ON THE UPPER BOUND OF THE NUMBER OF LIMIT CYCLES OBTAINED BY THE SECOND ORDER AVERAGING METHOD II

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ABSTRACT. For ε small we consider the number of limit cycles of the system $\dot{x} = -y(1+x)+\varepsilon F(x,y)$, $\dot{y} = x(1+x)+\varepsilon G(x,y)$, where F and G are polynomials of degree n starting with terms of degree 1. We prove for $n = 1, 2, 3, 4$ that at most $2n-1$ limit cycles can bifurcate from the periodic orbits of the unperturbed system ($\varepsilon = 0$) using the averaging theory of second order, and that there are systems realizing this upper bound for $n = 4$.

1. Introduction

In the research on the existence and distribution of the limit cycles of planar polynomial differential systems one of the main tools is the study of the limit cycles which can bifurcate from the periodic orbits of a center when we perturb it. For example, for the system

$$\begin{aligned}\dot{x} &= -y + \varepsilon p(x,y),\\ \dot{y} &= x + \varepsilon q(x,y),\end{aligned}$$

where $p(x,y)$, $q(x,y)$ are polynomials of degree n, it is well known that at most $[(n-1)/2]$ limit cycles can bifurcate from the periodic orbits of the linear center $\dot{x} = -y,\ \dot{y} = x$, using techniques of first order, see for instance [2].

In [4] the authors perturb the quadratic center, formed by the linear center and a straight line of singular points:

$$\begin{aligned}\dot{x} &= -y(1+x) + \varepsilon p(x,y),\\ \dot{y} &= x(1+x) + \varepsilon q(x,y).\end{aligned}$$

They obtain at least n limit cycles bifurcating from the periodic orbits of the center, using averaging theory of first order.

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Moreover, in [1] and [5] the authors extend these results to three dimensional systems, studying the perturbation of the systems $\dot{x} = -y,\ \dot{y} = x,\ \dot{z} = 0$ and $\dot{x} = -y(1+x),\ \dot{y} = x(1+x),\ \dot{z} = 0$ inside the class of polynomial vector fields of degree n, respectively. Thus they obtain that these systems can have at least $n(n-1)/2$ and n^2 limit cycles, respectively, using averaging theory of first order.

In this paper we consider the two dimensional polynomial differential systems

$$\begin{array}{rcl} \dot{x} & = & -y(1+x) + \varepsilon F(x,y), \\ \dot{y} & = & x(1+x) + \varepsilon G(x,y), \end{array} \tag{1}$$

where $F(x,y)$ and $G(x,y)$ are polynomials of degree n starting with terms of degree 1. Using the averaging theory, if the first averaged function vanishes, the number of limit cycles of system (1) depends on the second averaged function. In [6] it is proved the following result.

Theorem 1. *Applying the averaging theory of second order to system* (1) *with F and G polynomials of degree n, we can obtain at most $2n-1$ limit cycles bifurcating from the periodic orbits of the center of system* (1) *for* $\varepsilon = 0$.

We do not know if the upper bound on the number of limit cycles in Theorem 1 can be reached. In general, this is a difficult problem. The main goal of the present paper is to show that this upper bound is reached for $n = 4$, instead of for $n = 1, 2, 3$. We describe the homogenous polynomials F and G of system (1) as $F = F_1 + F_2 + \cdots + F_n$ and $G = G_1 + G_2 + \cdots + G_n$, where

$$\begin{array}{rcl} F_k(x,y) & = & \displaystyle\sum_{i+j=k} a_{i,j}\, x^i y^j, \\ G_k(x,y) & = & \displaystyle\sum_{i+j=k} b_{i,j}\, x^i y^j, \end{array}$$

for $k = 1, \cdots, n$.

Theorem 2. *For $n = 1$ system* (1) *has at most one limit cycle using averaging theory of first order and there are systems* (1) *having one limit cycle. Moreover, $a_{1,0} = b_{0,1} = 0$ if and only if the first order averaged function is zero. If $a_{1,0} = b_{0,1} = 0$, then the origin of system* (1) *is a center.*

If $n = 2$ the conditions in order that the first averaged function vanishes are

$$a_{1,0} = a_{2,0}, \quad b_{0,1} = -a_{2,0}, \quad b_{1,1} = -(a_{2,0} + a_{0,2}). \tag{2}$$

Theorem 3. *For $n = 2$ assume that (2) holds. Then system (1) can have at most 1 limit cycles using averaging theory of second order. Moreover, there are systems (1) satisfying (2) with 1 limit cycles.*

If $n = 3$ the conditions in order that the first averaged function vanishes are

(3) $b_{0,3} = 0,\ \ a_{3,0} = a_{2,0} - a_{1,0},\ \ b_{2,1} = b_{1,1} + a_{0,2} - a_{1,2} + a_{1,0},\ \ b_{0,1} = -a_{1,0}.$

Theorem 4. *For $n = 3$ assume that (3) holds. then system (1) can have at most 3 limit cycles using averaging theory of second order. Moreover, there are systems (1) satisfying (3) with 3 limit cycles.*

If $n = 4$ the conditions in order that the first averaged function vanishes are

(4)
$$\begin{array}{l} a_{0,2} = b_{2,1} + a_{1,2} - b_{1,1} + 3b_{0,3} + 3a_{4,0} - a_{1,0},\ \ a_{0,4} = -b_{1,3} + b_{0,3}, \\ b_{0,1} = -a_{1,0},\ \ a_{2,0} = -a_{4,0} + a_{1,0} + a_{3,0},\ \ a_{2,2} = -3b_{0,3} - 3a_{4,0} - b_{3,1}. \end{array}$$

Theorem 5. *For $n = 4$ assume that (4) holds. then system (1) can have at most 7 limit cycles using averaging theory of second order. Moreover, there are systems (1) satisfying (4) with 7 limit cycles.*

In short, the averaging theory of second order applied to systems (1) does not provide a better number on their limit cycles when $n = 1, 2$ and $n = 3$. But for $n = 4$, the results using the second order are clearly better than the ones obtained using first order.

This paper is organized as follows. In Section 2 and 3 we present the first averaged function and some degenerated conditions. In Section 4 we present the proof of the results for $n = 3$ (i.e. Theorem 4 and examples). In Section 5, we give the proof of Theorems 2 and 3. Finally, in Section 6, we show that for $n = 4$ there are 7 limit cycles in system (1) by averaging theory of second order (i.e. Theorem 5).

2. Preliminary results

Here we use the notation on the averaging theory of the first and second order introduced in Subsection 2.1 of [6]. By means of the change of variables $x = r\cos\theta, y = r\sin\theta$, system (1) in the region $r > 0$ can be written as

$$\begin{aligned} \dot{r} &= \varepsilon\left(\cos\theta\, F + \sin\theta\, G\right), \\ \dot{\theta} &= 1 + r\cos\theta + \frac{\varepsilon}{r}\left(\cos\theta\, G - \sin\theta\, F\right), \end{aligned} \tag{5}$$

Here and in what follows F and G will denote $F(r\cos\theta, r\sin\theta)$ and $G(r\cos\theta, r\sin\theta)$, respectively. System (5) in the region $r>0$ is equivalent to the system

$$\frac{dr}{d\theta} = \varepsilon f(\theta,r)+\varepsilon^2 g(\theta,r)+O(\varepsilon^3), \tag{6}$$

where

$$f(\theta,r) = \frac{\cos\theta\, F+\sin\theta\, G}{1+r\cos\theta},$$

$$g(\theta,r) = -\frac{(\cos\theta F+\sin\theta G)(\cos\theta G-\sin\theta F)}{r(1+r\cos\theta)^2}.$$

In this paper we consider system (6), when the first averaged function vanishes. So applying the averaging theory to system (6), we must assume that,

$$f^0(r)=\frac{1}{2\pi}\int_0^{2\pi} f(\theta,r)d\theta\equiv 0, \tag{7}$$

which naturally implies that $\displaystyle\int_0^{2\pi}\frac{\partial f(\theta,r)}{\partial r}d\theta\equiv 0$. Then we have the second averaging system associated with system (6)

$$\frac{d\,r}{d\,\theta}=\varepsilon^2 f^{10}(r)+\varepsilon^2 g^0(r), \tag{8}$$

where $g^0=\displaystyle\frac{1}{2\pi}\int_0^{2\pi} g(\theta,r)d\theta$ and

$$\begin{aligned} f^{10} &= \frac{1}{2\pi}\int_0^{2\pi}\frac{\partial f(s,r)}{\partial r}\left(\int_0^s f(\theta,r)d\theta\right)ds+\frac{1}{2\pi}\int_0^{2\pi}\frac{\partial f(s,r)}{\partial r}z(r)ds,\\ &= \frac{1}{2\pi}\int_0^{2\pi}\frac{\partial f(s,r)}{\partial r}\left(\int_0^s f(\theta,r)d\theta\right)ds, \end{aligned} \tag{9}$$

with $z(r)=-\displaystyle\frac{1}{2\pi}\int_0^{2\pi}\left(\int_0^s f(\theta,r)d\theta\right)ds$. We denote the second averaged function by

$$A_2=f^{10}+g^0.$$

According with Corollary 4 in [6], every simple equilibrium point of system (8), that is, a simple zero of the function A_2 implies a limit cycle of system (1).

To prove Theorem 3 we shall need the following Lemma.

Lemma 6. *The following equalities hold.*

$$
\begin{aligned}
I_1 &= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{(1+r\cos\theta)^2}\, d\theta = \frac{1}{(1-r^2)^{3/2}},\\
I_2 &= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{1+r\cos\theta}\, d\theta = \frac{1}{\sqrt{1-r^2}},\\
I_3 &= \frac{1}{2\pi}\int_0^{2\pi} \cos\theta \ln(1+r\cos\theta)\, d\theta = \frac{1-\sqrt{1-r^2}}{r},\\
I_4 &= \frac{1}{2\pi}\int_0^{2\pi} \cos^3\theta \ln(1+r\cos\theta)\, d\theta = \frac{3r^2+2-2\sqrt{1-r^2}(1+2r^2)}{6r^3},\\
I_5 &= \frac{1}{2\pi}\int_0^{2\pi} \cos 2\theta \ln(1+r\cos\theta)\, d\theta = \frac{2\sqrt{1-r^2}+r^2-2}{2r^2},\\
I_6 &= \frac{1}{2\pi}\int_0^{2\pi} \cos 4\theta \ln(1+r\cos\theta) d\theta = -\frac{(4r^2-8)\sqrt{1-r^2}+r^4-8r^2+8}{4r^4}.
\end{aligned}
$$

Proof: The integrals can be computed by the Residue Theorem. See Lemma 8 in [6]. □

3. The averaged function of second order for $n = 3$

First we consider the case $n = 3$. We obtain easily the first averaged function

$$f^0 = \frac{p(r)\sqrt{1-r^2}+q(r)}{2r\sqrt{1-r^2}} = \frac{\phi(r)}{2r\sqrt{1-r^2}},$$

where

$$
\begin{aligned}
p(r) &= (\eta-\xi+3b_{0,3})r^2 - 2e_0,\\
q(r) &= 2r^4 b_{0,3} + 2(a_{1,0}-\eta-e_0-b_{0,3})r^2 + 2e_0,\\
e_0 &= a_{0,2}-a_{2,0}+b_{1,1}+b_{0,3}-b_{2,1}-a_{1,2}+a_{3,0}+a_{1,0}-b_{0,1},\\
\eta &= a_{2,0}-a_{3,0}, \quad \xi = -a_{0,2}-b_{1,1}+a_{1,2}+b_{2,1}.
\end{aligned}
$$

The number of zeros of $f^0(r) = 0$ is the same as for $\phi(r)$. Let $\rho = \sqrt{1-r^2}$, we get $\phi(r) = \hat{\phi}(\rho)$, where

$$\hat{\phi} = (\rho-1)(2\rho^3 b_{0,3} - (\eta-\xi+b_{0,3})\rho^2 + (2e_0+\xi+\eta-3b_{0,3}-2a_{1,0})\rho + 2(\eta-a_{1,0})).$$

Since $a_{1,0}, \xi, \eta$ and $b_{0,3}$ are arbitrary parameters, it is clear that there can exist three zeros of $\hat{\phi}(\rho)$ in $(0, 1)$, which means that three is the upper bound on the number of limit cycles in system (1) for $n = 3$, when we consider the averaged function up to the first order. This result can be referred to [4]. Moreover, we have the degenerated condition of (7) $e_0 = b_{0,3} = 0$ and $\eta = \xi = a_{1,0}$, i.e.

$$b_{0,3} = 0, a_{3,0} = a_{2,0} - a_{1,0}, b_{2,1} = b_{1,1} + a_{0,2} - a_{1,2} + a_{1,0}, b_{0,1} = -a_{1,0},$$

such that $p(r) \equiv 0, q(r) \equiv 0$. Therefore, system (1) is simplified into
(10)

$$\begin{array}{rcl}\dot{x} & = & -y(1+x)+\varepsilon(a_{1,0}x+a_{0,1}y+a_{2,0}x^2+a_{1,1}xy+a_{0,2}y^2+\\ & & (a_{2,0}-a_{1,0})x^3+a_{2,1}x^2y+a_{1,2}xy^2+a_{0,3}y^3),\\ \dot{y} & = & x(1+x)+\varepsilon(b_{1,0}x-a_{1,0}y+b_{2,0}x^2+b_{1,1}xy+b_{0,2}y^2+\\ & & b_{3,0}x^3+(a_{1,0}+a_{0,2}-a_{1,2}+b_{1,1})x^2y+b_{1,2}xy^2).\end{array}$$

In polar coordinates we have the corresponding function $f(\theta, r)$. If we write $\sin^2\theta = 1-\cos^2\theta$, it can be written as

$$f(\theta,r) = q_0 + q_1(\cos\theta, r) + q_2(\cos\theta, r)\sin\theta + q_3(r)\frac{\sin\theta}{1+r\cos\theta}, \tag{11}$$

where

$$\begin{array}{ll} q_0 & = a_{1,0}r\cos 2\theta,\ \ q_1 = r^2(-N\cos^3\theta + M\cos\theta),\\ q_2 & = e_2r^2\cos^2\theta + e_1r\cos\theta + (a_{2,1}+b_{3,0}-e_2)r^2 + e_4,\\ N & = 2a_{1,0}-a_{2,0}+a_{0,2}+b_{1,1},\ \ M = a_{0,2}+b_{1,1}+a_{1,0},\\ e_1 & = b_{2,0}+a_{1,1}-b_{0,2}+a_{0,3}-a_{2,1}-b_{3,0}+b_{1,2},\\ e_2 & = -a_{0,3}+a_{2,1}+b_{3,0}-b_{1,2},\ \ e_4 = a_{0,1}+b_{1,0}-e_1,\end{array}$$

and

$$q_3 = e_3r^2 + e_4, \quad e_3 = b_{0,2} - b_{1,2} - a_{0,3}.$$

Since $f^0(r) = 0$, we know that $q_1(\cos\theta, r)$ is always an odd function in the variable $\cos\theta$. By calculation we have

$$\int_0^t f(\theta,r)d\theta = \bar{q}_{01}(\cos t, r)\sin t + \bar{q}_2(\cos t, r) + q_3(r)I_0,$$

where

$$\begin{array}{rcl} I_0 & = & \displaystyle\int_0^t \frac{\sin\theta}{1+r\cos\theta}d\theta = \frac{\ln(1+r)-\ln(1+r\cos t)}{r},\\ \bar{q}_{01} & = & \displaystyle\frac{1}{\sin t}\int_0^t (q_0+q_1)d\theta = \frac{1}{3}r^2(N\cos^2 t - 2N + 3M) + a_{1,0}r\cos(t),\\ \bar{q}_2 & = & \displaystyle\int_0^t q_2(\cos\theta, r)\sin\theta d\theta.\end{array}$$

Hence, the first term appearing in the expression $A_2(r)$ is
(12)

$$f^{10} = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\partial(q_0+q_1)}{\partial r}(\bar{q}_2+q_3I_0)+(\frac{\partial q_2}{\partial r}+\frac{\partial}{\partial r}(\frac{q_3}{1+r\cos t}))\bar{q}_{01}\sin^2 t\right)dt.$$

If $q_1(\cos\theta, r) \equiv 0$, which implies $N = M = a_{1,0} = 0$, that is,

$$a_{1,0} = 0, \quad a_{2,0} = 0, \quad a_{0,2} + b_{1,1} = 0, \tag{13}$$

then $f^{10} \equiv 0$. Moreover, in this case we have

$$A_2(r) = \frac{1}{2\pi}\int_0^{2\pi} g(\theta, r)d\theta.$$

Now, we assume that $q_1(\cos\theta, r) \not\equiv 0$. Noting that $q_i, \bar{q}_i, i = 1, 2$ are trigonometric polynomials , we have by a direct calculation,

$$\frac{1}{2\pi}\int_0^{2\pi} (\frac{\partial(q_0+q_1)}{\partial r}\bar{q}_2 + \bar{q}_{01}\frac{\partial q_2}{\partial r}\sin^2 t)dt = \frac{1}{4}(3N-4M)e_4 r.$$

It is easy to know that the other integrals in f^{10} can be expressed as a combination of five basic integrals $I_i, i = 1, 2, \cdots, 5$ shown in Lemma 6. Substituting them into f^{10}, we can obtain

$$f^{10} = \frac{1}{12r^3\sqrt{1-r^2}}(f_1\sqrt{1-r^2} + g_1), \tag{14}$$

where

$$f_1 = m_4 r^4 + m_2 r^2 - 24(2a_{1,0}+N)e_4,\ g_1 = n_6 r^6 + n_4 r^4 + n_2 r^2 + 24(2a_{1,0}+N)e_4$$

with

$$\begin{aligned}
m_4 &= 3(3e_4+4e_3)N - 12(e_4+2e_3)M,\\
m_2 &= 12(3e_4-2e_3)M + 4(10e_3-3e_4)N + 12a_{2,0}(e_4+2e_3),\\
n_6 &= 4(2N-3M)e_3,\ n_4 = 8(e_3-3e_4)N + 12(e_3+3e_4)M) + 12a_{2,0}e_3,\\
n_2 &= -12(e_4-2e_3)M - 4(10e_3+6e_4)N - 12a_{2,0}(3e_4+2e_3).
\end{aligned}$$

Next we consider the function $g(\theta, r)$. We write $\sin^2\theta = 1 - \cos^2\theta$, and denote

$$g(\theta, r) = \frac{1}{r}H_1(\theta, r)H_2(\theta, r), \tag{15}$$

where

$$\begin{aligned}
H_1 &= \frac{\cos\theta F + \sin\theta G}{(1+r\cos\theta)^2} = \frac{f(\theta, r)}{1+r\cos\theta} = Q_1(\cos\theta, r)\sin\theta + Q_0(\cos\theta, r)\\
&= \left(q_{10}(\cos\theta, r) + \frac{q_{11}(r)}{1+r\cos\theta} + \frac{q_{12}(r)}{(1+r\cos\theta)^2}\right)\sin\theta + \\
&\qquad q_{00}(\cos\theta, r) + \frac{q_{01}(\cos\theta, r)}{1+r\cos\theta},
\end{aligned}$$

with

$$\begin{aligned}
q_{00} &= -rN\cos^2\theta + (N+2a_{1,0})\cos\theta + rM - (N+2a_{1,0})/r,\\
q_{01} &= -(M+a_{1,0})r + (N+2a_{1,0})/r,\\
q_{10} &= e_2 r\cos\theta - e_2 + e_1,\\
q_{11} &= (b_{0,2}-e_3)r^2 + e_4 + e_2 - e_1,\\
q_{12} &= e_3 r^2 - e_4,
\end{aligned}$$

and

$$H_2 = \sin\theta F - \cos\theta G = P_0(\cos\theta, r) + P_1(\cos\theta, r)\sin\theta,$$

where

$$\begin{aligned} P_0 &= e_2 r^3 \cos^4\theta + (e_1+e_2) r^2 \cos^3\theta + a_1\cos^2\theta - e_5 r^2\cos\theta - a_0, \\ P_1 &= N r^3\cos^3\theta + (N - 2a_{1,0}) r^2\cos^2\theta - (a_{1,2} r^3 + 2a_{1,0} r)\cos\theta - a_{0,2} r^2, \end{aligned}$$

with

$$\begin{aligned} a_0 &= a_{0,3} r^3 + a_{0,1}, \\ a_1 &= e_6 r^3 + (e_1+e_4) r, \\ e_6 &= b_{3,0} + a_{0,3} - e_2, \\ e_5 &= a_{1,1} - b_{0,2}. \end{aligned}$$

Hence from (8) the average of the function $g(\theta, r)$ is

$$g^0 = \frac{1}{2\pi}\int_0^{2\pi} g(\theta, r) d\theta = \frac{1}{2\pi r}\int_0^{2\pi} (P_0 Q_0 + P_1 Q_1 \sin^2\theta) d\theta. \tag{16}$$

The function g^0 can be expressed as a combination of the basic integrals I_1 and I_2 shown in Lemma 6. Substituting them into g^0 and taking into account the parity of $P_i, Q_i, i = 0, 1$ in the variable $\cos\theta$, we can obtain

$$g^0 = -\frac{1}{8r^3\sqrt{1-r^2}}(f_2(r)\sqrt{1-r^2} + g_2(r)) = \frac{1}{8r^3\sqrt{1-r^2}}\phi(r), \tag{17}$$

where

$$\begin{aligned} f_2 &= (t_{6m} M + t_{6n} N + t_{60}) r^6 + (t_{4m} M + t_{4n} N + t_{40}) r^4 + \\ &\quad (8Me_4 + t_{2n} N + t_{2a} a_{1,0} + t_{20}) r^2 - 16(2a_{1,0} + N) e_4 \\ t_{6m} &= (3e_2 - 8a_{0,3} + 4e_6), \\ t_{6n} &= (-e_3 - 2e_2 - 3e_6 + 4a_{0,3} + b_{0,2}), \\ t_{60} &= -(4b_{0,2} - 4e_3 + e_2) a_{1,2}, \\ t_{4m} &= (4e_4 + 8e_5 - 8a_{0,1} + 8e_6), \\ t_{4n} &= (-2e_4 - 4e_3 + 4a_{0,1} - 4e_5 + 8a_{0,3} - 4e_6), \\ t_{40} &= (-8e_3 + 16a_{0,3}) a_{1,0} + (16e_3 - 8b_{0,2} + 4e_2 - 4e_1) a_{0,2} + 4a_{1,2} t_{4a}, \\ t_{4a} &= (-6e_3 + 2b_{0,2} + e_1 - e_4 - e_2), \\ t_{2n} &= (8a_{0,1} - 8e_5 + 8e_3 - 8e_6), \\ t_{2a} &= (-16e_6 - 16e_5 + 8e_4 + 16a_{0,1} + 16e_3), \\ t_{20} &= (-16e_4 + 8e_1 - 8e_2) a_{0,2} - 8a_{1,2}(e_1 - e_2 - 3e_4), \end{aligned}$$

and

$$
\begin{aligned}
g_2 &= (8Ma_{0,3}+s_{60})r^6+(s_{4m}M+s_{4n}N+s_{40})r^4+\\
&\quad(-8Me_4+s_{2n}N+s_{2a}a_{1,0}+s_{20})r^2+16(a_{1,0}+N)e_4\\
s_{60} &= 8a_{1,0}a_{0,3}+(8e_3-8b_{0,2})a_{0,2}+8a_{1,2}(-2e_3+b_{0,2}),\\
s_{4m} &= (8a_{0,1}-8e_6-8e_5),\\
s_{4n} &= -8a_{0,3}+8e_3\\
s_{40} &= s_{4a}a_{1,0}+s_{4b}a_{0,2}-8a_{1,2}(e_1-3e_3+b_{0,2}-2e_4-e_2),\\
s_{4a} &= (8a_{0,1}+16e_3-8e_5-8e_6-16a_{0,3}),\\
s_{4b} &= (8e_1-8e_4+8b_{0,2}-8e_2-16e_3),\\
s_{2n} &= (-8e_4-8a_{0,1}+8e_5-8e_3+8e_6),\\
s_{20} &= (8e_2-8e_1+16e_4)a_{0,2}+8a_{1,2}(e_1-e_2-3e_4),\\
s_{2a} &= (16e_6-16a_{0,1}-16e_3+16e_5-24e_4).
\end{aligned}
$$

The functions f_2 and g_2 are two even polynomials of degree 6 in the variable r. In the view of (14) and (17), we get

$$
A_2=\frac{(2f_1+3f_2)\sqrt{1-r^2}+2g_1+3g_2}{24r^3\sqrt{1-r^2}}=\frac{\Phi(r)}{24r^3\sqrt{1-r^2}}. \tag{18}
$$

4. Proof of Theorems 4

Using the second averaging theory, it is easy from formula (18) to know that the number of simple zeros of $\Phi(r)$ in $(0,1)$ is equal to the number of limit cycles bifurcating form system (1) considering up to second order averaging function.

Proof of statement (a) of Theorem 4 : Condition (13) implies that $a_{2,0}=M=N=0$. So, $f^{10}\equiv 0$ and we have the function

$$
A_2=g^0=-\frac{(\rho-1)^2}{8\rho\sqrt{1-\rho^2}}\hat{\phi}(\rho),
$$

where

$$
\begin{aligned}
\hat{\phi}(\rho) \quad = \quad & a_{1,2}(4b_{0,2}-4e_3+e_2)\rho^3+(8(b_{0,2}-e_3)a_{0,2}+2(e_2+4e_3)a_{1,2})\rho^2+\\
& (4(2b_{0,2}+e_2-e_1)a_{0,2}+(4e_3-3e_2-4e_4-4b_{0,2}+4e_1)a_{1,2})\rho+\\
& 8(e_3-e_4)a_{0,2}+8(e_4-e_3)a_{1,2}.
\end{aligned}
$$

Hence by the second averaging theory, system (1) has at most 3 limit cycles. Moreover, note that the following facts,

(i) $e_i, i=1,2,3,4$ and $b_{0,2}$ can take arbitrary values because they are formed by different coefficients $a_{i,j}$ and $b_{i,j}$;
(ii) $a_{0,2}$ and $a_{1,2}$ are not included in $e_i, i=1,2,3,4$.

Thus, $b_{0,2}, e_2, e_1$ and e_3 can allow to choose the coefficients of $\hat{\phi}(\rho)$ arbitrarily. Hence $\varphi(r)$ can have at most 3 zeros in $(0,1)$. Hence system (1) can have at most 3 limit cycles. This completes the proof of statement (a). □

For example we can take the system

$$\begin{aligned}\dot{x} &= -y(1+x)+\varepsilon(\frac{1}{2}y^2+xy^2+\frac{43}{6}y^3),\\ \dot{y} &= x(1+x)+\varepsilon(-\frac{45}{4}x^2-\frac{1}{2}xy-\frac{13}{4}y^2-\frac{21}{2}x^3-x^2y),\end{aligned}$$

which satisfies the degenerated condition (13), $f^0=f^{10}=0$ and

$$\hat{\phi}(\rho)=32(\rho-\frac{1}{4})(\rho-\frac{1}{2})(\rho-\frac{3}{4}).$$

Hence, this system has at least 3 limit cycles near $\rho=1/4$, $\rho=1/2$ and $\rho=3/4$ for ε small by the averaging theory of second order.

Proof of statement (b) of Theorem 4 : If we write $\sqrt{1-r^2}=\rho$, we can know that

$$A_2=f^{10}+g^0=\frac{(\rho-1)^3(\rho+1)}{24\rho(1-\rho^2)^{3/2}}\hat{\Phi}(\rho), \tag{19}$$

where $\hat{\Phi}(\rho)$ is a polynomial of degree 3 in the variable ρ. We are only interested in the zeros of $A_2(\rho)$ with $\rho\in(0,1)$. Note that f^{10} and g^0 are given in (14) and (17), respectively. More exactly,

$$\begin{aligned}\hat{\Phi}(\rho) &= (3c_{3,1}a_{1,0}+3c_{3,2}a_{2,0}+3c_{3,3}(a_{0,2}+b_{1,1})+3c_{3,4}a_{1,2})\rho^3+\\ &\quad (c_{2,1}a_{1,0}+c_{2,2}a_{2,0}+c_{2,3}a_{0,2}+c_{3,4}a_{1,2}+2c_{2,5}b_{1,1})\rho^2+\\ &\quad (c_{1,1}a_{1,0}+c_{1,2}a_{2,0}+c_{1,3}a_{0,2}+c_{1,4}a_{1,2}+c_{1,5}b_{1,1})\rho+\\ &\quad 24(2a_{1,0}-a_{2,0})c_{0,1}+(a_{1,2}-a_{0,2})c_{0,2},\end{aligned}$$

where

$$
\begin{aligned}
c_{3,1} &= -a_{0,3} + b_{1,2} + a_{2,1} - b_{3,0},\\
c_{3,2} &= -a_{2,1} + 2b_{3,0} - a_{0,3},\\
c_{3,3} &= -2a_{0,3} + b_{1,2} + b_{3,0},\\
c_{3,4} &= -a_{2,1} - 3b_{1,2} - 3a_{0,3} - b_{3,0},\\
c_{2,1} &= 50a_{0,3} + 14b_{1,2} - 8b_{0,2} + 6a_{2,1} - 6b_{3,0},\\
c_{2,2} &= -6a_{2,1} + 16b_{0,2} - 16b_{1,2} - 22a_{0,3} + 12b_{3,0},\\
c_{2,3} &= 8b_{0,2} - 20a_{0,3} - 26b_{1,2} + 6b_{3,0},\\
c_{2,5} &= 2a_{0,3} + 4b_{0,2} + 3b_{3,0} - b_{1,2},\\
c_{2,4} &= -24b_{0,2} + 30b_{1,2} - 6a_{2,1} + 30a_{0,3} - 6b_{3,0},\\
c_{1,1} &= 3a_{2,1} - 47a_{0,3} + 12b_{1,0} + 44b_{0,2} - 41b_{1,2} + 9b_{3,0} - 12b_{2,0} - 12c_{0,1},\\
c_{1,2} &= 12c_{0,1} - 3a_{2,1} + 25a_{0,3} - 24b_{1,0} - 18b_{3,0} - 16b_{0,2} + 24b_{2,0} + 16b_{1,2},\\
c_{1,3} &= -33b_{3,0} + 24b_{2,0} + 26a_{0,3} - 24a_{2,1} - 32b_{0,2} + 12a_{1,1} - 12b_{1,0} + 23b_{1,2},\\
c_{1,4} &= 24b_{0,2} - 24b_{2,0} + 33a_{2,1} - 24a_{1,1} - 21a_{0,3} + 33b_{3,0} - 21b_{1,2} + 12m_0,\\
m_0 &= a_{0,1} + b_{1,0},\\
c_{1,5} &= -b_{1,2} + 2a_{0,3} - 12b_{1,0} + 4b_{0,2} + 12b_{2,0} - 9b_{3,0},\\
c_{0,1} &= a_{1,1} - a_{2,1} - a_{0,1},\\
c_{0,2} &= a_{1,1} + b_{2,0} - a_{0,1} - a_{2,1} - b_{1,0} - b_{3,0}.
\end{aligned}
$$

Hence the second averaged function can have at most 3 zeros in $(0,1)$. Next, we take $a_{1,0} = a_{2,0} = a_{1,2} = b_{1,1} = 0$. Then we have

$$\hat{\Phi}(\rho) = a_{0,2}(3c_{3,3}\rho^3 + c_{2,3}\rho^2 + c_{1,3}\rho - c_{0,2}). \tag{20}$$

It is easy to check that there are enough coefficients $a_{i,j}$ and $b_{i,j}$ such that the coefficients $c_{3,3}$, $c_{2,3}$, $c_{1,3}$ and $c_{0,2}$ of function (20) are arbitrary. □

For example we can take the system

$$
\begin{aligned}
\dot{x} &= -y(1+x) + \varepsilon(\frac{937}{24}y + y^2 + 14y^3),\\
\dot{y} &= x(1+x) + \varepsilon(-\frac{467}{6}x + x^2y + \frac{116}{3}x^3).
\end{aligned}
$$

Then formula (19) becomes

$$\hat{\Phi}(\rho) = 32(\rho - \frac{1}{4})(\rho - \frac{1}{2})(\rho - \frac{3}{4}).$$

Hence, this system has at least 3 limit cycles near $\rho = 1/4$, $\rho = 1/2$ and $\rho = 3/4$ for ε small by the averaging theory of second order.

5. Proofs of Theorems 2 and 3

Proof of Theorem 2 : For $n = 1$ we can obtain easily the first averaged function

$$f^0 = \frac{(b_{0,1} - a_{1,0})\sqrt{1-r^2} + b_{0,1}r^2 + a_{1,0} - b_{0,1}}{r\sqrt{1-r^2}}.$$

Writing $\rho = \sqrt{1-r^2}$, we have

$$f^0 = \frac{1-\rho}{\rho\sqrt{1-\rho^2}}(b_{0,1}\rho + a_{1,0}).$$

It is clear that f^0 has a unique root in $(0,1)$, when we consider the averaging theory of first order. Then if $b_{0,1} = a_{1,0} = 0$, we need to consider the second averaged function $A_2(r)$. We say $A_2(r) \equiv 0$, because the singular point $(0,0)$ is a center having the first integral

$H(x, y) =$

$$= \frac{x^2+y^2}{2} + \varepsilon(a_{0,1}+b_{1,0})x + \varepsilon(\varepsilon a_{0,1}^2 + \varepsilon a_{0,1}b_{1,0} - b_{1,0} - a_{0,1})\ln(1 + x - \varepsilon a_{0,1}).$$

In fact Theorem 2 is proved. □

For $n = 2$ the first averaged function is

$$f^0 = \frac{(r^2\xi + 2(\beta - \xi + 2\alpha))\sqrt{1-r^2} + 2(\beta - \xi + \alpha)r^2 + 2(\xi - \beta - 2\alpha)}{2r\sqrt{1-r^2}},$$

where $\alpha = a_{2,0} - a_{1,0}$, $\beta = b_{0,1} + a_{1,0}$ and $\xi = a_{0,2} + a_{2,0} + b_{1,1}$. Taking $\sqrt{1-r^2} = \rho$, we have

$$f^0 = \frac{1-\rho}{2\rho\sqrt{1-\rho^2}}(\xi\rho^2 + (2\beta - \xi + 2\alpha)\rho - 2\alpha).$$

Since α, β and ξ are arbitrary parameters, it is obvious that f^0 can have at most 2 zeros in $(0,1)$, which means that system (1) for $n = 2$ can have at most two limit cycles by averaging theory of first order. Moreover, we consider the averaged function up to second order. From the degenerated condition of (7), we have $\alpha = \beta = \xi = 0$, i.e.

$$a_{1,0} = a_{2,0}, \quad b_{0,1} = -a_{2,0}, \quad b_{1,1} = -(a_{2,0} + a_{0,2}).$$

Hence, system (1) is simplified into

(21)
$$\begin{aligned} \dot{x} &= -y(1+x) + \varepsilon(a_{2,0}x + a_{0,1}y + a_{2,0}x^2 + a_{1,1}xy + a_{0,2}y^2) \\ \dot{y} &= x(1+x) + \varepsilon(b_{1,0}x - a_{2,0}y + b_{2,0}x^2 - (a_{0,2} + a_{2,0})xy + b_{0,2}y^2). \end{aligned}$$

Proof of statement of Theorem 3 : From (11) we get the function $f(\theta, r)$ with

$$q_0 = a_{1,0}r\cos 2\theta, \; q_1 = 0, \; q_2 = \eta r\cos\theta - \gamma, \; q_3 = b_{0,2}r^2 + \gamma$$

where

$$\eta = a_{1,1} + b_{2,0} - b_{0,2},\ \ \delta = b_{1,0} + a_{0,1},\ \ \gamma = \eta - \delta.$$

Then, by formula (12), we can obtain

$$f^{10} = \frac{a_{2,0}}{r^3\sqrt{1-r^2}}(((2b_{0,2}-\gamma)r^2+4\gamma)\sqrt{1-r^2}+r^4b_{0,2}+(3\gamma-2b_{0,2})r^2-4\gamma).$$

If $q_1(\cos(\theta, r) \equiv 0$, i.e. $a_{2,0} = 0$, we have $f^{10} \equiv 0$. Then $A_2 = g^0$. In the same way as before, we have $rg(\theta, r) = H_1H_2$, where

$$H_1 = \frac{f(\theta, r)}{1 + r\cos\theta} = Q_0(\cos\theta, r) + Q_1(\cos\theta, r)\sin\theta,$$

with

$$Q_0 = \frac{q_0 + q_1}{1 + r\cos\theta},\ \ Q_1 = \eta - \frac{\eta + \gamma}{1 + r\cos\theta} + \frac{q_3}{(1 + r\cos\theta)^2},$$

and

$$H_2 = \cos\theta G - \sin\theta F = P_0(\theta, r) + P_1(\theta, r)\sin\theta$$

with

$$\begin{aligned} P_0 =&\ \ \eta r^2\cos^3\theta + \delta r\cos^2\theta + (b_{2,0} - \eta)r^2\cos\theta + (b_{1,0} - \delta)r, \\ P_1 =&\ \ -(2a_{2,0}r^2\cos^2\theta + 2a_{2,0}r\cos\theta + a_{0,2}r^2). \end{aligned}$$

Applying formula (16) and the integrals I_1, I_2 and I_5 of Lemma 6, we get

$$g^0 = -\frac{1}{4r^3\sqrt{1-r^2}}((t_4r^4 + t_2r^2 + t_0)\sqrt{1-r^2} + s_4r^4 + s_2r^2 + s_0),$$

where

$$\begin{aligned} t_4 &= (2b_{0,2} - \eta)a_{0,2} - 2b_{0,2}a_{2,0}, \\ t_2 &= (6\gamma + 2\delta)a_{0,2} - 2a_{2,0}(3\gamma - 2b_{0,2} + 2b_{1,0} - 2b_{2,0})), \\ s_4 &= (2\delta + 4\gamma - 2b_{0,2})a_{0,2} - 2a_{2,0}(b_{1,0} - 2b_{0,2} - b_{2,0}), \\ s_2 &= -(6\gamma + 2\delta)a_{0,2} + 2a_{2,0}(2b_{1,0} - 2b_{0,2} - 2b_{2,0} + 5\gamma), \\ t_0 &= -s_0 = 8a_{2,0}\gamma. \end{aligned}$$

If $a_{2,0} = 0$, taking $\rho = \sqrt{1-r^2}$ we easily get

$$A_2 = g^0 = \frac{a_{0,2}(\rho - 1)^2}{2\rho\sqrt{1-\rho^2}}((2b_{0,2} - \eta)\rho + 2(\gamma + b_{0,2})). \tag{22}$$

Since $b_{0,2}, \eta$ and γ can be chosen arbitrarily, the function A_2 may have at most 1 zeros in $(0, 1)$. If $a_{2,0} \neq 0$, the second averaged function becomes

$$A_2 = \frac{(\rho-1)^2}{2\rho\sqrt{1-\rho^2}}((a_{0,2}(2b_{0,2}-\eta)-2b_{0,2}a_{2,0})\rho-2(a_{1,1}+b_{1,0})a_{2,0}+2a_{0,2}(\gamma+b_{0,2})).$$

It is easy to check that $a_{2,0}$, $b_{0,2}$, η and γ are arbitrary parameters. Hence A_2 can have also at most one zeros in $(0, 1)$. □

6. Proofs of Theorem 5

For $n = 4$ we can obtain easily the first averaged function

$$f^0 = \frac{p_4(r)\sqrt{1-r^2} + q_4(r)}{8r\sqrt{1-r^2}} = \frac{\phi(r)}{8r\sqrt{1-r^2}},$$

where p_4 and q_4 are two even polynomials in r of degree 4. Writing $\rho = \sqrt{1-r^2}$, we get $\phi(r) = \hat{\phi}(\rho)$, where

$$\begin{aligned}
\hat{\phi} &= (\rho - 1)(c_0\rho^4 + c_1\rho^3 + c_2\rho^2 + c_3\rho - c_4),\\
c_0 &= 3a_{0,4} + a_{2,2} + 3a_{4,0} + b_{3,1} + 3b_{1,3},\\
c_1 &= a_{2,2} - 5b_{1,3} + b_{3,1} + 8b_{0,3} - 5a_{0,4} + 3a_{4,0},\\
c_2 &= 4b_{2,1} + 4a_{1,2} - 4b_{1,1} - 5b_{3,1} + b_{1,3} - 4b_{0,3} - 4a_{2,0} -\\
&\quad 7a_{4,0} + 4a_{3,0} + a_{0,4} - 5a_{2,2} - 4a_{0,2},\\
c_3 &= (-4b_{0,3} - 4b_{2,1} + b_{1,3} - 7a_{4,0} + 3a_{2,2} + 3b_{3,1} - 8b_{0,1} +\\
&\quad 4b_{1,1} + 4a_{0,2} + a_{0,4} - 4a_{2,0} - 4a_{1,2} + 4a_{3,0}),\\
c_4 &= 8a_{1,0} + 8a_{2,0} - 8a_{3,0} + 8a_{4,0}.
\end{aligned}$$

Since the coefficients c_i are arbitrary for $i = 0, 1, \cdots, 4$, the function $\hat{\phi}(\rho)$ can have at most 4 zeros in $(0, 1)$, which means that system (1) for $n = 4$ can have at most four limit cycles by averaging theory of first order. Moreover, if the first averaged function vanishes, we get the degenerated condition (4) from $\hat{\phi}(\rho) \equiv 0$. Using (4) system (1) can be simplified into

(23)

$$\begin{aligned}
\dot{x} &= -y(1+x) + \varepsilon(a_{1,0}x + a_{0,1}y + (b_{2,1} + (-a_{4,0} + a_{1,0} + a_{3,0})x^2 +\\
&\quad a_{1,1}xy + a_{1,2} - b_{1,1} + 3b_{0,3} + 3a_{4,0} - a_{1,0})y^2 + a_{3,0}x^3 +\\
&\quad a_{1,2}xy^2 + a_{2,1}x^2y + a_{0,3}y^3 + (-b_{1,3} + a_{4,0}x^4 + a_{1,3}xy^3 +\\
&\quad (-3b_{0,3} - 3a_{4,0} - b_{3,1})x^2y^2 + a_{3,1}x^3y + b_{0,3})y^4),\\
\dot{y} &= x(1+x) + \varepsilon(b_{1,0}x - a_{1,0}y + b_{2,0}x^2 + b_{1,1}xy + b_{0,2}y^2 + b_{3,0}x^3 +\\
&\quad b_{1,2}xy^2 + b_{2,1}x^2y + b_{0,3}y^3 + b_{4,0}x^4 + b_{1,3}xy^3 + b_{2,2}x^2y^2 +\\
&\quad b_{3,1}x^3y + b_{0,4}y^4).
\end{aligned}$$

From (11) the function $f(\theta, r)$ is given by

$$\begin{aligned}
q_0 &= \frac{1}{2}(b_{0,3} + a_{4,0})r^3\cos 4\theta + \frac{1}{2}(a_{4,0} - b_{0,3})r^3\cos 2\theta + a_{1,0}r\cos 2\theta,\\
q_1 &= (a_{3,0} - a_{4,0} - n_5)r^2\cos^3\theta + n_5 r^2\cos\theta,\\
q_2 &= n_3 r^3\cos^3\theta + (n_0 + n_4)r^2\cos^2\theta +\\
&\quad (n_1 r^3 + (b_{1,0} + a_{0,1} - n_0)r)\cos\theta + n_2 r^2 + n_0,\\
q_3 &= b_{0,4}r^4 + (b_{0,2} - n_2)r^2 - n_0,
\end{aligned}$$

where

$$
\begin{aligned}
n_0 &= a_{2,1} - a_{0,3} - b_{1,2} + b_{3,0} - n_3 - n_4, \\
n_1 &= a_{1,3} - 2b_{0,4} + b_{2,2}, \\
n_2 &= b_{1,2} - a_{1,3} - b_{2,2} + a_{0,3} + 2b_{0,4}, \\
n_3 &= -a_{1,3} + a_{3,1} - b_{2,2} + b_{4,0} + b_{0,4}, \\
n_4 &= a_{1,1} - b_{0,2} + b_{2,0} - a_{0,1} - b_{1,0}, \\
n_5 &= 3b_{0,3} + b_{2,1} + a_{1,2} + 3a_{4,0}.
\end{aligned}
$$

Moreover, from (15) we rewrite the function $g(\theta, r)$ into the form $rg(\theta, r) = H_1 H_2$, where

$$
\begin{aligned}
H_1 &= \frac{f(\theta, r)}{1 + r\cos\theta} = Q_1(\cos\theta, r)\sin\theta + Q_0(\cos\theta, r) \\
&= \left(q_{10} + \frac{q_{11}}{1 + r\cos\theta} + \frac{q_{12}}{(1 + r\cos\theta)^2}\right)\sin\theta + q_{00} + \frac{q_{01}}{1 + r\cos\theta}, \\
H_2 &= \cos\theta G - \sin\theta F = P_0(\cos\theta, r) + P_1(\cos\theta, r)\sin\theta.
\end{aligned}
$$

After computing, we have that

$$
\begin{aligned}
q_{00} &= 4(b_{0,3} + a_{4,0})r^2\cos^3\theta + (\zeta + 2a_{1,0})r\cos^2\theta - \\
&\quad ((3a_{4,0} + 5b_{0,3})r^2 + \zeta)\cos\theta + (3a_{4,0} + 5b_{0,3} + n_5)r + \frac{\zeta}{r}, \\
q_{01} &= \frac{1}{r}(b_{0,3}r^4 - (3a_{4,0} + n_5 + 5b_{0,3} + a_{1,0})r^2 - \zeta), \\
q_{10} &= n_3 r^2\cos^2\theta + \sigma r\cos\theta + n_1 r^2 + b_{1,0} + a_{0,1} + \sigma - n_0, \\
q_{11} &= (n_2 - n_1)r^2 + 2n_0 + \sigma - b_{1,0} - a_{0,1}, \\
q_{12} &= b_{0,4}r^4 + (b_{0,2} - n_2)r^2 - n_0, \\
\zeta &= a_{3,0} - 5a_{4,0} - n_5 - 4b_{0,3} - 2a_{1,0}, \\
\sigma &= n_0 - n_3 + n_4,
\end{aligned}
$$

and

$$
\begin{aligned}
P_0 &= (m_3 + m_2 - b_{4,0})r^4\cos^5\theta - (a_{0,3} + b_{3,0} + m_4)r^3\cos^4\theta - \\
&\quad ((m_3 + 2m_2)r^4 + (m_1 + b_{2,0})r^2)\cos^3\theta + (r^4 m_2 + r^2 m_1)\cos\theta + \\
&\quad (m_4 r^3 - (a_{0,1} + b_{1,0})r)\cos^2\theta + a_{0,3}r^3 + r a_{0,1} \\
P_1 &= 4s_0 r^4\cos^4\theta + s_1 r^3\cos^3\theta + (s_2 r^4 + (s_1 - 4s_0 + 2a_{1,0})r^2)\cos^2\theta + \\
&\quad (r^3(a_{1,2} - b_{0,3}) + 2a_{1,0}r)\cos\theta + (-b_{1,3} + b_{0,3})r^4 + \\
&\quad (a_{3,0} - s_1 - b_{1,1} + b_{0,3} + 3s_0 - a_{1,0})r^2,
\end{aligned}
$$

where

$$\begin{aligned} s_0 &= b_{0,3}+a_{4,0},\\ s_1 &= a_{3,0}+b_{0,3}-a_{1,2}-b_{2,1},\\ s_2 &= b_{1,3}-b_{3,1}-5b_{0,3}-3a_{4,0},\\ m_1 &= a_{1,1}-b_{0,2},\\ m_2 &= a_{1,3}-b_{0,4},\\ m_3 &= b_{2,2}-a_{3,1},\\ m_4 &= a_{2,1}-b_{1,2}-2a_{0,3}. \end{aligned}$$

Again applying formula (16) and the integrals I_1, I_2, I_5 and I_6 of Lemma 6, we obtain the averaged function of second order with respect to system (23), more precisely

$$A_2 = \frac{f_3\sqrt{1-r^2}+g_3}{96r^3\sqrt{1-r^2}} = \frac{\Phi(r)}{96r^3\sqrt{1-r^2}}, \tag{24}$$

where f_3 and g_3 are two even polynomials in r of degree 8. In order to evaluate the number of zeros of function (24), we write $r=\sqrt{1-\rho^2}$, and have $\Phi(r)=\hat{\phi}(\rho)$, with

$$\hat{\phi} = (\rho-1)^2\varphi(\rho), \tag{25}$$

where φ is a polynomial in ρ of degree 7. Hence, system (23) can have at most 7 limit cycles by averaging theory of second order. This number coincides with the upper bound $2n-1$ given by $n=4$ in [6]. The functions Φ and φ can be computed by Maple, we omit them here because they are very large. We claim that the upper bound 7 can be attained doing an example.

For example we can take the system

$$\begin{aligned} \dot{x} &= -y(1+x)+\varepsilon(-x-\tfrac{4052055}{608}y-2x^2+\tfrac{831173039}{20064}xy-\tfrac{165746021}{10032}y^3\\ &\qquad -6xy^2+x^4-\tfrac{964825019}{20064}x^3y-\tfrac{30137463}{3344}xy^3),\\ \dot{y} &= x(1+x)+\varepsilon(y+xy-\tfrac{17206151}{3344}y^2+\tfrac{13712275}{1672}x^3+y^3-6x^3y-\\ &\qquad \tfrac{270361841}{10032}x^2y^2+xy^3-\tfrac{65009403}{6688}y^4). \end{aligned}$$

Then it is easy to check that function $\varphi(\rho)$ in formula (23) is

$$\varphi = 131072(\rho-\frac{1}{8})(\rho-\frac{2}{8})(\rho-\frac{3}{8})(\rho-\frac{4}{8})(\rho-\frac{5}{8})(\rho-\frac{6}{8})(\rho-\frac{7}{8}).$$

Hence, this system has at least 7 limit cycles near $\rho=1/8$, $\rho=2/8$, $\cdots$, and $\rho=7/8$ for ε small by the averaging theory of second order.

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