ON THE UPPER BOUND OF THE NUMBER OF LIMIT CYCLES OBTAINED BY THE SECOND ORDER AVERAGING METHOD I

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ABSTRACT. For ε small we consider the number of limit cycles of the system $\dot{x} = -y(1+x)+\varepsilon F(x,y)$, $\dot{y} = x(1+x)+\varepsilon G(x,y)$, where F and G are polynomials of degree n starting with terms of degree 1. We prove that at most $2n-1$ limit cycles can bifurcate from the periodic orbits of the unperturbed system ($\varepsilon = 0$) using the averaging theory of second order.

1. Introduction

This paper is concerned with the number of limit cycles that can bifurcate from the periodic orbits of a class of planar quadratic systems under small polynomial perturbation of degree $n \in \mathbb{N}$. We assume that the unperturbed system is the linear center with a straight line of singular points. More explicitly, we consider the two dimensional polynomial differential system

$$\begin{array}{rcl} \dot{x} & = & -y(1+x)+\varepsilon F(x,y), \\ \dot{y} & = & x(1+x)+\varepsilon G(x,y), \end{array} \tag{1}$$

where F and G are polynomials of degree n starting with terms of degree 1. We note that system (1) for $\varepsilon = 0$ is not Hamiltonian.

One often analyze the number of limit cycles bifurcating from a center by the first return map,

$$\mathcal{P}(h,\varepsilon) - h = \varepsilon M_1(h) + \varepsilon^2 M_2(h) + \cdots + \varepsilon^k M_k(h) + \cdots ,$$

where $M_k(h)$ is called the k–order Poincaré–Pontryagin function (also called Melnikov function). If $M_k \not\equiv 0$, and $M_i \equiv 0$ for $i = 1,2,\cdots,k-1$ in some open segments, then the maximum number of simple zeros of $M_k(h)$ give an upper bound of the number of limit cycles up to k order. For example, many authors give using the first Poincaré–Pontryagin function see [6], (i.e. the Abelian integral) linear estimations in n on the number

1991 *Mathematics Subject Classification.* 37G15, 37D45.

Key words and phrases. limit cycle, averaging theory, polynomial differential system.

of limit cycles of degree n polynomial perturbations of different class of polynomial differential systems. Also many authors give the exact upper bound for some degree 2 or 3 perturbations of some systems, see [1], [5] and the references therein. In [3] the authors study degree n polynomial perturbations of quadratic reversible Hamiltonian systems with one center and one saddle point using the second Poincaré–Pontryagin function, and obtain that the exact upper bound is $2(n-1)$. But there are few results for degree n polynomial perturbations.

Another method used often is the averaging theory, see [2] and [7]–[12]. There are many works involved in the averaging method of first order, for example in [8] the authors obtain for system (1) that at most n limit cycles can bifurcate from the periodic orbits of the center with the averaging theory of first order, and that this number is realizable for convenient polynomial F and G of degree n.

Using the averaging theory, if the first averaged function vanishes, the number of limit cycles of perturbed systems depends on the second averaged function. In general, the number of limit cycles will increase double for such more degenerate case.

In this paper we study system (1) by the averaging method of second order, and we have the following result.

Theorem 1. *Applying the averaging theory of second order to system* (1) *with F and G polynomials of degree n, we can obtain at most $2n-1$ limit cycles bifurcating from the periodic orbits of the center of system* (1) *for* $\varepsilon = 0$.

In [10] which is a natural continuation of this paper, applying again the averaging method of second order, the authors prove that the exact upper bound on the number of limit cycles of system (1) vanishing the first averaged function zero is $2n-1$ if the degree of the perturbation is $n=4$, while the number are $0, 1$ and 3 for $n = 1, 2$ and 3, respectively. So the upper bound given in Theorem 1 is, in general, the best possible.

This paper is organized as follows. In Section 2 first we recall some fundamental results on averaging theory. After we give some basic results. The main results of this paper is to find the expression of the second order averaged function (introduced in Section 2) and to bound the number of its zeros. In Section 3 we compute the averaged function. In Section 4 we prove Theorem 1 and we do some remarks.

2. Basic results

This section is divided into

2.1. **The averaging theory.** In this section we summarize the main results on the theory of averaging that we will apply to systems (1). The next theorem provides a first order approximation for the periodic solutions of a periodic differential system, for a proof see Theorem 2.6.1 of Sanders Verhulst [11] and Theorems 11.5 and 11.6 of Verhulst [12]. The original theorems are given for a system of differential equations, but since we will use them only for one differential equation we state them in this case.

Theorem 2. *We consider the following two initial value problems*

$$\dot{x} = \varepsilon f(t,x) + \varepsilon^2 g(t,x,\varepsilon) , \qquad x(0) = x_0 , \tag{2}$$

and

$$\dot{y} = \varepsilon f^0(y) , \qquad y(0) = x_0 , \tag{3}$$

x, y, $x_0 \in D$ an open subset of $\mathbb{R}$, $t \in [0,\infty)$, $\varepsilon \in (0,\varepsilon_0]$, f and g are periodic of period T in the variable t, and

$$f^0(y) = \frac{1}{T}\int_0^T f(t,y)dt . \tag{4}$$

Suppose

(i) *f, $\partial f/\partial x$, $\partial^2 f/\partial x^2$, g and $\partial g/\partial x$ are defined, continuous and bounded by a constant independent on ε in $[0,\infty)\times D$ and $\varepsilon \in (0,\varepsilon_0]$;*
(ii) *T is independent on ε;*
(iii) *$y(t)$ belongs to D on the time–scale $\frac{1}{\varepsilon}$.*

Then the following statements hold.

(a) *On the time scale $\frac{1}{\varepsilon}$ we have that*
$$x(t) - y(t) = O(\varepsilon) ,$$
as $\varepsilon \to 0$.
(b) *If p is an equilibrium point of the averaged system (3) such that*
$$\left.\frac{\partial f^0}{\partial y}\right|_{y=p} \neq 0 , \tag{5}$$
then there exists a T–periodic solution $\phi(t,\varepsilon)$ of equation (2) which is close to p such that $\phi(t,\varepsilon) \to p$ as $\varepsilon \to 0$.
(c) *If (5) is negative, then the corresponding periodic solution $\phi(t,\varepsilon)$ of equation (2) in the space (t,x) is asymptotically stable for ε sufficiently small. If (5) is positive, then it is unstable.*

The next theorem provides a second order approximation for the solutions of a periodic differential system, for a proof see Theorem 3.5.1 of Sanders and Verhulst [11].

Theorem 3. *We consider the following two initial value problems*

$$\dot{x} = \varepsilon f(t,x) + \varepsilon^2 g(t,x) + \varepsilon^3 R(t,x,\varepsilon) \ , \qquad x(0) = x_0 \ , \tag{6}$$

and

$$\dot{y} = \varepsilon f^0(y) + \varepsilon^2 f^{10}(y) + \varepsilon^2 g^0(y) \ , \qquad y(0) = x_0 \ , \tag{7}$$

with $f,\ g : [0,\infty) \times D \to \mathbb{R}^m$*,* $R : [0,\infty) \times D \times (0,\varepsilon_0] \to \mathbb{R}^m$*,* D *an open subset of* $\mathbb{R}^m$*,* f*,* g *and* R *periodic of period* T *in the variable* t*,*

$$f^1(t,x) = \frac{\partial f}{\partial x} y^1(t,x) - \frac{\partial y^1}{\partial x} f^0(x) \ ,$$

where

$$y^1(t,x) = \int_0^t \big(f(s,x) - f^0(x)\big)\, ds + z(x),$$

with $z(x)$ *a* C^1 *function such that the averaged of* y^1 *is zero. Of course,* f^0*,* f^{10} *and* g^0 *denote the averaged functions of* f*,* f^1 *and* g*, respectively, defined as in (4). Suppose*

- (i) $\partial f/\partial x$ *is Lipschitz in* x*,* g *and* R *are Lipschitz in* x *and all these functions are continuous on their domain of definition;*
- (ii) $|R(t,x,\varepsilon)|$ *is bounded by a constant uniformly in* $[0, L/\varepsilon) \times D \times (0,\varepsilon_0]$*;*
- (iii) T *is independent on* ε*;*
- (iv) $y(t)$ *belongs to* D *on the time–scale* $\dfrac{1}{\varepsilon}$*.*

Then

$$x(t) = y(t) + \varepsilon y^1(t, y(t)) + O(\varepsilon^2)$$

on the time–scale $\dfrac{1}{\varepsilon}$*.*

An easy corollary of Theorem 3 is the following one, see [7]

Corollary 4. *Under the assumptions of Theorem 3 we assume that* $f^0(y) \equiv 0$*. Then the following statements hold.*

- (a) *If* p *is an equilibrium point of the averaged system (7) such that*

$$\frac{\partial}{\partial y}\left(f^{10}(y) + g^0(y)\right)\Big|_{y=p} \neq 0 \ , \tag{8}$$

  then there exists a T*–periodic solution* $\phi(t,\varepsilon)$ *of equation (6) which is close to* p *such that* $\phi(t,\varepsilon) \to p$ *as* $\varepsilon \to 0$*.*
- (b) *If (8) is negative, then the corresponding periodic solution* $\phi(t,\varepsilon)$ *of equation (6) in the space* (t,x) *is asymptotically stable for* ε *sufficiently small. If (8) is positive, then it is unstable.*

2.2. **Second order averaged function for system** (1)**.** We write the homogenous polynomials F and G appearing in (1) as $F = F_1+F_2+\cdots+F_n$ and $G = G_1 + G_2 + \cdots + G_n$, where

$$\begin{aligned} F_k = F_k(x, y) &= \sum_{i+j=k} a_{i,j}\, x^i y^j, \\ G_k = G_k(x, y) &= \sum_{i+j=k} b_{i,j}\, x^i y^j, \end{aligned}$$

for $k = 1, \cdots, n$. By means of the change of variables $x = r\cos\theta, y = r\sin\theta$, system (1) in the region $r > 0$ can be written as

$$\begin{aligned} \dot{r} &= \varepsilon\,(\cos\theta\, F + \sin\theta\, G)\,, \\ \dot{\theta} &= 1 + r\cos\theta + \frac{\varepsilon}{r}\,(\cos\theta\, G - \sin\theta\, F)\,. \end{aligned} \tag{9}$$

Here and in what follows F and G will denote $F(r\cos\theta, r\sin\theta)$ and $G(r\cos\theta, r\sin\theta)$, respectively. System (9) in the region $r > 0$ is equivalent to the system

$$\frac{dr}{d\theta} \;=\; \varepsilon f(\theta, r) + \varepsilon^2 g(\theta, r) + O(\varepsilon^3), \tag{10}$$

where

$$\begin{aligned} f(\theta, r) &= \frac{\cos\theta\, F + \sin\theta\, G}{1 + r\cos\theta}, \\ g(\theta, r) &= -\frac{(\cos\theta F + \sin\theta G)(\cos\theta G - \sin\theta F)}{r(1 + r\cos\theta)^2}. \end{aligned}$$

In this paper, we consider the case, when the first averaged function vanishes. So applying the averaging theory to system (10), we must assume that,

$$f^0(r) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta, r)d\theta \equiv 0, \tag{11}$$

which naturally implies that $\displaystyle\int_0^{2\pi} \frac{\partial f(\theta, r)}{\partial r} d\theta \equiv 0$. Then the averaging system associated to system (10) becomes

$$\frac{d\,r}{d\,\theta} = \varepsilon^2 f^{10}(r) + \varepsilon^2 g^0(r), \tag{12}$$

where $g^0 = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} g(\theta, r)d\theta$ and

$$
\begin{aligned}
f^{10} &= \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial f(s,r)}{\partial r}\left(\int_0^s f(\theta,r)d\theta\right)ds + \frac{1}{2\pi}\int_0^{2\pi}\frac{\partial f(s,r)}{\partial r}z(r)ds, \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial f(s,r)}{\partial r}\left(\int_0^s f(\theta,r)d\theta\right)ds + \frac{z(r)}{2\pi}\int_0^{2\pi}\frac{\partial f(s,r)}{\partial r}ds, \qquad (13)\\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial f(s,r)}{\partial r}\left(\int_0^s f(\theta,r)d\theta\right)ds,
\end{aligned}
$$

with $z(r) = -\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\left(\int_0^s f(\theta,r)d\theta\right)ds$. We denote the second averaged function by

$$A_2(r) = f^{10}(r) + g^0(r). \qquad (14)$$

According with Corollary 4, every simple equilibrium point of (12), that is, every simple zero point of the function A_2 provides a limit cycle of system (1). Hence, the main result of this paper is to compute the second averaged function (see (32)), and to bound its number of simple zeros, (see Section 4).

2.3. **Preliminary results for computing $A_2(r)$.** We must compute the averaged function of $f(\theta, r)$ and $g(\theta, r)$. First we start to analyze their numerators.

Lemma 5. *We consider*

$$
\begin{aligned}
M_k(\theta,r) &= \cos\theta F_k + \sin\theta G_k = r^k A_{k,0}(\cos\theta) + r^k A_{k,1}(\cos\theta)\sin\theta, \\
N_k(\theta,r) &= \cos\theta G_k - \sin\theta F_k = r^k B_{k,0}(\cos\theta) + r^k B_{k,1}(\cos\theta)\sin\theta.
\end{aligned} \qquad (15)
$$

Then $A_{k,0}$ and $B_{k,0}$ are the odd (even) polynomials of degree $k+1$ in the variable $\cos\theta$, respectively, if k is even number (odd number); while $A_{k,1}$ and $B_{k,1}$ are the even (odd) polynomials of degree k in the variable $\cos\theta$, respectively,if k is even number (odd number). Moreover $k-q \geq -1$ odd for every term $r^k\cos^q\theta$ in $r^kA_{k,0}$ and $r^kB_{k,0}$, while $k-q \geq 0$ even for every term $r^k\cos^q\theta$ in $r^kA_{k,1}$ and $r^kB_{k,1}$.

Proof : We consider

$$\begin{aligned}
M_k &= r^k \sum_{i+j=k} (a_{i,j}\cos^{i+1}\theta\sin^j\theta + b_{i,j}\cos^i\theta\sin^{j+1}\theta) \\
&= r^k \sum_{\substack{j\,=\,0\\ j\ even}}^{k} a_{k-j,j}\cos^{k-j+1}\theta\sin^j\theta + r^k \sum_{\substack{j\,=\,0\\ j\ odd}}^{k} b_{k-j,j}\cos^{k-j}\theta\sin^{j+1}\theta + \\
&\quad r^k \sum_{\substack{j\,=\,0\\ j\ odd}}^{k} a_{k-j,j}\cos^{k-j+1}\theta\sin^j\theta + r^k \sum_{\substack{j\,=\,0\\ j\ even}}^{k} b_{k-j,j}\cos^{k-j}\theta\sin^{j+1}\theta \\
&= M_{1k} + M_{2k},
\end{aligned}$$

where M_{1k} and M_{2k} denote the first two sums of series and the last two sums, respectively. Furthermore, writing $\sin^2\theta = 1-\cos^2\theta$, we have

$$\begin{aligned}
M_{1k} &= r^k \sum_{\substack{j\,=\,0\\ j\ even}}^{k} a_{k-j,j} \sum_{\substack{m\,=\,0\\ m\ even}}^{j} d_{j,m}\cos^{k-j+1+m}\theta + \\
&\quad r^k \sum_{\substack{j\,=\,0\\ j\ odd}}^{k} b_{k-j,j} \sum_{\substack{m\,=\,0\\ m\ even}}^{j+1} d_{j+1,m}\cos^{k-j+m}\theta,
\end{aligned}$$

and

$$\begin{aligned}
\frac{M_{2k}}{\sin\theta} &= r^k \sum_{\substack{j\,=\,0\\ j\ odd}}^{k} a_{k-j,j} \sum_{\substack{m\,=\,0\\ m\ even}}^{j-1} d_{j-1,m}\cos^{k-j+1+m}\theta + \\
&\quad r^k \sum_{\substack{j\,=\,0\\ j\ even}}^{k} b_{k-j,j} \sum_{\substack{m\,=\,0\\ m\ even}}^{j} d_{j,m}\cos^{k-j+m}\theta,
\end{aligned}$$

where $d_{j,m} = (-1)^{m/2}C_{j/2}^{m/2}$. We denote $A_{k,0}(\cos\theta)$ and $A_{k,1}(\cos\theta)$ as M_{1k}/r^k and $M_{2k}/(r^k\sin\theta)$, respectively. Observing the power exponents of r and $\cos\theta$ in the above two formulas, we can easily get the first formula of the lemma. The second formula can be obtained in the same way. □

The following results are easy to prove.

Lemma 6. *The following equalities hold.*

$$\begin{aligned}
\frac{z^k}{1+rz} &= (-1)^k\frac{1}{r^k(1+rz)} + \sum_{v=1}^{k}(-1)^{v-1}\frac{z^{k-v}}{r^v}, \\
\frac{z^k}{(1+rz)^2} &= (-1)^k\frac{1}{r^k(1+rz)^2} + (-1)^{k-1}\frac{k}{r^k(1+rz)} + \sum_{v=2}^{k}(-1)^v\frac{(v-1)z^{k-v}}{r^v}.
\end{aligned}$$

Lemma 7. *We define*

$$f_k(\theta,r) = \frac{M_k(\theta,r)}{1+r\cos\theta}.$$

Then

$$f_k(\theta,r) = q_{k,0} + q_{k,1} + q_{k,2}\sin\theta + q_{k,3}\frac{\sin\theta}{1+r\cos\theta}, \tag{16}$$

where $q_{k,l} = q_{k,l}(r,\cos\theta)$ *are polynomials in* r *and* $\cos\theta$ *for* $l = 0,1,2$, *and* $q_{k,3} = q_{k,3}(r)$ *is a polynomial in* r. *Additionally*

(a) $q_{k,1}$ *is an even polynomial of degree* $k-2$ *or* $k-1$ *in the variable* r *for* k *even or odd, while it is an odd polynomial of degree* $k-1$ *or* k *in the variable* $\cos\theta$ *for* k *even or odd. Moreover for every term of the form* $r^p\cos^q\theta$ *in* $q_{k,1}$, *we have that* $q-p\le 1$.

(b) $q_{k,2}$ *is a polynomial of degree* $k-1$ *in both variables* r *and* $\cos\theta$. *Moreover for every term* $r^i\cos^j\theta$, *we have that* $i-j\ge 0$ *is an even number.*

(c) $q_{k,3}$ *is an even polynomial of degree* k *or* $k-1$ *in the variable* r *for* k *even or odd.*

(d) *The function*

$$q_{k,0} = \frac{1}{r}f_k^e(\cos\theta,r) + \frac{\hat f_k^e(r)}{r(1+r\cos\theta)},$$

where

(d.1) f_k^e *is an even polynomial of degree* k *or* $k-1$ *in the variables* r *and* $\cos\theta$ *for* k *even or odd. Moreover for every term* $r^p\cos^q\theta$ *in* f_k^e, *we have that* $p-q\ge 0$.

(d.2) $\hat f_k^e$ *is an even polynomial of degree* k *or* $k+1$ *in the variable* r *for* k *even or odd.*

Proof: Using Lemmas 5 and 6, we have

$$\frac{M_{1k}}{1+r\cos\theta} = \frac{1}{r}f_k^e(\cos\theta,r) + \frac{\hat f_k^e(r)}{r(1+r\cos\theta)} + f_k^d(r,\cos\theta),$$

where

$$\begin{aligned} f_k^e \;=\; & \sum_{\substack{j=0\\ j\ even}}^{k} a_{k-j,j} \sum_{\substack{m=0\\ m\ even}}^{j} d_{j,m} \sum_{\substack{v=1\\ k-v\ odd}}^{k+1-\xi} (-1)^{v-1} r^{k-v+1}\cos^{k-v+1-\xi}\theta + \\ & \sum_{\substack{j=0\\ j\ odd}}^{k} b_{k-j,j} \sum_{\substack{m=0\\ m\ even}}^{j+1} d_{j+1,m} \sum_{\substack{v=1\\ k-v\ odd}}^{k+1-\eta} (-1)^{v-1} r^{k-v+1}\cos^{k-v+1-\eta}\theta, \end{aligned}$$

$$
\begin{aligned}
f_k^d &= \sum_{\substack{j=0 \\ j\ even}}^{k} a_{k-j,j} \sum_{\substack{m=0 \\ m\ even}}^{j} d_{j,m} \sum_{\substack{v=1 \\ k-v\ even}}^{k+1-\xi} (-1)^{v-1} r^{k-v} \cos^{k-v+1-\xi}\theta + \\
& \sum_{\substack{j=0 \\ j\ odd}}^{k} b_{k-j,j} \sum_{\substack{m=0 \\ m\ even}}^{j+1} d_{j+1,m} \sum_{\substack{v=1 \\ k-v\ even}}^{k+1-\eta} (-1)^{v-1} r^{k-v} \cos^{k-v+1-\eta}\theta,
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{f}_k^e &= \sum_{\substack{j=0 \\ j\ even}}^{k} a_{k-j,j} \sum_{\substack{m=0 \\ m\ even}}^{j} d_{j,m} \sum_{v=1}^{k+1-\xi} (-1)^{k+1-\xi} r^{\xi} + \\
& \sum_{\substack{j=0 \\ j\ odd}}^{k} b_{k-j,j} \sum_{\substack{m=0 \\ m\ even}}^{j+1} d_{j+1,m} \sum_{v=1}^{k+1-\eta} (-1)^{k+1-\eta} r^{\eta},
\end{aligned}
$$

with $\xi = j - m$ and $\eta = j - m + 1$. Clearly, ξ and η in the above formulas are even numbers. Denoting f_k^d as $q_{k,1}$ and observing the power exponents of r and $\cos\theta$ in the above formulas, we can easily get statements (a), (d.1) and (d.2) of the lemma. Statements (b) and (c) can be obtained in the same way. $\square$

In the following lemma we list some useful results on integrals.

Lemma 8. *The following equalities holds.*

$$
\begin{aligned}
E_k &= \frac{1}{2\pi}\int_0^{2\pi} \cos^k\theta \, d\theta = \begin{cases} 0 & \text{if } k \text{ odd}, \\ C_k^{k/2} 2^{-k} & \text{if } k \text{ even}. \end{cases} \\
I_1 &= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{(1+r\cos\theta)^2} \, d\theta = \frac{1}{(1-r^2)^{3/2}}, \\
I_2 &= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{1+r\cos\theta} \, d\theta = \frac{1}{\sqrt{1-r^2}},
\end{aligned}
$$

$$
\begin{aligned}
I_3 &= \frac{1}{2\pi}\int_0^{2\pi} \cos 2t \ln(1+r\cos t) \, dt = \frac{2\sqrt{1-r^2}+r^2-2}{2r^2}, \\
I_{2k,m} &= \frac{1}{2\pi}\int_0^{2\pi} \cos^{2m-1}(2kt)\ln(1+r\cos t) dt = \frac{\varphi(r)\sqrt{1-r^2}+\psi(r)}{r^{2k(2m-1)}}, \\
J_{2k-1} &= \frac{1}{2\pi}\int_0^{2\pi} \cos^{2k-1}(t)\ln(1+r\cos t) \, dt = \frac{p(r)\sqrt{1-r^2}+q(r)}{r^{2k-1}},
\end{aligned}
$$

where $k, m \in \mathbb{N}$. The functions $\varphi(r)$ and $\psi(r)$ are even polynomials of degree $2k(2m-1)-2$ and $2k(2m-1)$, respectively. The functions $p(r)$ and $q(r)$ are even polynomials of degree $2k-2$.

Proof : We can deal with I_1 and I_2 in the complex plane by the transformation $z = e^{i\theta}$, i.e.

$$I_k = -\frac{2^k i}{2\pi} \int_{|z|=1} \frac{z^{k-1}}{(rz + 2z + 2)^k} dz, \quad i = 1, 2.$$

Then applying the Residue Theorem, we get the results of the statement of the lemma for $I_k, k = 1, 2$. Again E_k can be also obtained by applying the Residue Theorem.

Now we compute J_{2k-1}. We first derivate it with respect to the variable r,

$$\frac{dJ_{2k-1}}{d\,r} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\cos^{2k}(t)}{1 + r\cos t} dt.$$

Using Lemma 6, we get

$$\begin{aligned} \frac{dJ_{2k-1}}{d\,r} &= \int_0^{2\pi} \sum_{v=1}^{2k} ((-1)^{v-1} \frac{\cos^{2k-v}(t)}{r^v} + \frac{r^{-2k}}{1 + r\cos t}) dt \\ &= -\sum_{\substack{v\,=\,1 \\ v\ even}}^{2k} E_{2k-v} \frac{1}{r^v} + \frac{1}{r^{2k}\sqrt{1 - r^2}}. \end{aligned}$$

Hence integrating it with respect to r from $\varepsilon > 0$ to r, we obtain using integration by parts that

$$J_{2k-1}(r) = \sum_{\substack{v\,=\,1 \\ v\ even}}^{2k} \frac{E_{2k-v}}{v - 1} \frac{1}{r^{v-1}} - \phi_0(\varepsilon) + h_{2k}, \tag{17}$$

where

$$\begin{aligned} h_{2k} &= \int_\varepsilon^r \frac{1}{r^{2k}\sqrt{1 - r^2}} dr = \int_\varepsilon^r \frac{\sqrt{1 - r^2}}{r^{2k}} dr + \int_\varepsilon^r \frac{1}{r^{2(k-1)}\sqrt{1 - r^2}} dr \\ &= -\frac{1}{2k - 1} \frac{\sqrt{1 - r^2}}{r^{2k-1}} + (1 - \frac{1}{2(k - 1)}) h_{2(k-1)} + \phi_k(\varepsilon), \end{aligned}$$

with $h_2 = -\sqrt{1 - r^2}/r - \phi_1(\varepsilon)$. Noting that $J_{2k-1}(0) = 0$, it implies that $\sum \phi_k(\varepsilon) \to 0$, as $\varepsilon \to 0$. Hence, from (17), it follows the expression of J_{2k-1}, which appears in the statement of the lemma.

Now I_3 can be obtained in a similar way.

Finally, we compute $I_{2k,m}$. We just prove the case $I_{2k,1}$. The other cases can be proved as we did for J_{2k-1}. Derivate $I_{2k,1}$ with respect to the variable r, we have

$$I'_{2k,1} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\cos 2kt \cos t}{1 + r\cos t} dt = -\frac{1}{2\pi} \int_0^{2\pi} \frac{\cos 2kt}{r(1 + r\cos t)} dt.$$

In fact, $I_{2,1} = I_3$. By induction, the formula of I_{2k} holds for $k = v$. Then noting $\int_0^{2\pi} \cos(2vt)dt = 0$ for $v \in \mathbb{N}$, we have using Lemma 6 and integration by parts that

$$\begin{aligned} I'_{2v+2,1} &= -\frac{1}{2\pi}\int_0^{2\pi} \frac{\cos 2vt(2\cos^2 t - 1) - 2\sin 2vt \cos t \sin t}{r(1 + r\cos t)} dt \\ &= -I'_{2v,1} - \frac{2}{2\pi r}\int_0^{2\pi} \cos 2vt \left(\frac{\cos t}{r} - \frac{1}{r^2} + \frac{1}{r^2(1 + r\cos t)}\right) dt + \\ &\quad \frac{4v}{2\pi r}\int_0^{2\pi} \cos 2vt \left(\frac{\cos t}{r} - \frac{\ln(1 + r\cos t)}{r^2}\right) dt. \end{aligned}$$

Hence we get a recurrence formula

$$I'_{2v+2,1} = (\frac{2}{r^2} - 1)I'_{2v,1} - \frac{4v}{r^3} I_{2v,1}. \tag{18}$$

Since $I_{2k+2,1}(0) = 0$, we integrate both sides of formula (18) with respect to r from $\varepsilon > 0$ to r. Then, taking $\varepsilon \to 0$, it is easy to obtain the formula for I_{2k} with for $k = v + 1$. $\square$

3. The averaged function of seconder order

Since $\int_0^{2\pi} \cos^i \theta d\theta = 0$ for i odd, we obtain easily the first averaged function from Lemmas 7 and 8,

$$\begin{aligned} f^0 &= \frac{1}{2\pi}\int_0^{2\pi} \sum_{k=1}^{n} f_k(\theta, r)d\theta = \frac{1}{2\pi}\int_0^{2\pi} \sum_{k=1}^{n} q_{k,0}(\cos\theta, r)d\theta \\ &= \frac{\sum_{k=1}^{n} (\bar{f}_k^e(r)\sqrt{1 - r^2} + \hat{f}_k^e(r))}{r\sqrt{1 - r^2}} = \frac{\phi(r)}{2r\sqrt{1 - r^2}}, \end{aligned} \tag{19}$$

where

$$\bar{f}_k^e(r) = \int_0^{2\pi} f_k^e(\cos\theta, r)d\theta.$$

The number of zeros of $f^0(r) = 0$ is the same as that of $\phi(r)$. Take $\rho = \sqrt{1 - r^2}$, we get a polynomial of degree $n + 1$ in the variable ρ, $\hat{\phi}(\rho) = \phi(r)$. In the view of the definition of $f(\theta, r)$, we know $f^0(0) = 0$ implies $\phi(0) = 0$. Hence, if the first averaged function vanishes, we get $n + 1$ conditions vanishing the coefficients of the polynomial $\hat{\phi}(\rho)$. Every one of these conditions is a linear combination of some coefficients of system (1).

In the following we consider the second averaged function and still we denote by system (1) the system for which the first averaged function vanishes.

In the polar coordinates, we have the corresponding function $f(\theta, r)$. If we write $\sin^2\theta = 1 - \cos^2\theta$, it can be denoted by

$$f(\theta, r) = q_0(\cos\theta, r) + q_1(\cos\theta, r) + q_2(\cos\theta, r)\sin\theta + q_3(r)\frac{\sin\theta}{1 + r\cos\theta}, \tag{20}$$

where

$$q_0(\cos\theta, r) = \sum_{k=1}^{n} f_k(\theta, r) = \sum_{k=1}^{n} q_{k,0}(\cos\theta, r),$$

and

$$q_i = \sum_{k=1}^{n} q_{k,i}, \quad i = 1, 2, 3.$$

Since $f^0(r) \equiv 0$, we know that $q_1(\cos\theta, r)$ is always an odd function in the variable $\cos\theta$. And $q_0(\cos\theta, r)$ have the following property.

Lemma 9. *The function $rq_0(\cos\theta, r)$ can be denoted as a polynomial in the variables r and $\cos 2k\theta$ for $k \in \mathbb{N}$, and is an even function in r of degree n or $n-1$ if n is even or odd, respectively. While it is an odd function in $\cos 2k\theta$. Moreover for every term $r^\xi \cos^\eta 2k\theta$ in rq_0, we have that $\xi - 2k\eta > 0$.*

Proof: From [8], we know that $\hat{\phi}(\rho)$ is a polynomial of degree $n+1$ and which can have at most n zeros in $(0, r)$. It is also shown in our paper that the first averaged function $f^0(r)$ has $n+2$ coefficients, one of them depending on the others $n+1$ coefficients. That is, $\hat{\phi}(\rho)$ has $n+2$ coefficients, which are linear combinations of some coefficients of system (1). However from $\hat{\phi}(\rho) \equiv 0$, we can get only $n+1$ independent equations.

In fact, from (19), we know that $\bar{f}_k^e(r)\sqrt{1-r^2}$ corresponds to the odd terms of $\hat{\phi}(\rho)$, while $\hat{f}_k^e(r)$ corresponds to the even ones. The function $q_0(\cos\theta, r)$ is even in $\cos\theta$ and satisfies that its integration from 0 to 2π vanishes. So it can be written as an odd function of $\cos 2k\theta$, $k \in \mathbb{N}$. Then $\hat{\phi}(\rho) \equiv 0$ implies that $\hat{f}_k^e(r) \equiv 0$ and

$$q_0 = \frac{1}{r}\sum_{k=0}^{n} f_k^e(\cos\theta, r) = \frac{1}{r}\sum_{2k\eta \le n} * r^\xi \cos^\eta 2k\theta, \tag{21}$$

where ξ is even, η is odd and $*$ denotes linear combinations of some coefficients of system (1). Since $\cos 2k\theta$ can be expressed as a homogenous polynomial in $\cos\theta$ and $\sin\theta$, the other properties of the function q_0 of this lemma can be obtained directly from statement (d.1) in Lemma 7. $\square$

By calculation we have

$$\int_0^t f(\theta, r)d\theta = \bar{q}_0(\cos t, r)\sin t + \bar{q}_1(\cos t, r)\sin t + \bar{q}_2(\cos t, r) + q_3(r)I_0,$$

where

$$
\begin{aligned}
& I_0 = \int_0^t \frac{\sin\theta}{1+r\cos\theta} d\theta = \frac{\ln(1+r) - \ln(1+r\cos t)}{r}, \\
(22) \qquad & \bar{q}_0 = \frac{1}{\sin t}\int_0^t q_0(\cos 2k\theta, r) d\theta = \frac{1}{r\sin t}\sum * r^{\xi}\cos^{2k\eta-1} t, \\
& \bar{q}_1 = \frac{1}{\sin t}\int_0^t q_1(\cos\theta, r)d\theta, \quad \bar{q}_2 = \int_0^t q_2(\cos\theta, r)\sin\theta d\theta.
\end{aligned}
$$

We know that $\bar{q}_1$ is an even polynomial of degree $n-2$ or $n-1$ in the variables r and $\cos t$, if n even or odd, respectively. Hence, the first term appearing in the expression of $A_2(r)$ is

$$
f^{10} = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\partial q_1}{\partial r}(\bar{q}_2 + q_3 I_0) + (\frac{\partial q_2}{\partial r} + \frac{\partial}{\partial r}(\frac{q_3}{1+r\cos t}))\bar{q}_1 \sin^2 t\right) dt + \hat{f}^{10},
$$

where

$$
\hat{f}^{10} = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\partial q_0}{\partial r}(\bar{q}_2 + q_3 I_0) + (\frac{\partial q_2}{\partial r} + \frac{\partial}{\partial r}(\frac{q_3}{1+r\cos t}))\bar{q}_0 \sin^2 t\right) dt.
$$

Theorem 10. *Considering the second averaged function of system* (1)*, we have*

$$
f^{10} = \frac{1}{r^3\sqrt{1-r^2}}(f_1\sqrt{1-r^2} + g_1), \tag{23}
$$

where f_1 and g_1 are even polynomials of degree at most $2n$ in the variable r.

Proof : In the following we denote the degree of a polynomial $f(r)$ by $\deg(f)$. We assume that n is an odd number, and using Lemma 7 we list some properties of q_i for $i = 1, 2, 3$ in Table 1.

First, noting that $q_i, \bar{q}_i$ for $i = 1, 2$ are trigonometric polynomials, we consider

$$
\mathcal{J}_1(r) = \frac{1}{2\pi}\int_0^{2\pi}(\frac{\partial q_1}{\partial r}\bar{q}_2 + \bar{q}_1\frac{\partial q_2}{\partial r}\sin^2 t)dt = \mathcal{J}_{11}(r) + \mathcal{J}_{12}(r),
$$

and we claim that $J_1(r)$ is an odd polynomial of degree at most $2n-3$. For example, we assume

$$
\frac{\partial q_1}{\partial r} = \sum c_{p,q} r^p \cos^q t, \quad \bar{q}_2 = \sum e_{i,j} r^i \cos^j t, \quad q_3 = \sum l_k r^k. \tag{24}
$$

Since $\int_0^{2\pi}\cos^k t dt = 0$ for k odd, then we have the following polynomial

$$
\mathcal{J}_{11} = \int_0^{2\pi}\frac{\partial q_1}{\partial r}\bar{q}_2 dt = \sum\sum_{q+j \text{ even}} c_{p,q} e_{i,j} r^{p+i}\int_0^{2\pi}\cos^{q+j} t dt.
$$

It is easy to get from Table 1 that $0 \le p+i \le 2n-3$ and $p+i$ is odd from the facts that p, q and $i+j$ are odd, and $q+j$ is even. Hence, the claim is true for $\mathcal{J}_{11}$, and can be proved for $\mathcal{J}_{12}$ in the same way.

	q_1	$\bar{q}_1$	$\frac{\partial q_1}{\partial r}$	q_2	$\bar{q}_2$	$\frac{\partial q_2}{\partial r}$
r	$n-1$, even	$n-1$, even	$n-2$, odd	$n-1$	$n-1$	$n-2$
$\cos t$	n, odd	$n-1$, even	n, odd	$n-1$	n	$n-1$
$\xi+\eta$	odd	even	even	even	odd	odd

Table 1: Here ξ and η are the power exponents of $r^\xi \cos^\eta t$. Every box of the second and third line describes the degree and parity in the variables r and $\cos t$ of the functions given in the top of every column. The fourth line describes the parity of $\xi+\eta$ appearing in every term of the form $r^\xi \cos^\eta t$.

Second, we claim that

$$\mathcal{J}_2 = \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial q_1}{\partial r} q_3 I_0 dt = \frac{1}{r^3}(P(r)\sqrt{1-r^2}+Q(r)), \tag{25}$$

where $P(r)$ and $Q(r)$ are two even polynomials of degree at most $2n-2$. In fact, we know from Lemma 7 that $q-p \le 2$ in (24). Then

$$\begin{aligned} \mathcal{J}_2 &= \sum\sum l_k c_{p,q} r^{k+p-1} \frac{1}{2\pi}\int_0^{2\pi} \cos^q t(\ln(1+r)-\ln(1+r\cos t))dt \\ &= \sum\sum l_k c_{p,q} r^{k+p-1-q}(p(r)\sqrt{1-r^2}+q(r)). \end{aligned}$$

From Lemma 8, we get that $\deg(r^{-q}p(r))$ and $\deg(r^{-q}q(r))$ are odd, and

$$-q \le \deg(r^{-q}p(r)), \deg(r^{-q}q(r)) \le -1.$$

According with the properties listed in Table 1, we have $0 \le k \le n-1$ and $-2 \le p-q \le n-2$. Then we obtain

$$-3 \le k-1+p+\deg(r^{-q}p(r)) \le 2n-5,$$

$$-3 \le k-1+p+\deg(r^{-q}q(r)) \le 2n-5,$$

both of which are odd because k is even and p is odd by Lemma 7. Hence $\mathcal{J}_2$ can be written as in (25).

Third, we claim that

$$\mathcal{J}_3 = \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial}{\partial r}\left(\frac{q_3}{1+r\cos t}\right)\bar{q}_1 \sin^2 t dt = \frac{P\sqrt{1-r^2}+Q}{r^3\sqrt{1-r^2}}, \tag{26}$$

where $P(r)$ and $Q(r)$ are even polynomials of degree at most $2n$. Here, we denote

$$\mathcal{J}_3 \quad = \quad \mathcal{J}_{31} + \mathcal{J}_{32} = \frac{1}{2\pi}\int_0^{2\pi} \frac{q_3'(r)\bar{q}_1 \sin^2 t}{1 + r\cos t} dt + \frac{1}{2\pi}\int_0^{2\pi} \frac{q_3 \bar{q}_1 \cos t \sin^2 t}{(1 + r\cos t)^2} dt,$$

and assume that $\bar{q}_1$ has also the form $\bar{q}_1 = \sum c_{p,q} r^p \cos^q t$, with p and q even and $q \le p$. Hence

$$\mathcal{J}_{31} = \sum \sum_{k \ge 2} k l_k c_{p,q} r^{k+p-1} \frac{1}{2\pi}\int_0^{2\pi} (\frac{\cos^q t}{1 + r\cos t} - \frac{\cos^{q+2} t}{1 + r\cos t}) dt.$$

For example, we consider the second term of $\mathcal{J}_{31}$. Using Lemma 6, we have

$$r^{k+p-1}\frac{1}{2\pi}\int_0^{2\pi} \frac{\cos^{q+2} t}{1 + r\cos t} dt = -\sum_{\substack{v\,=\,1 \\ v\ even}}^{q+2} E_{q+2-v} r^{k+p-v-1} + \frac{r^{k+p-1-(q+2)}}{\sqrt{1 - r^2}}.$$

Since, from Table 1, we have that $2 \le k \le n-1$, $0 \le p, q \le n-1$ and $0 \le p - q$ are even. Due to the fact that $2 \le v \le q+2$ is even, we have $-1 \le k + p - q - 3 \le 2n - 5$ and $-1 \le k + p - v - 1 \le 2n - 5$ are odd. In the same way we know that the first term in $\mathcal{J}_{31}$ is an odd function of r with power exponents between 1 and $2n - 3$. Thus, $\mathcal{J}_{31}$ can be written as in (25).

Now we compute $\mathcal{J}_{32}$. We have

$$\mathcal{J}_{32} = \sum \sum l_k c_{p,q} r^{k+p} \frac{1}{2\pi}\int_0^{2\pi} \frac{\cos^{q+1} t \sin^2 t}{(1 + r\cos t)^2} dt,$$

and we claim that $\mathcal{J}_{32}$ can be written as in (26) with P and Q of degree at most $2n$. Since

$$\frac{\sin^2 t}{(1 + r\cos t)^2} = \frac{2}{r^2(1 + r\cos t)} - \frac{1}{r^2} + \frac{r^2 - 1}{r^2(1 + r\cos t)^2}, \tag{27}$$

we here just consider the last term denoted by $\tilde{\mathcal{J}}_{32}$, i.e.

$$\tilde{\mathcal{J}}_{32} = \sum \sum l_k c_{p,q} r^{k+p} \frac{1}{2\pi}\int_0^{2\pi} \frac{(r^2 - 1)\cos^{q+1} t}{r^2(1 + r\cos t)^2} dt.$$

the others can be studied as before. For every term in $\tilde{\mathcal{J}}_{32}$, applying Lemmas 6 and 8, we obtain

$$
\begin{aligned}
T(r) &= \\
&= r^{k+p-2}(r^2-1)\frac{1}{2\pi}\int_0^{2\pi}\frac{\cos^{q+1}t}{(1+r\cos t)^2}dt \\
&= \sum_{\substack{v\,=\,2\\ v\ odd}}^{q+1} E_{q+1-v}r^{k+p-2-v}(r^2-1)+(q+1)\frac{r^{k+p-q-3}(1-r^2)}{\sqrt{1-r^2}}-\frac{r^{k+p-q-3}}{\sqrt{1-r^2}}.
\end{aligned}
$$

Since $0\le k\le n-1$, $0\le p,q\le n-1$ and $0\le p-q$ are even, while $3\le v\le q+1$ odd, we have $-3\le k+p-q-3\le 2n-5$ and $-3\le k+p-v-2\le 2n-7$ are odd, which implies that $T(r)$ can be written as in (26).

In short, we can easily rearrange $\mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3$ into the form (23).

Finally, it can be proved that $\hat{f}^{10}$ can be written as in (23). For example, we consider the first integral of $\hat{f}^{10}$,

$$\hat{\mathcal{J}}_1=\frac{1}{2\pi}\int_0^{2\pi}\frac{\partial q_0}{\partial r}(\bar{q}_2+q_3I_0)dt.$$

Since for any $k,m\in\mathbb{N}$, if $\eta\in\mathbb{N}$ or m is odd, then

$$\int_0^{2\pi}\cos^\eta kt\cos^m t dt=0.$$

Hence the first integral of $\hat{\mathcal{J}}_1$ vanishes. Take q_0 as in (21), then we get from Lemma 8

$$
\begin{aligned}
\hat{\mathcal{J}}_1 &= \frac{1}{2\pi}\int_0^{2\pi}\frac{\partial q_0}{\partial r}q_3I_0dt \\
&= \sum\sum *l_kr^{k+\xi-3}\frac{1}{2\pi}\int_0^{2\pi}\cos^\eta 2kt(\ln(1+r)-\ln(1+r\cos t))dt \\
&= \sum\sum *l_kr^{k+\xi-3-2k\eta}(\varphi(r)\sqrt{1-r^2}+\psi(r)).
\end{aligned}
$$

From the properties of q_0 and q_3, we have for n odd that $0\le k,\xi\le n-1$ and $\deg(\varphi),\deg(\psi)$ are even, and

$$-2k\eta\le\deg(r^{-2k\eta}\varphi),\deg(r^{-2k\eta}\psi)\le 0.$$

Hence

$$
\begin{aligned}
-3\le k-1+p+\deg(r^{-2k\eta}\varphi(r))\le 2n-5,\\
-3\le k-1+p+\deg(r^{-2k\eta}\psi(r))\le 2n-5,
\end{aligned}
$$

both of which are odd. This implies that $\hat{\mathcal{J}}_1$ can be written as in (25).

Next, we compute

$$\begin{aligned}\hat{\mathcal{J}}_3 &= \frac{1}{2\pi}\int_0^{2\pi}\frac{\partial}{\partial r}(\frac{q_3}{1+r\cos t})\bar{q}_0\sin^2 t dt,\\ &= \frac{1}{2\pi}\int_0^{2\pi}\frac{q_3'(r)\bar{q}_0\sin^2 t}{1+r\cos t}dt+\frac{1}{2\pi}\int_0^{2\pi}\frac{q_3\bar{q}_0\cos t\sin^2 t}{(1+r\cos t)^2}dt.\end{aligned}$$

Now we work with the first integral of $\hat{\mathcal{J}}_3$. In a similar way, we can work with the second integral of $\hat{\mathcal{J}}_3$. Taking $\bar{q}_0$ and q_3' as in (22) and (24), we get

$$\sum\sum_{k\geq 2} *kl_k r^{k+\xi-2}\frac{1}{2\pi}\int_0^{2\pi}(\frac{\cos^{2k\eta-1}t}{1+r\cos t}-\frac{\cos^{2k\eta+1}t}{1+r\cos t})dt.$$

Here we consider the first term of the above sum. Using Lemma 6, we have

$$r^{k+\xi-2}\frac{1}{2\pi}\int_0^{2\pi}\frac{\cos^{2k\eta-1}t}{1+r\cos t}dt=-\sum_{\substack{v=1\\ v\ odd}}^{2k\eta-1}E_{2k\eta+1-v}r^{k+\xi-v-2}+\frac{r^{k+\xi-2k\eta-1}}{\sqrt{1-r^2}}.$$

From Table 1 and Lemma 9, we obtain that $2\leq k\leq n-1$, $0\leq\xi\leq n-1$ and $\xi-2k\eta\geq 0$ are even, while $1\leq v\leq 2k\eta-1$ is odd, we have $1\leq k+\xi-2k\eta-1\leq 2n-3$ and $1\leq k+\xi-v-2\leq 2n-5$ are odd. Hence $\hat{\mathcal{J}}_3$ can be written as in (25). In the same way we can consider the other integrals of $\hat{f}^{10}$.

Up to now, we have completed the proof of the theorem for n odd. The expression (23) still holds for n even using the same arguments, we omit them here. □

Next, we consider the function $g(\theta,r)$. We write $\sin^2\theta=1-\cos^2\theta$, and denote $rg(\theta,r)=H_1(r,\theta)H_2(r,\theta)$, where

$$H_1=\frac{\cos\theta F+\sin\theta G}{(1+r\cos\theta)^2}=\frac{f(\theta,r)}{1+r\cos\theta}=Q_0(\cos\theta,r)+Q_1(\cos\theta,r)\sin\theta$$

with

$$Q_0=\frac{q_0(\cos\theta,r)}{1+r\cos\theta}+\sum_{k=1}^{n}\frac{q_{k,1}}{1+r\cos\theta},\quad Q_1=\sum_{k=1}^{n}\frac{q_{k,2}}{1+r\cos\theta}+\sum_{k=1}^{n}\frac{q_{k,3}}{(1+r\cos\theta)^2},$$

and in the view of Lemma 5, we have

$$H_2=\sin\theta F-\cos\theta G=P_0(\cos\theta,r)+P_1(\cos\theta,r)\sin\theta,$$

where

$$P_0=\sum_{k=1}^{n}r^k B_{k,0},\quad P_1=\sum_{k=1}^{n}r^k B_{k,1}.$$

Hence,

$$g^0 = \frac{1}{2\pi}\int_0^{2\pi} g(\theta, r)d\theta = \frac{1}{2\pi r}\int_0^{2\pi}(P_0Q_0 + P_1Q_1\sin^2\theta)d\theta. \tag{28}$$

Function g^0 can be expressed as a combination of the basic integrals I_1 and I_2 given in Lemma 8.

Theorem 11. *Considering the second averaged function of system* (1)*, we have*

$$g^0 = \frac{1}{r^3\sqrt{1-r^2}}(f_2\sqrt{1-r^2} + g_2), \tag{29}$$

where f_2 and g_2 are even polynomials of degree at most $2n$ in the variable r.

Proof: Here we assume that n is odd. First, we consider

$$\mathcal{I}_1 = \frac{1}{2\pi r}\int_0^{2\pi} P_0 \sum_{k=1}^{n} \frac{q_{k,1}}{1 + r\cos\theta}d\theta = \sum_{k=1}^{n}\sum_{k=1}^{n}\frac{1}{2\pi r}\int_0^{2\pi}\frac{r^k B_{k,0} q_{k,1}}{1 + r\cos\theta}d\theta.$$

By Lemmas 5 and 7, we have that

$$q_{k,1} = \sum c_{p,q} r^p \cos^q t, \quad B_{k,0} = \sum h_l \cos^l t,$$

with

$$-1 \le p - q \le n - 2, \quad -1 \le k - l \le n, \tag{30}$$

and p even, q odd, while l odd if k even; l even if k odd. For every term in $\mathcal{I}_1$, using Lemma 6 and 8, we have

$$\frac{1}{2\pi r}\int_0^{2\pi}\frac{r^k B_{k,0} q_{k,1}}{1 + r\cos\theta}d\theta = \sum\sum c_{p,q} h_l \frac{1}{2\pi}\int_0^{2\pi}\frac{r^{k+p-1}\cos^{q+l}\theta}{1 + r\cos\theta}d\theta$$

$$= \sum\sum c_{p,q}h_l\Big(\sum_{\substack{v\,=\,1\\ l-v\ odd}}^{q+l}(-1)^{v-1}E_{q+l-v}r^{k+p-v-1} + \frac{(-1)^{q+l}r^{k+p-q-l-1}}{\sqrt{1-r^2}}\Big).$$

It is easy to see that $k+p-v-1$ and $k+p-q-l-1$ are odd. From Table 1, Lemmas 5 and 7, we know that $0 \le p \le n-1$, $1 \le q \le n$, $1 \le k \le n$. Therefore, by (30), we get

$$-3 \le k + p - v - 1 \le 2n - 3, \quad -3 \le k + p - q - l - 1 \le 2n - 3.$$

Hence $\mathcal{I}_1$ can be written as in (29).

Second, we consider

$$\mathcal{I}_2 = \sum_{k=1}^{n}\sum_{k=1}^{n}\frac{1}{2\pi r}\int_0^{2\pi}\frac{r^k B_{k,1} q_{k,2}\sin^2\theta}{1 + r\cos\theta}d\theta.$$

Note that
$$\frac{\sin^2\theta}{1+r\cos\theta} = -\frac{\cos\theta}{r} + \frac{1}{r^2} + \frac{r^2-1}{r^2(1+r\cos\theta)}.$$
We claim that the last term of the previous equality substituted in $\mathcal{I}_2$, which is
$$\mathcal{U}_1(r) = \frac{1}{2\pi r}\int_0^{2\pi} r^k B_{k,1} q_{k,2} \frac{r^2-1}{r^2(1+r\cos\theta)} d\theta,$$
can be written as in (29). Again by Lemmas 5 and 7, we have that
$$q_{k,2} = \sum c_{p,q} r^p \cos^q t, \quad B_{k,1} = \sum h_l \cos^l t,$$
with

$$0 \le p-q \le n-1, \quad 0 \le k-l \le n-1, \quad 1 \le k \le n. \tag{31}$$

Hence
$$\mathcal{U}_1(r) = \sum\sum h_l c_{p,q}(r^2-1) r^{k+p-3} \frac{1}{2\pi}\int_0^{2\pi} \frac{\cos^{q+l}\theta}{1+r\cos\theta} d\theta.$$
Using Lemma 8, we calculate every term in $\mathcal{U}_1(r)$ and we get
$$h_l c_{p,q}(r^2-1)\Big(\sum_{\substack{v\,=\,1 \\ q+l-v\,even}}^{q+l} (-1)^{v-1} E_{q+l-v} r^{k+p-v-3} + \frac{(-1)^{q+l} r^{k+p-q-l-3}}{\sqrt{1-r^2}}\Big).$$
Since $k-l$ and $p-q$ are even numbers according with Lemmas 5 and 7, $k+p-v-3$ and $k+p-q-l-3$ are odd numbers. Moreover, from (31) and $1 \le v \le q+l$, we have that
$$-3 \le k+p-v-3 \le 2n-3, \quad -3 \le k+p-q-l-3 \le 2n-3,$$
which completes our claim.

The other term of the function $\mathcal{I}_2$ has the form
$$\mathcal{U}_2(r) = \frac{1}{2\pi r}\int_0^{2\pi} r^k B_{k,1} q_{k,2}\Big(-\frac{\cos\theta}{r} + \frac{1}{r^2}\Big) d\theta = \sum\sum h_l c_{p,q}\,\omega(r),$$
where
$$\omega(r) = r^{k+p-1}\frac{1}{2\pi}\int_0^{2\pi} \cos^{q+l}\theta\Big(-\frac{\cos\theta}{r} + \frac{1}{r^2}\Big) d\theta$$
$$= \sum_{\substack{v\,=\,1 \\ q+l-v\,even}}^{q+l} (-1)^{v-1} E_{q+l-v} r^{k+p-v-3} - \sum_{\substack{v\,=\,1 \\ q+l-v\,odd}}^{q+l+1} (-1)^{v-1} E_{q+l+1-v} r^{k+p-v-2}.$$
Using the same arguments that in the case $\mathcal{U}_1$, we know that $\omega(r)$ is an odd polynomial satisfying
$$-3 \le \deg(\omega) \le 2n-3.$$
Hence $\mathcal{I}_2$ can be written as in (29).

Third, we consider

$$\mathcal{I}_3 = \sum_{k=1}^{n}\sum_{k=1}^{n}\Omega(r) = \sum_{k=1}^{n}\sum_{k=1}^{n}\frac{1}{2\pi r}\int_0^{2\pi}\frac{r^k B_{k,1}q_{k,3}\sin^2\theta}{(1+r\cos\theta)^2}d\theta.$$

Recall $q_3 = \sum_{m=1}^{n-1} c_m r^m$ with m even. Then, from (27), we arrange Ω as

$$\Omega = \sum\sum h_l c_m r^{k+m-1}\frac{1}{2\pi}\int_0^{2\pi}(\frac{2\cos^l\theta}{r^2(1+r\cos\theta)} - \frac{\cos^l\theta}{r^2} + \frac{(r^2-1)\cos^l\theta}{r^2(1+r\cos\theta)^2})d\theta.$$

We just consider the last term denoted by $\tilde{\Omega}$,

$$\tilde{\Omega} = \sum\sum h_l c_m r^{k+m-1}\frac{1}{2\pi}\int_0^{2\pi}\frac{(r^2-1)\cos^l\theta}{r^2(1+r\cos\theta)^2}dt.$$

The other terms can be written as in (29) using the same method as before. Applying Lemmas 6–8, we estimate the general term of Ω and we get

$$\sum_{\substack{v=2\\ l-v\,even}}^{l} E_{l-v}r^{\eta-v}(r^2-1) + (-1)^{l-1}l\frac{r^{\eta-l}(r^2-1)}{\sqrt{1-r^2}} + (-1)^l\frac{r^{\eta-l}}{\sqrt{1-r^2}},$$

where $\eta = k+m-3$. Since $k-l$ and m are even numbers according with Lemmas 5 and 7, $k+m-v-3$ and $k+m-l-3$ are odd numbers. Moreover, from (31) and $2 \le v \le l$, we get

$$-3 \le k+m-v-3 \le 2n-3, \quad -3 \le k+m-l-3 \le 2n-3,$$

which ensures that the function $\tilde{\Omega}$ can be written as in (29).

Finally, we can easily prove that the integral $\mathcal{I}_4$ has the form (29) using the same method as above, where

$$\mathcal{I}_4 = \frac{1}{2\pi r}\int_0^{2\pi} P_0\frac{q_0(\cos\theta, r)}{1+r\cos\theta}d\theta.$$

Up to now we affirm that $g^0 = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4$ is of the form (29) if n is odd. For n even, we can prove the theorem in the same way. □

4. The proof of theorem 1

Using the averaging theory, if the first averaged function f^0 vanishes, then we have to consider the second averaged function $A_2 = f^{10} + g^0$. It is easy from (23) and (29) to see that

$$A_2(r) = f^{10} + g^0 = \frac{1}{r^3\sqrt{1-r^2}}(f(r)\sqrt{1-r^2} + g(r)) = \frac{\Phi(r)}{r^3\sqrt{1-r^2}}, \tag{32}$$

where f and g are even polynomials of degree at most $2n$ in the variable r. Then according with Corollary 4(a) every simple zero of $\Phi(r)$ in the interval $(0,1)$ provides a limit cycle bifurcating from system (1).

Moreover, we have $A_2(0) = 0$. It is clear from (10) that $f(0, \theta) \equiv 0$ and $g(0, \theta) \equiv 0$, because F and G are polynomials in the variable r, starting with terms of degree one. Since $A_2(r)$ is analytic at $r = 0$, then $\Phi(r)$ in (32) has a quadruple root at $r = 0$. That is

$$\Phi(r) = r^4 \beta(r), \ r \in (0, 1),$$

where β is an analytic function in $(0, 1)$. Taking

$$\bar{\Phi}(r) = -f(r)\sqrt{1 - r^2} + g(r),$$

we know that the following function

$$\hat{\Phi} = \Phi\bar{\Phi} = -f^2(r)(1 - r^2) + g^2(r)$$

is an even polynomial in r of degree $4n + 2$. And the polynomial $\hat{\Phi}(r)$ has at least the quadruple root $r = 0$, that is

$$\hat{\Phi}(r) = r^4 \phi(r) \text{ where } \phi(r) = \beta(r)\bar{\Phi}(r),$$

where ϕ is an even polynomial of degree at most $4n - 2$ in the variable r. Hence $\phi(r)$ can have at most $2n - 1$ zeros in $(0, 1)$. Then we know that $\Phi(r)$ has at most $2n - 1$ zeros in $(0, 1)$, which completes the proof of Theorem 1.

Remark: In [10] we prove that the upper bounds of Theorem 1 are reached for $n = 4$. By induction, when n increases up in one unity, the roots of $A_2(r)$ increases in two units, if the highest terms of f and g have arbitrary coefficients. At the same time, system (1) increase up to $2n + 1$ arbitrary coefficients. So, it is reasonable to believe that the two coefficients of the highest terms are arbitrary. Then we can make the conjecture that for $n \geq 4$ the number $2n - 1$ is the lowest upper bound for the number of limit cycles in system (1) which can be found using the averaging theory of second order. However, the difficulty for proving such a conjecture lies in the complicated form of the averaged function of second order obtained when $f^0 \equiv 0$.

Acknowledgements

The first author is partially supported by a MEC grant number MTM2005-06098-C02-01 and by a CICYT grant number 2005SGR 00550. The second author is partially supported by Grants NSFC-10371072 of China, and Grant G63009138 of Spain. He thanks to CRM and to Department of Mathematics of the Universitat Autònoma de Barcelona for their support and hospitality.

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