LOWER BOUNDS FOR THE NUMBER OF LIMIT CYCLES OF TRIGONOMETRIC ABEL EQUATIONS

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ABSTRACT. We consider the Abel equation $\dot{x} = A(t)x^3 + B(t)x^2$, where $A(t)$ and $B(t)$ are trigonometric polynomials of degree n and m, respectively, and we give lower bounds for its number of isolated periodic orbits for some values of n and m. These lower bounds are obtained by two different methods: the study of the perturbations of some Abel equations having a continuum of periodic orbits and the Hopf-type bifurcation of periodic orbits from the solution $x = 0$.

1. INTRODUCTION AND MAIN RESULTS

In this paper we deal with the Abel equation

$$\frac{dx}{dt} = A(t)x^3 + B(t)x^2, \tag{1.1}$$

where $A(t)$ and $B(t)$ are $2\pi-$trigonometric polynomials of degrees n and m, respectively. A solution of the previous equation satisfying $x(0) = x(2\pi)$ is called a periodic orbit. It is easy to prove from the results of [7] that if either $A(t)$ or $B(t)$ does not change sign, the previous equation has at most two isolated periodic orbits. The same result also holds if there exist two real numbers a and b such that the function $aA(t) + bB(t)$ does not change sign, see [1]. But if none of the above mentioned conditions is satisfied, it is not known how to bound the number of periodic orbits that the equation (1.1) can have. It is neither known how to obtain this bound depending only on the degrees of $A(t)$ and $B(t)$, see [9].

These type of bounds are usually called Hilbert type numbers because of their relationship with the Hilbert's 16th problem, see [8]. We will denote them by $H(n, m) \in \mathbb{N} \cup \{\infty\}$. A connection between planar vector fields and Abel equations comes from the fact that some planar vector fields can be transformed, after and adequate change of variables, into an Abel equation, see [4].

The function $H(n, m)$ is far from being known even for $n = m = 1$. Only some particular cases have been studied, see [1] and [2]. In our paper we mainly focus on studying lower bounds of $H(n,m)$ for two cases: $m=1$ or $n=1$. Our main result is the following:

Theorem A. *Set $H(n,m)$ for the number of isolated periodic orbits of the Abel equation (1.1). Then*

(i) $H(n,0)=H(0,m)=2$,
(ii) $H(n,1)\geq n+2$,
(iii) $H(1,m)\geq 2m+1$.

The first statement of above result can be easily proved by using the results of [10] and [7]. The second one was also proved by Lins in [10]. His proof as well as our approach, are based on studying the number of zeros in $(-1,1)$ of a first order Melnikov function

$$W(\rho)=\rho^3\sum_{k=0}^{n}a_k\int_0^{2\pi}\frac{\sin^k(t)}{1-\rho\sin(t)}\,dt,$$

associated to a perturbed Abel equation. Lins' result gives a lower bound for the number of zeros of W in a neighborhood of $\rho=0$ while our study gives a sharp upper bound for the number of zeros in the interval $(-1,1)$, see Theorem 3.1, which coincides with the number of zeros obtained by Lins.

The third statement of the theorem is a new result. Its proof is also based in computing the number of zeros in $(-1,1)$ of another first order Melnikov function. In this case this function is

$$W(\rho)=\rho^2\sum_{k=0}^{m}b_k\int_0^{2\pi}\sin^k(t)\sqrt{1-\rho^2\sin(t)}\,dt,$$

and we have been able to obtain a lower and an upper bound for its number of zeros. Nevertheless, in this case, we have not been able to prove that our upper bound is sharp, see Theorem 3.3. This case is much more difficult than the previous one because, while in the first case the expression of $W(\rho)$ can be obtained in terms of elementary functions, in the second one it involves elliptic functions.

Finally we have also studied two concrete Hilbert numbers, $H(3,1)$ and $H(2,2)$, by the method of computing several Lyapunov constants associated to the solution $x=0$. The method that we use for our computations is based on the results of [2]. For the first one we have got a higher lower bound than the one given in Theorem A, see Theorem 4.1. Our results are summarized in Table 1.

2. Preliminary results

We first characterize a family of centers inside (1.1).

$deg(A(t))$ / $deg(B(t))$	0	1	2	3	4	$\cdots$	n
0	2	2	2	2	2	$\cdots$	2
1	2	≥ 3	≥ 4	$\geq 7^*$	≥ 6	$\cdots$	$\geq n+2$
2	2	≥ 5	≥ 7				
3	2	≥ 7					
$\vdots$	$\vdots$	$\vdots$					
m	2	$\geq 2m+1$					

Table 1. Values of $H(n,m)$. For the value of $H(3,1)$ marked with an asterisk we have two lower bounds: the general one $n+2=5$ and a best one, 7.

Lemma 2.1. *Consider the differential equation in the cylinder*

$$\frac{dx}{dt} = D(t)x^n, \tag{2.1}$$

with $\int_0^{2\pi} D(t)\,dt = 0$ *and* $n \in \mathbb{N} \setminus \{1\}$. *Then* $x=0$ *is a center, i.e. there exists a neighborhood of* $x=0$ *where all the orbits are periodic.*

Proof. The solution of equation (2.1) with initial condition $x=\rho$ at $t=0$ is given by

$$x(t;\rho) = \frac{\rho}{\sqrt[n-1]{1+(1-n)\rho^{n-1}\int_0^t D(s)\,ds}}.$$

Note that it is well defined for all $t \in [0,2\pi]$ if $|\rho|$ is small enough. Furthermore, as $\int_0^{2\pi} D(s)\,ds = 0$, it satisfies $x(2\pi;\rho) = x(0;\rho) = \rho$. Then, for any initial condition near $\rho = 0$ all the orbits are periodic as we wanted to prove. □

Now we make a general perturbation to the previous equation,

$$\frac{dx}{dt} = D(t)x^n + \varepsilon f(t,x), \tag{2.2}$$

and we prove a result that indicates for which initial conditions the orbits are still periodic after the perturbation. Our proof is an adaptation for our equation of the method used in [10].

Proposition 2.2. *Consider the perturbed differential equation (2.2). Let I be the open interval of initial conditions for which the solutions of the unperturbed equation are periodic. Then, the simple zeros in $I \setminus \{0\}$ of the function*

$$W(\rho) = \int_0^{2\pi} f(t, x_0(t;\rho)) \left(1 + (1-n)\rho^{n-1} \int_0^t D(s)\,ds\right)^{\frac{n}{n-1}} dt, \qquad (2.3)$$

where $x_0(t;\rho) = \rho / \sqrt[n-1]{1 + (1-n)\rho^{n-1}\int_0^t D(s)\,ds}$, give rise to initial conditions of periodic orbits of (2.2) that tend to these zeros when ε tends to 0.

Proof. The solution of (2.2) can be expanded in a neighborhood of $\varepsilon = 0$ in powers of ε as

$$x_\varepsilon(t;\rho) = x_0(t;\rho) + \varepsilon S(t,\rho) + \varepsilon^2 R(t,\rho,\varepsilon),$$

where $S(0,\rho) \equiv 0$ and x_0 is the solution of the unperturbed equation, computed in Lemma 2.1.

Set $\widetilde{W}(\rho) = S(2\pi,\rho)$ and suppose that there exists a simple root ρ_0 of the previous function. Thus, by the Implicit Function Theorem applied to $(x_\varepsilon(2\pi,\rho)-\rho)/\varepsilon = \widetilde{W}(\rho) + \varepsilon R(2\pi,\rho,\varepsilon) = 0$, there exist $\varepsilon_0 > 0$ and a smooth function $\bar\rho(\varepsilon)$ defined in $\varepsilon \in (-\varepsilon_0,\varepsilon_0)$, such that $\widetilde{W}(\bar\rho(\varepsilon)) + \varepsilon R(1,\bar\rho(\varepsilon),\varepsilon) \equiv 0$ and $\bar\rho(0) = \rho_0$. Then $x_\varepsilon(t,\bar\rho(\varepsilon))$ is a periodic orbit of (2.2) for $\varepsilon \in (-\varepsilon_0,\varepsilon_0)$ having an initial condition which tends to ρ_0 when ε goes to zero. Hence, we can find as many periodic orbits for (2.2) as simple roots of $\widetilde{W}(\rho)$.

Now we compute $\widetilde{W}(\rho)$ in terms of $f(t,x)$ and $D(t)$. The variational equations associated to (2.2) are:

$$\frac{\partial}{\partial t} S(t,\rho) = n(x_0(t;\rho))^{n-1} D(t) S(t,\rho) + f(t, x_0(t;\rho)).$$

Since $D(t) = (\partial x_0(t;\rho)/\partial t)/(x_0(t;\rho))^n$, the above equality writes as

$$\frac{\partial}{\partial t}\left(\frac{S(t, x_0(t;\rho))}{(x_0(t;\rho))^n}\right) = \frac{f(t, x_0(t;\rho))}{(x_0(t;\rho))^n}.$$

By integrating both sides with respect to t between 0 and 2π we get

$$W(\rho) = \frac{\widetilde{W}(\rho)}{\rho^n} = \int_0^{2\pi} f(t, x_0(t;\rho)) \left(1 + (1-n)\rho^{n-1}\int_0^t D(s)\,ds\right)^{\frac{n}{n-1}} dt.$$

Since the simple zeros of $W(\rho)$ in $I \setminus \{0\}$ coincide with the ones of $\widetilde{W}(\rho)/\rho^n$ the result follows.

□

Definition 2.3. The integral expression (2.3) will be called the *first order Melinkov function* associated to the perturbed equation (2.2)

3. Proof of Theorem A

The proof of Theorem A follows from the following two results.

Theorem 3.1. *Consider the Abel equation*

$$\frac{dx}{dt} = \cos(t)x^2 + \varepsilon\left[b_0x^2 + \left(\sum_{k=0}^{n} a_k \sin^k(t)\right)x^3\right]. \tag{3.1}$$

Its associated first order Melnikov function is

$$W(\rho) = 2\pi b_0\rho^2 + \left[\sum_{k=0}^{n} a_k \int_0^{2\pi} \frac{\sin^k(t)}{1-\rho\sin(t)}\,dt\right]\rho^3, \tag{3.2}$$

and it has at most $n+1$ simple roots in the interval $I = (-1,1)\setminus\{0\}$. Furthermore this bound is sharp.

Proof. According to Lemma 2.1, the unperturbed equation

$$\frac{dx}{dt} = \cos(t)x^2,$$

has a center at $x = 0$ and all solutions starting at $(-1,1)$ are periodic. From Proposition 2.2 we get that its first order Melnikov function is the one given in (3.2). We write it as

$$W(\rho) = 2\pi b_0\rho^2 + \rho^3\sum_{k=0}^{n} a_k I_k, \quad \text{where} \quad I_k = \int_0^{2\pi} \frac{\sin^k(t)}{1-\rho\sin(t)}\,dt. \tag{3.3}$$

We first prove that when $b_0 = 0$ it has at most n simple roots in the set I and that this bound is sharp. We write $W_0(\rho)$ for the Melnikov function given in (3.3) when $b_0 = 0$.

The integrals I_k can be computed explicitly. Indeed they satisfy the relationship

$$-\rho I_k + I_{k-1} = \int_0^{2\pi} \sin^{k-1}(t)\,dt = f(k-1),$$

where

$$f(k) = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \frac{2\pi(k-1)!!}{2^{k/2}\left(\frac{k}{2}\right)!} & \text{if } k \text{ even.} \end{cases}$$

Then, it turns out that

$$I_k = \frac{I_{k-1} - f(k-1)}{\rho} = \ldots = \frac{I_0 - f(0) - \rho f(1) - \ldots - \rho^{k-1} f(k-1)}{\rho^k}.$$

Using that $I_0 = \int_0^{2\pi} \frac{1}{1-\rho\sin(t)}\,dt = \frac{2\pi}{\sqrt{1-\rho^2}}$, and making some computations it is easy to prove that

$$I_k = 2\pi \left(\frac{1}{\rho^k \sqrt{1-\rho^2}} - \sum_{\substack{l=1 \\ l \equiv k (\mathrm{mod} 2)}}^{k} 2^{l-k} C_{k-l}^{(k-l)/2} \rho^{-l} \right), \qquad (3.4)$$

where $C_{k-l}^{(k-l)/2} = \binom{k-l}{(k-l)/2}$.

We first assume that $n = 2m$ and we compute the simple roots of $\widehat{W}_0(\rho) = W_0(\rho)/\rho^3$ in I that coincide with the ones of $W_0(\rho)$. Substituting (3.4) into (3.3) we get

$$\begin{aligned} \widehat{W}_0(\rho) \quad = \quad & \frac{2\pi}{\sqrt{1-\rho^2}} \left(\sum_{k=0}^{2m} a_k \rho^{-k} \right) - 2\pi \left(\sum_{k=1}^{m} \rho^{-2k} \sum_{l=k}^{m} a_{2l}\, 2^{2(k-l)}\, C_{2(l-k)}^{\,l-k} + \right. \\ & \left. + \sum_{k=0}^{m-1} \rho^{-(2k+1)} \sum_{l=k}^{m-1} a_{2l+1}\, 2^{2(k-l)}\, C_{2(l-k)}^{\,l-k} \right). \end{aligned} \qquad (3.5)$$

We study separately the cases $\rho \in (0,1)$ and $\rho \in (-1,0)$. In the first case, multiplying the previous expression by $\rho^{2m}\sqrt{1-\rho^2}/2\pi$ and making the change of variables $r = \sqrt{1-\rho^2}$, with $r \in (0,1)$, we arrange (3.5) as

$$f(r) = \widehat{W}_0\left(\sqrt{1-r^2}\right) \frac{(1-r^2)^m r}{2\pi} = p(r) + q(r)\sqrt{1-r^2}, \qquad (3.6)$$

where

$$p(r) = \sum_{k=0}^{m} a_{2k}(1-r^2)^{m-k} - \sum_{k=1}^{m} a_{2k} \sum_{l=1}^{k} 2^{2(l-k)} C_{2(k-l)}^{\,k-l} r(1-r^2)^{m-l},$$

$$q(r) = \sum_{k=0}^{m-1} a_{2k+1}(1-r^2)^{m-1-k} - \sum_{k=0}^{m-1} a_{2k+1} \sum_{l=0}^{k} 2^{2(l-k)} C_{2(k-l)}^{\,k-l} r(1-r^2)^{m-1-l}.$$

We note that each simple zero of $f(r)$ in $(0,1)$ will give rise to a simple zero of $W_0(\rho)$ in $(0,1)$ and conversely.

We claim that

$$f(r) = (1-r)^m(\hat{p}(r) + \hat{q}(r)\sqrt{1-r^2}), \qquad (3.7)$$

where $\hat{p}(r)$ is an arbitrary polynomial of degree m in r that only depends on the coefficients of the even terms of $\widehat{W}_0(r)$, while $\hat{q}(r)$ is an arbitrary polynomial of degree $m-1$ that only depends on the ones of the odd terms.

In the following we prove by induction the claim for $p(r)$. In the above expression of $p(r)$ we denote by $p_v(r)$ the function obtained when the summands range from $k=0$ until $k=v$. In particular $p_m(r)=p(r)$. Obviously, $p_1(1)=0$. Assume that the claim holds for $m=v-1$. Hence, we have

$$p_v(r) = g_v(r) + (1-r^2)p_{v-1}(r),$$

where

$$g_v(r) = a_{2v} - a_{2v}\sum_{l=1}^{v} 2^{2(l-v)} C_{2(v-l)}^{\,v-l} r(1-r^2)^{v-l}.$$

Substituting $\eta = 1-r$ and expanding in series in η, we have

$$g_v(\eta) = a_{2v}\left(1 - \sum_{l=1}^{v}\left(C_{2(v-l)}^{\,v-l}(1-\eta)\eta^{v-l}\sum_{i=0}^{v-l}(-1)^i 2^{l-v-i} C_{v-l}^{\,i}\eta^i\right)\right).$$

Let $u = v-l+i$ and rearrange $g_v(\eta)/a_{2v}$ into a series of η as follows,

$$\begin{aligned}\frac{g_v(\eta)}{a_{2v}} &= 1-(1-\eta)\left(\sum_{u=0}^{v-1}\eta^u\sum_{i=0}^{[u/2]}(-1)^i 2^{-u} C_{u-i}^{\,i} C_{2(u-i)}^{\,u-i} - \right.\\ &\quad \left. - \sum_{u=v}^{2(v-1)}\eta^u \sum_{i=u-v+1}^{[u/2]}(-1)^i 2^{-u} C_{u-i}^{\,i} C_{2(u-i)}^{\,u-i}\right).\end{aligned}$$

By calculations, we know $\sum_{i=0}^{[u/2]}(-1)^i 2^{-u} C_{u-i}^{\,i} C_{2(u-i)}^{\,u-i} = 1$. Then, it turns out that $g_v(\eta) = a_{2v}\eta^v\tilde{g}_v(\eta)$, and thus $p_v(r) = (1-r)^v\hat{p}_v(r)$, where $\hat{p}_v$ is also a polynomial. Hence $p(r) = (1-r)^m\hat{p}(r)$ and only remains to prove that $\hat{p}(r)$ is an arbitrary polynomial of degree m.

Now observe that the first summand in the definition of $p(r)$ is even in r and depends on $a_{2k},\ k=0,1,\dots,m$, while the second summand is odd in r. Hence, it is easy to see that the coefficients of the even terms of $p(r)$ are arbitrary. From (3.7), we will prove that the coefficients of r in $\hat{p}(r)$ can be expressed as a linear invertible combination of the ones of the even terms in $p(r)$, which we already know that are arbitrary. Concretely, by writing

$\hat{p}(r)=\sum\limits_{j=0}^{m} d_j r^j$ we have that

$$\begin{aligned}(1-r)^m\hat{p}(r) &= \sum_{s=0}^{m}(-1)^s C_m^s r^s \sum_{j=0}^{m} d_j r^j = \\ &= \sum_{v=0}^{m} r^v \sum_{s=0}^{v}(-1)^s d_{v-s} C_m^s + \sum_{v>m}^{2m} r^v \sum_{s=v-m}^{m} (-1)^s d_{v-s} C_m^s,\end{aligned}$$

where we have performed the transformation $v=s+j$. Comparing the even terms in r of $p(r)$ with the ones of the above function, we have that $\alpha=\mathrm{M}\beta$, where $\beta=(d_m,d_{m-1},\ldots,d_0)^T$, $\alpha=(a_{2m},a_{2(m-1)},\ldots,a_0)^T$, and M is the following invertible matrix for m even and odd, respectively

$$\begin{pmatrix} C_m^m & 0 & 0 & \cdots & 0 & 0 & 0 \\ C_m^{m-2} & -C_m^{m-1} & C_m^m & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ C_m^0 & -C_m^1 & C_m^2 & \cdots & C_m^{m-2} & -C_m^{m-1} & C_m^m \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & C_m^0 & -C_m^1 & C_m^2 \\ 0 & 0 & 0 & \cdots & 0 & 0 & C_m^0 \end{pmatrix},$$

$$\begin{pmatrix} -C_m^m & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -C_m^{m-2} & C_m^{m-1} & -C_m^m & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ -C_m^1 & C_m^2 & -C_m^3 & C_m^4 & \cdots & C_m^{m-1} & -C_m^m & 0 \\ 0 & C_m^0 & -C_m^1 & C_m^2 & \cdots & C_m^{m-3} & -C_m^{m-2} & C_m^{m-1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & C_m^0 & -C_m^1 & C_m^2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & C_m^0 \end{pmatrix}.$$

Hence, $\hat{p}(r)$ is an arbitrary polynomial of degree m. Similarly, we can prove that $q(r)=(1-r)^m\hat{q}(r)$, being $\hat{q}(r)$ an arbitrary polynomial of degree $m-1$.

In short we have proved that for $\rho\in(0,1)$ if $r=\sqrt{1-\rho^2}$ we obtain that

$$\frac{\rho^{2m}\sqrt{1-\rho^2}}{2\pi}\widehat{W}_0(\rho)=(1-r)^m\left(\hat{p}(r)+\sqrt{1-r^2}\hat{q}(r)\right),$$

with $\hat{p}$ and $\hat{q}$ arbitrary polynomials of degree m and $m-1$, respectively. Similarly, when $\rho\in(-1,0)$ we introduce a new variable $s\in(0,1)$ as $\rho=-\sqrt{1-s^2}$ and we obtain that

$$\frac{\rho^{2m}\sqrt{1-\rho^2}}{2\pi}\widehat{W}_0(\rho)=(1-s)^m\left(\hat{p}(s)-\sqrt{1-s^2}\hat{q}(s)\right),$$

where $\hat{p}$ and $\hat{q}$ are the same polynomials than above. We also introduce two functions $f^{\pm}(r) = \hat{p}(r) \pm \sqrt{1-r^2}\hat{q}(r)$ defined for $r \in (0,1)$. We want to prove that the number of solutions of the equation $f^{+}(r) = 0$ in $(0,1)$ plus the number of solutions of the equation $f^{-}(r) = 0$ also in $(0,1)$ is at most n. Let $r_1, \ldots, r_\ell$ be the number of real common zeros of $\hat{p}$ and $\hat{q}$, which clearly are solutions of both equations. Write $f^{\pm}(r) = \hat{p}(r) \pm \sqrt{1-r^2}\hat{q}(r) = \left(\prod_{i=1}^{\ell}(r-r_i)\right)\left(\hat{p}_\ell(r) \pm \sqrt{1-r^2}\hat{q}_\ell(r)\right)$, being p_ℓ and q_ℓ polynomials of degree $m-\ell$ and $m-1-\ell$, respectively. Finally note that the solutions of either $f^{+}(r) = 0$ or $f^{-}(r) = 0$ different from $r_1, \ldots, r_\ell$ both satisfy

$$\hat{p}_\ell^2(r) = \hat{q_\ell}^2(r)(1-r^2),$$

which is a polynomial equation of degree $2(m-\ell)$. Furthermore each of these solutions is either a solution of $f^{+}(r) = 0$ or of $f^{-}(r) = 0$ but not of both equations simultaneously. Thus, the number of solutions of either $f^{+}(r) = 0$ or $f^{-}(r) = 0$ is at most $2(m-\ell)+\ell$, being the first $2(m-\ell)$ numbers solution of only one of the equations and the second set of ℓ numbers solution of both equations. Transforming these solutions into the variable $\rho \in I$ we obtain that $\widehat{W}_0(\rho)$ has at most $n = 2m$ solutions because each r_i for $i = 1, \ldots, \ell$ gives rise to two zeros of $\widehat{W}_0(\rho)$, $\rho_i^{\pm} = \pm\sqrt{1-r_i^2}$.

To see that the bound is sharp it suffices for instance to give an example with n zeros in $(0,1)$. Recall that $\hat{p}(r)$ and $\hat{q}(r)$ are two arbitrary polynomials. This fact assures that for any given $r_i \in (0,1)$, $i = 1, 2, \cdots n$, there exist d_k, $k = 0, 1, 2, \cdots n$, not all identically zero and such that the following n-dimensional linear system holds,

$$\hat{p}(r_i)+\hat{q}(r_i)\sqrt{1-r_i^2} = \sum_{j=0}^{m} d_j r_i^j + \sum_{j=0}^{m-1} d_{j+m+1} r_i^j \sqrt{1-r_i^2} = 0, \; i = 1, 2, \cdots, n. \tag{3.8}$$

Thus, $f(r) \not\equiv 0$ has exactly n zeros in the interval $(0,1)$, which have to be simple roots, as we wanted to prove. Thus in the case $n = 2m$, $W_0(\rho)$ with $b_0 = 0$ has at most n simple zeros in the set I and this bound is sharp, as desired.

For the case of $n = 2m+1$, we have the corresponding formulas

$$f^{\pm}(r) = (1-r)^m\sqrt{1-r^2}\left(\hat{p}(r) \pm \hat{q}(r)\sqrt{\frac{1-r}{1+r}}\right), \tag{3.9}$$

where $\hat{p}(r)$ and $\hat{q}(r)$ are also two arbitrary polynomials of degree m in r, only depending on the coefficients of the even and the odd terms of $\widehat{W}_0$, respectively. Then, in the same way as in the even case, we can prove that

$W_0(\rho)$ has at most $n = 2m+1$ simple zeros in the set I and that this bound is sharp.

To end the proof we have to prove that when we consider the whole first order Melnikov function given in (3.2) the upper bound increases by one and that it is also sharp.

In the same way as before, we first assume $n = 2m$ and that $\rho \in (0,1)$. Hence, according to (2.3), its first order Melnikov function is $W(\rho) = 2\pi b_0\rho^2 + \widehat{W}_0(\rho)\rho^3$ where $\widehat{W}(\rho)$ is given in expressions (3.5), (3.6) and (3.7). Multiplying it by $\rho^{2m-2}\sqrt{1-\rho^2}/2\pi$, writing $r = \sqrt{1-\rho^2}$, and defining

$$\widehat{W}(r) = \frac{r(1-r^2)^{m-1}}{2\pi} W(\sqrt{1-r^2}),$$

we have

$$\begin{aligned}\widehat{W}(r) &= (1-r^2)^m r\, b_0 + \sqrt{1-r^2}\, p(r) + (1-r^2)q(r) = \\ &= (1-r)^m(1+r)\left(b_0 r(1+r)^{m-1} + \hat{q}(r)(1-r) + \hat{p}(r)\sqrt{\frac{1-r}{1+r}}\right) = \\ &= (1-r)^m(1+r)\left(\eta(r) + \hat{p}(r)\sqrt{\frac{1-r}{1+r}}\right)\end{aligned}$$

where $\eta(r) = e_0 + \sum_{i=1}^{m-1}(e_i - e_{i-1} + b_0\, C_{m-1}^{i-1})r^i + (e_{m-1}+b_0)r^m$ and e_i are the coefficients of r^i in $\hat{q}(r)$. So, $\hat{p}(r)$ and $\eta(r)$ are two arbitrary polynomials of degree m. Observe that the expression in the parenthesis is exactly the same as (3.9). Doing a similar study when $\rho \in (-1,0)$ and joining the results, we obtain that $W(\rho)$ has at most $n+1$ simple zeros in I, being this bound also sharp.

For $n = 2m+1$ odd the result can be proved in a similar way.

□

Remark 3.2. Observe that if instead of the equation studied in the previous theorem we consider the more general equation

$$\frac{dx}{dt} = (a\cos(t) + b\sin(t))x^2 + \varepsilon\left[b_0 x^2 + \left(\sum_{i+j=0}^{n} a_{ij}\cos^i(t)\sin^j(t)\right)x^3\right],$$

we get the same number of zeros for its first order Melnikov function. This is because, firstly, the equation with $\varepsilon = 0$ can be transformed into $dx/dt = \cos(t)x^2$ through a translation in the time and a scaling of x and these transformations do not modify the form of the perturbation. Secondly, if we do the same process as in the previous proof, we get that

$$\begin{aligned} W(\rho) &= 2\pi b_0\rho^2 + \left[\sum_{i+j=0}^{n} a_{ij} \int_0^{2\pi} \frac{\cos^i(t)\sin^j(t)}{1-\rho\sin(t)}\,dt\right]\rho^3 = \\ &= 2\pi b_0\rho^2 + \left[\sum_{i+j=0}^{n} a_{ij} I_{ij}\right]\rho^3. \end{aligned}$$

It is easy to prove that $I_{ij} = 0$ if i is odd and then we can arrange the previous expression as

$$W(\rho) = 2\pi b_0\rho^2 + \left[\sum_{k=0}^{n} c_k \int_0^{2\pi} \frac{\sin^k(t)}{1-\rho\sin(t)}\,dt\right]\rho^3,$$

where c_k are a linear combination of $a_{2i,j}$ and this expression coincides with (3.2).

Theorem 3.3. *Consider the Abel equation*

$$\frac{dx}{dt} = \frac{\cos(t)}{2}x^3 + \varepsilon\left(\sum_{k=0}^{m} b_k \sin^k(t)\right)x^2 = A(t)x^3 + \varepsilon B(t)x^2. \qquad (3.10)$$

Its associated first order Melnikov function is

$$W(\rho) = \rho^2\sum_{k=0}^{m} b_k \int_0^{2\pi} \sin^k(t)\sqrt{1-\rho^2\sin(t)}\,dt, \qquad (3.11)$$

and it has at most $4m+2$ simple roots in the set $I = (-1,1)\setminus\{0\}$.

Furthermore there are Abel equations of the form (3.10) such that its associated $W(\rho)$ has at least $2m$ simple roots in I.

Proof. By Lemma 2.1 we know that the unperturbed equation

$$\frac{dx}{dt} = \frac{\cos(t)}{2}x^3$$

has a center at $x = 0$. By using Proposition 2.2 we obtain that its first order Melnikov function is

$$W(\rho) = \rho^2\sum_{k=0}^{m} b_k \int_0^{2\pi} \sin^k(t)\sqrt{1-\rho^2\sin(t)}\,dt = \rho^2\sum_{k=0}^{m} b_k I_k, \qquad (3.12)$$

as we wanted to prove. We claim that $I_0 = 4\sqrt{\frac{2}{2-r^2}}\mathcal{E}(r)$ and that for $k = 1,\ldots,m$

$$I_k = \sqrt{\frac{2}{2-r^2}}\sum_{i=0}^{k}\Big(P_i(\frac{1}{r^2})\mathcal{K}(r) + Q_i(\frac{1}{r^2})\mathcal{E}(r)\Big), \qquad (3.13)$$

where P_i and Q_i are polynomials of exactly degree i, $r = \sqrt{2}|\rho|/\sqrt{1+\rho^2}$, and $\mathcal{K}(r)$ and $\mathcal{E}(r)$ are the two following elliptic functions

$$\mathcal{K}(r) = \int_0^1 \frac{1}{\sqrt{1-\mu^2}\sqrt{1-r^2\mu^2}}\, d\mu, \quad \mathcal{E}(r) = \int_0^1 \frac{\sqrt{1-r^2\mu^2}}{\sqrt{1-\mu^2}}\, d\mu,$$

see [3].

To prove the claim, consider the integral

$$\begin{aligned} I_k &= \int_0^{2\pi} \sin^k(t)\sqrt{1-\rho^2\sin(t)}\, dt = \int_0^{2\pi} \cos^k(t)\sqrt{1-\rho^2\cos(t)}\, dt = \\ &= \sqrt{1+\rho^2}\int_0^{2\pi} \cos^k(t)\sqrt{1-\frac{2\rho^2}{1+\rho^2}\frac{\cos(t)+1}{2}}\, dt = \\ &= \sqrt{\frac{2}{2-r^2}}\int_0^{2\pi} \sum_{i=0}^{k}(-1)^{k-i}C_k^i 2^i \cos^{2i}\left(\frac{t}{2}\right)\sqrt{1-r^2\cos^2\left(\frac{t}{2}\right)}\, dt, \end{aligned}$$

where $r = \sqrt{2}|\rho|/\sqrt{1+\rho^2}$. Making the change of variables $\cos(t/2) = \mu$, we have

$$\begin{aligned} I_k &= 4\sqrt{\frac{2}{2-r^2}}\sum_{i=0}^{k}(-1)^{k-i}C_k^i 2^i \int_0^1 \frac{\mu^{2i}\sqrt{1-r^2\mu^2}}{\sqrt{1-\mu^2}} d\mu = \\ &= 4\sqrt{\frac{2}{2-r^2}}\sum_{i=0}^{k}(-1)^{k-i}C_k^i 2^i\ J_i. \end{aligned}$$

By [3, formula 320.05], we get a recurrence formula for J_i,

$$J_i = \int_0^1 \frac{\mu^{2i}\sqrt{1-r^2\mu^2}}{\sqrt{1-\mu^2}}\, d\mu = O_{2i} - r^2 O_{2i+2},$$

where

$$\begin{aligned} O_{2i+2} &= \int_0^1 \frac{\mu^{2i+2}}{\sqrt{1-\mu^2}\sqrt{1-r^2\mu^2}}\, d\mu = \frac{2i(1+r^2)O_{2i} + (1-2i)O_{2i-2}}{(2i+1)r^2} \\ O_0 &= \mathcal{K}(r), \quad O_2 = \frac{1}{r^2}(\mathcal{K}(r) - \mathcal{E}(r)). \end{aligned}$$

Hence $J_0 = \mathcal{E}(r)$ and by induction we have for $i = 1, \ldots, k$,

$$J_i = p_i\left(\frac{1}{r^2}\right)\mathcal{K}(r) + q_i\left(\frac{1}{r^2}\right)\mathcal{E}(r), \tag{3.14}$$

where each polynomial p_i and q_i has exactly degree i. From the above expression the claim easily follows.

Now we can obtain explicitly $\widetilde{W}(\rho) = W(\rho)/\rho^2$,

$$\begin{aligned}\widetilde{W}(\rho) &= \sum_{k=0}^{m} b_k I_k = 4\sqrt{1+\rho^2}\sum_{k=0}^{m} b_k \sum_{i=0}^{k}\Big(P_i(\frac{1}{r^2})\mathcal{K}(r) + Q_i(\frac{1}{r^2})\mathcal{E}(r)\Big) = \\ &= 4\sqrt{1+\rho^2}\sum_{k=0}^{m} b_k \Big(\tilde{P}_k(\frac{1}{r^2})\mathcal{K}(r) + \tilde{Q}_k(\frac{1}{r^2})\mathcal{E}(r)\Big) = \\ &= 4\sqrt{1+\rho^2}\sum_{k=0}^{m} b_k\, \eta_k(r^2),\end{aligned}$$

where again $r = \sqrt{2}|\rho|/\sqrt{1+\rho^2}$ and $\eta_0(r^2) = \mathcal{E}(r)$. Observe that the functions $\eta_k(r^2)$, $k = 1, 2, \cdots, m$ are functionally independent because the degrees of the polynomials $\tilde{P}_k$ and $\tilde{Q}_k$ strictly increase.

Now we prove that $W(\rho)$ (and then $\widetilde{W}(\rho)$) has at least $2m$ simple roots. For any given $r_i \in (0,1)$, $i = 1, 2, \cdots, m$, consider the linear system

$$\sum_{k=0}^{m} b_k\, \eta_k(r_i^2) = 0, \quad i = 1, 2, \cdots, m. \tag{3.15}$$

Since there are $m+1$ arbitrary coefficients b_k, there is a non-zero solution $\{b_k^*\}$ of system (3.15). Moreover, in view of the functionally independence of $\eta_k(r^2)$, we have that

$$\sum_{k=0}^{m} b_k^*\, \eta_k(r^2) \not\equiv 0.$$

Because of the definition of $W(\rho)$ it is an even function; hence, if $\rho^* \in (0,1)$ is a root of $\widetilde{W}(\rho)$ then $-\rho^* \in (-1,0)$ is also. Thus, $W(\rho)$ has at least $2m$ isolated roots. To obtain a new equation with at least $2m$ simple roots a key point is that $\eta_0(r^2) = \mathcal{E}(r) \neq 0$ in $(-1,1)$, see [5]. By taking ϵ small enough and with the suitable sign, it is not difficult to see that the new function

$$\sum_{k=0}^{m} b_k^*\, \eta_k(r_i^2) + \epsilon\eta_0(r^2)$$

has the desired property.

To get an upper bound for the number of zeros of the Melnikov function, we rewrite $\widetilde{W}(\rho)$ as

$$\begin{aligned}\widetilde{W}(\rho) &= 4\sqrt{1+\rho^2}\sum_{k=0}^{m} b_k \Big(\tilde{P}_k(\frac{1}{r^2})\mathcal{K}(r) + \tilde{Q}_k(\frac{1}{r^2})\mathcal{E}(r)\Big) = \\ &= 4\sqrt{1+\rho^2}\, r^{-2m}\Big(f(r^2)\mathcal{K}(r) + g(r^2)\mathcal{E}(r)\Big),\end{aligned}$$

where $r = \sqrt{2}|\rho|/\sqrt{1+\rho^2}$ and f and g are polynomials of degree m.

In [6] it is proved that a function of the form $P(r)\mathcal{K}(r) + Q(r)\mathcal{E}(r)$ with P and Q polynomials of degree p and q respectively has at most $p+q+2$ zeros in $(-1,1)$. By applying this result to our situation we get the upper bound of $4m+2$ zeros. □

Remark 3.4. A similar remark than Remark 3.2 can be done. If we study the more complete equation

$$\frac{dx}{dt} = (a\cos(t) + b\sin(t))x^3 + \varepsilon \left(\sum_{i+j=0}^{n} a_{ij} \cos^i(t) \sin^j(t) \right) x^2,$$

we obtain the same lower and upper bounds than the ones given in the case studied in the Theorem 3.3.

¿From the previous results we can easily prove our main theorem.

Proof of Theorem A.
(i) If either $A(t)$ or $B(t)$ is a constant function, by using the results of [7] we know that the Abel equation will have at most three periodic orbits, taking into account their multiplicities. But it is obvious that the solution $x = 0$ is at least a double periodic orbit of the equation. Hence the Abel equation will have, at most two periodic orbits.
(ii) The result follows by applying Proposition 2.2 and Theorem 3.1 taking into account that $x = 0$ is also an isolated periodic orbit.
(iii) The result follows by applying Proposition 2.2 and Theorem 3.3 taking again into account that $x = 0$ is also an isolated periodic orbit.

□

4. Lower bounds for $\mathbf{H(3,1)}$

In this section we are going to compute a new lower bound for $H(3,1)$ using a kind of Lyapunov constants associated to the solution $x = 0$. This bound is bigger than the one given in Theorem A.

Recall that the Lyapunov constants are essentially the coefficients of the Taylor expansion at the origin of the Poincaré map defined by the flow of (1.1) between the lines $t = 0$ and $t = 2\pi$. We will compute these coefficients by following the method proposed in [2]. We write (1.1) when $n = 3$ and

$m = 1$ as

$$\begin{aligned} \frac{dx}{dt} &= A(t)x^3 + B(t)x^2 = \Big(a_0 + a_{10}\cos(t) + a_{01}\sin(t) + \\ &+a_{20}\cos^2(t) + a_{11}\cos(t)\sin(t) + a_{02}\sin^2(t) + a_{30}\cos^3(t) + \\ &+a_{21}\cos^2(t)\sin(t) + a_{12}\cos(t)\sin^2(t) + a_{03}\sin^3(t)\Big)x^3 + \\ &+\Big(b_0 + b_{10}\cos(t) + b_{01}\sin(t)\Big)x^2, \end{aligned} \tag{4.1}$$

and we prove next result:

Theorem 4.1. *There are equations inside the family (4.1) with at least 7 isolated periodic orbits. Moreover, equation (4.1) has a center at $x = 0$ if and only if $b_0 = 0, a_0 = a_{02}, a_{10} = -a_{12}, a_{20} = a_{02}, a_{30} = a_{12}$.*

Proof. Without loss of generality we can assume that $b_{10} = 0$ and $b_{01} = 1$. In this case, the Lyapunov constants associated to the solution $x = 0$ are

$$\begin{aligned} V_2 &= 2\pi b_0, \\ V_3 &= \pi(a_{20} + a_{02} + 2a_0), \\ V_4 &= \frac{-\pi}{4}(a_{12} + 3a_{30} + 4a_{10}), \\ V_5 &= \frac{\pi}{4}(a_{20} - a_{02}), \\ V_6 &= \frac{\pi}{96}(a_{30} - a_{12})(a_{11} - 6), \\ V_7 &= \frac{\pi}{48}(-a_{30} + a_{12})(4a_{01} + a_{21} + 3a_{03}), \\ V_8 &= \frac{-11\pi}{64}(-a_{30} + a_{12}). \end{aligned}$$

If we now make the following election of the other parameters

$$\begin{aligned} &a_0 = \beta - \frac{\delta}{2}, \quad a_{10} = \gamma - \frac{1}{4}, \quad a_{01} = \xi - \frac{1}{4}, \\ &a_{20} = 0, \quad a_{11} = 6 + \eta, \quad a_{02} = -\delta, \\ &a_{30} = 0, \quad a_{21} = 1, \quad a_{12} = 1, \\ &a_{03} = 0, \quad b_0 = -\alpha, \end{aligned}$$

we get that the Lyapunov constants are

$$V_2 = -2\pi\,\alpha, \quad V_3 = 2\pi\,\beta, \quad V_4 = -\pi\,\gamma, \quad V_5 = \frac{\pi}{4}\,\delta,$$

$$V_6 = -\frac{\pi}{96}\,\eta, \quad V_7 = \frac{\pi}{12}\,\xi, \quad V_8 = -\frac{11\pi}{64}.$$

If we choose $\alpha, \beta, \gamma, \delta, \eta$ and ξ positive and such that $|\alpha| \ll |\beta| \ll |\gamma| \ll |\delta| \ll |\eta| \ll |\xi| \ll 1$ then we can ensure that a multiple Hopf bifurcation takes place. Therefore we know that there will appear at least 6 non-zero isolated periodic orbits bifurcating from the periodic orbit $x = 0$, which also remains as an isolated one. Hence an example with at least seven isolated periodic orbits is constructed as desired.

Let us prove that equation (4.1) has a center if and only if the conditions given in the statement hold. By imposing that the first seven Lyapunov constants vanish we get that they are necessary conditions. We prove in the following that they are also sufficient. If we impose the conditions to equation (4.1), with $b_{10} = 0$ and $b_{01} = 1$, it turns out that the functions A and B write as

$$\begin{aligned} A(t) &= a_{01}\sin(t) + a_{11}\cos(t)\sin(t) + a_{21}\cos^2(t)\sin(t) + a_{03}\sin^3(t), \\ B(t) &= \sin(t). \end{aligned}$$

It is well known that if there exist smooth functions F, G, w such that $\int_0^t A(s)\,ds = F(w(t))$ and $\int_0^t B(s)\,ds = G(w(t))$, being $w(t)$ a $2\pi-$periodic function, then $x = 0$ is a center, see [2]. In our case $\int_0^t A(s)\,ds = F(\cos(t))$ and $\int_0^t B(s)\,ds = G(\cos(t))$ and hence the result follows. □

5. About $H(n, 2)$

We have also tried to get a lower bound for $H(n, 2)$ but we have not been able to improve our lower bound of $H(n, 1)$. A way of approaching consists in studying the equation

$$\begin{aligned} \dot{x} &= \varepsilon\left(\sum_{i+j=0}^{n} a_{i,j}\sin^i(t)\cos^j(t)\right)x^3 + \left(\sum_{i+j=0}^{2} b_{i,j}\sin^i(t)\cos^j(t)\right)x^2 \\ &= \varepsilon A(t)x^3 + B(t)x^2, \end{aligned}$$

with a center at $x = 0$, *i.e.* with $2b_{00} + b_{20} + b_{02} = 0$. By applying Proposition 2.2 we get that we have to control the zeros of

$$W(\rho) = \rho^3 \sum_{i+j=0}^{n} a_{i,j} \int_0^{2\pi} \frac{\sin^i(t)\cos^j(t)}{1 - \rho\int_0^t B(s)\,ds}\,dt,$$

but we have not been able to perform this study. We have only worked in some special cases in which several of the $b_{i,j}$ are zero. For instance, if $B(t) = 2\sin(t)\cos(t)$ we can compute the previous integral and get at most $[n/2]$ zeros and that this bound can be attained. Unfortunately this bound is worst than our bound of $H(n, 1)$.

We have obtained a lower bound for $H(2, 2)$ by computing the Lyapunov constants associated to $x = 0$, as we have done in Section 4 to study $H(3, 1)$. We prove:

Proposition 5.1. *$H(2, 2) \geq 7$.*

Proof. Consider the following equation

$$\begin{aligned}\frac{dx}{dt} &= \Big(a_0 + a_{10}\cos(t) + a_{01}\sin(t) + a_{20}\cos^2(t) + a_{11}\cos(t)\sin(t) + \\ &\quad + a_{02}\sin^2(t)\Big)x^3 + \Big(b_0 + b_{10}\cos(t) + b_{01}\sin(t) + b_{20}\cos^2(t) + \\ &\quad + b_{11}\cos(t)\sin(t) + b_{02}\sin^2(t)\Big)x^2. \end{aligned} \tag{5.1}$$

In this case the computation of the Lyapunov constants is much more costly than for (4.1). For this reason we fix the coefficients $b_{01} = b_{02} = 0, a_{20} = b_{10} = 1$ and we make the following choice for the other coefficients of the equation (the ones that do not appear in the previous list are free),

$$a_0 = -\frac{1}{2} + \beta, \quad a_{10} = \frac{5+\xi}{2\sqrt{2}} - \eta, \quad a_{01} = \frac{-\delta - 2\sqrt{2}}{8} - \gamma, \quad a_{11} = -8 - \xi,$$

$$b_0 = -\alpha, \qquad b_{11} = -2\sqrt{2} - \delta.$$

The first seven Lyapunov constants are

$$V_2 = -2\pi\alpha, \quad V_3 = 2\pi\beta, \quad V_4 = -\pi\gamma, \qquad V_5 = \frac{\pi(4\sqrt{2}+\delta)}{32}\,\delta,$$

$$V_6 = -\frac{\pi}{8}\,\eta, \quad V_7 = \frac{\pi}{16}\,\xi, \quad V_8 = -\frac{883\pi}{768\sqrt{2}}.$$

If we choose $\alpha, \beta, \gamma, \delta, \eta$ and ξ positive and such that $|\alpha| \ll |\beta| \ll |\gamma| \ll |\delta| \ll |\eta| \ll |\xi| \ll 1$ then a multiple Hopf bifurcation appears and we can ensure that at least 6 non-zero isolated periodic orbits born from the solution $x = 0$, which also remains as an isolated periodic orbit. $\square$

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