GEOMETRIC TOOLS TO DETERMINE THE HYPERBOLICITY OF LIMIT CYCLES

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ABSTRACT. In this paper we present a new method to study limit cycles' hyperbolicity. The main tool is the function $\nu = ([V, W] \wedge V)/(V \wedge W)$, where V is the vector field under investigation and W a transversal one. Our approach gives a high degree of freedom for choosing operators to study the stability. It is related to the divergence test, but provides more information on the system's dynamics. We extend some previous results on hyperbolicity and apply our results to get limit cycles' uniqueness. Liénard systems and conservative+dissipative systems are considered among the applications.

1. Introduction

In this paper we are concerned with plane differential systems,

$$z' = V(z), \qquad z \in \Omega \subset \mathbb{R}^2, \tag{1}$$

with Ω open connected, $V(z) = (P(z), Q(z)) \in C^2(\Omega, \mathbb{R}^2)$, $z = (x, y) \in \Omega$. We denote by $\phi_V(t, z)$ the local flow defined by (1). We say that γ is a *limit cycle* of (1) if it is an isolated periodic orbit.

It is well-known that not every stable (unstable) limit cycle is structurally stable; that is, not every stable (unstable) limit cycle remains and preserves stability (unstability) under any small perturbation. For a limit cycle to be structurally stable in a general sense, it needs to be hyperbolic; in other words, it needs that each of its characteristic multipliers is different from 1.

† Partially supported by the DGES grant number MTM2005-06098-C02-1 and CONACIT grant number 2005SGR-986.

‡ Partially supported by the PRIN group "Dinamica anolonoma, hamiltoniana, delle popolazioni e dei sistemi piani"and by the GNAMPA group "Analisi qualitativa e comportamento asintotico di equazioni differenziali ordinarie e di equazioni alle differenze ".

In $\mathbb{R}^2$, the hyperbolicity of a limit cycle $\gamma = \{(x(t), y(t)),\, t \in [0, T) \subset \mathbb{R}\}$ is usually characterized as some integral of type $\int_0^T \Psi(V)(x(t), y(t))\, dt$ being not zero, where Ψ belongs to a short list of well-known operators like the divergence of V or the curvature of $V^\perp$ (see [4] for a survey and discussion of methods). Since γ is, in general, unknown explicitly, it turns out that a practical way to prove hyperbolicity is proving that γ belongs to some region $\Omega \subset \mathbb{R}^2$ where a suitable $\Psi(V)$ does not change sign.

As far as we know, the most recent way to find a $\Psi(V)$ has been given in [4] in case that there exists a vector field W transversal to V such that $[W, V] = \mu W$ for some function μ. Then, $\Psi(V) = \mu$. The main obstruction of the method of [4] is that it is necessary to find a vector field W for which V is an infinitesimal generator. In this paper, following this line and the ideas used in [11] for the period function of centres, we show how to manage to obtain a candidate for Ψ with the unique restriction that W is transversal to V. This fact implies that our new Ψ is as easy to compute as div V, thus eliminating the main handicap of the method given in [4]. Let us denote by $[V, W]$ is the Lie bracket of V and W, and by $V \wedge W$ the determinant of the matrix having V and W as rows. Defining

$$\nu = \frac{[V, W] \wedge V}{V \wedge W},$$

the main result we present here consists in the equality

$$\int_0^T \text{div } V(x(t), y(t))\, dt = \int_0^T \nu(x(t), y(t))\, dt. \tag{2}$$

We remark that if V normalizes W, that is if $[W, V] = \mu W$ for some function μ, then $\nu = \mu$, so that our result reduces to that one presented in [4].

It is well-known that the divergence test has a degree of freedom: a cycle's hyperbolicity can be studied by replacing V with $L V$, where L is a suitable non-vanishing function. The new system has the same orbits as the old one, but div $L V = \partial_V L + L \,\text{div } V \neq \text{div } V$, so that one can try to find a function L such that div $L V \neq 0$ on a suitable region. When this occurs, L is said to be a *Dulac function*. A common strategy for proving hyperbolicity consists in looking for suitable Dulac functions, thus avoiding integration. However, there are not algorithms for that and so methods to obtain new operators Ψ are welcome. Our approach also allows to give a new way to look for Dulac functions. In fact, if $\nu \neq 0$ in a region, then the function $\frac{1}{V \wedge W}$ is a Dulac function for the system (1). This fact gives the possibility to widen the range of "natural" candidates for Dulac functions and turns out to be useful in several situations. Moreover, our approach also provides additional information about the location of limit cycles; in fact, if $[V, W] \wedge V \neq 0$ on a region, then a limit cycle cannot intersect the curve $V \wedge W = 0$.

Notice that the freedom to choose W looking for a ν that does not change sign is equivalent to the freedom of choosing multiples of V when looking for Dulac functions. This can be interesting also in relation to the different limitations on the location of the limit cycles that different choices of W provide.

In Section 2, the main result (Theorem 1) is presented and related to results (mainly [4]) and methods (Dulac functions, orthogonal curvatures) already known. Theorem 1, then, is applied to obtain general formulas for two big classes of vector fields: (1) those which admit a decomposition $V = AU + BW$, being U a conservative vector field and W one of its normalizers; and, (2) Liénard systems, expressed in several forms, for which a list of "natural" operators is given.

In Section 3, we choose the more suitable operators studied in Section 2 to give some results on uniqueness of limit cycles for the two families mentioned in the previous paragraph. The main result of this section is Theorem 2, which enriches Theorem 1. The key point is an observation that allows to use vector fields W that loss transversality with respect to V on Jordan curves. This result enlarges the set of systems to which our method can be applied. We also provide some specific examples to illustrate these features.

2. The main result and computational strategies

If f is a function defined on an open subset of Ω, we denote by $\partial_V f$ the derivative of f along the solutions of V, that is $\partial_V f = \nabla f\, V$. Similarly for vector fields, that is $\partial_V W = DW\, V$.

In connection to (1), we consider also a second vector system

$$z' = W(z), \qquad z \in \Omega \subset \mathbb{R}^2, \tag{3}$$

where $W(z) \in C^2(\Omega, \mathbb{R}^2)$, $z = (x, y) \in \Omega$. We denote by $\phi_W(t, z)$ the local flow defined by (3).

We set $V \wedge W = \det(V, W)$. Denoting by $[V, W] = \partial_V W - \partial_W V$ the Lie bracket of V and W, we set

$$\nu = \frac{[V, W] \wedge V}{V \wedge W}. \tag{4}$$

Theorem 1. *Let $V \wedge W \neq 0$ at non-critical points of V. Let $\gamma(t)$ be a T-periodic non-trivial cycle of (1). Then one has*

$$\int_0^T \nu(\gamma(\mathrm{t}))\ dt = \int_0^T div\ V\ (\gamma(t))\ dt.$$

Proof. Without loss of generality, we may assume that $V \wedge W > 0$. In [13], Walcher proved that

$$[V, W] = (-\partial_W \ln(V \wedge W) + \text{div } W)\, V + (\partial_V \ln(V \wedge W) - \text{div } V)\, W.$$

Then one has

$$\nu = \frac{[V, W] \wedge V}{V \wedge W} = -\partial_V \ln(V \wedge W) + \text{div } V.$$

Integrating along γ, one has

$$\int_0^T \nu(\gamma(\mathrm{t}))\ dt = \int_0^T (-\partial_V \ln(V \wedge W) + \text{div } V)\,(\gamma(t))\ dt = \int_0^T \text{div } V\ (\gamma(t))\ dt.$$

♣

Remark 1. *In the case that V is a normalizer of W, that is, $[W, V] = \mu\, W$, then $\nu = \dfrac{-\mu\, W \wedge V}{V \wedge W} = \mu$ and the result of Theorem 1 coincides with that of* [4, Theorem 2].

Remark 2. *In the proof of Theorem 1, we have stated that $\nu = -\partial_V \ln(V \wedge W) + div\ V$ or, equivalently, that*

$$\nu = (V \wedge W)\ div\ \left(\frac{V}{V \wedge W}\right).$$

In other words: if ν does not vanish on a certain region Ω, then $\dfrac{1}{V \wedge W}$ is a Dulac function in $\Omega \setminus \{Z\}$, where Z is the set of critical points of V in Ω. Notice that, since $V \wedge W$ vanishes on Z, in general we cannot apply Bendixson-Dulac criterion even if ν does not vanish at non-critical points.

Also, for the specific case that $W = \dfrac{V^\perp}{||V||}$, the operator $\dfrac{\nu}{||V||}$ coincides with the curvature of the orthogonal vector field, which is another operator used in the literature to study stability of orbits (see [1] *or* [14]*).*

In the literature, Dulac functions have been found mostly by an algebraic approach, looking for norms (e.g., $1/(x^2+y^2)$, $x^\alpha\, y^\beta$,...), co-factors or other functions suggested by the analytic expression of the vector field. In this paper we propose a more geometrical approach: we think of which will be the optimal transversal vector fields to be tested according to the geometry of the problem. For instance, a very natural choice would be taking the orthogonal vector field, $W = (-Q, P)$. Then

$$\nu = \frac{(Q_y - P_x)P^2 - 2(P_y + Q_x)PQ - (Q_y - P_x)Q^2}{P^2 + Q^2},$$

where subscripts denote partial derivatives. The numerator we get is a quadratic form in P and Q, which is obviously indefinite, since not all limit cycles are hyperbolic.

Although the orthogonal vector field (equivalent to $1/(P^2 + Q^2)$ being candidate for Dulac function) can be a useful choice, our best experience comes from writing V as the sum of a conservative vector field and a dissipative one. Starting from the fact that every conservative vector field U has a non-trivial normalizer W (see [11, Lemma 1]), that is $[U, W] = \mu U$, the computations needed to find ν get simpler. Since U and W are transversal, every vector field V can be written in a unique way as a linear combination of U and W: $V = AU + BW$, where $A = \frac{V \wedge W}{U \wedge W}$ and $B = \frac{V \wedge U}{W \wedge U}$. If, additionally, V is transversal to W, then W can be used for the calculation of a suitable ν. The form of ν is given by next corollary.

Corollary 1. *Let W be a normalizer of U, $[U, W] = \mu U$. Let V be transversal to W at non-critical points, and A, B be such that*

$$V = AU + BW.$$

Then

$$\nu = B\left(\mu - \frac{\partial_W A}{A}\right) + \partial_W B. \tag{5}$$

Proof. From the transversality of V and W one has

$$A = \frac{V \wedge W}{U \wedge W} \neq 0.$$

Then one can write

$$\begin{aligned} [V, W] &= [AU + BW, W] \\ &= A[U, W] - (\partial_W A)U - (\partial_W B)W \\ &= (A\mu - (\partial_W A))\, U - (\partial_W B)W. \end{aligned}$$

Hence

$$
\begin{aligned}
[V,W]\wedge V &= \Big((A\mu-(\partial_W A))U-(\partial_W B)W\Big)\wedge\Big(AU+BW\Big)\\
&= \Big(B(A\mu-(\partial_W A))+A(\partial_W B)\Big)U\wedge W.
\end{aligned}
$$

Concluding,

$$
\nu=\frac{\Big(B(A\mu-(\partial_W A))+A(\partial_W B)\Big)U\wedge W}{(AU+BW)\wedge W}=B\left(\mu-\frac{\partial_W A}{A}\right)+\partial_W B.
$$

♣

One practical way to read the above corollary is that every Hamiltonian system with a known normalizer is a suggestion for a family of systems to obtain a suitable ν.

Remark 3. *Denote by $\sigma = U\wedge W$ the wedge product among the conservative and the dissipative part. From Walcher's formula (see* [13]*), we know that*

$$
\mu=\sigma\, div\,\left(\frac{W}{\sigma}\right).
$$

On the other hand, as in Remark 7.1 in [3]*, we know that W is also a normalizer of U/σ. In fact,*

$$
[U/\sigma,W]=div(W)\,(U/\sigma).
$$

So, decomposing $V=(\sigma\,A)\,(U/\sigma)+B\,W$ and applying Corollary 1 to this new decomposition, we have that

$$
\begin{aligned}
\nu &= B\left(div(W)-\frac{\partial_W(A\,\sigma)}{A\,\sigma}\right)+\partial_W B=B\left((A\,\sigma)div\left(\frac{W}{A\,\sigma}\right)\right)+\partial_W B\\
&= A\,\sigma\;div\left(\frac{B}{A\,\sigma}W\right).
\end{aligned}
$$

In general, since $A\neq 0$, one can divide V by A and consider the new ν. As written in the introduction, replacing V with $\frac{V}{A}$ leads to replace the old ν with $\frac{\nu}{A}$. Equivalently, one can consider $A\equiv 1$, that is $V=U+\frac{B}{A}W=:U+\overline{B}W$. Then,

$$
\overline{\nu}=\overline{B}\,\mu+\partial_W\overline{B}=\frac{B}{A}\mu+\partial_W\left(\frac{B}{A}\right).
$$

Another strong reason to use the approach "conservative + dissipative" is that there are already some results on the literature providing normalizers for some families of Hamiltonian vector fields. In Section 2.1 we take advantage of this fact.

2.1. **Conservative part with separable variables.** We obtain a remarkable class of examples by taking a Hamiltonian system with separable variables as the conservative system U,

$$x' = E'(y), \qquad y' = -C'(x), \tag{6}$$

where E and C are $\mathcal{C}^1$ functions with $E(0) = E'(0) = C(0) = C'(0) = 0$, $x\,C(x) > 0$ for $x \neq 0$, $y\,E(y) > 0$ for $y \neq 0$, so that $\frac{C(x)}{C'(x)}$, $\frac{E(y)}{E'(y)}$ exist on all of Ω.

Such a system has the following system W as a normalizer (see [3]):

$$x' = \frac{C(x)}{C'(x)}, \qquad y' = \frac{E(y)}{E'(y)}. \tag{7}$$

Hence, the vector field $V = AU + BW$ is

$$x' = A(x,y)E'(y) + \frac{B(x,y)C(x)}{C'(x)}, \qquad y' = -A(x,y)C'(x) + \frac{B(x,y)E(y)}{E'(y)}. \tag{8}$$

The normalizing function μ, as proved in [3], has the form

$$\mu(x,y) = \left(\frac{E(y)}{E'(y)}\right)' + \left(\frac{C(x)}{C'(x)}\right)' - 1.$$

Hence one has

$$\nu = B\left(\left(\frac{E}{E'}\right)' + \left(\frac{C}{C'}\right)' - 1 - \frac{\partial_W A}{A}\right) + \partial_W B. \tag{9}$$

One natural question arising from considering this special class of conservative systems is which systems in the plane can be written in form (8). Straightforward linear algebra and adaptation of (9) give that any planar system (1) can be written in form (8), being (C, E) any pair of one variable functions satisfying the hypotheses after formula (6).

In this notation, ν is given by formula (9) with

$$\begin{aligned} A(x,y) &= \frac{P(x,y)\,E(y)/E'(y) - Q(x,y)\,C(x)/C'(x)}{C(x)+E(y)}, \\ B(x,y) &= \frac{P(x,y)\,C'(x) + Q(x,y)\,E'(y)}{C(x)+E(y)}. \end{aligned} \tag{10}$$

Unfortunately, this expression of ν gives little information for general P and Q. It is only when we restrict ourselves to special families of systems when we can obtain operators ν easier to handle, as the next examples show.

Example 1. *If we choose $E(y) = y^2/2$ and take A and B depending only on x, then we are restricting to a class of systems equivalent to second order differential equations. For the corresponding system,*

$$x' = yA(x) + B(x)\frac{C(x)}{C'(x)}, \qquad y' = -A(x)C'(x) + B(x)\frac{y}{2}, \tag{11}$$

one obtains that the function ν is independent of y,

$$\nu = B\left(\left(\frac{C}{C'}\right)' - \frac{1}{2} - \frac{A'C}{AC'}\right) + B'\frac{C}{C'} = \left(\frac{BC}{C'}\right)' - \frac{A'}{A}\frac{BC}{C'} - \frac{B}{2}.$$

In this case using ν may be more convenient than using the divergence of (11), which depends on both variables,

$$div\, V(x,y) = yA'(x) + \left(B(x)\frac{C(x)}{C'(x)}\right)' + \frac{B(x)}{2}.$$

Taking $A \equiv 1$ and computing $\overline{\nu}$ gives

$$\overline{\nu} = \overline{B}\left(\left(\frac{C}{C'}\right)' - \frac{1}{2}\right) + \overline{B}'\frac{C}{C'} = \left(\frac{\overline{B}\,C}{C'}\right)' - \frac{\overline{B}}{2}.$$

In this case also the divergence is independent of y, and is related to $\overline{\nu}$ by a simple relationship,

$$div\, V = \left(\frac{\overline{B}C}{C'}\right)' + \frac{\overline{B}}{2} = \overline{\nu} + \overline{B}.$$

Example 2. *In general, the system (11) is equivalent to a second order differential equation of the type*

$$x'' + f(x)x' + h(x){x'}^2 + g(x) = 0, \tag{12}$$

where

$$\begin{cases} f &= -\dfrac{B}{2} + \dfrac{A'BC}{AC'} - \left(\dfrac{BC}{C'}\right)' \\ g &= C'A^2 + \dfrac{B^2C}{2C'} \\ h &= -\dfrac{A'}{A}, \end{cases} \tag{13}$$

It would be useful to compute A, B, C starting from f, g, h, in order to study the equation (12) by means of the system (11), but we were not able to do that. On the other hand, the above equalities can be used to reduce the system to a class of equations whose qualitative properties have been widely studied. This is the case for boundedness properties, that can be used together with some repelling property of a critical point in order to prove the existence of limit cycles. Comparing the above expression of f with that one obtained for ν, we see that

$$\nu = -f - B.$$

Example 3. *When $A \equiv 1$, that is when $h \equiv 0$, equation (12) becomes of Liénard type: consider the system*

$$x' = y + \frac{C(x)B(x)}{C'(x)}, \qquad y' = -C'(x) + \frac{yB(x)}{2}, \tag{14}$$

where $B(x)$, $C(x)$ are functions of class C^1 on a suitable interval. In order to have monodromy of the solutions close to the origin, we assume that $x\,C'(x) > 0$ for $x \neq 0$ in a neighbourhood of $x = 0$. As a consequence, we have $C(x) > 0$ for $x \neq 0$, in a neighbourhood of $x = 0$. The equation equivalent to (14) is then

$$x'' + \left[-\frac{B}{2} - \left(\frac{BC}{C'}\right)'\right]x' + C' + \frac{B^2C}{2C'} = 0.$$

Here again we were not able to express f and g as functions of B and C.

Choosing $W = \left(\dfrac{C(x)}{C'(x)}, \dfrac{y}{2}\right)$, one has

$$\nu = \frac{(-2BCC'' + BC'^2 + 2B'CC')y^2 - 4C^2C''B + 4B'C^2C' + 2BCC'^2}{4C'^2(\frac{y^2}{2} + C)}.$$

Transversality holds for $C'^2(\frac{y^2}{2} + C) > 0$. The function ν is positive for

$$(-2BCC'' + BC'^2 + 2B'CC')(-4C^2C''B + 4B'C^2C' + 2BCC'^2) > 0.$$

Example 4. *As an example, taking* $A(x) = 1$, $\overline{B}(x) = 2(1-x^2)$, $C(x) = \frac{x^2}{2}$, *we obtain the system*

$$x' = y + x(1 - x^2), \qquad y' = -x + y(1 - x^2), \tag{15}$$

equivalent to the Liénard equation

$$x'' + (4x^2 - 2)x' + x(x^4 - 2x^2 + 2) = 0.$$

Its divergence, $2 - 4x^2$*, is independent of* y *but does not have constant sign. A limit cycle of (15) has to encircle the unique critical point* $(0, 0)$*, and cannot be contained in the region of positive divergence. On the other hand, one has* $\nu = -2x^2 \leq 0$*, so that every limit cycle is attracting and hyperbolic. This also gives the uniqueness of the limit cycle, that will be considered in greater detail in Section 3.*

In Section 2.2, we explore other possibilities for the Liénard equation.

2.2. **Liénard equation.** As it is well-known, the Liénard equation $x'' + f(x)\,x' + g(x) = 0$ is usually transformed into the two following forms:

- *Phase plane* form:

$$\begin{cases} x' &= y, \\ y' &= -g(x) - y\,f(x). \end{cases} \tag{16}$$

- *Liénard plane* form:

$$\begin{cases} x' &= y - F(x), \\ y' &= -g(x), \end{cases} \tag{17}$$

where $F'(x) = f(x)$. For special purposes, it has also been considered (see [10]) the form:

$$\begin{cases} x' &= y - x\,S(x), \\ y' &= -R(x) - y\,S(x), \end{cases} \tag{18}$$

$R(x)$ and $S(x)$ are continuous functions such that, setting $I(x) = \int_0^x s f(s) ds$, for $x \neq 0$ one has

$$S(x) = \frac{I(x)}{x^2}, \qquad R(x) = g(x) - x\,S(x)^2.$$

In this section we just give different options for ν corresponding to different "natural" transversal vector fields: the trivial radial vector field, the orthogonal one, the orthogonal to the conservative part and others obtained from different choices of the "conservative + dissipative" structure explored above. We summarize them in Table 1.

$V(x,y)$	$W(x,y)$	$\nu(x,y)$
(17)	(x,y)	$\dfrac{(g-xg')y-xfg+xg'F}{y^2-yF+xg}$
(17)	$(g(x),y)$	$\dfrac{(-2g'g+2g)y-gF+2gg'F-fg^2}{y^2-yF+g^2}$
(17)	$(g(x),y-F(x))$	$\dfrac{\alpha_2(x)\,y^2+\alpha_1(x)\,y+\alpha_0(x)}{(y-F)^2+g^2}$ $\alpha_2=f$, $\alpha_1=-2g'g-2fF+2g$, $\alpha_0=-2gF+2gg'F-fg^2+fF^2$.
(17)	$\left(\dfrac{G(x)}{g(x)},\dfrac{y}{2}\right)$	$\dfrac{g(x)\,F(x)-2\,f(x)\,G(x)}{y^2-yF(x)+2\,G(x)}$, $G(x)=\int_0^x g(s)\,ds$
(18)	(x,y)	$\dfrac{\alpha_0(x)+\alpha_1(x)\,y+\alpha_2(x)\,y^2}{y^2+x\,R(x)}$ $\alpha_0(x)=x^3\,S(x)^2\left(\dfrac{R(x)}{x\,S(x)}\right)'$, $\alpha_1(x)=-x^2\left(\frac{R(x)}{x}\right)'$, $\alpha_2(x)=-x\,S'(x)$.

TABLE 1. Choice of transversal vector fields for two of the three forms of the Liénard equation: (17) and (18). For the sake of conciseness we do not consider equation (16) in this table. The function ν (third column) is obtained from the vector fields V (first column) and W (second column) through $\nu=\frac{[V,W]\wedge V}{V\wedge W}$. Observe that the transversality conditions are the denominators of $\nu(x,y)$ being different from zero.

For all the cases, the transversality conditions are the denominators of $\nu(x,y)$ being different from zero. In all the above cases, taking $g(x) = \kappa x$, $\kappa \neq 0$, simplifies the expressions, since all the terms $g - xg'$, $-2g'g + 2g$ vanish.

The choice of W being the orthogonal vector field (the sixth row in Table 1), for instance, turns out to be disappointing. Although it gives transversality for free, the numerator of $\nu(x,y)$ is an indefinite form. In fact, computing the discriminant Δ of the numerator, thought as a quadratic polynomial in y, one has

$$\Delta = (-2g'g - 2fF + 2g)^2 - 4f(-2gF + 2gg'F - fg^2 + fF^2) = 4g^2((g'-1)^2 + f^2).$$

On the other hand, the last but one option of the list has shown to be the most appropriate to ensure transversality and non-vanishing of ν simultaneously. This fact will be exploited in Section 3 to provide hypotheses for proving the uniqueness of limit cycles.

We would like to emphasize that each of the options for W considered in Table 1 gives a different Dulac function in the regions where the numerator does not vanish.

3. Applications to uniqueness of limit cycles

As observed in the previous section, if ν has constant sign, then every limit cycle is hyperbolic, with the same stability character. In this section we apply such a principle in order to give some applications of the results proved in Section 2 to limit cycles' uniqueness. First, we take advantage of the decompositions and operators obtained in Section 2.1. Second, we use suitable transversal vector fields to obtain results for the Liénard equations.

A key issue when applying Theorem 1 is the control of the vanishing set of the denominator of ν, that is, the set $\Gamma := \{(x,y) : (V \wedge W)(x,y) = 0\}$. It is obvious that ν is not defined on Γ and that Γ is the locus of the plane where V and W are not transversal. In next theorem we show that, actually, such curves are not an obstacle for giving results on limit cycles. We denote by Γ^c the complement of Γ, $\Gamma^c := \{(x,y) : (V \wedge W)(x,y) \neq 0\}$.

Theorem 2. *Consider a couple of $\mathcal{C}^1$ vector fields, V and W, defined in $\mathbb{R}^2$. Suppose that the set $\Gamma_0 = \Gamma \setminus \{(x,y) : V(x,y) = 0\}$ is a union of Jordan curves.*

(1) *Then,*
$$\partial_V \ (V \wedge W)|_\Gamma = - \ [V, W] \wedge V|_\Gamma \,.$$
(2) *If* $[V, W] \wedge V$ *does not vanish at non-critical points, then every limit cycle is contained in a connected component of* Γ^c*.*
(3) *If* $[V, W] \wedge V$ *does not vanish at non-critical points, and a simply connected component of* Γ_0^c *contains no more than one critical point, then it contains at most one limit cycle.*
(4) *If* $[V, W] \wedge V$ *does not vanish at non-critical points, and an annular region of* Γ_0^c *does not contain any critical point, then it contains at most one limit cycle.*

Proof of Theorem 2.

(1) One has that
$$\begin{aligned}
\partial_V \,(V \wedge W) &= (\partial_V \, V) \wedge W + V \wedge (\partial_V \, W) \\
&= (\partial_V \, V) \wedge W + V \wedge ([V, W] + \partial_W \, V) \\
&= V \wedge [V, W] + (\partial_V \, V) \wedge W + V \wedge (\partial_W \, V) \\
&= -[V, W] \wedge V + \operatorname{div} V \,(V \wedge W).
\end{aligned}$$
The result follows immediately.

(2) Assume, by absurd, that an orbit γ of system (1) intersects two different adjacent connected components Ω_1 and Ω_2 of the complement, Γ^c, of Γ. Call p one of the corresponding intersection points on Γ. Since $\partial_V(V \wedge W) = -[V, W] \wedge V \neq 0$ on Γ, the vector field V is transversal to Γ. Without loss of generality we may suppose that V points onto Ω_1 and so, that γ enters into Ω_1 through p.

However, since the component of Γ containing p is a Jordan curve and the orbit γ is closed, there must exist another point $q \in \Gamma \subset \gamma$, with $q \neq p$, through which the cycle leaves from Ω_1 to Ω_2. This is a contradiction with the fact that $\partial_V(V \wedge W)$ does not change sign.

(3) Assume, by absurd, that Ω_1, a simply connected component of Γ_0^c, contains two distinct limit cycles γ_1, γ_2. Since there is only one critical point p in Ω_1, γ_1 and γ_2 are concentric. They have the same stability character, because ν does not change sign, hence in the annular region bounded by γ_1 and γ_2 there should exist either another limit cycle, with opposite stability character, or a critical point. This contradicts the fact that ν does not change sign.

(4) The same argument as in point 3 works in this case.

♣

When the system has a unique critical point O, the above theorem gives also some information about the limit cycle location. It is contained in the connected component of Γ^c containing O. This kind of information cannot be obtained via a Dulac function.

We consider now some applications to the systems of Table 1. We start with the last but one normalizer.

Corollary 2. *Consider the vector field (17), where F and g are $\mathcal{C}^1$ functions, with $x\,g(x)>0$ everywhere but at zero. Suppose that $\Big(F(x)/\sqrt{G(x)}\Big)'>0$ (<0) for all $x \in \mathbb{R}\setminus\{0\}$. Then, the system (17) has at most one limit cycle, which is hyperbolic and stable (unstable). If it exists, such a cycle is contained in the connected component of $\big\{(x,y) : y^2 - yF(x) + 2G(x) > 0\big\}$ whose closure contains the origin.*

Proof. Let us observe that ν's numerator $\mathcal{N} = [V,W] \wedge V$ satisfies

$$\mathcal{N}(x) = \frac{1}{2}g(x)F(x) - f(x)G(x) = -G(x)^{3/2}\frac{d}{dx}\left(\frac{F(x)}{\sqrt{G(x)}}\right) > 0$$

for $x \neq 0$. Moreover, the components of the curve $V \wedge W = 0$, that is $\big\{(x,y) : y^2 - yF(x) + 2G(x) = 0\big\}$, are Jordan curves because every line $x = const$ meets such a curve at 0, 1 or 2 points according to the sign of $F(x)^2 - 8\,G(x)$. Then, applying point 3 of Theorem 2 we get the thesis. ♣

As for the uniqueness, such a result has been already proved in [2, Th. C]. On the other hand, in such a paper no information about the limit cycle location was given.

Corollary 3. *Consider the vector field (17), where F and g are $\mathcal{C}^1$ functions, with $x\,g(x) > 0$ for $|x| > k$, $k \in \mathbb{R}$, and $G(x) > 0$ for every $x \neq 0$. Suppose that $\Big(F(x)/\sqrt{G(x)}\Big)' > 0$ (< 0) for all $x \in \mathbb{R}\setminus\{0\}$. Then, every annular region free of critical points contains at most one limit cycle of system (17), which is hyperbolic. If it exists, such a cycle does not intersect the curve $\big\{(x,y) : y^2 - yF(x) + 2G(x) = 0\big\}$ and it is stable (unstable) if*

$$\Big(F(x)/\sqrt{G(x)}\Big)'\,(V \wedge W) > 0 \qquad (< 0).$$

Proof. All the critical points lie on $y = 0$. Moreover, $V \wedge W = 0$ intersects the line $y = 0$ only when $G(x) = 0$; that is, only at $(0,0)$, which is an isolated point of $V \wedge W = 0$. The proof, then, follows as in the previous corollary, now applying point 4 of Theorem 2 to get the thesis. ♣

This result could be used in order to give other estimates of the number of limit cycles in the line of the paper by Gasull and Giacomini (see [6]).

Adding to the theorem a set of hypotheses that ensure boundedness of solutions (see Graef, [7]), we can get also a theorem that guarantees the existence of limit cycles. It is necessary to introduce first the notion of *uniformly ultimately bounded* system:

Definition 1. *The solutions of a system are said to be* uniformly ultimately bounded *if there exists a constant $K > 0$ such that for any solution, there is a time T such that for all $t > T$ we have $||(x(t), y(t))|| < K$.*

Corollary 4. *If the hypothesis of corollary 2 are satisfied and, additionally, $F'(0) < 0$ and the following hold:*

> *There exist positive constants k and c such that:*
> *(a) $x\,F(x) > 0$ if $|x| \geq k$;*
> *(b) $F(x) \geq c > 0$ if $x \geq k$ or $F(x) \leq c < 0$ if $x \leq -k$;*
> *(c) $\displaystyle\int_0^{\infty} (f(x) + |g(x)|)\, dx = \pm\infty$;*
> *then, there exists a unique limit cycle for system (17).*

Proof. We only have to apply Theorem 3.1. given in [7]. In such a way, we know that a sufficiently big neighbourhood of the origin will be positively invariant. This fact, together with the repulsive character of the origin due to $F'(0) < 0$, allows to apply Poincaré-Bendixson theorem and ensure the existence of the limit cycle. ♣

We consider now the last example of Table 1.

Corollary 5. *Consider the vector field (18) where R and S are $\mathcal{C}^1$ functions, with $x\,R(x) > 0$ everywhere but at zero. Suppose that*

$$\psi(x) := \left(\left(\frac{R(x)}{x} \right)' \right)^2 + \frac{4}{3}\, \left(S(x)^3\right)' \left(\frac{R(x)}{x\,S(x)} \right)' < 0$$

for all $x \in \mathbb{R} \setminus \{0\}$. Then, the system (18) has at most one limit cycle, which is hyperbolic and stable (unstable) if $x\,S'(x) > 0$ (< 0) for all $x \neq 0$.

Proof. It is sufficient to observe that ψ is the discriminant of the numerator of ν (see Table 1). Since $\nu < 0$ and its denominator $y^2 + x\,R(x)$ does not vanish except at the origin, one can apply Theorem 2 (in this case, $\Gamma = \{(0,0)\}$) to get the thesis. To determine the stability character it suffices to see that the leading term of the numerator of ν is $(-x\,S'(x))$. ♣

Next example is an application of Corollary 5.

Example 5. *Consider system (18) with $R(x) = x\,\exp(-x^2)$ and $S(x) = 1-x^2$. It is equivalent to a Liénard equation, according to the formulas given after systems (18). Graef' hypotheses for boundedness in negative time are satisfied, and the origin is a stable critical point, since $f(0) = 2 > 0$. Hence a limit cycle exists.*

The function $\psi(x)$ in the statement of Corollary 5 is

$$\psi(x) = -4\,x^2 \left(4 + 4\,x^2\,\exp(-x^2) - \exp(-2\,x^2)\right),$$

which is negative for all $x \in \mathbb{R}^2$. So, there is at most one limit cycle in $\mathbb{R}^2$; if it exists, is unstable because $x\,S'(x) = -2\,x^2 < 0$.

We now consider some vector fields in the form given in Corollary 1, taking $U = (y, -x)$, $W = \frac{1}{2}(x, y)$. As seen in Section 2, every differential system in the plane can be written as

$$x' = P(x,y) = yA(x,y) + xB(x,y), \quad y' = Q(x,y) = -xA(x,y) + yB(x,y), \quad (19)$$

where

$$A(x,y) = \frac{yP(x,y) - xQ(x,y)}{x^2+y^2}, \qquad B(x,y) = \frac{xP(x,y) + yQ(x,y)}{x^2+y^2}. \quad (20)$$

.

If $A(x,y)$ does not vanish on its domain, one can divide the vector field by $A(x,y)$, obtaining a new system,

$$x' = P(x,y) = y + x\frac{A(x,y)}{B(x,y)}, \qquad y' = Q(x,y) = -x + y\frac{A(x,y)}{B(x,y)},$$

with constant angular speed. Such a system normalizes the vector field (x,y), so that it can be treated as in [4]. On the other hand, if $A(x,y)$ vanishes somewhere, a different approach is required.

Corollary 6. *Consider the vector field (19), with A, $B \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R})$. Assume every connected component of the set $\Gamma_A := \{(x,y) : A(x,y) = 0\}$ to be a Jordan curve. Suppose $(P(xQ_x + yQ_y) - Q(xP_x + yP_y)) < 0 (> 0)$ for all $(x,y) \in \mathbb{R}^2$. Then, every connected component of $\Gamma_A^c :=$*

$\{(x,y):\, A(x,y) \neq 0\}$ *contains at most one limit cycle of the system (19), which is hyperbolic and stable (unstable).*

Proof. Computing ν gives

$$\nu = \frac{P(xQ_x + yQ_y) - Q(xP_x + yP_y)}{yP - xQ}. \tag{21}$$

Under the hypothesis' assumption, ν's numerator is negative. Then the thesis comes from Theorem 2. ♣

Systems with $A(x,y) \neq 0$ appeared several times in the literature. In this case one can write

$$\nu = xB_x + yB_y - B\left[x(\ln A)_x + y(\ln A)_y\right].$$

Such an expression might appear more complicated than the divergence of (P,Q).

$$P_x + Q_y = yA_x - xA_y + xB_x + yB_y + 2B,$$

but it may have constant sign in some cases in which the divergence changes sign (see Example 6 below). Also, it admits a nice geometric interpretation. The numerator of the fraction in (21) has the same sign as

$$x\left(\frac{Q(x,y)}{P(x,y)}\right)_x + y\left(\frac{Q(x,y)}{P(x,y)}\right)_y = r\left(\frac{Q(r\cos\theta, r\sin\theta)}{P(r\cos\theta, r\sin\theta)}\right)_r.$$

The ratio $\dfrac{Q}{P}$ is the trigonometric tangent of the angle between the vector V and the direction of the semi-axis $x > 0$. A sign condition on the radial derivative of $\dfrac{Q}{P}$ is equivalent to a condition on the rotation of V along rays. Such a hypothesis was considered by Sansone ([12]) and Massera ([9]) in a uniqueness theorem for limit cycles of Liénard equation. Their result, based on a geometric argument, required g to be linear and f to be increasing on $(0,+\infty)$, decreasing on $(-\infty,0)$. The Corollary 6 is an extension of their result.

Another special case arises when A and B depend only on x. Then (19) is equivalent to the second order equation of type (12) with $C(x) = x^2/2$. From Example 2, we know that

$$\begin{cases} f &= -2B + \frac{xA'B}{A} - xB', \\ g &= x(A^2 + B^2), \\ h &= -\frac{A'}{A}. \end{cases} \tag{22}$$

In this special case it is possible to compute A and B starting from f, g, h, but this can be done only if f, g, h satisfy a particular relationship. Setting $H(x) = \int_0^x h(s)\, ds$, $K(s) = \int_0^x sf(s) \exp(H(s))\, ds$, the above equations lead to

$$\begin{array}{ll} A(x) & = H_0 \exp\left(-H(x)\right), \qquad H_0 \in \mathbb{R}, \\ B(x) & = -K(x) \exp(-H(x))/x^2. \end{array}$$

The functions H and K allow to express the relationship necessary for (2) to be represented by a system of the form (19), that is

$$g(x) = x \exp(-2H(x)) \left[1 + \frac{K(x)^2}{x^4}\right].$$

In other words, every second-order differential equation of type (2) can be written in the form (19) if and only if $g(x)$ satisfies the above equality.

Example 6. *As a particular example, taking* $A(x) = x^2+1$*,* $B(x) = 1-x^4$*, one has the system*

$$\left\{ \begin{array}{ll} x' & = y(x^2+1) + x(1-x^4) = x + y + x^2 y - x^5, \\ y' & = -x(x^2+1) + y(1-x^4) = -x + y - x^3 - x^4 y, \end{array} \right. \tag{23}$$

for which

$$\nu = -2x^4 - 2x^2,$$

while the system's divergence is $2 + 2xy - 6x^4$.

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