LIMIT CYCLES FOR A CLASS OF THREE DIMENSIONAL POLYNOMIAL DIFFERENTIAL SYSTEMS

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Abstract. Perturbing the system $\dot{x} = -y(1+x)$, $\dot{y} = x(1+x)$, $\dot{z} = 0$, inside the family of polynomial differential systems of degree n in $\mathbb{R}^3$, we obtain at most n^2 limit cycles using averaging theory of first order. Moreover, there are such perturbed system having at least n^2 limit cycles.

1. Introduction and statement of the main results

We perturb the system $\dot{x} = -y(1+x)$, $\dot{y} = x(1+x)$, $\dot{z} = 0$ inside the class of polynomial differential systems of degree n in $\mathbb{R}^3$. The unperturbed system has the straight line $x = 0$, $y = 0$ and the plane $x = -1$ fulfilled of singular points, and on each plane $z = \bar{z} =$ constant the flow is invariant. In fact, on every plane $z = \bar{z}$ the singular point $(0, 0, \bar{z})$ is a center.

Theorem 1. *We consider the family of systems*

$$\begin{array}{rcl} \dot{x} & = & -y(1+x) + \varepsilon(ax + F(x,y,z)), \\ \dot{y} & = & x(1+x) + \varepsilon(ay + G(x,y,z)), \\ \dot{z} & = & \varepsilon(cz + R(x,y,z)), \end{array} \tag{1}$$

where $F(x,y,z)$, $G(x,y,z)$ and $R(x,y,z)$ are polynomials of degree n starting with terms of degree 2. Then there exists an $\varepsilon_0 > 0$ sufficiently small such that for $|\varepsilon| < \varepsilon_0$ there are systems (1) having at least n^2 limit cycles bifurcating from the periodic orbits of the system $\dot{x} = -y(1+x)$, $\dot{y} = x(1+x)$, $\dot{z} = 0$.

Theorem 1 improves the results of [1] where perturbing the system $\dot{x} = -y$, $\dot{y} = x$, $\dot{z} = 0$ inside the same class of polynomial vector fields the averaging method up to first oder only can obtain at most $n(n-1)/2$ limit cycles. Preliminary results in this direction where obtained by Żołądek in [5, 6, 7]. His main result is that the number of limit cycles that he can

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obtain from the periodic orbits of the center at the origin of the invariant plane $z = 0$ is of the order a constant$\cdot n$. But we study the limit cycles bifurcating from the periodic orbits at any plane $z =$ constant, not only from the plane $z = 0$. In this way we get that the number of limit cycles is n^2.

2. Limit cycles via averaging theory

In few words we can say that the averaging method [3, 4] gives a quantitative relation between the solutions of some non–autonomous periodic differential system and the solutions of its averaged differential system, which is an autonomous one. The next theorem provides a first order approximation in ε for the limit cycles of a periodic differential system, for a proof see Theorem 2.6.1 of Sanders and Verhulst [3] and Theorem 11.5 of Verhulst [4].

Theorem 2. *We consider the following two initial value problems*

$$\dot{x} = \varepsilon\, f(t,x) + \varepsilon^2\, g(t,x,\varepsilon), \qquad x(0) = x_0, \tag{2}$$

and

$$\dot{y} = \varepsilon\, f^0(y), \qquad y(0) = x_0, \tag{3}$$

where x, y, $x_0 \in D$ *an open subset of* $\mathbb{R}^n$, $t \in [0,\infty)$, $|\varepsilon| \le \varepsilon_0$, f *and* g *are periodic of period* T *in the variable* t, *and* $f^0(y)$ *is the averaged function of* $f(t,x)$ *with respect to* t, *i.e.,*

$$f^0(y) = \frac{1}{T}\int_0^T f(t,y)dt.$$

Suppose: (i) f, *its Jacobian* $\partial f/\partial x$, *its Hessian* $\partial^2 f/\partial x^2$, g *and its Jacobian* $\partial g/\partial x$ *are defined, continuous and bounded by a constant independent on* ε *in* $[0,\infty)\times D$ *and* $|\varepsilon| \le \varepsilon_0$; *(ii)* T *is a constant independent of* $|\varepsilon|$; *and (iii)* $y(t)$ *belongs to* D *on the interval of time* $[0, 1/|\varepsilon|]$. *Then the following statements hold.*

(a) *On the time scale* $1/|\varepsilon|$ *we have that* $x(t) - y(t) = O(\varepsilon)$, *as* $\varepsilon \to 0$.
(b) *If* p *is a singular point of the averaged system* (3) *such that the determinant of the Jacobian matrix* $\partial f^0/\partial y|_{y=p}$ *is not zero, then there exists a limit cycle* $\phi(t,\varepsilon)$ *of period* T *for the system* (2) *which is close to* p *and such that* $\phi(t,\varepsilon) \to p$ *as* $\varepsilon \to 0$.
(c) *The stability or instability of the limit cycle* $\phi(t,\varepsilon)$ *is given by the stability or instability of the singular point* p *of the averaged system* (3). *In fact, the singular point* p *has the stability behavior of the Poincaré map associated to the limit cycle* $\phi(t,\varepsilon)$.

3. Preliminary results

To prove Theorem 1 we shall need the following three lemmas which are proved in [2].

Lemma 3. *For $i, j \in \mathbb{N}$, we define*

$$I_{i,j} = \frac{1}{2\pi}\int_0^{2\pi} \frac{\cos^i\theta\ \sin^j\theta}{1 + r\cos\theta}\,dt.$$

Then $I_{i,j} \neq 0$ if and only if j is even number.

Lemma 4. *For $i, j \in \mathbb{N}$, and j even,* $I_{i,j} = \sum_{\substack{s=0 \\ s\ even}}^{j} (-1)^{s/2} \binom{j/2}{s/2} I_{i+s,0}$.

Lemma 5. *For $i \in \mathbb{N}$, we have*

$$\begin{aligned} I_{i,0} &= \frac{1}{2\pi}\int_0^{2\pi} \frac{\cos^i\theta}{1 + r\cos\theta}\,dt. \\ &= \frac{(-1)^i}{r^i\sqrt{1-r^2}} + \sum_{\substack{l=1 \\ l \equiv i(\mathrm{mod} 2)}}^{i} (-1)^{l-1} 2^{l-i} \binom{i-l}{(i-l)/2} r^{-l}. \end{aligned} \tag{4}$$

4. Proof of Theorem 1

Let

$$\begin{aligned} F(x,y,z) &= F_2(x,y,z) + F_3(x,y,z) + \cdots + F_n(x,y,z), \\ G(x,y,z) &= G_2(x,y,z) + G_3(x,y,z) + \cdots + G_n(x,y,z), \\ R(x,y,z) &= R_2(x,y,z) + R_3(x,y,z) + \cdots + R_n(x,y,z), \end{aligned}$$

be polynomials such that F_i, G_i and R_i are the homogeneous parts of F, G and R of degree i, respectively. In cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, system (1) in the region $r > 0$ can be written as

$$\begin{aligned} \dot{r} &= \varepsilon\,(a\,r + \cos\theta\,F + \sin\theta\,G)\,, \\ \dot{\theta} &= 1 + r\cos\theta + \frac{\varepsilon}{r}\,(\cos\theta\,G - \sin\theta\,F)\,, \\ \dot{z} &= \varepsilon\,(cz + R)\,. \end{aligned} \tag{5}$$

Here and in what follows F, G and R will denote $F(r\cos\theta, r\sin\theta, z)$, $G(r\cos\theta,\ r\sin\theta, z)$ and $R(r\cos\theta, r\sin\theta, z)$, respectively. System (5) in the region $r > 0$ is equivalent to system

$$
\begin{aligned}
\frac{dr}{d\theta} &= \varepsilon \frac{ar + \cos\theta\, F + \sin\theta\, G}{1 + r\cos\theta} + O(\varepsilon^2), \\
\frac{dz}{d\theta} &= \varepsilon \frac{cz + R}{1 + r\cos\theta} + O(\varepsilon^2).
\end{aligned} \tag{6}
$$

Let D be an arbitrary ball of radius smaller than 1 centered at the origin of $\mathbb{R}^2$ and ε_0 be a positive number. Then, system (6) satisfies the assumptions of Theorem 2 if ε_0 is sufficiently small and D is fixed. In order to apply the averaging theory to system (6) we have to compute the averaged functions

$$
\begin{aligned}
f = f(r, z) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{a\, r + \cos\theta\, F + \sin\theta\, G}{1 + r\cos\theta} d\theta, \\
g = g(r, z) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{cz + R}{1 + r\cos\theta} d\theta.
\end{aligned} \tag{7}
$$

For each $k = 2, \cdots, n$ we write

$$
\begin{aligned}
F_k(x, y, z) &= \sum_{i+j+l=k} a^k_{i,j,l}\, x^i y^j z^l, \\
G_k(x, y, z) &= \sum_{i+j+l=k} b^k_{i,j,l}\, x^i y^j z^l, \\
R_k(x, y, z) &= \sum_{i+j+l=k} c^k_{i,j,l}\, x^i y^j z^l.
\end{aligned}
$$

Now using the notation introduced in Lemma 3, the averaged functions write as

$$
f = ar + \sum_{k=2}^{n} \sum_{i+j+l=k} r^{i+j} z^l (a^k_{i,j,l} I_{i+1,j} + b^k_{i,j,l} I_{i,j+1}), \tag{8}
$$

and

$$
g = g(r, z) = czI_{0,0} + \sum_{k=2}^{n} \sum_{i+j+l=k} c^k_{i,j,l} r^{i+j} z^l I_{i,j}. \tag{9}
$$

Consequently, f and g are polynomials in the variables r, z of degree n. Now we obtain by the lemmas

$$
\begin{aligned}
g &= czI_{0,0} + \sum_{k=2}^{n}\sum_{m=0}^{k} r^m z^{k-m} \sum_{\substack{i+j=m \\ j\ even}} c^k_{i,j,k-m} I_{i,j} \\
&= czI_{0,0} + \sum_{m=0}^{n} r^m \sum_{k=m}^{n} z^{k-m} \sum_{\substack{i=0 \\ m-i\ even}}^{m} c^k_{i,m-i,k-m} I_{i,m-i} - \\
&\quad \sum_{m=0}^{1} r^m \sum_{m\le k<2} z^{k-m} \sum_{\substack{i=0 \\ m-i\ even}}^{m} c^k_{i,m-i,k-m} I_{i,m-i} \\
&= czI_{0,0} + g_1(r,z) + g_2(r,z),
\end{aligned}
$$

with

$$
\begin{aligned}
g_1 &= \sum_{m=0}^{n} r^m \sum_{k=m}^{n} z^{k-m} \sum_{\substack{i=0 \\ m-i\ even}}^{m} c^k_{i,m-i,k-m} \sum_{\substack{s=0 \\ s\ even}}^{m-i} d_{s,m-i} \frac{r^{-(i+s)}}{\sqrt{1-r^2}}, \\
g_2 &= \sum_{m=0}^{n} r^m \sum_{k=m}^{n} z^{k-m} \sum_{\substack{i=0 \\ m-i\ even}}^{m} c^k_{i,m-i,k-m} \sum_{\substack{s=0 \\ s\ even}}^{m-i} d_{s,m-i} \sum_{\substack{l=1 \\ i-l\ even}}^{i+s} e_{i+s,l} r^{-l},
\end{aligned}
$$

where

$$
\begin{aligned}
d_{s,m-i} &= (-1)^{s/2} \binom{(m-i)/2}{s/2}, \\
e_{i+s,l} &= (-1)^{l-1} 2^{l-(i+s)} \binom{i+s-l}{(i+s-l)/2}.
\end{aligned}
$$

We note that $c^k_{i,j,k} = 0$ for $k < 2$.

We rearrange the order of r, z in g_i for $i = 1, 2$. Thus we have

$$
\begin{aligned}
g_1 &= \frac{1}{\sqrt{1-r^2}} \sum_{m=0}^{n} r^m \sum_{\substack{v=0 \\ m-v\ even}}^{m} r^{-v} \sum_{k=m}^{n} z^{k-m} \sum_{\substack{s=0 \\ s\ even}}^{v} c^k_{v-s,m-v+s,k-m} d_{s,m-v+s} \\
&= \frac{1}{\sqrt{1-r^2}} \sum_{\substack{w=0 \\ w\ even}}^{n} r^w \sum_{k=w}^{n} \sum_{v=0}^{k-w} z^{k-w-v} \sum_{\substack{s=0 \\ s\ even}}^{v} c^k_{v-s,w+s,k-w-v} d_{s,w+s}
\end{aligned}
$$

$$= \frac{1}{\sqrt{1-r^2}} \sum_{\substack{w=0\\ w\ even}}^{n} r^w \sum_{\alpha=w}^{n} z^{\alpha-w} \sum_{v=0}^{n-\alpha} \sum_{\substack{s=0\\ s\ even}}^{v} c^{\alpha+v}_{v-s,w+s,\alpha-w} d_{s,w+s}.$$

In the previous first equality $v = i + s$, in the second one $w = m - v$, and finally in the third $\alpha = k - v$. Moreover,

$$\begin{aligned}
g_2 &= \sum_{m=0}^{n} r^m \sum_{k=m}^{n} z^{k-m} \sum_{\substack{i=0\\ m-i\ even}}^{m} c^k_{i,m-i,k-m} \sum_{\substack{\delta=i\\ \delta-i\ even}}^{m} d_{\delta-i,m-i} \sum_{\substack{l>0\\ \delta-l\ even}}^{\delta} e_{\delta,l} r^{-l} \\
&= \sum_{m=0}^{n} r^m \sum_{k=m}^{n} z^{k-m} \sum_{\substack{l>0\\ m-l\ even}}^{m} r^{-l} \sum_{\substack{\delta=l\\ \delta-l\ even}}^{m} \sum_{\substack{i=0\\ \delta-i\ even}}^{\delta} d_{\delta-i,m-i} e_{\delta,l} c^k_{i,m-i,k-m} \\
&= \sum_{\substack{w=0\\ w\ even}}^{n} r^w \sum_{k=w}^{n} \sum_{l>0}^{k-w} z^{k-w-l} \sum_{\substack{\delta=l\\ \delta-l\ even}}^{w+l} \sum_{\substack{i=0\\ \delta-i\ even}}^{\delta} d_{\delta-i,w+l-i} e_{\delta,l} c^k_{i,w+l-i,k-w-l} \\
&= \sum_{\substack{w=0\\ w\ even}}^{n} r^w \sum_{\alpha=w}^{n} z^{\alpha-w} \sum_{l>0}^{n-\alpha} \sum_{\substack{u=0\\ u\ even}}^{w} \sum_{\substack{i=0\\ l-i\ even}}^{u+l} {}_{u+l-i,w+l-i} e_{u+l,l} c^{\alpha+l}_{i,w+l-i,\alpha-w}.
\end{aligned}$$

In the previous first equality $\delta = i + s$, in the second one $w = m - l$, in the third $\alpha = k - l$, and finally in the fourth $u = \delta - l$.

Defining $t = \sqrt{1-r^2}$ we have

$$\begin{aligned}
\sqrt{1-r^2}\, g_1(r,z) &= t g_1(t,z) \\
&= \sum_{\substack{w=0\\ w\ even}}^{n} (1-t^2)^{\frac{w}{2}} \sum_{\alpha=w}^{n} z^{\alpha-w} g^1_{w,\alpha} \\
&= \sum_{\substack{w=0\\ w\ even}}^{n} \sum_{\substack{\rho=0\\ \rho\ even}}^{w} (-1)^{\frac{\rho}{2}} \binom{\frac{w}{2}}{\frac{\rho}{2}} t^\rho \sum_{\alpha=w}^{n} z^{\alpha-w} g^1_{w,\alpha} \\
&= \sum_{\substack{\rho=0\\ \rho\ even}}^{n} t^\rho \sum_{\xi=0}^{n-\rho} z^\xi \sum_{\substack{w=\rho\\ w\ even}}^{n-\xi} (-1)^{\frac{\rho}{2}} \binom{\frac{w}{2}}{\frac{\rho}{2}} g^1_{w,w+\xi},
\end{aligned}$$

in the last equality we have $\xi = \alpha - w$, and

$$g_2(r,z) = g_2(t,z) = \sum_{\substack{\rho=0\\ \rho\ even}}^{n} t^\rho \sum_{\xi=0}^{n-\rho} z^\xi \sum_{\substack{w=\rho\\ w\ even}}^{n-\xi} (-1)^{\frac{\rho}{2}} \binom{\frac{w}{2}}{\frac{\rho}{2}} g^2_{w,w+\xi},$$

where

$$g^1_{w,w+\xi} = \sum_{v=0}^{n-w-\xi} \sum_{\substack{s=0 \\ s\ even}}^{v} c^{w+\xi+v}_{v-s,w+s,\xi} d_{s,w+s},$$

$$g^2_{w,w+\xi} = \sum_{l>0}^{n-w-\xi} \sum_{\substack{u=0 \\ u\ even}}^{w} \sum_{\substack{i=0 \\ l-i\ even}}^{u+l} d_{u+l-i,w+l-i} e_{u+l,l} c^{w+\xi+l}_{i,w+l-i,\xi}.$$

We have $g(r,z)=(t,z)=\frac{1}{t}P(t,z)$, where $P(t,z)=cz+\sum_{\rho=0}^{n+1} P_\rho(z)t^\rho$ and

$$(10)\quad P_\rho(z) = \begin{cases} \displaystyle\sum_{\xi=0}^{n-\rho} z^\xi \sum_{\substack{w=\rho \\ w\ even}}^{n-\xi} (-1)^{\frac{\rho}{2}} \begin{pmatrix} \frac{w}{2} \\ \frac{\rho}{2} \end{pmatrix} g^1_{w,w+\xi} & \text{if } \rho \text{ even}, \\ \displaystyle\sum_{\xi=0}^{n-\rho+1} z^\xi \sum_{\substack{w=\rho-1 \\ w\ even}}^{n-\xi} (-1)^{\frac{\rho-1}{2}} \begin{pmatrix} \frac{w}{2} \\ \frac{\rho-1}{2} \end{pmatrix} g^2_{w,w+\xi} & \text{if } \rho \text{ odd}. \end{cases}$$

In fact $P(t,z)$ is a polynomial of degree n in the variables t and z, respectively, which can be written as

$$(11)\quad P = \begin{cases} cz + P^0_n + \hat{P}^1_{n-1}t + P^2_{n-2}t^2 + \hat{P}^3_{n-3}t^3 + \cdots + P^n_0 t^n & n \text{ even}, \\ cz + P^0_n + \hat{P}^1_{n-1}t + P^2_{n-2}t^2 + \hat{P}^3_{n-3}t^3 + \cdots + \hat{P}^n_0 t^n & n \text{ odd}, \end{cases}$$

where the polynomials $P^\rho_\zeta = P^\rho_\zeta(z)$ or $\hat{P}^\rho_\zeta = \hat{P}^\rho_\zeta(z)$ of degree ζ denote the coefficient of t^ρ. Then, we expand it and arrange the coefficients of $z^m t^\rho$ into the following table:

n	$n-1$	$n-2$	$n-3$	$n-4$	$\cdots$	0	m/ρ
$P^0_{n,n}$	$P^0_{n,n-1}$	$P^0_{n,n-2}$	$P^0_{n,n-3}$	$P^0_{n,n-4}$	$\cdots$	$P^0_{n,0}$	0
$\hat{P}^1_{n,n}$	$\hat{P}^1_{n,n-1}$	$\hat{P}^1_{n,n-2}$	$\hat{P}^1_{n,n-3}$	$\hat{P}^1_{n,n-4}$	$\cdots$	$\hat{P}^1_{n,0}$	1
		$P^2_{n-2,n-2}$	$P^2_{n-2,n-3}$	$P^2_{n-2,n-4}$	$\cdots$	$P^2_{n-2,0}$	2
		$\hat{P}^3_{n-2,n-2}$	$\hat{P}^3_{n-2,n-3}$	$\hat{P}^3_{n-2,n-4}$	$\cdots$	$\hat{P}^3_{n-2,0}$	3
				$\dots$	$\dots$	$\dots$	$\vdots$

where

$$\begin{aligned}
P^{\rho}_{n-\rho,\xi} &= \sum_{\substack{w=\rho \\ w\ even}}^{n-\xi} \sum_{v=0}^{n-\xi-w} \sum_{\substack{s=0 \\ s\ even}}^{v} *c^{\xi+w+v}_{v-s,w+s,\xi} \\
&= \sum_{\substack{w=\rho \\ w\ even}}^{n-\xi} \sum_{v=0}^{n-\xi-w} \sum_{\substack{i=0 \\ v-i\ even}}^{v} *c^{\xi+w+v}_{i,w+v-i,\xi}, \\
\hat{P}^{\rho+1}_{n-\rho,\xi} &= \sum_{\substack{w=\rho \\ w\ even}}^{n-\xi} \sum_{l>0}^{n-\xi-w} \sum_{\substack{u=0 \\ u\ even}}^{w} \sum_{\substack{i=0 \\ l-i\ even}}^{u+l} *c^{\xi+w+l}_{i,w+l-i,\xi} \\
&= \sum_{\substack{w=\rho \\ w\ even}}^{n-\xi} \sum_{l>0}^{n-\xi-w} \Big(\sum_{\substack{i=0 \\ l-i\ even}}^{l} c^{\xi+w+l}_{i,w+l-i,\xi} * + \sum_{\substack{i>l \\ i-l\ even}}^{l+w} c^{\xi+w+l}_{i,w+l-i,\xi} * \Big),
\end{aligned}$$

where ρ is even and the $*$'s denote some constants. From the formula of $\hat{P}^{\rho+1}_{n-\rho,\xi}$, it is easy to calculate that $\hat{P}^{\rho+1}_{n-\rho,n-\rho} = 0$, which means that $P(t,z)$ is a polynomial in the variables of t and z of degree n, and of degree n in t or z, respectively.

In the following we consider the function $f = f(r,z)$. Taking $G(x,y,z) \equiv 0$ we have

$$\begin{aligned}
f &= ar + \sum_{k=2}^{n} \sum_{m=0}^{k} r^m z^{k-m} \sum_{\substack{i+j=m \\ j\ even}} a^k_{i,j,k-m} I_{i+1,j} \\
&= ar + \sum_{m=0}^{n} r^m \sum_{k=m}^{n} z^{k-m} \sum_{\substack{i=0 \\ m-i\ even}}^{m} a^k_{i,m-i,k-m} I_{i+1,m-i} \\
&\qquad - \sum_{m=0}^{1} r^m \sum_{m\le k<2} z^{k-m} \sum_{\substack{i=0 \\ m-i\ even}}^{m} a^k_{i,m-i,k-m} I_{i,m-i} \\
&= ar + f_1(r,z) + f_2(r,z),
\end{aligned}$$

where

$$\begin{aligned}
f_1 &= \sum_{m=0}^{n} r^m \sum_{k=m}^{n} z^{k-m} \sum_{\substack{i=0 \\ m-i\ even}}^{m} a^k_{i,m-i,k-m} \sum_{\substack{s=0 \\ s\ even}}^{m-i} A_{s,m-i} \frac{r^{-(i+s+1)}}{\sqrt{1-r^2}}, \\
f_2 &= \sum_{m=0}^{n} r^m \sum_{k=m}^{n} z^{k-m} \sum_{\substack{i=0 \\ m-i\ even}}^{m} a^k_{i,m-i,k-m} \sum_{\substack{s=0 \\ s\ even}}^{m-i} A_{s,m-i} \sum_{\substack{l=0 \\ i-l\ even}}^{i+s} E_{i+s,l} r^{-l-1},
\end{aligned}$$

where

$$
\begin{aligned}
A_{s,m-i} &= (-1)^{s/2}\begin{pmatrix} (m-i)/2 \\ s/2 \end{pmatrix}, \\
E_{i+s,l} &= (-1)^{l}2^{l-(i+s)}\begin{pmatrix} i+s-l \\ (i+s-l)/2 \end{pmatrix}.
\end{aligned}
$$

Note that $A_{s,m-i}$ and $E_{i+s,l}$ are the same as $d_{s,m-i}$ and $e_{i+s,l}$, we also use them later on. Moreover, note that $a^k_{i,j,k} = 0$ for $k < 2$.

We rearrange the order of r, z in $f_i, i = 1, 2$, thus we have

$$
\begin{aligned}
f_1 &= \frac{1}{r\sqrt{1-r^2}}\sum_{m=0}^{n} r^m \sum_{\substack{v=0\\ m-v\ even}}^{m} r^{-v}\sum_{k=m}^{n} z^{k-m}\sum_{\substack{s=0\\ s\ even}}^{v} a^k_{v-s,m-v+s,k-m}d_{s,m-v+s} \\
&= \frac{1}{r\sqrt{1-r^2}}\sum_{\substack{w=0\\ w\ even}}^{n} r^w \sum_{k=w}^{n}\sum_{v=0}^{k-w} z^{k-w-v}\sum_{\substack{s=0\\ s\ even}}^{v} a^k_{v-s,w+s,k-w-v}d_{s,w+s} \\
&= \frac{1}{r\sqrt{1-r^2}}\sum_{\substack{w=0\\ w\ even}}^{n} r^w \sum_{\alpha=w}^{n} z^{\alpha-w}\sum_{v=0}^{n-\alpha}\sum_{\substack{s=0\\ s\ even}}^{v} a^{\alpha+v}_{v-s,w+s,\alpha-w}d_{s,w+s},
\end{aligned}
$$

In the previous first equality $v = i + s$, in the second one $w = m - v$, and finally in the third $\alpha = k - v$. Moreover,

$$
\begin{aligned}
f_2 &= \frac{1}{r}\sum_{m=0}^{n} r^m \sum_{k=m}^{n} z^{k-m}\sum_{\substack{i=0\\ m-i\ even}}^{m} a^k_{i,m-i,k-m}\sum_{\substack{\delta=i\\ \delta-i\ even}}^{m} d_{\delta-i,m-i}\sum_{\substack{l=0\\ \delta-l\ even}}^{\delta} e_{\delta,l}r^{-l} \\
&= \frac{1}{r}\sum_{m=0}^{n} r^m \sum_{k=m}^{n} z^{k-m}\sum_{\substack{l=0\\ m-l\ even}}^{m} r^{-l}\sum_{\substack{\delta=l\\ \delta-l\ even}}^{m}\sum_{\substack{i=0\\ \delta-i\ even}}^{\delta} d_{\delta-i,m-i}e_{\delta,l}a^k_{i,m-i,k-m} \\
&= \frac{1}{r}\sum_{\substack{w=0\\ w\ even}}^{n} r^w \sum_{k=w}^{n}\sum_{l=0}^{k-w} z^{k-w-l}\sum_{\substack{\delta=l\\ \delta-l\ even}}^{w+l}\sum_{\substack{i=0\\ \delta-i\ even}}^{\delta} d_{\delta-i,w+l-i}e_{\delta,l}a^k_{i,w+l-i,k-w-l} \\
&= \frac{1}{r}\sum_{\substack{w=0\\ w\ even}}^{n} r^w \sum_{\alpha=w}^{n} z^{\alpha-w}\sum_{l=0}^{n-\alpha}\sum_{\substack{u=0\\ u\ even}}^{w}\sum_{\substack{i=0\\ l-i\ even}}^{u+l} d_{u+l-i,w+l-i}e_{u+l,l}a^{\alpha+l}_{i,w+l-i,\alpha-w},
\end{aligned}
$$

In the previous first equality $\delta = i + s$, in the second one $w = m - l$, in the third $\alpha = k - l$, and finally in the fourth $u = \delta - l$. Taking $t = \sqrt{1-r^2}$ we obtain

$$\begin{aligned} r\sqrt{1-r^2}f_1(r,z) &= rtf_1(t,z) \\ &= \sum_{\substack{w=0 \\ w\ even}}^{n} (1-t^2)^{\frac{w}{2}} \sum_{\alpha=w}^{n} z^{\alpha-w} f^1_{w,\alpha} \\ &= \sum_{\substack{w=0 \\ w\ even}}^{n} \sum_{\substack{\rho=0 \\ \rho\ even}}^{w} (-1)^{\frac{\rho}{2}} \binom{\frac{w}{2}}{\frac{\rho}{2}} t^{\rho} \sum_{\alpha=w}^{n} z^{\alpha-w} f^1_{w,\alpha} \\ &= \sum_{\substack{\rho=0 \\ \rho\ even}}^{n} t^{\rho} \sum_{\xi=0}^{n-\rho} z^{\xi} \sum_{\substack{w=\rho \\ w\ even}}^{n-\xi} (-1)^{\frac{\rho}{2}} \binom{\frac{w}{2}}{\frac{\rho}{2}} f^1_{w,w+\xi}, \end{aligned}$$

in the last equality we have $\xi = \alpha - w$, and

$$rf_2(r,z) = rf_2(t,z) = \sum_{\substack{\rho=0 \\ \rho\ even}}^{n} t^{\rho} \sum_{\xi=0}^{n-\rho} z^{\xi} \sum_{\substack{w=\rho \\ w\ even}}^{n-\xi} (-1)^{\frac{\rho}{2}} \binom{\frac{w}{2}}{\frac{\rho}{2}} f^2_{w,w+\xi},$$

where

$$\begin{aligned} f^1_{w,w+\xi} &= \sum_{v=0}^{n-\xi-w} \sum_{\substack{s=0 \\ s\ even}}^{v} a^{\xi+w+v}_{v-s,w+s,\xi} d_{s,w+s} \\ &= \sum_{v=0}^{n-\xi-w} \sum_{\substack{i=0 \\ v-i\ even}}^{v} a^{\xi+w+v}_{i,w+v-i,\xi} d_{v-i,w+v-i}, \\ f^2_{w,w+\xi} &= \sum_{l=0}^{n-\xi-w} \sum_{\substack{u=0 \\ u\ even}}^{w} \sum_{\substack{i=0 \\ l-i\ even}}^{u+l} d_{u+l-i,w+l-i} e_{u+l,l} a^{\xi+w+l}_{i,w+l-i,\xi} \end{aligned}$$

$$= \sum_{l=0}^{n-\xi-w} \Big(\sum_{\substack{i=0 \\ l-i\ even}}^{l} * a^{\xi+w+l}_{i,w+l-i,\xi} + \sum_{\substack{i>l \\ l-i\ even}}^{l+w} * a^{\xi+w+l}_{i,w+l-i,\xi} \Big),$$

where the *'s denote some constants. We have $f(r,z) = f(t,z) = \frac{1}{t\sqrt{1-t^2}} Q(t,z)$, where $Q(t,z) = a(1-t^2) + \sum_{\rho=0}^{n+1} Q_{\rho}(z) t^{\rho}$ and

(12)

$$Q_{\rho}(z) = \begin{cases} \displaystyle\sum_{\xi=0}^{n-\rho} z^{\xi} \sum_{\substack{w=\rho \\ w\ even}}^{n-\xi} (-1)^{\frac{\rho}{2}} \binom{\frac{w}{2}}{\frac{\rho}{2}} f^1_{w,w+\xi} & \text{if } \rho \text{ even}, \\ \displaystyle\sum_{\xi=0}^{n-\rho+1} z^{\xi} \sum_{\substack{w=\rho-1 \\ w\ even}}^{n-\xi} (-1)^{\frac{\rho-1}{2}} \binom{\frac{w}{2}}{\frac{\rho-1}{2}} f^2_{w,w+\xi} & \text{if } \rho \text{ odd}. \end{cases}$$

In fact $Q(t,z)$ is a polynomial in the variables of t and z of degree $n+1$, of degree n in z, which can be written as

$$(13)\quad Q = \begin{cases} a(1-t^2) + Q_n^0 + \hat{Q}_n^1 t + Q_{n-2}^2 t^2 + \cdots + \hat{Q}_0^{n+1} t^{n+1} & n \text{ even}, \\ a(1-t^2) + Q_n^0 + \hat{Q}_n^1 t + Q_{n-2}^2 t^2 + \cdots + \hat{Q}_1^n t^n & n \text{ odd}, \end{cases}$$

where the polynomials $Q_\zeta^\rho = Q_\zeta^\rho(z)$ or $\hat{Q}_\zeta^\rho = \hat{Q}_\zeta^\rho(z)$ of degree ζ denote the coefficient of t^ρ.

Moreover, from (7), we have $f(0,z) = 0$ which implies $Q(1,z) = 0$, i.e. $Q(t,z) = (t-1)\bar{Q}(t,z)$, where $\bar{Q}(t,z)$ is a polynomial in the variables of t and z of degree n, and of degree at most n in t or z, respectively. Hence, by Bezout's Theorem the maximum number of the common solution of $P(t,z)$ and $Q(t,z)$ is at most n^2 for $0 < t < 1$, because $P(t,z)$ and $\bar{Q}(t,z)$ are the polynomials in the variables of t and z of degree at most n, respectively. Thus, by Theorem (2), the maximum number of limit cycles bifurcating form system (1) is n^2.

Next, we shall provide a system having n^2 limit cycles. Here, we just consider n even, and we take $G(x,y,z) = 0$ and

$$(14)\quad \begin{aligned} F(x,y,z) &= \sum_{k=2}^{n} a_{0,0,k}^k z^k + a_{1,0,1}^2 xz, \\ R(x,y,z) &= \sum_{\substack{k=2 \\ k \text{ even}}}^{n} (c_{k,0,0}^k x^k + c_{0,k,0}^k y^k). \end{aligned}$$

Computing the averaged functions and taking $t = \sqrt{1-r^2}$, we have

$$\begin{aligned} r\sqrt{1-r^2} f(r,z) &= ar^2 + (a_{1,0,1}^2 z - \sum_{k=2}^{n} a_{0,0,k}^k z^k)(1 - \sqrt{1-r^2}) \\ &= (1-t)(a(1+t) + a_{1,0,1}^2 z - \sum_{k=2}^{n} a_{0,0,k}^k z^k) \\ &= (1-t)(a(1+t) - \bar{Q}(z)), \end{aligned}$$

where $\bar{Q}(z)$ is an arbitrary polynomial in z of degree n, such that $\bar{Q}(0) = 0$. At the same time, the averaged function corresponding to $R(x,y,z)$ satisfies

$$\sqrt{1-r^2} g(r,z) = cz + \sum_{\substack{k=2 \\ k \text{ even}}}^{n} r^k (c_{k,0,0}^k I_{k,0} + c_{0,k,0}^k I_{0,k}).$$

Using (4) we obtain that $\sqrt{1-r^2}g(r,z) = cz + g_1(r) + g_2(r)$, where

$$g_1 = \sum_{\substack{k=2\\k\ even}}^{n} c^k_{k,0,0} - \sqrt{1-r^2}\sum_{\substack{m=0\\m\ even}}^{n-1} r^m \sum_{\substack{k=m+1\\k\ even}}^{n} c^k_{k,0,0}2^{-m}\begin{pmatrix} m\\ \frac{m}{2}\end{pmatrix},$$

$$g_2 = \sum_{\substack{m=0\\m\ even}}^{n} A_m r^m + \sqrt{1-r^2}\sum_{\substack{m=0\\m\ even}}^{n-1} B_m r^m,$$

with

$$A_m = \sum_{\substack{k=m\\k\ even}}^{n} c^k_{0,k,0} d_{k-m,k}, \quad B_m = \sum_{\substack{k=m\\k\ even}}^{n} c^k_{0,k,0} d_{k-m,k} \sum_{\substack{l>0\\l\ even}}^{n} e_{k-m,l}.$$

Writing $t = \sqrt{1-r^2}$ from the definition (7) the polynomials $P_i(t) = g_i(r)$, $i = 1, 2$, satisfy $P_i(1) = g_i(0) = 0$. Then we can define a polynomial in t of degree n, $\bar{P}(t) = P_1(t) + P_2(t) = (t-1)\tilde{P}(t)$. We claim that $\bar{P}(t)$ is an arbitrary polynomial such that $\bar{P}(1) = 0$. It is obvious to know that g_1, g_2 have $n/2$ parameters, respectively, where the $n/2$ coefficients $c^k_{0,k,0}$ allow to choose the first term of g_2 arbitrarily but the term with $m = 0$, implying that the even terms of $\bar{P}(t)$ are arbitrary but the constant term; while another $n/2$ coefficients $c^k_{k,0,0}$ allow to choose the second term in g_1 arbitrarily, implying that the odd terms of $\bar{P}(t)$ are arbitrary. So, the polynomial $\bar{P}(t)$ of degree n satisfies $\bar{P}(1) = 0$, and has n arbitrary coefficients, which completes our claim. (see also [2])

In short, the number of the solutions of $f(r,z) = 0, g(r,z) = 0$ is equal to the number of the intersection points of the curves

$$\ell_1 : \quad cz + \bar{P}(t) = 0, \quad \ell_2 : \quad a(1+t) - \bar{Q}(z) = 0. \tag{15}$$

Proposition 6. *System* (15) *has at least* n^2 *common solutions* (t,z) *in* $(0,1) \times (0, z^*)$ *for any given positive number* z^*.

Proof We know that the point $(t,z) = (-1,0)$ lies on the curve ℓ_2. Hence, we can choose some suitable coefficients in (14), such that for any given positive number $z^* > 0$, ℓ_2 intersects the line $t = 0$ n times in the interval $(0, z^*)$, and the smallest maximum,

$$\hat{t} = \min\{t = \bar{Q}(z)/a - 1 \mid \bar{Q}'(z) = 0, \bar{Q}''(z)/a < 0, z \in (0, z^*)\},$$

is larger than 1. Hence, there are n intersection points between ℓ_2 and $t = \bar{t}$ for all $\bar{t} \in (0,1)$.

On the other hand, we know that the point $(1,0)$ lies on the curve ℓ_1. Hence, we take ℓ_1, such that it intersects the line $z = 0$ n times in the interval $(0,1]$ and the smallest maximum,

$$\hat{z} = \min\{z = -\bar{P}(t)/c \mid \bar{P}'(z) = 0, -\bar{P}''(t)/c < 0, t \in (0,1)\},$$

is larger than z^*. Hence, there are n^2 intersection points between ℓ_1 and ℓ_2 contained in the rectangle $(0,1) \times (0,z^*)$.

We can consider the case n odd in a similar way. Provided that we take in (14),

$$R(x,y,z) = c^n_{n,0,0}x^n + \sum_{\substack{k=2 \\ k\ even}}^{n} (c^k_{k,0,0}x^k + c^k_{0,k,0}y^k),$$

we can get the same curves (15) and the same result. □

The proof of Proposition 6 implies that for any integer $n > 2$, we can find n^2 intersection points on $f(r,z) = 0$ with $g(r,z) = 0$ for $r \in (r_0, 1)$, $0 < r_0 \ll 1$, which (using the averaging theory , see Theorem 2) give rise to n^2 limit cycles bifurcating from the periodic orbits of the system $\dot{x} = -y(1+x)$, $\dot{y} = x(1+x)$, $\dot{z} = 0$.

Furthermore, if $G(x,y,z) \not\equiv 0$, then we can know that $Q(t,z)$ in (13) is also a polynomial in the variables t and z of degree $n+1$, and of degree at most n in z. And we also have $Q(1,z) = 0$, which implies $Q(t,z) = (t-1)\bar{Q}(t,z)$, where $\bar{Q}(t,z)$ is a polynomial in the variables t and z of degree n. Hence, by Bezout's Theorem the maximum number of the common solution of $P(t,z)$ and $Q(t,z)$ is at most n^2 for $0 < t < 1$. Thus, the maximum number of limit cycles bifurcating form system (1) is n^2, if we use the averaging theory up to first order in ε.

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