq n - 2$  and  $\bar{w}$  is a word in the  $B_{n+1}$  generators. Similarly,  $w$  may also be replaced by either  $\bar{w}$  or  $x_i\bar{w}$  where  $\bar{w}$  is a word in the  $B_{n+1}$  generators.*

*Proof.* Consider the leftmost occurrence of  $\tau_{n-1}$  in the word  $w$ , that is,  $w = w_1\tau_{n-1}^{\pm 1}w_2$ , where  $w_1$  has only  $\sigma$  generators. Using relations (D2) or (D3) depending on the exponent of  $\tau_{n-1}$ , replace  $w$  by  $w_1\sigma_{n-1}^{\pm 1}\tau_n^{\pm 1}x_{n-2}^{-1}w_2$ , and then apply Lemma 2.5 to  $x_{n-2}^{-1}w_2$  to replace it with either  $\bar{w}_2x_i^{-1}$  or  $\bar{w}_2$  with  $i \leq n - 2$  and where  $\bar{w}_2$  is a word in the  $B_{n+1}$  generators. Then the desired  $\bar{w}$  is  $w_1\sigma_{n-1}^{\pm 1}\tau_n^{\pm 1}\bar{w}_2$ . Similarly, working from the right,  $w$  can be replaced by either  $x_i\bar{w}$  or  $\bar{w}$ .  $\square$

The two previous lemmas will now be used to show that the relators allow us to transform the product of two blocks to a new product of two blocks where the combined length of the middle two of the 6 subwords involved is reduced.

**Lemma 2.7.** *Let  $w = w_1w_2w_3^{-1}$  and  $v = v_1v_2v_3^{-1}$  be two blocks. Then the relations in  $G$  allow us to replace the word  $wv$  by  $w'_1w'_2w'_3{}^{-1}v'_1v'_2v'_3{}^{-1}$ ,*

the product of two blocks  $w' = w'_1 w'_2 w_3'^{-1}$  and  $v' = v'_1 v'_2 v_3'^{-1}$ , where  $l(v'_1) + l(w_3') < l(v_1) + l(w_3)$ .

*Proof.* Let  $x_{i_w}$  and  $x_{i_v}$  be the first letters in  $w_3$  and  $v_1$ . If they are the same, we can delete the pair  $x_{i_w}^{-1} x_{i_v}$  and we are done. If not, suppose  $i_w < i_v$  (if  $i_v < i_w$  a similar argument works, truncating  $v_3$  and absorbing  $x_{i_v}$  into  $w$ ). Let  $w'_1 = w_1$ ,  $w'_2 = w_2$ , and let  $w'_3$  be  $w_3$  with  $x_{i_w}$  deleted. Now we use the relations (A) to replace  $x_{i_w}^{-1} v_1$  by  $v'_1 x_{i_w}^{-1}$ . Note that  $v'_1$  and  $v_1$  have the same length, but  $N(v'_1) = N(v_1) + 1$ , since each index in  $v_1$  is increased by 1 as  $x_{i_w}^{-1}$  moves past it (see Theorem 3 of [5]). Next, suppose  $v_2$  is a word in  $\tau_{n-1}^{\pm 1}$  and  $\sigma_j^{\pm 1}$  with  $1 \leq j \leq n-2$ . Then  $N(v_1) \leq n$  and  $N(v_3) \leq n$ , so  $N(v'_1) \leq n+1$ . We must replace  $x_{i_w}^{-1} v_2 v_3^{-1}$  by  $v'_2 v_3'^{-1}$  so that  $v'_1 v'_2 v_3'^{-1}$  is a block. We will consider two cases.

*Case 1:* If  $i_w \leq n-2$ , we use Lemma 2.5 to replace  $x_{i_w}^{-1} v_2$  by  $v'_2 x_i^{-1}$  with  $i \leq n-2$  and  $v'_2$  a word in  $\tau_n^{\pm 1}$  and  $\sigma_j^{\pm 1}$ . Then  $(v_3 x_i)^{-1}$  can be rewritten using relations (A) as a word  $(v'_3)^{-1}$  so that  $v'_3$  is a positive word in the generators of  $F$  with increasing indices from left to right. Then it again follows from [5] that  $N(v'_3) = \text{Max}(N(v_3) + 1, i + 2)$ . Hence  $N(v'_3) \leq n+1$ , since  $N(v_3) + 1 \leq n+1$  and  $i + 2 \leq n$ , and this implies that  $v'_1 v'_2 (v'_3)^{-1}$  is a block as desired.

*Case 2:* If  $i_w > n-2$ , it is necessary to first use Lemma 2.6 to replace  $v_2$  by either  $\bar{v}_2$  or  $\bar{v}_2 x_i^{-1}$  where  $i \leq n-2$ . If  $x_i^{-1}$  is present, we use relations (A) to replace  $(v_3 x_i)^{-1}$  by  $\bar{v}_3^{-1}$ , and we see that  $N(\bar{v}_3) = \text{Max}(N(v_3) + 1, i + 2) \leq n+1$ . We continue applying Lemma 2.6 and absorbing any resulting  $x_i^{-1}$  letters into the  $v_3^{-1}$  part of the word in this manner, and after  $i_w - (n-2)$  repetitions we have replaced  $x_{i_w}^{-1} v_1 v_2 v_3^{-1}$  by  $v'_1 x_{i_w}^{-1} \bar{v}_2 \bar{v}_3^{-1}$ , where  $\bar{v}_2$  a word in the  $B_{i_w+2}$  generators, and  $\bar{v}_3$  is a word in the  $x_i$  with indices increasing from left to right with  $N(\bar{v}_3) \leq n + (i_w - (n-2)) = i_w + 2$ . Now just as before we can apply Lemma 2.5 to replace  $v'_1 x_{i_w}^{-1} \bar{v}_2 \bar{v}_3^{-1}$  by  $v'_1 v'_2 x_i^{-1} \bar{v}_3^{-1}$  where  $i \leq i_w$ , and  $v'_2$  is a word in the  $B_{i_w+3}$  generators. When we use relations (A) to replace  $(\bar{v}_3 x_i)^{-1}$  by  $v_3'^{-1}$ ,  $N(v'_3) = \text{Max}(N(\bar{v}_3) + 1, i + 2)$ . But since  $N(\bar{v}_3) + 1 \leq n + (i_w - (n-2)) = i_w + 3$  and  $i + 2 \leq i_w + 2$ ,  $N(v'_3) \leq i_w + 3$ , and hence  $v'_1 v'_2 v_3'$  is a block.  $\square$

Now we are in a position to prove the final lemma which completes the proof of Theorem 2.4.

**Lemma 2.8.** *The product of two blocks may be rewritten, using the relations of  $G$ , as a single block.*

*Proof.* Let  $w = w_1 w_2 w_3^{-1}$  and  $v = v_1 v_2 v_3^{-1}$  be two blocks. We apply Lemma 2.7, at most  $l(w_3) + l(v_1)$  times, to replace  $wv$  by  $w'_1 w'_2 v'_2 (v'_3)^{-1}$  where  $w'_2$  is

a word in the  $B_n$  generators,  $v'_2$  is a word in the  $B_{n'}$  generators,  $N(w'_1) \leq n$ , and  $N(v'_3) \leq n'$ . If  $n = n'$ , declaring  $u_1 = w'_1$ ,  $u_2 = w'_2 v'_2$ , and  $u_3 = v'_3$  shows that  $u_1 u_2 u_3^{-1} = w'_1 w'_2 v'_2 (v'_3)^{-1}$  is a block. If not, say  $n' < n$ , we apply Lemma 2.6  $n - n'$  times to replace  $v'_2$  by a word  $\bar{v}_2$ , a word in the  $B_n$  generators, followed by some new  $x^{-1}$  generators, so that  $v'_2 v'_3^{-1}$  has been replaced by  $\bar{v}_2 (v'_3 x_{i_1} x_{i_2} \cdots x_{i_{n-n'}})^{-1}$  where  $i_j \leq n' + j - 3$  for  $1 \leq j \leq n - n'$ . Now  $N(v'_3 x_{i_1}) = \text{Max}(N(v_3) + 1, i_1 + 2) \leq n' + 1$ . So inductively, we have that  $N(v'_3 x_{i_1} x_{i_2} \cdots x_{i_{n-n'}}) \leq n$ , and hence declaring  $u_1 = w'_1$ ,  $u_2 = w'_2 \bar{v}_2$ , and  $u_3 = \bar{v}_3$ , shows that  $u_1 u_2 u_3^{-1} = w'_1 w'_2 \bar{v}_2 \bar{v}_3^{-1}$  is a block. Of course, if  $n' > n$ , a similar argument works, pumping up indices in  $w'_2$  instead.  $\square$

We remark that an infinite presentation of Thompson's group  $V$  is easily obtained from the presentation for  $BV$  by adding two more infinite families of relators,  $\sigma_i^2 = 1$  and  $\tau_i^2 = 1$  for all  $i$ .

### 3. A FINITE PRESENTATION FOR $BV$

It is common for these type of infinite presentations for Thompson-type groups to reduce to finite presentations. For example, in [7] an inductive argument is spelled out which obtains the standard two generator-two relator presentation for  $F$  from the standard infinite presentation. Brin uses similar arguments to obtain finite presentations for  $BV$  and  $\widehat{BV}$  from his infinite ones. In a similar manner, the infinite presentation for  $BV$  in the previous section reduces to a finite presentation with 4 generators and only 18 relators, an improvement over the presentation in [3], which has 4 generators and 26 relators.

**Theorem 3.1.** *The group  $BV$  admits a finite presentation with generators  $x_0, x_1, \sigma_1, \tau_1$  and relators*

- (a)  $x_2 x_0 = x_0 x_3, x_3 x_1 = x_1 x_4$
- (c1)  $\sigma_1 x_2 = x_2 \sigma_1, \sigma_1 x_3 = x_3 \sigma_1, \sigma_2 x_3 = x_3 \sigma_2, \sigma_2 x_4 = x_4 \sigma_2$
- (c3)  $\sigma_2 x_0 = x_0 \sigma_3, \sigma_3 x_1 = x_1 \sigma_4$
- (c4)  $\sigma_1 x_0 = x_1 \sigma_1 \sigma_2, \sigma_2 x_1 = x_2 \sigma_2 \sigma_3$
- (d1)  $\tau_2 x_0 = x_0 \tau_3, \tau_3 x_1 = x_1 \tau_4$
- (d2)  $\tau_1 x_0 = \sigma_1 \tau_2, \tau_2 x_1 = \sigma_2 \tau_3$
- (b1)  $\sigma_1 \sigma_3 = \sigma_3 \sigma_1$
- (b2)  $\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2$
- (b3)  $\sigma_1 \tau_3 = \tau_3 \sigma_1$
- (b4)  $\sigma_1 \tau_2 \sigma_1 = \tau_2 \sigma_1 \tau_2$

where the letters in the relators not in the set of 4 generators are defined inductively by  $x_{i+2} = x_i^{-1} x_{i+1} x_i$  for  $i \geq 0$ ,  $\sigma_{i+1} = x_{i-1}^{-1} \sigma_i x_i \sigma_i^{-1}$  for  $i \geq 1$ , and  $\tau_{i+1} = x_{i-1}^{-1} \tau_i \sigma_i^{-1}$  for  $i \geq 1$ .

*Proof.* That the two (a) relators yield inductively all (A) relators in the infinite presentation is a standard argument, given in [7]. Notice that the relators (C2) and (D3) in the infinite presentation are precisely the relations used to inductively define the higher index generators in the infinite presentation. Now a straightforward induction yields the (C1) relators in the infinite presentation from the (c1) relators, then the (C3) relators from the (c3) relators, and so on, in the order the groups of relators are listed in the finite presentation above. As an example, we spell out the induction for the (B3) relators. So suppose we have (b3), or  $\sigma_1\tau_3 = \tau_3\sigma_1$ . Then suppose inductively that we have established  $\sigma_1\tau_i = \tau_i\sigma_1$  for  $3 \leq i < k$ , where  $k \geq 4$ . Then  $\sigma_1\tau_k = \sigma_1(x_{k-2}^{-1}\tau_{k-1}\sigma_{k-1}^{-1})$  using the relator defining  $\tau_k$ . Now we can move the  $\sigma_1$  to the right, first using the (C1) relators, then the inductive hypothesis, and finally the (B1) relators, and then use the defining relation for  $\tau_k$  in the other direction, to obtain the relator  $\sigma_1\tau_k = \tau_k\sigma_1$ . Therefore, by induction,  $\sigma_1\tau_k = \tau_k\sigma_1$  for all  $k \geq 3$ . Now suppose that we have  $\sigma_i\tau_j = \tau_j\sigma_i$  for  $j - i \geq 2$ ,  $1 \leq i < k$ , and  $k \geq 2$ . Then it follows that  $\sigma_k\tau_j = \tau_j\sigma_k$  for  $j - k \geq 2$ . We replace  $\sigma_k$  in the word  $\sigma_k\tau_j$  by  $x_{k-2}^{-1}\sigma_{k-1}x_{k-1}\sigma_{k-1}^{-1}$ , and then moves the  $\tau_j$  to the left, first using the inductive hypothesis to obtain  $x_{k-2}^{-1}\sigma_{k-1}x_{k-1}\tau_j\sigma_{k-1}^{-1}$ , then using the (D1) relators to obtain  $x_{k-2}^{-1}\sigma_{k-1}\tau_{j-1}x_{k-1}\sigma_{k-1}^{-1}$ , and finally using the inductive hypothesis again to obtain  $x_{k-2}^{-1}\tau_{j-1}\sigma_{k-1}x_{k-1}\sigma_{k-1}^{-1}$ . Now use (D1) relators to move  $\tau_{j-1}$  left, to obtain  $\tau_j x_{k-2}^{-1}\sigma_{k-1}x_{k-1}\sigma_{k-1}^{-1}$ . But now the rightmost four letters can be replaced by  $\sigma_k$  using the defining relation for  $\sigma_k$  in reverse, showing that  $\sigma_k\tau_j = \tau_j\sigma_k$ . Hence, by induction, all (B3) relators hold.  $\square$

For the corresponding finite presentation for  $V$ , we note that the relations  $\tau_i^2 = 1$  and  $\sigma_i^2 = 1$  for  $i \geq 2$  can be deduced inductively from the two relations  $\tau_1^2 = \sigma_1^2 = 1$ , using the (C2) and (C3) relators in the case of  $\sigma_i$ , and the (D2) and (D3) relators for  $\tau_i$ . This yields a presentation for  $V$  with 4 generators and 20 relators, not quite as efficient as the presentation in [7] with 14 relators.

#### 4. THE GROUP $PBV$

The pure braid groups  $P_n$  are the groups of braids where the  $i$ th strand is braided with the other strands but returns to the  $i$ th position. There are several possible ways to construct analogous subgroups of  $BV$ . One way is by considering the standard short exact sequences for the braid groups, involving the pure braid groups and the permutation groups. For each  $n$ ,

we have:

$$1 \longrightarrow P_n \longrightarrow B_n \xrightarrow{\phi_n} S_n \longrightarrow 1$$

which maps a braid to its permutation, and whose kernel is the pure braid group  $P_n$ . This family of maps  $\phi_n$  collectively induces a map

$$\bar{\phi}: BV \longrightarrow V$$

defined by  $\bar{\phi}(T_-, b, T_+) = (T_-, \phi_n(b), T_+)$ , where we use the appropriate  $\phi_n$  for the number of leaves in either tree.

Let  $PBV = \ker \bar{\phi}$ . By definition, a diagram  $(T_-, b, T_+)$  represents an element in  $PBV$  if it maps to the identity in  $V$ , that is, if  $\phi(b) = \text{id}$ , and if  $T_- = T_+$ . Hence,  $PBV$  is the subgroup of  $BV$  which consists of those elements which admit a representative  $(T, p, T)$  on which the two trees are the same and the braid is pure. If an element admits one representative where the two trees are equal, then every representative will have the trees being equal. So  $PBV$  is a subgroup, because the product of such two elements also has representatives where the two trees are equal. Note that it is crucial in this construction that the braid is a pure braid.

The main result concerning the group  $PBV$  is the following.

**Theorem 4.1.** *The group  $PBV$  is not finitely generated.*

*Proof.* Given two elements of  $PBV$  by their diagrams  $(T, p, T)$  and  $(S, q, S)$ , their product always admits a representative diagram  $(R, r, R)$ , where  $R$  is the *least common multiple* of  $S$  and  $T$ —that is, the minimal tree which contains both  $S$  and  $T$  as subtrees. Hence, if  $PBV$  were to be generated by a finite set  $(T_i, p_i, T_i)$ , for  $i = 1, \dots, k$ , every tree in  $PBV$  would admit a representative whose tree would be the least common multiple of the  $T_i$ . There are elements whose smallest representatives are of increasing size and thus  $PBV$  cannot be finitely generated.  $\square$

## 5. THE BRAIDED THOMPSON'S GROUP $BF$

From the map  $\bar{\phi}$  defined above, since  $F$  is the subgroup of  $V$  of those elements whose permutation is the identity, we can define the group  $BF = \bar{\phi}^{-1}(F)$ . The group  $BF$  is the subgroup of  $BV$  of those elements which admit a representative  $(T_-, p, T_+)$ , where  $p$  is a pure braid. Note the contrast with  $PBV$ , since here the two trees are not necessarily equal. In fact,  $PBV$  is a subgroup of  $BF$ , and observing the restriction to  $BF$  of the map  $\bar{\phi}$  above, it is easy to see that  $PBV$  is also the kernel of  $\bar{\phi}|_{BF}$ . Thus the diagram below is commutative:

$$\begin{array}{ccccccc}
1 & \longrightarrow & PBV & \longrightarrow & BF & \xrightarrow{\bar{\phi}|_{BF}} & F & \longrightarrow & 1 \\
& & \parallel & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & PBV & \longrightarrow & BV & \xrightarrow{\bar{\phi}} & V & \longrightarrow & 1
\end{array}$$

The main goal of the remainder of this paper is to prove that  $BF$  is finitely presented and to find both finite and infinite presentations.

Finding generators for  $BF$  is not difficult. Just as for  $BV$ , an element of  $BF$  is given by a triple  $(T_-, p, T_+)$ , where this time  $p$  is a pure braid. Again, we factor the element into three pieces

$$(T_-, id, R_n) \quad (R_n, p, R_n) \quad (R_n, id, T_+),$$

where the individual diagrams may not be reduced. Hence, we can always think of an element of  $BF$  as if it were composed of two elements of Thompson's group  $F$ , one positive and one negative, and one pure braid. Again, just as for  $BV$ , we take as a set of generators of  $BF$ , the set of generators for  $F$  and the set of generators for the pure braid groups, interpreted as braided tree pair diagrams between all-right trees. We consider the element  $(R_n, p, R_n)$  as an element of the appropriate group  $P_n$  of pure braids. To generate these groups  $P_n$  we would like to use the braids  $A_{ij}$ , for  $i < j$ , which wrap the  $i$ -th strand around the  $j$ -th one. See Hansen [10] for details of these generating sets. The process of obtaining the generators of  $BF$  from the generators  $A_{ij}$  of  $P_n$  is the same than the process specified above for  $BV$  from the standard braid generators.

We will denote by  $\alpha_{ij}$  the element  $(R_{j+1}, A_{ij}, R_{j+1})$ , and by  $\beta_{ij}$  the element  $(R_j, A_{ij}, R_j)$ . As in  $BV$ , the differences between these two families of generators are whether or not the last strand is involved in the braiding. Figure 4 shows an example of the two generators of  $BF$  corresponding to a generator  $A_{ij}$ .

The proof of the following proposition is analogous to the proof of Proposition 2.1.

**Proposition 5.1.** *The elements  $x_i$ , for  $i \geq 0$ ,  $\alpha_{ij}$ , for  $1 \leq i < j$ , and  $\beta_{ij}$ , for  $1 \leq i < j$ , form a set of generators of  $BF$ .*

In the next theorem we will give a presentation for the group  $BF$ . The relators are going to be divided into four families. The family (A) is obtained from the relators of  $F$ . The family (B) is obtained from the presentation of the pure braid group:

- $A_{rs}^{-1} A_{ij} A_{rs} = A_{ij}$ , if  $1 \leq r < s < i < j \leq n$  or  $1 \leq i < r < s < j \leq n$
- $A_{rs}^{-1} A_{ij} A_{rs} = A_{rj} A_{ij} A_{rj}^{-1}$ , if  $1 \leq r < s = i < j \leq n$

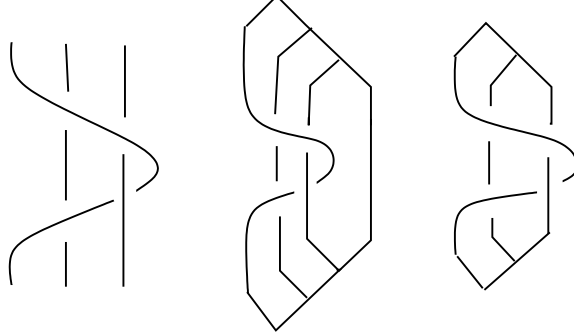


FIGURE 4. The generator  $A_{13}$  in  $P_3$ , with its corresponding generators  $\alpha_{13}$  and  $\beta_{13}$  in  $BF$ .

- $A_{rs}^{-1}A_{ij}A_{rs} = (A_{ij}A_{sj})A_{ij}(A_{ij}A_{sj})^{-1}$ , if  $1 \leq r = i < s < j \leq n$
- $A_{rs}^{-1}A_{ij}A_{rs} = (A_{rj}A_{sj}A_{rj}^{-1}A_{sj}^{-1})A_{ij}(A_{rj}A_{sj}A_{rj}^{-1}A_{sj}^{-1})^{-1}$ , if  $1 \leq r < i < s < j \leq n$

obtained from [10]. The family (D) reflects the interactions between generators of  $F$  and pure braids. In [2], Brin constructs these relators using the structure of Zappa-Szép product of the monoid associated to  $\widehat{BV}$ . This construction is not possible here because the presentations for the groups  $P_\infty$  are not monoid presentations. In fact, the monoid of pure positive braids is not finitely generated as shown by Burillo, Gutierrez, Krstić and Nitecki. [4].

The family (C) of relators is given by the special way that the pure braid groups are embedded into each other inside  $BF$ . To embed  $P_n$  into  $P_{n+1}$  we split the last strand in two. If the last strand is not braided, this does not affect the element, but if the last strand takes part in the actual braiding, then these elements in  $P_n$  change when embedded in  $P_{n+1}$ . When a generator  $\beta_{ij}$  has its last strand split, now the  $i$ -th strand wraps around two strands (the  $j$ -th and the  $(j+1)$ -th), and the element is now a product of two generators.

**Theorem 5.2.** *The group  $BF$  admits a presentation with generators:*

- $x_i$ , for  $i \geq 0$ ,
- $\alpha_{ij}$ , for  $1 \leq i < j$ ,
- $\beta_{ij}$ , for  $1 \leq i < j$

and relators:

- (A)  $x_j x_i = x_i x_{j+1}$ , if  $i < j$
- (B1)  $\alpha_{rs}^{-1} \alpha_{ij} \alpha_{rs} = \alpha_{ij}$ , if  $1 \leq r < s < i < j$  or  $1 \leq i < r < s < j$

- (B2)  $\alpha_{rs}^{-1}\alpha_{ij}\alpha_{rs} = \alpha_{rj}\alpha_{ij}\alpha_{rj}^{-1}$ , if  $1 \leq r < s = i < j$   
(B3)  $\alpha_{rs}^{-1}\alpha_{ij}\alpha_{rs} = (\alpha_{ij}\alpha_{sj})\alpha_{ij}(\alpha_{ij}\alpha_{sj})^{-1}$ , if  $1 \leq r = i < s < j$   
(B4)  $\alpha_{rs}^{-1}\alpha_{ij}\alpha_{rs} = (\alpha_{rj}\alpha_{sj}\alpha_{rj}^{-1}\alpha_{sj}^{-1})\alpha_{ij}(\alpha_{rj}\alpha_{sj}\alpha_{rj}^{-1}\alpha_{sj}^{-1})^{-1}$ , if  $1 \leq r < i < s < j$   
(B5)  $\alpha_{rs}^{-1}\beta_{ij}\alpha_{rs} = \beta_{ij}$ , if  $1 \leq r < s < i < j$  or  $1 \leq i < r < s < j$   
(B6)  $\alpha_{rs}^{-1}\beta_{ij}\alpha_{rs} = \beta_{rj}\beta_{ij}\beta_{rj}^{-1}$ , if  $1 \leq r < s = i < j$   
(B7)  $\alpha_{rs}^{-1}\beta_{ij}\alpha_{rs} = (\beta_{ij}\beta_{sj})\beta_{ij}(\beta_{ij}\beta_{sj})^{-1}$ , if  $1 \leq r = i < s < j$   
(B8)  $\alpha_{rs}^{-1}\beta_{ij}\alpha_{rs} = (\beta_{rj}\beta_{sj}\beta_{rj}^{-1}\beta_{sj}^{-1})\beta_{ij}(\beta_{rj}\beta_{sj}\beta_{rj}^{-1}\beta_{sj}^{-1})^{-1}$ , if  $1 \leq r < i < s < j$   
(C)  $\beta_{ij} = \beta_{i,j+1}\alpha_{ij}$ , if  $i < j$   
(D1)  $\alpha_{ij}x_k = x_k\alpha_{i+1,j+1}$ , if  $k < i - 1$   
(D2)  $\alpha_{ij}x_k = x_k\alpha_{i+1,j+1}\alpha_{i,j+1}$ , if  $k = i - 1$   
(D3)  $\alpha_{ij}x_k = x_k\alpha_{i,j+1}$ , if  $i - 1 < k < j - 1$   
(D4)  $\alpha_{ij}x_k = x_k\alpha_{i,j+1}\alpha_{ij}$ , if  $k = j - 1$   
(D5)  $\alpha_{ij}x_k = x_k\alpha_{ij}$ , if  $k > j - 1$   
(D6)  $\beta_{ij}x_k = x_k\beta_{i+1,j+1}$ , if  $k < i - 1$   
(D7)  $\beta_{ij}x_k = x_k\beta_{i+1,j+1}\beta_{i,j+1}$ , if  $k = i - 1$   
(D8)  $\beta_{ij}x_k = x_k\beta_{i,j+1}$ , if  $i - 1 < k < j - 1$   
(D9)  $\beta_{ij}x_k = x_k\beta_{ij}$ , if  $k \geq j - 1$

*Proof.* As in the proof of Theorem 2.4, we consider the algebraic and the geometric group, establish a homomorphism between them, which is well-defined and surjective. For instance, one needs to check geometrically the relators to see it is well defined. See Figure 5 for an example. After this, it only remains to check the injectivity of the map.

To prove that the homomorphism is injective, we consider an element of  $G$ , given as a word  $w(x_i, \alpha_{ij}, \beta_{ij})$ , and imagine that it is mapped to the identity in  $G$ . We need to prove that it is consequence of the relators listed above.

As a first step, we can see that the relators (D1) to (D9) and (A) can be used to transform any such word into the product of three words,

$$w_1(x_i) w_2(\alpha_{ij}, \beta_{ij}) w_3(x_i)^{-1}.$$

We can arrange this in such a way that the words  $w_1$  and  $w_3$  contain only generators  $x_i$  and not their inverses.

This special expression is then particularly useful for studying the element, because it corresponds easily to the representative  $(T_-, p, T_+)$ . We will use now the following lemma whose proof is analogous to the proof of Lemma 2.3:



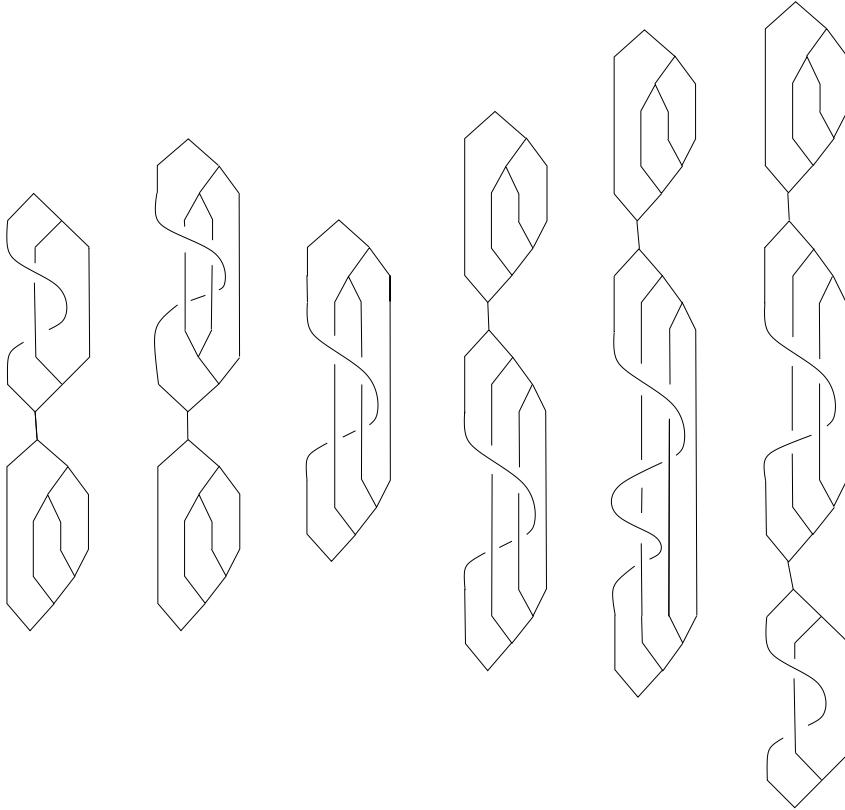


FIGURE 5. The process of checking the relator (D4). It starts on the left with  $\alpha_{12}x_1$  and the following steps are the multiplication processes which transform it into  $x_1\alpha_{13}\alpha_{12}$ .

**Lemma 5.3.** *A triple  $(T_-, p, T_+)$  in  $BF$  represents the identity element if and only if the braid  $p$  is the trivial braid and the element  $(T_-, T_+)$  represents the identity in  $F$ .*

With this lemma, we can assume now that the two words  $w_1w_3^{-1}$  and  $w_2$  map to the identity, and we must prove that they are consequence of the relators. The word  $w_1w_3^{-1}$  lies in the subgroup isomorphic to  $F$ . So if it is the identity, it is consequence of the relators (A).

The word  $w_2$  is a product of some  $\alpha$  and  $\beta$  generators. We would like to consider this word inside some  $P_n$ , for a fixed  $n$ . The image in  $BF$  of  $\beta_{ij}$

has  $j$  strands, and the image of  $\alpha_{ij}$  has  $j + 1$  strands. So the appropriate  $n$  to use is the maximum of the following set:

$$\{j \mid \beta_{ij} \text{ appears in } w_2\} \cup \{j + 1 \mid \alpha_{ij} \text{ appears in } w_2\}.$$

If we have a  $\beta_{ij}$  with  $j < n$ , we use the relators (C) to increase the second index of that  $\beta_{ij}$  to  $n$ . This way, the only generators involved are  $\alpha_{ij}$ , for  $1 < i < j < n$ , and  $\beta_{in}$ , for  $1 < i < n$ , which generate a copy of  $P_n$  inside  $BF$ . So, we can use the relators (C) to have our word expressed in this small set of generators and assume that it is a word in  $P_n$ . So if the word is the identity, it is consequence of the relators of  $P_n$ . But these relators correspond to the relators (B1) to (B8).  $\square$

## 6. A FINITE PRESENTATION FOR $BF$

As is common in the groups of the Thompson family, the infinite presentations are interesting and useful because of their symmetry and associated normal forms, but often it turns out that there are finite presentations from which the infinitely many generators and relations can be constructed and deduced. In this section, we construct a finite presentation for  $BF$ .

Thompson's group  $F$  admits a finite presentation which is merely the first two generators  $x_0$  and  $x_1$  and the first two non-trivial relations. We construct  $x_n$  from  $x_0$  and  $x_1$  as  $x_n = x_0^{-n+1}x_1x_0^{n-1}$  and from the two first two non-trivial relations we can deduce all of the relations in (A) above. This is the first building block for our finite presentation.

In a similar way we can construct all generators  $\alpha_{ij}$  from a few ones. The generators needed are  $\alpha_{12}$ ,  $\alpha_{13}$ ,  $\alpha_{23}$ ,  $\alpha_{24}$ . The idea is that conjugating a braid with  $x_k$  has the effect of splitting the  $(k - 1)$  strand, so from these four generators and the generators for  $F$ , we can construct any generator  $\alpha_{ij}$  by the process of splitting as many strands as necessary to produce the strands before the  $i$ -th and between the  $i$ -th and  $j$ -th. This process is as follows:

- Given  $\alpha_{i+1,j+1}$ , with  $j \geq i + 3$ , we use the relators (D3) to decrease the distance between  $i$  and  $j$  until 2:

$$\alpha_{i,j+1} = x_{j-2}^{-1}\alpha_{ij}x_{j-2}.$$

If  $i = 1$  then this process brings any generator  $\alpha_{1j}$  down to  $\alpha_{13}$ . For any other value of  $i$  it reduces to  $\alpha_{i,i+2}$ .

- We reduce the generators  $\alpha_{i+1,i+3}$  with  $i \geq 2$  to  $\alpha_{24}$  with relators of type (D1):

$$\alpha_{i+1,i+3} = x_{i-2}^{-1}\alpha_{i,i+2}x_{i-2}.$$

- And finally, we reduce generators of type  $\alpha_{i+1,i+2}$  to  $\alpha_{23}$  by again using (D1):

$$\alpha_{i+1,i+2} = x_{i-2}^{-1} \alpha_{i,i+1} x_{i-2}.$$

The generators  $\beta$  are constructed in exactly the same way, where we replace  $\alpha$  by  $\beta$ , and with the same constraints on indices.

To see which relators to include in the finite presentation, we see which can be used to get the full families. We consider the relators (D) first and we will use those to help with the other families.

In each of the relators of (D1), there are three strands which are important: the strands labelled  $i$ ,  $j$  and  $k$ . The idea is that between those, we only need to have one strand, because by splitting it, we can get to any number of strands in that position. Thus, we can get all of the relators (D1) from the following

$$(d1.1) \alpha_{34} = x_0^{-1} \alpha_{23} x_0$$

$$(d1.2) \alpha_{35} = x_0^{-1} \alpha_{24} x_0$$

$$(d1.3) \alpha_{45} = x_0^{-1} \alpha_{34} x_0$$

$$(d1.4) \alpha_{46} = x_0^{-1} \alpha_{35} x_0$$

$$(d1.5) \alpha_{45} = x_1^{-1} \alpha_{34} x_1$$

$$(d1.6) \alpha_{46} = x_1^{-1} \alpha_{35} x_1$$

$$(d1.7) \alpha_{56} = x_1^{-1} \alpha_{45} x_1$$

$$(d1.8) \alpha_{57} = x_1^{-1} \alpha_{46} x_1$$

Every relator of the type (D1) is a consequence of the definitions above and of these eight relators. As an example, we will show the relator  $\alpha_{i,i+1} x_0 = x_0 \alpha_{i+1,i+2}$ . If  $i = 2$ , the relator is (D1.1) above. If  $i > 2$ , then we use the definitions:

$$\alpha_{i+1,i+2} = x_{i-2}^{-1} x_{i-3}^{-1} \dots x_2^{-1} \alpha_{45} x_2 \dots x_{i-3} x_{i-2}$$

And using (d1.3) we get to

$$x_{i-2}^{-1} x_{i-3}^{-1} \dots x_2^{-1} (x_0^{-1} \alpha_{34} x_0) x_2 \dots x_{i-3} x_{i-2}$$

which is

$$x_0^{-1} x_{i-3}^{-1} x_{i-4}^{-1} \dots x_1^{-1} \alpha_{34} x_1 \dots x_{i-4} x_{i-3} x_0$$

finally equal to

$$x_0^{-1} \alpha_{i,i+1} x_0.$$

All the other relators of type (D) are very similar to this case, and we leave the details to the reader as they are straightforward but tedious.

The families of relators (D1) and (D3) are especially important, because they are used to split or combine adjacent strands which are not involved in the braiding. Any two adjacent strands which are not braided can be joined using a relator from one of these families. This is useful for the families (B1) to (B9).

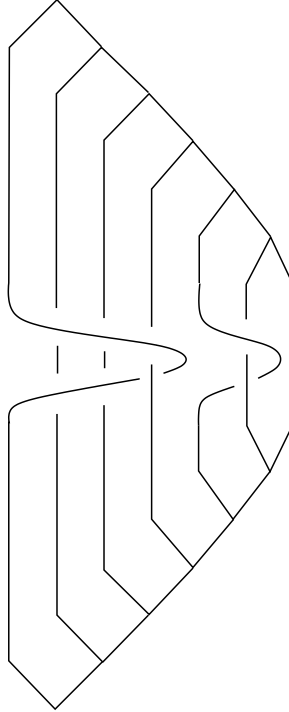


FIGURE 6. The element involved in the relator  $\alpha_{14}\alpha_{56} = \alpha_{56}\alpha_{14}$ . The commutativity is apparent.

For instance, the relators (B1) show that two braids  $\alpha_{ij}$  and  $\alpha_{rs}$  commute if  $1 \leq r < s < i < j$ . This relation only involves the strands  $r$ ,  $s$ ,  $i$  and  $j$ . If there are strands in between, they are uninvolved in the braiding. If there is more than one strand, we can apply the conjugating relations (D1) or (D3) to bring the relators down to a simple one where there is just one strand between. For instance, we show the relator

$$\alpha_{14}\alpha_{56} = \alpha_{56}\alpha_{14}$$

in Figure 6. We see that the two strands between the first and the fourth are straight. It is clear they can be obtained by splitting a single strand from an analogous relator whose braided strands are 1, 3, 4 and 5. A conjugation then by  $x_1$  brings it down, according to the definitions above. Now we have that

$$\alpha_{14} = x_1^{-1}\alpha_{13}x_1,$$

which is of the type (D3), and

$$\alpha_{56} = x_1^{-1} \alpha_{45} x_1,$$

which is of the type (D1). So the relator is a consequence of the relator

$$\alpha_{13} \alpha_{45} = \alpha_{45} \alpha_{13},$$

using only (D1) and (D3) relators. In this way we see that the only relators that we need to construct all the relators in (B1) are those that have either zero or one strand at the beginning, or between the  $i$ ,  $j$ ,  $r$ , and  $s$ -th strands. These are:

- $\alpha_{12} \alpha_{34} = \alpha_{34} \alpha_{12}$
- $\alpha_{12} \alpha_{35} = \alpha_{35} \alpha_{12}$
- $\alpha_{12} \alpha_{45} = \alpha_{45} \alpha_{12}$
- $\alpha_{12} \alpha_{46} = \alpha_{46} \alpha_{12}$
- $\alpha_{13} \alpha_{45} = \alpha_{45} \alpha_{13}$
- $\alpha_{13} \alpha_{46} = \alpha_{46} \alpha_{13}$
- $\alpha_{13} \alpha_{56} = \alpha_{56} \alpha_{13}$
- $\alpha_{13} \alpha_{57} = \alpha_{57} \alpha_{13}$
- $\alpha_{23} \alpha_{45} = \alpha_{45} \alpha_{23}$
- $\alpha_{23} \alpha_{46} = \alpha_{46} \alpha_{23}$
- $\alpha_{23} \alpha_{56} = \alpha_{56} \alpha_{23}$
- $\alpha_{23} \alpha_{57} = \alpha_{57} \alpha_{23}$
- $\alpha_{24} \alpha_{56} = \alpha_{56} \alpha_{24}$
- $\alpha_{24} \alpha_{57} = \alpha_{57} \alpha_{24}$
- $\alpha_{24} \alpha_{67} = \alpha_{67} \alpha_{24}$
- $\alpha_{24} \alpha_{68} = \alpha_{68} \alpha_{24}$

The other families, including the ones which involve  $\beta$  generators, are completely analogous.

We are left only with the family (C). The relators are all of the type  $\beta_{ij} = \beta_{i,j+1} \alpha_{ij}$ , with  $i < j$ . As before, it is clear that if  $i \geq 3$ , we can conjugate by  $x_0$  to get a relator with lower indices. So only ones with  $i = 1$  or  $i = 2$  are needed. Again we see that all are consequences of a fundamental finite set of relations:

- $\beta_{12} = \beta_{13} \alpha_{12}$
- $\beta_{13} = \beta_{14} \alpha_{13}$
- $\beta_{14} = \beta_{15} \alpha_{14}$
- $\beta_{23} = \beta_{24} \alpha_{23}$
- $\beta_{24} = \beta_{25} \alpha_{24}$
- $\beta_{25} = \beta_{26} \alpha_{25}$

For example, we see that

$$\beta_{1j} = x_{j-3}^{-1} \beta_{1,j-1} x_{j-3} = x_{j-3}^{-1} \beta_{1j} \alpha_{1,j-1} x_{j-3} = \beta_{1,j+1} \alpha_{1j}$$

if  $j \geq 5$ . And the cases for  $i = 2$  are exactly similar.

Hence, we have proved the following theorem:

**Theorem 6.1.** *The group  $BF$  is finitely presented*

The finite presentation for  $BF$  is the following:

Generators:  $x_0, x_1, \alpha_{12}, \alpha_{13}, \alpha_{23}, \alpha_{24}, \beta_{12}, \beta_{13}, \beta_{23}, \beta_{24}$

Relators:

- (A)  $x_j x_i = x_i x_{j+1}$ , for  $(i, j) = (1, 2), (1, 3)$   
 (B1)  $\alpha_{rs}^{-1} \alpha_{ij} \alpha_{rs} = \alpha_{ij}$ , for

$$(r, s, i, j) = \begin{cases} (1, 2, 3, 4) & (1, 2, 3, 5) & (1, 2, 4, 5) & (1, 2, 4, 6) \\ (1, 3, 4, 5) & (1, 3, 4, 6) & (1, 3, 5, 6) & (1, 3, 5, 7) \\ (2, 3, 4, 5) & (2, 3, 4, 6) & (2, 3, 5, 6) & (2, 3, 5, 7) \\ (2, 4, 5, 6) & (2, 4, 5, 7) & (2, 4, 6, 7) & (2, 4, 6, 8) \\ (2, 3, 1, 4) & (2, 3, 1, 5) & (2, 4, 1, 5) & (2, 4, 1, 6) \\ (3, 4, 1, 5) & (3, 4, 1, 6) & (3, 5, 1, 6) & (3, 5, 1, 7) \\ (3, 4, 2, 5) & (3, 4, 2, 6) & (3, 5, 2, 6) & (3, 5, 2, 7) \\ (4, 5, 2, 6) & (4, 5, 2, 7) & (4, 6, 2, 7) & (4, 6, 2, 8) \end{cases}$$

- (B2)  $\alpha_{rs}^{-1} \alpha_{ij} \alpha_{rs} = \alpha_{rj} \alpha_{ij} \alpha_{rj}^{-1}$ , for

$$(r, s, i, j) = \begin{cases} (1, 2, 2, 3) & (1, 2, 2, 4) & (1, 3, 3, 4) & (1, 3, 3, 5) \\ (2, 3, 3, 4) & (2, 3, 3, 5) & (2, 4, 4, 5) & (2, 4, 4, 6) \end{cases}$$

- (B3)  $\alpha_{rs}^{-1} \alpha_{ij} \alpha_{rs} = (\alpha_{ij} \alpha_{sj}) \alpha_{ij} (\alpha_{ij} \alpha_{sj})^{-1}$ , for

$$(r, s, i, j) = \begin{cases} (1, 2, 1, 3) & (1, 2, 1, 4) & (1, 3, 1, 4) & (1, 3, 1, 5) \\ (2, 3, 2, 4) & (2, 3, 2, 5) & (2, 4, 2, 5) & (2, 4, 2, 6) \end{cases}$$

- (B4)  $\alpha_{rs}^{-1} \alpha_{ij} \alpha_{rs} = (\alpha_{rj} \alpha_{sj} \alpha_{rj}^{-1} \alpha_{sj}^{-1}) \alpha_{ij} (\alpha_{rj} \alpha_{sj} \alpha_{rj}^{-1} \alpha_{sj}^{-1})^{-1}$ , for

$$(r, s, i, j) = \begin{cases} (1, 3, 2, 4) & (1, 3, 2, 5) & (1, 4, 2, 5) & (1, 4, 2, 6) \\ (1, 4, 3, 5) & (1, 4, 3, 6) & (1, 5, 3, 6) & (1, 5, 3, 7) \\ (2, 4, 3, 5) & (2, 4, 3, 6) & (2, 5, 3, 6) & (2, 5, 3, 7) \\ (2, 5, 4, 6) & (2, 5, 4, 7) & (2, 6, 4, 7) & (2, 6, 4, 8) \end{cases}$$

(B5)  $\alpha_{rs}^{-1}\beta_{ij}\alpha_{rs} = \beta_{ij}$ , for

$$(r, s, i, j) = \begin{cases} (1, 2, 3, 4) & (1, 2, 3, 5) & (1, 2, 4, 5) & (1, 2, 4, 6) \\ (1, 3, 4, 5) & (1, 3, 4, 6) & (1, 3, 5, 6) & (1, 3, 5, 7) \\ (2, 3, 4, 5) & (2, 3, 4, 6) & (2, 3, 5, 6) & (2, 3, 5, 7) \\ (2, 4, 5, 6) & (2, 4, 5, 7) & (2, 4, 6, 7) & (2, 4, 6, 8) \\ (2, 3, 1, 4) & (2, 3, 1, 5) & (2, 4, 1, 5) & (2, 4, 1, 6) \\ (3, 4, 1, 5) & (3, 4, 1, 6) & (3, 5, 1, 6) & (3, 5, 1, 7) \\ (3, 4, 2, 5) & (3, 4, 2, 6) & (3, 5, 2, 6) & (3, 5, 2, 7) \\ (4, 5, 2, 6) & (4, 5, 2, 7) & (4, 6, 2, 7) & (4, 6, 2, 8) \end{cases}$$

(B6)  $\alpha_{rs}^{-1}\beta_{ij}\alpha_{rs} = \beta_{rj}\beta_{ij}\beta_{rj}^{-1}$ , for

$$(r, s, i, j) = \begin{cases} (1, 2, 2, 3) & (1, 2, 2, 4) & (1, 3, 3, 4) & (1, 3, 3, 5) \\ (2, 3, 3, 4) & (2, 3, 3, 5) & (2, 4, 4, 5) & (2, 4, 4, 6) \end{cases}$$

(B7)  $\alpha_{rs}^{-1}\beta_{ij}\alpha_{rs} = (\beta_{ij}\beta_{sj})\beta_{ij}(\beta_{ij}\beta_{sj})^{-1}$ , for

$$(r, s, i, j) = \begin{cases} (1, 2, 1, 3) & (1, 2, 1, 4) & (1, 3, 1, 4) & (1, 3, 1, 5) \\ (2, 3, 2, 4) & (2, 3, 2, 5) & (2, 4, 2, 5) & (2, 4, 2, 6) \end{cases}$$

(B8)  $\alpha_{rs}^{-1}\beta_{ij}\alpha_{rs} = (\beta_{rj}\beta_{sj}\beta_{rj}^{-1}\beta_{sj}^{-1})\beta_{ij}(\beta_{rn}\beta_{sj}\beta_{rj}^{-1}\beta_{sj}^{-1})^{-1}$ , for

$$(r, s, i, j) = \begin{cases} (1, 3, 2, 4) & (1, 3, 2, 5) & (1, 4, 2, 5) & (1, 4, 2, 6) \\ (1, 4, 3, 5) & (1, 4, 3, 6) & (1, 5, 3, 6) & (1, 5, 3, 7) \\ (2, 4, 3, 5) & (2, 4, 3, 6) & (2, 5, 3, 6) & (2, 5, 3, 7) \\ (2, 5, 4, 6) & (2, 5, 4, 7) & (2, 6, 4, 7) & (2, 6, 4, 8) \end{cases}$$

(C)  $\beta_{ij} = \beta_{i,j+1}\alpha_{ij}$ , for

$$(i, j) = (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (2, 5)$$

(D1)  $\alpha_{ij}x_k = x_k\alpha_{i+1,j+1}$ , for

$$(i, j, k) = \begin{cases} (2, 3, 0) & (2, 4, 0) & (3, 4, 0) & (3, 5, 0) \\ (3, 4, 1) & (3, 5, 1) & (4, 5, 1) & (4, 6, 1) \end{cases}$$

(D2)  $\alpha_{ij}x_k = x_k\alpha_{i+1,j+1}\alpha_{i,j+1}$ , for

$$(i, j, k) = (1, 2, 0), (1, 3, 0), (2, 3, 1), (2, 4, 1)$$

(D3)  $\alpha_{ij}x_k = x_k\alpha_{i,j+1}$ , for

$$(i, j, k) = \begin{cases} (1, 3, 1) & (1, 4, 1) & (1, 4, 2) & (1, 5, 2) \\ (2, 4, 2) & (2, 5, 2) & (2, 5, 3) & (2, 6, 3) \end{cases}$$

$$(D4) \quad \alpha_{ij}x_k = x_k\alpha_{i,j+1}\alpha_{ij}, \text{ for} \\ (i, j, k) = (1, 2, 1), (1, 3, 2), (2, 3, 2), (2, 4, 3)$$

$$(D5) \quad \alpha_{ij}x_k = x_k\alpha_{ij}, \text{ for} \\ (i, j, k) = \begin{cases} (1, 2, 2) & (1, 2, 3) & (1, 3, 3) & (1, 3, 4) \\ (2, 3, 3) & (2, 3, 4) & (2, 4, 4) & (2, 4, 5) \end{cases}$$

$$(D6) \quad \beta_{ij}x_k = x_k\beta_{i+1,j+1}, \text{ for} \\ (i, j, k) = \begin{cases} (2, 3, 0) & (2, 4, 0) & (3, 4, 0) & (3, 5, 0) \\ (3, 4, 1) & (3, 5, 1) & (4, 5, 1) & (4, 6, 1) \end{cases}$$

$$(D7) \quad \beta_{ij}x_k = x_k\beta_{i+1,j+1}\beta_{i,j+1}, \text{ for} \\ (i, j, k) = (1, 2, 0), (1, 3, 0), (2, 3, 1), (2, 4, 1)$$

$$(D8) \quad \beta_{ij}x_k = x_k\beta_{i,j+1}, \text{ for} \\ (i, j, k) = \begin{cases} (1, 3, 1) & (1, 4, 1) & (1, 4, 2) & (1, 5, 2) \\ (2, 4, 2) & (2, 5, 2) & (2, 5, 3) & (2, 6, 3) \end{cases}$$

$$(D9) \quad \beta_{ij}x_k = x_k\beta_{ij}, \text{ for} \\ (i, j, k) = (1, 2, 1), (1, 3, 2), (2, 3, 2), (2, 4, 3)$$

This gives a total of 10 generators and 192 relators.

## 7. THE BRAIDED THOMPSON GROUP $\widehat{BF}$

In the previous section we constructed a presentation of  $BF$ . Brin [2] described both  $BV$  and  $\widehat{BV}$ . Brin describes the group  $\widehat{BV}$  via a Zappa-Szép product of  $F$  and  $B_\infty$ , and describes  $BV$  as a subgroup of  $\widehat{BV}$ . The group  $\widehat{BV}$  is the group of braided forest diagrams, where all but finitely many of the forests are trivial, and the group  $BV$  is the subgroup of  $\widehat{BV}$  where all of the trees in the forest pairs are trivial except the first pair, which have the same number of leaves. Not only is  $BV$  a subgroup of  $\widehat{BV}$ , but also  $\widehat{BV}$  is a subgroup of  $BV$ , as described by Brin [2]. We take the standard identification of the real line with the unit interval which is compatible with the relevant dyadic subdivisions which sends the interval  $[i, i+1]$  of  $\mathbf{R}$  with the interval  $[1-2^{-i}, 1-2^{-i-1}]$  and then we see that  $\widehat{BV}$  is the subgroup of  $BV$  in which the last strand is not braided with any other strands. Similarly, we have the group  $\widehat{BF}$  which can either be regarded as the supergroup of  $BF$  of pure braided forest diagrams, or as a subgroup of  $BF$  where the braiding does not involve the last strand.



Here, we easily describe the subgroup  $\widehat{BF}$  of  $BF$  by omitting the generators and relations from  $BF$  which involve braiding the last strand. So we obtain presentations for  $\widehat{BF}$  which are sub-presentations of the infinite and finite presentations for  $BF$  given in the earlier sections above.

**Proposition 7.1.** *The elements  $x_i$ , for  $i \geq 0$ ,  $\alpha_{ij}$ , for  $1 \leq i < j$  form a set of generators of  $\widehat{BF}$ .*

*Proof.* The proof is similar to the case for  $BF$ . Here we note that  $\widehat{BF}$  is exactly the subgroup where the last strand is not braided with any previous strands, and by omitting the  $\beta$  generators we guarantee that the last strand is not braided. An argument similar to the earlier one for  $BF$  shows that these generate  $\widehat{BF}$ .  $\square$

To find relators for a presentation of  $\widehat{BF}$  we use the same sets of generators and relators as for  $BF$ , deleting the generators in the  $\beta$  family and deleting all relations which include any of the  $\beta_{ij}$ .

We thus obtain the following:

**Theorem 7.2.** *The group  $\widehat{BF}$  admits a presentation with generators:*

- $x_i$ , for  $i \geq 0$ ,
- $\alpha_{ij}$ , for  $1 \leq i < j$ ,

and relators:

- (A)  $x_j x_i = x_i x_{j+1}$ , if  $i < j$
- (B1)  $\alpha_{rs}^{-1} \alpha_{ij} \alpha_{rs} = \alpha_{ij}$ , if  $1 \leq r < s < i < j$  or  $1 \leq i < r < s < j$
- (B2)  $\alpha_{rs}^{-1} \alpha_{ij} \alpha_{rs} = \alpha_{rj} \alpha_{ij} \alpha_{rj}^{-1}$ , if  $1 \leq r < s = i < j$
- (B3)  $\alpha_{rs}^{-1} \alpha_{ij} \alpha_{rs} = (\alpha_{ij} \alpha_{sj}) \alpha_{ij} (\alpha_{ij} \alpha_{sj})^{-1}$ , if  $1 \leq r = i < s < j$
- (B4)  $\alpha_{rs}^{-1} \alpha_{ij} \alpha_{rs} = (\alpha_{rj} \alpha_{sj} \alpha_{rj}^{-1} \alpha_{sj}^{-1}) \alpha_{ij} (\alpha_{rj} \alpha_{sj} \alpha_{rj}^{-1} \alpha_{sj}^{-1})^{-1}$ , if  $1 \leq r < i < s < j$
- (D1)  $\alpha_{ij} x_k = x_k \alpha_{i+1, j+1}$ , if  $k < i - 1$
- (D2)  $\alpha_{ij} x_k = x_k \alpha_{i+1, j+1} \alpha_{i, j+1}$ , if  $k = i - 1$
- (D3)  $\alpha_{ij} x_k = x_k \alpha_{i, j+1}$ , if  $i - 1 < k < j - 1$
- (D4)  $\alpha_{ij} x_k = x_k \alpha_{i, j+1} \alpha_{ij}$ , if  $k = j - 1$
- (D5)  $\alpha_{ij} x_k = x_k \alpha_{ij}$ , if  $k > j - 1$

*Proof.* Using the interpretation of the geometric group  $\widehat{BF}$  as the subgroup of  $BF$  where braiding never involves the last strand, the same proof used in the section 5 to establish the presentation for  $BF$  goes through in this situation. The only difference is that since we have no  $\beta$  generators, once we rearrange the word to have all  $\alpha$  generators in the middle, they are already generators for one copy of the pure braids on  $n$  strands with the rightmost strand unbraided, so there is no need for the step using relators of type (C).  $\square$

Note that the group  $\widehat{BF}$  will also be finitely presented. The arguments needed to see this are similar to those for  $BF$ . The finite presentation for  $\widehat{BF}$  can be easily obtained from the finite presentation for  $BF$  by deleting all generators  $\beta_{ij}$  and the relations where they appear.

## REFERENCES

- [1] James M. Belk. *Minimal Length Elements of Thompson's group F*. PhD thesis, Cornell Univ, 2004.
- [2] Matthew G. Brin. The algebra of strand splitting I. A braided version of Thompson's group  $V$ . Preprint.
- [3] Matthew G. Brin. The algebra of strand splitting II. A presentation for the braid group in one strand. Preprint.
- [4] J. Burillo, M. Gutierrez, S. Krstić, and Z. Nitecki. Crossing matrices and Thurston's normal form for braids. *Topology Appl.*, 118(3):293–308, 2002.
- [5] José Burillo, Sean Cleary, and Melanie Stein. Metrics and embeddings of generalizations of Thompson's group  $F$ . *Trans. Amer. Math. Soc.*, 353(4):1677–1689 (electronic), 2001.
- [6] José Burillo, Sean Cleary, Melanie Stein, and Jennifer Taback. Metric and combinatorial properties of Thompson's group  $T$ . to appear in *Trans. Amer. Math. Soc.*
- [7] J. W. Cannon, W. J. Floyd, and W. R. Parry. Introductory notes on Richard Thompson's groups. *Enseign. Math. (2)*, 42(3-4):215–256, 1996.
- [8] P. Dehornoy. The group of parenthesized braids.
- [9] S. Blake Fordham. Minimal length elements of Thompson's group  $F$ . *Geom. Dedicata*, 99:179–220, 2003.
- [10] V.L. Hansen. *Braids and Coverings*. London Mathematical Society Student Texts, 18. Cambridge University Press, 1989.

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