ALGEBRAIC EXTENSIONS IN FREE GROUPS

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Abstract. The aim of this paper is to unify the points of view of three recent and independent papers (Ventura 1997, Margolis, Sapir and Weil 2001 and Kapovich and Miasnikov 2002), where similar modern versions of a 1951 theorem of Takahasi were given. We develop a theory of algebraic extensions for free groups, highlighting the analogies and differences with respect to the corresponding classical field-theoretic notions, and we discuss in detail the notion of algebraic closure. We apply that theory to the study and the computation of certain algebraic properties of subgroups (*e.g.* being malnormal, pure, inert or compressed, being closed in certain profinite topologies) and the corresponding closure operators. We also analyze the closure of a subgroup under the addition of solutions of certain sets of equations.

1. Introduction

A well-known result by Nielsen and Schreier states that all subgroups of a free group F are free. A non-specialist in group theory could be tempted to guess from this pleasant result that the lattice of subgroups of F is simple, and easy to understand. This is however far from being the case, and a closer look quickly reveals the classical fact that inclusions do not respect rank. In fact, the free group of countably infinite rank appears many times as a subgroup of the free group of rank 2. There are also many examples of subgroups H, K of F such that the rank of $H \cap K$ is greater than the ranks of H and K. These are just a few indications that the lattice of subgroups of F is not easy.

Although the lattice of subgroups of free groups was already studied by earlier authors, Serre and Stallings in their seminal 1977 and 1983 papers [14, 16], introduced a powerful new technique, that has since turned out to be extremely useful in this line of research. It consists in thinking of F as the fundamental group of a bouquet of circles R, and of subgroups of F as covering spaces of R, i.e. some special types of graphs. With this idea in mind, one can understand and prove many properties of the lattice of subgroups of F using graph theory. These techniques are also very useful to solve algorithmic problems and to effectively compute invariants concerning subgroups of F.

The present paper offers a contribution in this direction, by analyzing a tool (an invariant associated to a given subgroup $H \leq F$) which is suggested by a 1951 theorem of Takahasi [17] (see Section 2.3). The algorithmic constructions involved in the computation of this invariant actually appeared in recent years, in three completely independent papers [20], [11] and [7], where the same notion was invented in independent ways. In chronological order, we refer:

- to the *fringe of a subgroup*, constructed in 1997 by Ventura (see [20]), and applied to the study of maximal rank fixed subgroups of automorphisms of free groups;
- to the *overgroups* of a subgroup, constructed in 2001 by Margolis, Sapir and Weil (see [11]), and applied to improve an algorithm of Ribes-Zaleskii for computing the pro-p topological closure of a finitely generated subgroup of a free group, among other applications; and
- to the *algebraic extensions* constructed in 2002 by Kapovich and Miasnikov (see [7]), in the context of a paper where the authors surveyed, clarified and extended the list of Stallings graphical techniques.

Turner also used the same notion, restricted to the case of cyclic subgroups, in his paper [18] (again, independently) when trying to find examples of test elements for the free group.

The terminology and the notation used in the above mentioned papers are different, but the basic concept – that of algebraic extension for free groups – is the same. Although aimed at different applications, the underlying basic result in these three papers is a modern version of an old theorem by Takahasi [17]. It states that, for every finitely generated subgroup H of a free group F, there exist finitely many subgroups $H_0, \ldots, H_n$ canonically associated to H, such that every subgroup of F containing H is a free multiple of H_i for some $i = 0, \ldots, n$. The original proof was combinatorial, while the proof provided in [20], [11] and [7] (which is the same up to technical details) is graphical, algorithmic, simpler and more natural.

The aim of this paper is to unify the points of view in [20], [11] and [7], and to systematize the study of the concept of algebraic extensions in free groups. We show how algebraic extensions intervene in the computation of certain abstract closure properties for subgroups, sometimes making these properties decidable. This was the idea behind the application of algebraic extensions to the study of profinite topological closures in [11], but it can be applied in other contexts. In particular, we extend the discussion of the notions of pure closure, malnormal closure, inert closure, etc (a discussion that was initiated in [7]).

A particularly interesting application concerns the property of being closed under the addition of the solutions of certain sets of equations. In this case, new results are obtained, and in particular one can show that the rank of the closure of a subgroup H is at most equal to $\mathsf{rk}(H)$.

The paper is organized as follows.

In section 2, we remind the readers of the fundamentals of the representation of finitely generated subgroups of a free group F by finite labeled graphs. This method, which was initiated by Serre and Stallings at the end of the 1970s, quickly became one of the major tools of the combinatorial theory of free groups. This leads us to the short, algorithmic proof of Takahasi's theorem discussed above (see Section 2.3).

Section 3 introduces algebraic extensions, essentially as follows: the algebraic extensions of a finitely generated subgroup H are the minimum family that can be associated to H by Takahasi's theorem. We also discuss the analogies that arise between this notion of algebraic extensions and classical field-theoretic notions, and we discuss in detail the corresponding notion of algebraic closure.

Section 4 is devoted to the applications of algebraic extensions. We show that whenever an abstract property of subgroups of free groups is closed under free products and finite intersections, then every finitely generated subgroup H admits a unique closure with respect to this property, which is finitely generated and is one of the algebraic extensions of H. Examples of such properties include malnormality, purity or inertness, as well as the property of being closed for certain profinite topologies. In a number of interesting situations, this leads to simple decidability results. Equations over a subgroup, or rather the property of being closed under the addition of solutions of certain sets of equations, provide another interesting example of such an abstract property of subgroups, which we discuss in Section 4.4.

Finally, in section 5, we collect the open questions and conjectures suggested by previous sections.

2. Preliminaries

Throughout this paper, A is a finite non-empty set and $F(A)$ (or simply F if no confusion may arise) is the free group on A.

In the algorithmic or computational statements on subgroups of free groups, we tacitly assume that the free group F is given together with a basis A, that the elements of F are expressed as words over A, and that finitely generated subgroups of F are given to us by finite sets of generators, and hence by finite sets of words.

2.1. **Representation of subgroups of free groups.** In his 1983 paper [16], Stallings showed how many of the algorithmic constructions introduced in the first half of the 20th century to handle finitely generated subgroups of free groups, can be clarified and simplified by adopting a graph-theoretic language. This method has been used since then in a vast array of articles, including work by the co-authors of this paper.

The fundamental notion is the existence of a natural, algorithmically simple one-to-one correspondence between subgroups of the free group F with basis A, and certain A-labeled graphs – mapping finitely generated subgroups to finite graphs and vice versa. This is nothing else than a particular case of the more general covering theory for topological spaces, particularized to graphs and free groups. We briefly describe this correspondence in the rest of this subsection. More detailed expositions can be found in the literature: see Stallings [16] or [20, 7] for a graph-oriented version, and see one of [4, 22, 11, 15] for a more combinatorial-oriented version, written in the language of automata theory.

By an *A-labeled graph* Γ we understand a directed graph (allowing loops and multiple edges) with a designated vertex written 1, and in which each edge is labeled by a letter of A. We say that Γ is *reduced* if it is connected (more precisely, the underlying undirected graph is connected), if distinct edges with the same origin (resp. with the same end vertex) have distinct labels, and if every vertex $v \neq 1$ is adjacent to at least two different edges.

In an A-labeled graph, we consider paths, where we are allowed to travel backwards along edges. The *label* of such a path p is the word obtained by reading consecutively the labels of the edges crossed by p, reading a^{-1} whenever an edge labeled $a \in A$ is crossed backwards. The path p is called *reduced* if it does not cross twice consecutively the same edge, once in one direction and then in the other. Note that if Γ is reduced then every reduced path labels a reduced word in $F(A)$.

The subgroup of $F(A)$ associated with a reduced A-labeled graph Γ is the set of (reduced) words, which label reduced paths in Γ from the designated vertex 1 back to itself. One can show that every subgroup of $F(A)$ arises in this fashion, in a unique way. That is, for each subgroup H of $F(A)$, there exists a unique reduced A-labeled graph, written $\Gamma_A(H)$, whose set of labels of reduced closed paths at 1 is exactly H.

Moreover, if the subgroup H is given together with a finite set of generators $\{h_1, \ldots, h_r\}$ (where the h_i are non-empty reduced words over the alphabet $A \sqcup A^{-1}$), then one can effectively construct $\Gamma_A(H)$, proceeding as follows. First, one constructs r subdivided circles around a common distinguished vertex 1, each labeled by one of the h_i (and following the above convention: an inverse letter, say a^{-1} with $a \in A$, in a word h_i gives rise to an a-labeled edge in the reverse direction on the corresponding circle). If h_i has length n_i, then the corresponding circle has n_i edges and $n_i - 1$ vertices, in addition to the vertex 1. Then, we iteratively identify identically labeled pairs of edges starting (resp. ending) at the same vertex. One shows that this process terminates, that it does not matter in which order identifications take place, and that the resulting A-labeled graph is reduced and equal to $\Gamma_A(H)$. In particular, it does not depend on the choice of a set of generators of H. Also, this shows that $\Gamma_A(H)$ is finite if and only if H is finitely generated (see one of [16, 20, 4, 22, 11, 7, 15] for more details).

Example 2.1. Let $A = \{a, b, c\}$. The above procedure applied to the subgroup $H = \langle aba^{-1}, aca^{-1}\rangle$ of $F(A)$ is represented in Figure 1, where the last graph is $\Gamma_A(H)$. □

Let Γ and Δ be reduced A-labeled graphs as above. A mapping φ from the vertex set of Γ to the vertex set of Δ (we write $\varphi\colon \Gamma \to \Delta$) is a *morphism of reduced (A-)labeled graphs* if it maps the designated vertex of Γ to the designated vertex of Δ and if, for each $a \in A$, whenever Γ has an a-labeled edge e from vertex u to vertex v, then Δ has an a-labeled edge f from vertex $\varphi(u)$ to vertex $\varphi(v)$. The edge f is uniquely defined since Δ is reduced. We then extend the domain and range of φ to the edge sets of the two graphs, by letting $\varphi(e) = f$.

Note that such a morphism of reduced A-labeled graphs is necessarily locally injective (an *immersion* in [16]), in the following sense: for each vertex v of Γ, distinct edges starting (resp. ending) at v have distinct images. Further following [16], we say that the morphism $\varphi\colon \Gamma \to \Delta$ is a *cover* if it is locally bijective, that is, if the following holds: for each vertex v of Γ, each edge of Δ starting (resp. ending) at $\varphi(v)$ is the image under φ of an edge of Γ starting (resp. ending) at v.

The graph with a single vertex, called 1, and with one a-labeled loop for each $a \in A$ is called the *bouquet of A circles.* It is a reduced graph, equal to $\Gamma_A(F(A))$, and every reduced graph admits a trivial morphism into it. One can show that a subgroup H of $F(A)$ has finite index if and only if this natural morphism from $\Gamma_A(H)$ to the bouquet of A circles is a cover, and in that case, the index of H in $F(A)$ is the number of vertices of $\Gamma_A(H)$. In particular, it is easily decidable whether a finitely generated subgroup of $F(A)$ has finite index.

This graph-theoretic representation of subgroups of free groups leads to many more algorithmic results, some of which are discussed at length in this paper. We will use some well-known facts (see [16]). If H is a finitely generated subgroup of $F(A)$, then the rank of H is given by the formula

$$\mathsf{rk}(H) = E - V + 1,$$

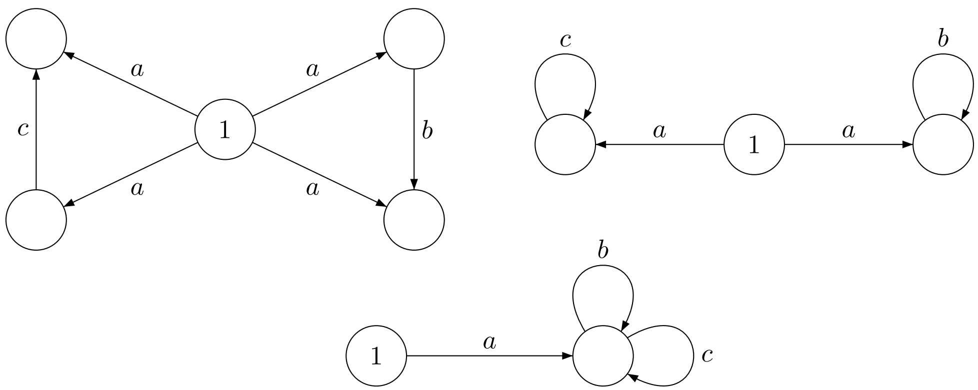

FIGURE 1. Computing the representation of $H = \langle aba^{-1}, aca^{-1} \rangle$

where E (resp. V) is the number of edges (resp. vertices) in $\Gamma_A(H)$. A more precise result shows how each spanning tree in $\Gamma_A(H)$ (a subtree of the graph $\Gamma_A(H)$ which contains every vertex) determines a basis of H. It is also interesting to note that if H and K are finitely generated subgroups of $F(A)$, then $\Gamma_A(H \cap K)$ can be easily constructed from $\Gamma_A(H)$ and $\Gamma_A(K)$: one first considers the A-labeled graph whose vertices are pairs (u, v) consisting of a vertex u of $\Gamma_A(H)$ and a vertex v of $\Gamma_A(K)$, with an a-labeled edge from (u, v) to (u', v') if and only if there are a-labeled edges from u to u' in $\Gamma_A(H)$ and from v to v' in $\Gamma_A(K)$. Finally, one considers the connected component of vertex $(1, 1)$ in this product, and we repeatedly remove the vertices of valence 1, other than the distinguished vertex $(1, 1)$ itself, to make it a reduced A-labeled graph.

To conclude this section, it is very important to observe that if we change the ambient basis of F from A to B, we may radically modify the labeled graph associated with a subgroup H of F, see Example 2.2 below. In fact, a clearer understanding of the transformation from $\Gamma_A(H)$ to $\Gamma_B(H)$ (put otherwise: of the action of the automorphism group of $F(A)$ on the A-labeled reduced graphs) is one of the challenges of the field.

Example 2.2. Let F be the free group with basis $A = \{a, b, c\}$, and let $H = \langle ab, acba \rangle$. Note that $B = \{a', b', c'\}$ is also a basis of F, where $a' = a$, $b' = ab$ and $c' = acba$. The graphs $\Gamma_A(H)$ and $\Gamma_B(H)$ are depicted in Figure 2. □

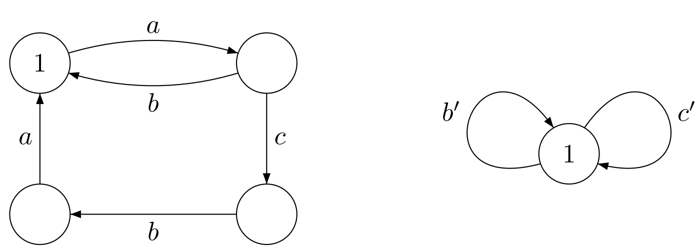

FIGURE 2. The graphs $\Gamma_A(H)$ and $\Gamma_B(H)$

2.2. **Subgroups of subgroups.** A pair of free groups $H \leq K$ is called an *extension* of free groups. If $H \leq M \leq K$ are free groups then $H \leq M$ will be referred to as a *sub-extension* of $H \leq K$.

If $H \leq K$ is an extension of free groups, we use the following shorthand notation: $H \leq_{\mathsf{fg}} K$ means that H is finitely generated; $H \leq_{\mathsf{fi}} K$ means that H has finite index in K; and $H \leq_{\mathsf{ff}} K$ means that H is a free factor of K.

Extensions can be characterized by means of the labeled graphs associated with subgroups as in Section 2.1. We first note the following simple result (see [11, Proposition 2.4] or [7, Section 4]).

Lemma 2.3. *Let H, K be subgroups of a free group F with basis A. Then $H \leq K$ if and only if there exists a morphism of labeled graphs $\varphi_{H,K}$ from $\Gamma_A(H)$ to $\Gamma_A(K)$. If it exists, this morphism is unique.*

Given an extension $H \leq K$ between subgroups of the free group with basis A, certain properties of the resulting morphism $\varphi_{H,K}$ have a natural translation on the relation between H and K. For instance, it is not difficult to verify that $\varphi_{H,K}$ is a covering if and only if H has finite index in K (and the index is the cardinality of each fibre). This generalizes the characterization of finite index subgroups of $F(A)$ given in the previous section.

If $\varphi_{H,K}$ is one-to-one (and that is, if and only if it is one-to-one on vertices), then H is a free factor of K. Unfortunately, the converse is far from holding since each non-cyclic free group has infinitely many free factors. Furthermore, given K, the particular collection of free factors $H \leq_{\mathsf{ff}} K$ such that $\varphi_{H,K}$ is one-to-one heavily depends on the ambient basis.

We recall here, for further reference, the following well-known properties of free factors (see [8] or [9]).

Lemma 2.4. *Let H, K, L, $(H_i)_{i\in I}$ and $(K_i)_{i\in I}$ be subgroups of a free group F.*

(i) *If $H \leq_{\mathsf{ff}} K \leq_{\mathsf{ff}} L$, then $H \leq_{\mathsf{ff}} L$.*
(ii) *If $H_i \leq_{\mathsf{ff}} K_i$ for each $i \in I$, then $\bigcap_i H_i \leq_{\mathsf{ff}} \bigcap_i K_i$.*

In particular, if H is a free factor of each K_i, then H is a free factor of their intersection; and an intersection of free factors of K is again a free factor of K.

Finally, in the situation $H \leq K$, we say that K is an *A-principal overgroup* of H if $\varphi_{H,K}$ is onto (both on vertices and on edges). We refer to the set of all A-principal overgroups of H as the *A-fringe* of H, denoted $\mathcal{O}_A(H)$. As seen later, this set strongly depends on A. The A-fringe of H is finite whenever H is finitely generated.

Principal overgroups were first considered under the name of *overgroups* in [11] (see [22] as well). They also appeared later as *principal quotients* in [7], and their first introduction is in the earlier [20], where $\mathcal{O}_A(H)$ was called the *fringe* of H, its *orla* in catalan. We shall use the phrase *principal overgroup* (to stress the fact that not every K containing H is a principal overgroup of H) and *fringe*, omitting the reference to the basis A when there is no risk of confusion. Both *orla* and *overgroup* justify the notation $\mathcal{O}_A(H)$.

Given a finitely generated subgroup $H \leq F(A)$, the fringe $\mathcal{O}_A(H)$ is computable: it suffices to compute $\Gamma_A(H)$, and to consider each equivalence relation $\sim$ on the set of vertices of $\Gamma_A(H)$. Say that such an equivalence relation $\sim$ is a congruence (with respect to the labeled graph structure of $\Gamma_A(H)$) if, whenever $p \sim q$ and there are a-labeled edges from p to p' and from q to q' (resp. from p' to p and from q' to q), then $p' \sim q'$. Then each congruence gives rise to a surjective morphism from $\Gamma_A(H)$ onto $\Gamma_A(H)/\sim$, and hence to a principal overgroup K of H such that $\Gamma_A(K) = \Gamma_A(H)/\sim$. Moreover, each principal overgroup $K \in \mathcal{O}_A(H)$ arises in this fashion. At the time of writing, a computer program is being developed with the purpose, among others, of efficiently computing the fringe of a finitely generated subgroup of a free group (see [13]).

Example 2.5. Let F be the free group with basis $A = \{a, b, c\}$, and let $H = \langle ab, acba \rangle \leq F$ (the graph $\Gamma_A(H)$ was constructed in Example 2.2). Successively identifying pairs of vertices of $\Gamma_A(H)$ and reducing the resulting A-labeled graph in all possible ways, one concludes that $\Gamma_A(H)$ has six congruences, whose corresponding quotient graphs are depicted in Figure 3.

Thus the A-fringe of H consists on $\mathcal{O}_A(H) = \{H_0, H_1, H_2, H_3, H_4, H_5\}$, where $H_0 = H$, $H_1 = \langle ab, ac, ba \rangle$, $H_2 = \langle ba, ba^{-1}, cb \rangle$, $H_3 = \langle ab, ac, ab^{-1}, a^2 \rangle$, $H_4 = \langle ab, aca, acba \rangle$ and $H_5 = \langle a, b, c \rangle = F(A)$.

However, with respect to the basis $B = \{a, ab, acba\}$ of F, the graph $\Gamma_B(H)$ has a single vertex, and hence the B-fringe of H is much simpler, $\mathcal{O}_B(H) = \{H\}$. $\square$

Finally we observe that, if $H \leq_{\mathsf{fi}} F(A)$, then $\mathcal{O}_A(H)$ consists of all the extensions of H. Indeed, suppose that $H \leq K \leq F(A)$ and $H \leq_{\mathsf{fi}} F(A)$. Since $\Gamma_A(H)$ is a cover of the bouquet of A circles (that is, each vertex of $\Gamma_A(H)$ is the origin and the end of an a-labeled edge for each $a \in A$), the range of $\varphi_{H,K}$ is also a cover of the bouquet of A circles. It follows that $\varphi_{H,K}$ is onto, since $\Gamma_A(K)$ is connected, and so $K \in \mathcal{O}_A(H)$. In particular, if $H \leq_{\mathsf{fi}} F(A)$, then $\mathcal{O}_A(H)$ does not depend on A, in contrast with what happens in general.

2.3. **Takahasi's theorem.** Of particular interest to our discussion is the following 1951 result by Takahasi (see [9, Section 2.4, Exercise 8], [17, Theorem 2] or [20, Theorem 1.7]).

Theorem 2.6 (Takahasi)**.** *Let $F(A)$ be the free group on A and $H \leq F(A)$ a finitely generated subgroup. Then, there exists a finite computable collection of extensions of H, say $H = H_0, H_1, \ldots, H_n \leq F(A)$ such that every extension K of H, $H \leq K \leq F(A)$, is a free multiple of one of the H_i.*

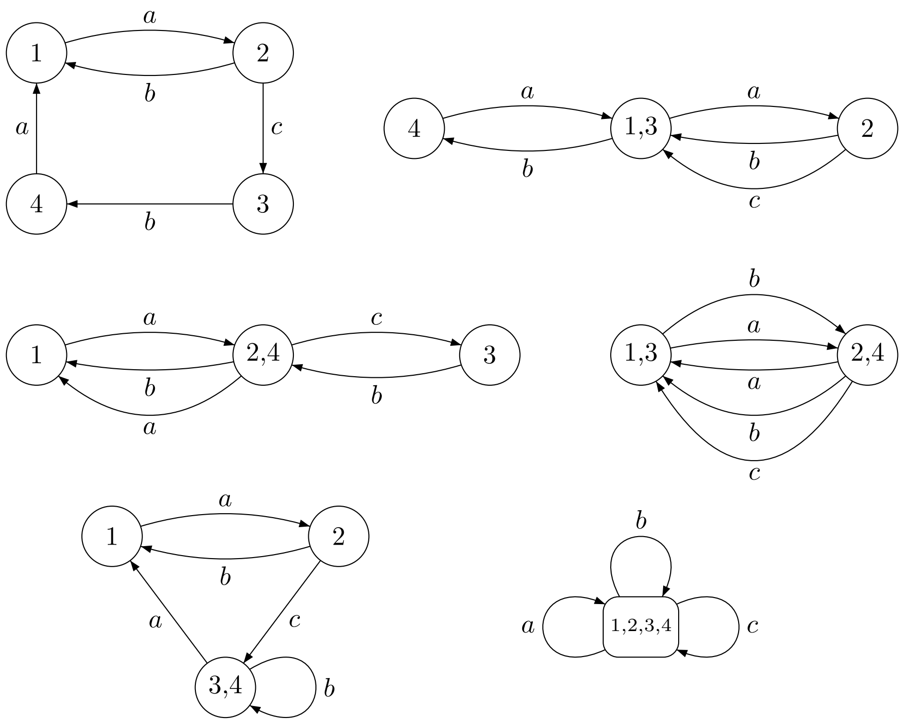

FIGURE 3. The six quotients of $\Gamma_A(\langle ab, acba\rangle)$

The original proof, due to M. Takahasi was combinatorial, using words and their lengths with respect to different sets of generators. The geometrical apparatus described in this section leads to a clear, concise and natural proof, which was discovered independently by Ventura in [20] and by Kapovich and Miasnikov in [7]. Margolis, Sapir and Weil, also independently considered the same construction in [11] for a slightly different purpose. Finally, we note that Turner considered a similar construction in the case of cyclic subgroups, in his work about test words [18]. We now give this proof of Takahasi's theorem.

Proof. Let K be an extension of H, and let $\varphi_{H,K}\colon \Gamma_A(H) \to \Gamma_A(K)$ be the resulting graph morphism. Note that the image of $\varphi_{H,K}$ is a reduced subgraph of $\Gamma_A(K)$, and let $L_{H,K}$ be the subgroup of $F(A)$ such that $\Gamma_A(L_{H,K}) = \varphi_{H,K}(\Gamma_A(H))$. By definition, $L_{H,K}$ is an A-principal overgroup of H and, by construction, $\Gamma_A(L_{H,K})$ is a subgraph of $\Gamma_A(K)$, which implies $L_{H,K} \leq_{\mathsf{ff}} K$ (see Section 2.2). It follows immediately that the A-fringe of H, $\mathcal{O}_A(H)$, satisfies the required conditions. $\square$

Thus, for a given $H \leq_{\mathsf{fg}} F(A)$, the A-principal overgroups of H form one possible collection of extensions that satisfy the requirements of Takahasi's theorem, let us say, a *Takahasi family for H*. This is certainly not the only one: firstly, we may add arbitrary subgroups to a Takahasi family; secondly, we observe that the statement of the theorem does not depend on the ambient basis, so if B is another basis of $F(A)$, then $\mathcal{O}_B(H)$ forms a Takahasi family for H as well. There does however exist a minimum Takahasi family for H (see Proposition 3.7 below), which in particular does not depend on the ambient basis. The main object of this paper is a discussion of this family, which is introduced in the next section.

3. Algebraic extensions

The notion of algebraic extension discussed in this paper was first introduced by Kapovich and Miasnikov [7]. It seems to be mostly of interest for finitely generated subgroups, but many definitions and results hold in general and we avoid restricting ourselves to finitely generated subgroups until that becomes necessary.

3.1. **Definitions.** Let $H \leq K$ be an extension of free groups and let $x \in K$. We say that x is *K-algebraic over H* if every free factor of K containing H, $H \leq L \leq_{\mathsf{ff}} K$, satisfies $x \in L$. Otherwise (i.e. if there exists $H \leq L \leq_{\mathsf{ff}} K$ such that $x \notin L$) we say that x is *K-transcendental over H*.

Example 3.1. If $H \leq K$, then every element $x \in H$ is obviously K-algebraic over H.

Every element $x \in K$ is K-algebraic over $\langle x^n \rangle$, for each integer $n \neq 0$. In fact, it is straightforward to verify that if x^n lies in a free factor L of K, then so does x.

If x is primitive in K (that is, if $\langle x \rangle \leq_{\mathsf{ff}} K$), then every element of $K \setminus \langle x \rangle$ is K-transcendental over the subgroup $\langle x \rangle$.

The notion of algebraicity over H is relative to K. For example, in $F = F(a,b)$, a^2 is $\langle a^2, b^2 \rangle$-transcendental over $H = \langle a^2 b^2 \rangle$ since $a^2 b^2$ is primitive in $\langle a^2, b^2 \rangle$. However, a^2 is F-algebraic over H because no proper free factor of F contains $a^2 b^2$. $\square$

The following is a trivial but useful observation.

Fact 3.2. *Let $H \leq K$ be an extension of free groups, and let $x, y \in K$.*

(i) *If x, y are K-algebraic over H then so are x^{-1} and xy.*
(ii) *If x, y are K-transcendental over H then so is x^{-1} (but not in general xy).*

We say that an extension of free groups $H \leq K$ is *algebraic*, and we write $H \leq_{\mathsf{alg}} K$, if every element of K is K-algebraic over H. It is called *purely transcendental* if every element of K is either in H or is K-transcendental over H. Naturally, there are extensions that are neither algebraic nor purely transcendental. These concepts were originally introduced in [7], and the following propositions further describe their properties.

Proposition 3.3. *Let $H \leq K$ be an extension of free groups. The following are equivalent:*

(a) *H is contained in no proper free factor of K;*
(b) *$H \leq_{\mathsf{alg}} K$, that is, every $x \in K$ is K-algebraic over H;*
(c) *there exists $X \subseteq K$ such that $K = \langle H \cup X \rangle$ and every $x \in X$ is K-algebraic over H (furthermore, if K is finitely generated, one may choose X to be finite).*

Proof. (b) follows from (a) by definition. If (b) holds, then (c) holds with X any system of generators for K. Finally, (a) follows from (c) in view of Fact 3.2 (i). □

Proposition 3.4. *Let $H \leq K$ be an extension of free groups. The following are equivalent:*

(a) *H is a free factor of K,*
(b) *$H \leq K$ is purely transcendental, that is, every $x \in K \setminus H$ is K-transcendental over H.*

Proof. (a) implies (b) by definition. To prove the converse, let M be the intersection of all the free factors of K containing H. By Lemma 2.4, M is a free factor of K containing H, and (b) implies that $M = H$. □

Example 3.5. It is easily verified (say, using Example 3.1) that if $1 \neq x \in F$ and $n \neq 0$, we have $\langle x^n \rangle \leq_{\mathsf{alg}} \langle x \rangle$.

By Proposition 3.4, an extension of the form $\langle x \rangle \leq F$ is purely transcendental if and only if x is a primitive element of F. Moreover, if F has rank two, then $\langle x \rangle \leq F$ is algebraic if and only if x is not a power of a primitive element of F.

Assuming again that F has rank two, $H \leq_{\mathsf{alg}} F$ for every non-cyclic subgroup H. Indeed, every proper free factor of F is cyclic and hence cannot contain H. □

We denote by $\mathsf{AE}(H)$ the set of algebraic extensions of H, and we observe that, in contrast with the definition of principal overgroups, this set does not dependent on the choice of an ambient basis. This same observation can be expressed as follows.

Fact 3.6. *Let $H \leq K \leq F$ be extensions of free groups and let $\varphi \in \mathsf{Aut}(F)$. Then $H \leq_{\mathsf{alg}} K$ if and only if $\varphi(H) \leq_{\mathsf{alg}} \varphi(K)$.*

We can now express the connection between algebraic extensions and Takahasi's theorem.

Proposition 3.7. *Let $H \leq_{\mathsf{fg}} F(A)$ be an extension of free groups. Then we have:*

(i) $\mathsf{AE}(H) \subseteq \mathcal{O}_A(H)$;
(ii) $\mathsf{AE}(H)$ *is finite (i.e., H admits only a finite number of algebraic extensions);*
(iii) $\mathsf{AE}(H)$ *is the set of $\leq_{\mathsf{ff}}$-minimal elements of every Takahasi family for H (see Section 2.3);*
(iv) $\mathsf{AE}(H)$ *is the minimum Takahasi family for H.*

Proof. Let K be an algebraic extension of H. The proof of Takahasi's theorem shows that K is a free multiple of some principal overgroup $L \in \mathcal{O}_A(H)$. Then, Proposition 3.3 implies that $L = K$ proving (i). Statement (ii) follows immediately.

Let $\mathcal{L}$ be a Takahasi family for H and let $K \in \mathsf{AE}(H)$. By definition of $\mathcal{L}$, there exists a subgroup $L \in \mathcal{L}$ such that $H \leq L \leq_{\mathsf{ff}} K$. By Proposition 3.3, it follows that $L = K$, so $K \in \mathcal{L}$. Thus $\mathsf{AE}(H)$ is contained in every Takahasi family for H. For the same reason, K is $\leq_{\mathsf{ff}}$-minimal in $\mathcal{L}$.

Now suppose that $K \in \mathcal{L}$ is $\leq_{\mathsf{ff}}$-minimal in $\mathcal{L}$, and let M be an extension of H such that $H \leq M \leq_{\mathsf{ff}} K$. By definition of a Takahasi family, there exists $L \in \mathcal{L}$ such that $H \leq L \leq_{\mathsf{ff}} M$, so $L \leq_{\mathsf{ff}} K$. Since K is $\leq_{\mathsf{ff}}$-minimal in $\mathcal{L}$, it follows that $L = K$, so $M = K$. Hence, $H \leq_{\mathsf{alg}} K$ concluding the proof of (iii).

Finally, it is immediate that the $\leq_{\mathsf{ff}}$-minimal elements of a Takahasi family for H again form a Takahasi family. Statement (iv) follows directly. □

Example 3.8. If $H \leq_{\mathsf{fi}} K$, then $H \leq_{\mathsf{alg}} K$. This follows immediately from the observation that a proper free factor of K has infinite index.

It follows that, if $H \leq_{\mathsf{fi}} F(A)$, then $\mathsf{AE}(H) = \mathcal{O}_A(H)$ is equal to the set of all extensions of H. Indeed, we have already observed at the end of Section 2.2 that every extension of H is an A-principal overgroup of H, and since H has finite index in each of its extensions, it is algebraic in each. □

Proposition 3.7 shows that $\mathsf{AE}(H)$ is contained in $\mathcal{O}_A(H)$ for each ambient basis A. We conjecture that $\mathsf{AE}(H)$ is in fact equal to the intersection of the sets $\mathcal{O}_A(H)$, when A runs over all the bases of F. Example 3.8 shows that the conjecture holds if H has finite index. It also holds if $H \leq_{\mathsf{ff}} F$, since in that case, $\mathsf{AE}(H) = \{H\}$, and F admits a basis B relative to which $\Gamma_B(H)$ is a graph with a single vertex.

We conclude with a simple but important statement.

Proposition 3.9. *Let $F(A)$ be the free group on A and $H \leq_{\mathsf{fg}} F(A)$. The set $\mathsf{AE}(H)$ is computable.*

Proof. Since every algebraic extension of H is in $\mathcal{O}_A(H)$, it suffices to compute $\mathcal{O}_A(H)$ and then, for each pair of distinct elements $K, L \in \mathcal{O}_A(H)$, to decide whether $L \leq_{\mathsf{ff}} K$: $\mathsf{AE}(H)$ consists of the principal overgroups of H that do not contain another principal overgroup as a free factor.

In order to conclude, we observe that deciding whether $L \leq_{\mathsf{ff}} K$ can be done, for example, using the first part the classical Whitehead's algorithm. More precisely, Whitehead's algorithm (see [8, Proposition 4.25]) shows how to decide whether a tuple of elements, say $u = (u_1, \ldots, u_r)$, of a free group K can be mapped to another tuple $v = (v_1, \ldots, v_r)$ by some automorphism of K. The first part of this algorithm reduces the sum of the length of the images of the u_i to its minimal possible value. And it is easy to verify that this minimal total length is exactly r if and only if $\{u_1, \ldots, u_r\}$ freely generates a free factor of K. We point out here that an alternative algorithm was recently proposed by Silva and Weil [15]. That algorithm is faster, and completely based on graphical tools. $\square$

The efficiency of the algorithm to compute $\mathsf{AE}(H)$ sketched in the proof of Proposition 3.9, is far from optimal. An upcoming paper by A. Roig, E. Ventura and P. Weil discusses better computation techniques for that purpose [13].

Remark 3.10. The terminology adopted for the concepts developed in this section is motivated by an analogy with the theory of field extensions. More precisely, if an element $x \in K$ is K-transcendental over H, then H is a free factor of $\langle H, x\rangle$ and $\langle H, x\rangle = H * \langle x\rangle$ (see Proposition 3.13 below). This is similar to the field-theoretic definition of transcendental elements: an element x is transcendental over H if and only if the field extension of H generated by x is isomorphic to the field of rational fractions $H(X)$.

However, the analogy is not perfect and in particular, the converse does not hold. For instance, a^2 is $\langle a, b\rangle$-algebraic over $\langle a^2b^2\rangle$ (see Example 3.1), but $\langle a^2b^2, a^2\rangle = \langle a^2b^2\rangle * \langle a^2\rangle$. This stems from the fact, noticed earlier, that the notion of an element x being K-algebraic over H, depends on K and not just on x.

It is natural to ask whether the analogy also extends to the definition of algebraic elements: in other words, is there a natural analogue in this context for the notion of roots of a polynomial with coefficients in H? The discussion of equations in Section 4.4 offers some insight into this question. □

3.2. **Composition of extensions.** We now consider compositions of extensions. Some of the results in the following proposition come from [7]. We restate and extend them here with simpler proofs. We also include in the statement well-known facts (the primed statements), in order to emphasize the dual properties of algebraic and purely transcendental extensions.

Proposition 3.11. *Let $H \leq K$ be an extension of free groups, and let $H \leq K_i \leq K$ be two sub-extensions, $i = 1, 2$.*

- (i) *If $H \leq_{\mathsf{alg}} K_1 \leq_{\mathsf{alg}} K$ then $H \leq_{\mathsf{alg}} K$.*
- (i') *If $H \leq_{\mathsf{ff}} K_1 \leq_{\mathsf{ff}} K$ then $H \leq_{\mathsf{ff}} K$.*
- (ii) *If $H \leq_{\mathsf{alg}} K$ then $K_1 \leq_{\mathsf{alg}} K$, while $H \leq K_1$ need not be algebraic.*
- (ii') *If $H \leq_{\mathsf{ff}} K$ then $H \leq_{\mathsf{ff}} K_1$, while $K_1 \leq K$ need not be purely transcendental.*
- (iii) *If $H \leq_{\mathsf{alg}} K_1$ and $H \leq_{\mathsf{alg}} K_2$ then $H \leq_{\mathsf{alg}} \langle K_1 \cup K_2 \rangle$, while $H \leq K_1 \cap K_2$ need not be algebraic.*
- (iii') *If $H \leq_{\mathsf{ff}} K_1$ and $H \leq_{\mathsf{ff}} K_2$ then $H \leq_{\mathsf{ff}} K_1 \cap K_2$, while $H \leq \langle K_1 \cup K_2 \rangle$ need not be purely transcendental.*

Proof. Statement *(i')* and the positive parts of statements *(ii')* and *(iii')* can be found in Lemma 2.4. The free group F on $\{a, b\}$ already contains counterexamples for the converse statements in *(ii')* and *(iii')*: for the first one, we have $\langle a \rangle \leq_{\mathsf{ff}} F$ while $\langle a \rangle \leq_{\mathsf{ff}} \langle a, b^2 \rangle \leq_{\mathsf{alg}} F$ (see Example 3.5). And for the second one, we have $\langle [a, b] \rangle \leq_{\mathsf{ff}} \langle a, [a, b] \rangle$ and $\langle [a, b] \rangle \leq_{\mathsf{ff}} \langle b, [a, b] \rangle$, whereas $\langle [a, b] \rangle \leq_{\mathsf{alg}} \langle a, [a, b], b \rangle = F$.

Now assume that $H \leq_{\mathsf{alg}} K_1 \leq_{\mathsf{alg}} K$ and let L be a free factor of K containing H. Then, $L \cap K_1$ is a free factor of K_1 containing H by Lemma 2.4. Since $H \leq K_1$ is algebraic, we deduce that $L \cap K_1 = K_1$, and hence $K_1 \leq L$. But $K_1 \leq_{\mathsf{alg}} K$, so $L = K$. Thus, the extension $H \leq K$ is algebraic, which proves *(i)*.

The first part of *(ii)* is clear. A counterexample for the second part in $F = F(a, b)$ is as follows: we have $\langle [a, b] \rangle \leq_{\mathsf{ff}} \langle a, [a, b] \rangle \leq F$, while $\langle [a, b] \rangle \leq_{\mathsf{alg}} F$ by Example 3.5.

Suppose now that $H \leq_{\mathsf{alg}} K_1$ and $H \leq_{\mathsf{alg}} K_2$, and let L be a free factor of $\langle K_1 \cup K_2 \rangle$ containing H. Then Lemma 2.4 shows that, for $i = 1, 2$, $L \cap K_i \leq_{\mathsf{ff}} K_i$ containing H. Since $H \leq_{\mathsf{alg}} K_i$, we deduce that $L \cap K_i = K_i$ and hence, $K_i \leq L$. Thus, $L = \langle K_1 \cup K_2 \rangle$, and the extension $H \leq \langle K_1 \cup K_2 \rangle$ is algebraic, thus proving the positive part of *(iii)*.

Finally, to conclude the proof of *(iii)*, it suffices to exhibit subgroups H, K_1, K_2 such that $H \leq_{\mathsf{alg}} K_i$ $(i = 1, 2)$ but $H \leq_{\mathsf{ff}} K_1 \cap K_2$. Again in

$F(a,b)$ take, for example, $K_1 = \langle a^2, b\rangle$ and $K_2 = \langle a^3, b\rangle$, whose intersection is $K_1 \cap K_2 = \langle a^6, b\rangle$. Letting $H = \langle a^6 b\rangle$, we have $H \leq_{\mathsf{ff}} K_1 \cap K_2$ but $H \leq_{\mathsf{alg}} K_1$ and $H \leq_{\mathsf{alg}} K_2$. $\square$

To close this section, let us note another natural property of algebraic extensions, which slightly generalizes a result of Kapovich and Miasnikov [7].

Proposition 3.12. *Let F be a free group. If $H_i \leq_{\mathsf{alg}} K_i \leq F$ $(i \in I)$, then $\langle \bigcup_i H_i\rangle \leq_{\mathsf{alg}} \langle \bigcup_i K_i\rangle$. The converse holds if $\langle \bigcup_i K_i\rangle = *_i K_i$.*

Proof. Suppose that $\langle \bigcup_i H_i\rangle \leq L \leq_{\mathsf{ff}} \langle \bigcup_i K_i\rangle$. Let $j \in I$. By Lemma 2.4, we have $L \cap K_j \leq_{\mathsf{ff}} \langle \bigcup_i K_i\rangle \cap K_j = K_j$. Moreover, $H_j \leq L \cap K_j$, so $K_j = L \cap K_j$ since $H_j \leq_{\mathsf{alg}} K_j$, and hence $K_j \subseteq L$. This holds for each $j \in I$, so $L = \langle \bigcup_i K_i\rangle$ and we have shown that $\langle \bigcup_i H_i\rangle \leq_{\mathsf{alg}} \langle \bigcup_i K_i\rangle$.

For the converse, suppose that $\langle \bigcup_i K_i\rangle = *_i K_i$. It follows that $\langle \bigcup_i H_i\rangle = *_i H_i$. Now we assume that $*_i H_i \leq_{\mathsf{alg}} *_i K_i$. Let $j \in I$. If $H_j \leq L \leq_{\mathsf{ff}} K_j$, then $*_i H_i \leq L * *_{i\neq j} K_i \leq_{\mathsf{ff}} *_i K_i$ and hence $L * *_{i\neq j} K_i = *_i K_i$. Taking the projection onto K_j, it follows that $L = K_j$. Thus $H_j \leq_{\mathsf{alg}} K_j$ for each $j \in I$. $\square$

Note that the converse of Proposition 3.12 does not hold in general, as can be seen from the counterexample provided in the proof of Proposition 3.11 (iii').

3.3. **Elementary extensions.** We say that an extension of free groups $H \leq K$ is *elementary* if $K = \langle H, x\rangle$ for some $x \in K$. Elementary extensions turn out to be either algebraic or purely transcendental, as we now see.

Proposition 3.13. *Let $H \leq F$ be an extension of free groups and let $x \in F$. Let also X be a new letter, not in F. The following are equivalent:*

- (a) *the morphism $H * \langle X\rangle \to F$ acting as the identity over H and sending X to x is injective;*
- (b) *H is a proper free factor of $\langle H, x\rangle$;*
- (c) *H is contained in a proper free factor of $\langle H, x\rangle$.*

If, in addition, H is finitely generated, then these are further equivalent to:

- (d) *$\mathsf{rk}(\langle H, x\rangle) = \mathsf{rk}(H) + 1$;*
- (e) *$\mathsf{rk}(\langle H, x\rangle) > \mathsf{rk}(H)$.*

Proof. It is immediately clear that statement (a) implies (b), and that (b) implies (c).

At this point, let us assume that H has finite rank. It is immediate that $\mathsf{rk}(\langle H, x\rangle) \leq \mathsf{rk}(H) + 1$, so (b) implies (d) and (d) and (e) are equivalent. Now consider the morphism from $H * \langle X\rangle$ to $\langle H, x\rangle$ mapping H identically to itself, and X to x. This morphism is surjective by construction, and if $\mathsf{rk}(\langle H,x\rangle) = \mathsf{rk}(H * \langle X\rangle)$, then it is injective by the hopfian property of finitely generated free groups. That is, (d) implies (a). Thus we have shown that if H has finite rank, then statements (a), (b), (d) and (e) are equivalent. It only remains to prove that (c) implies (a).

We now return to the general case, where H may have infinite rank, and we assume that (c) holds, that is, $\langle H,x\rangle = K * L$ for some $L \neq 1$ and $H \leq K$. We have $\langle H,x\rangle \leq \langle K,x\rangle$, and hence $\langle K,x\rangle = \langle H,x\rangle = K * L$. Moreover, $x \notin K$ and we let $x = k_0\ell_1k_1\cdots\ell_rk_r$ be the normal form of x in the free product $K * L$.

Let M be a finitely generated free factor of K containing the k_i, and let N be such that $K = M * N$. First we observe that

$$\langle M,x\rangle \leq M * L \leq_{\mathsf{ff}} K * L = M * N * L = \langle K,x\rangle = \langle M,N,x\rangle.$$

It follows that N is a free complement of $\langle M,x\rangle$ in $\langle K,x\rangle$, that is, $\langle K,x\rangle = \langle M,x\rangle * N$.

Next we note that $M \leq_{\mathsf{ff}} K \leq_{\mathsf{ff}} \langle K,x\rangle$, so $M \leq_{\mathsf{ff}} \langle K,x\rangle$ and hence $M \leq_{\mathsf{ff}} \langle M,x\rangle$. Since $x \notin K$, M is a finitely generated, proper free factor of $\langle M,x\rangle$, and we already know that this implies that the morphism from $M * \langle X\rangle$ to F mapping M identically to itself and mapping X to x, is injective. Since N is a free complement of the range of this morphism in $\langle H,x\rangle$, and also a free complement of M in K, it follows that the natural mapping from $K * \langle X\rangle$ to F mapping X to x is injective. Its restriction to $H * \langle X\rangle$ is therefore injective, and statement (a) holds, which completes the proof. $\square$

Proposition 3.13 immediately translates into the following.

Corollary 3.14. *Let F be a free group and $H \leq K$ be an elementary extension of subgroups of F. Then, either $H \leq_{\mathsf{alg}} K$ or $H \leq_{\mathsf{ff}} K$. Furthermore, if H is finitely generated then $\mathsf{rk}(K) \leq \mathsf{rk}(H) + 1$ with equality if and only if $H \leq_{\mathsf{ff}} K$.*

Let us say that an extension $H \leq K$ is *e-algebraic*, written $H \leq_{\mathsf{ealg}} K$, if it splits as a finite composition of algebraic, elementary extensions, $H \leq_{\mathsf{alg}} H_1 \leq_{\mathsf{alg}} \cdots \leq_{\mathsf{alg}} H_k = K$. Then Proposition 3.13 yields the following.

Corollary 3.15. *Let H be a finitely generated subgroup of a free group F and let $H \leq_{\mathsf{ealg}} K$ be an e-algebraic extension. Then $\mathsf{rk}(K) \leq \mathsf{rk}(H)$.*

Obviously, every extension $H \leq K$ with K finitely generated, splits into a composition of elementary extensions, but an algebraic extension $H \leq_{\mathsf{alg}} K$ cannot always be split into a composition of algebraic elementary extensions. In view of Corollary 3.15, this is the case for the algebraic extension $\langle [a,b]\rangle \leq_{\mathsf{alg}} F(a,b)$. Thus, $H \leq_{\mathsf{alg}} K$ does not imply $H \leq_{\mathsf{ealg}} K$.

3.4. **Algebraic closure of a subgroup.** If $H \leq K$ is an extension of free groups, there exists a greatest algebraic extension of H inside K. This can be deduced from Proposition 3.12, but the following theorem is a more precise statement.

Theorem 3.16. *Let $H \leq L \leq K$ be extensions of free groups. The following are equivalent.*

(a) $H \leq_{\mathsf{alg}} L \leq_{\mathsf{ff}} K$.
(b) *L is the intersection of the free factors of K containing H.*
(c) *L is the set of elements of K that are K-algebraic over H.*
(d) *L is the greatest algebraic extension of H contained in K.*

In this case, the subgroup L is uniquely determined by H and K.

Proof. Let $x \in K$. By definition, x is K-algebraic over H if and only if x sits in every free factor of K containing H. This is exactly the equivalence of statements (b) and (c). The equivalence of (c) and (d) is a direct consequence of the fact that the elements that are K-algebraic over H form a subgroup (Fact 3.2). Thus statements (b), (c) and (d) are equivalent.

Now let L be defined as in (b): by (d), $H \leq_{\mathsf{alg}} L$. Now let $x \in K \setminus L$. Since x is not algebraic over H, there exists a free factor $M \leq_{\mathsf{ff}} K$ containing H and missing x. But $L \leq M$, so x is not K-algebraic over L either. It follows that the extension $L \leq K$ is purely transcendental, and hence $L \leq_{\mathsf{ff}} K$ by Proposition 3.4. This proves (b) implies (a).

Finally, let us assume that $H \leq_{\mathsf{alg}} L \leq_{\mathsf{ff}} K$ for some L. Let M be such that $H \leq M \leq_{\mathsf{ff}} K$. Then $L \cap M \leq_{\mathsf{ff}} L$ by Lemma 2.4 (ii). But we also have $H \leq L \cap M \leq L$ and $H \leq_{\mathsf{alg}} L$. It follows that $L \cap M = L$, that is $L \leq M$, and (b) follows. This concludes the proof. □

Remark 3.17. It is interesting to compare Theorem 3.16 with M. Hall's Theorem, stating that every finitely generated subgroup $H \leq F$ is a free factor of a subgroup M of finite index in F. In other words, one can split the extension $H \leq F$ in two parts, $H \leq_{\mathsf{ff}} M \leq_{\mathsf{fi}} F$, the first being purely transcendental, and the second being finite index (and hence, algebraic). Note that the intermediate subgroup M is not unique in general. Theorem 3.16 yields a "dual" splitting of the extension $H \leq F$, where the order between the transcendental and the algebraic parts is switched around, and with the additional nice property that the intermediate extension is now uniquely determined by $H \leq F$. □

Let $H \leq K$ be an extension of free groups. The subgroup L characterized in Theorem 3.16 is called the *K-algebraic closure* of H, denoted $\mathsf{cl}_K(H)$. It is natural to consider the extremal situations, where $\mathsf{cl}_K(H) = H$ (we say that H is K-*algebraically closed*) and where $\mathsf{cl}_K(H) = K$ (we say that H is K-*algebraically dense*). Of course, these situations coincide with $H \leq K$ being purely transcendental and algebraic, respectively.

Fact 3.18. *Let $H \leq K$ be an extension of free groups. Then,*

(i) *H is K-algebraically closed if and only if $H \leq_{\mathsf{ff}} K$,*
(ii) *H is K-algebraically dense if and only if $H \leq_{\mathsf{alg}} K$.*

As established in the following proposition, maximal proper retracts of a finitely generated free group K are good examples of extremal subgroups, i.e. subgroups of K that are either K-algebraically closed or K-algebraically dense. Recall that a subgroup $H \leq K$ is a *retract* of K if the identity $\mathsf{id}\colon H \to H$ extends to a homomorphism $K \to H$, called a *retraction* (see [9] for a general description of retracts of finitely generated free groups); in particular, free factors of K are retracts of K. Note that if H is a retract of K then $\mathsf{rk}(H) \leq \mathsf{rk}(K)$. Moreover, if K is finitely generated, the hopfian property of finitely generated free groups shows that K is the unique retract of K with rank equal to $\mathsf{rk}(K)$. So, if H is a proper retract of K then $\mathsf{rk}(H) < \mathsf{rk}(K)$.

We also say that H is *compressed in* K (see [5]) if $\mathsf{rk}(H) \leq \mathsf{rk}(L)$ for each $H \leq L \leq K$. By restricting a retraction to L, it is clear that every retract of K (and, in particular, every free factor of K) is compressed in K.

Proposition 3.19. *Let K be a finitely generated free group. A maximal proper compressed subgroup (resp. a maximal proper retract) H of K is either K-algebraically dense, or K-algebraically closed. In the latter case, H is in fact a free factor of K, of rank $\mathsf{rk}(K) - 1$.*

Proof. The algebraic closure $\mathsf{cl}_K(H)$ is a free factor of K, and hence it is also a retract and a compressed subgroup. By definition of H, either $\mathsf{cl}_K(H) = K$, and H is K-algebraically dense; or $\mathsf{cl}_K(H) = H$ and H is K-algebraically closed and a free factor. Maximality then implies the announced rank property. $\square$

We now discuss the behavior of the algebraic closure operator.

Proposition 3.20. *Let $H_i \leq K$, $i = 1, 2$, be two extensions of free groups. Then, $\mathsf{cl}_K(H_1 \cap H_2) \leq_{\mathsf{ff}} \mathsf{cl}_K(H_1) \cap \mathsf{cl}_K(H_2)$, and the equality is not true in general.*

Proof. By Theorem 3.16, $\mathsf{cl}_K(H_i)$ is a free factor of K containing H_i, so $\mathsf{cl}_K(H_1) \cap \mathsf{cl}_K(H_2)$ is a free factor of K containing $H_1 \cap H_2$ (Lemma 2.4). Again by Theorem 3.16, $\mathsf{cl}_K(H_1 \cap H_2)$ is a free factor of $\mathsf{cl}_K(H_1) \cap \mathsf{cl}_K(H_2)$.

A counterexample for the reverse inclusion is as follows: let $K = F(a, b)$, $H_1 = \langle [a, b] \rangle$ and $H_2 = \langle [a, b^{-1}] \rangle$. Both these subgroups are K-algebraically dense (see Example 3.5) and their intersection is trivial. $\square$

Proposition 3.21. *Let $K_i \leq K$, $i = 1, 2$, be two extensions of free groups and let $H \leq K_1 \cap K_2$. Then, $\mathsf{cl}_{K_1 \cap K_2}(H) \leq_{\mathsf{ff}} \mathsf{cl}_{K_1}(H) \cap \mathsf{cl}_{K_2}(H)$, and the equality is not true in general.*

Proof. By Theorem 3.16, $\mathsf{cl}_{K_i}(H)$ is a free factor of K_i containing H, so $\mathsf{cl}_{K_1}(H) \cap \mathsf{cl}_{K_2}(H)$ is a free factor of $K_1 \cap K_2$ containing H (Lemma 2.4). Again by Theorem 3.16, $\mathsf{cl}_{K_1 \cap K_2}(H)$ is a free factor of $\mathsf{cl}_{K_1}(H) \cap \mathsf{cl}_{K_2}(H)$.

The following is a counter-example for the converse inclusion. Let $K = \langle a, b, c\rangle$ be a free group of rank 3, let $H = \langle [a, b], [a, c]\rangle$, $K_1 = \langle a, b, [a, c]\rangle$ and $K_2 = \langle a, c, [a, b]\rangle$. One can verify that $K_1 \cap K_2 = \langle a, [a, b], [a, c]\rangle$, so $\mathsf{cl}_{K_1 \cap K_2}(H) = H$. On the other hand, $H \leq_{\mathsf{alg}} K_i$ by Example 3.5 and Proposition 3.12, so $\mathsf{cl}_{K_i}(H) = K_i$ and $\mathsf{cl}_{K_1}(H) \cap \mathsf{cl}_{K_2}(H) = K_1 \cap K_2 \neq \mathsf{cl}_{K_1 \cap K_2}(H)$. □

Remark 3.22. If $H \leq K_1 \leq K_2$, Proposition 3.21 shows that $\mathsf{cl}_{K_1}(H) \leq \mathsf{cl}_{K_2}(H)$. If in addition $K_1 \leq_{\mathsf{ff}} K_2$, Proposition 3.16 shows that $\mathsf{cl}_{K_1}(H) = \mathsf{cl}_{K_2}(H)$. However, in general, even the inclusion $\mathsf{cl}_{K_1}(H) \leq K_1 \cap \mathsf{cl}_{K_2}(H)$ may be strict, as the following counterexample shows.

Let $K_2 = \langle a, b\rangle$ be a free group of rank 2, and let $H = \langle [a, b]\rangle$ and $K_1 = \langle a, [a, b]\rangle$. Then $H \leq_{\mathsf{ff}} K_1 \leq_{\mathsf{alg}} F$ and $H \leq_{\mathsf{alg}} F$. So, $\mathsf{cl}_{K_1}(H) = H$ is properly contained in $K_1 \cap \mathsf{cl}_F(H) = K_1 \cap F = K_1$. □

Finally, let us consider e-algebraic extensions. There too, there exists a greatest e-algebraic extension, at least for finitely generated subgroups. We first prove the following technical lemma.

Lemma 3.23. *Let $H \leq K \leq F$ be extensions of free groups and let $x \in F$. If $H \leq_{\mathsf{alg}} \langle H, x\rangle$, then $K \leq_{\mathsf{alg}} \langle K, x\rangle$.*

Proof. Assume $H \leq_{\mathsf{alg}} \langle H, x\rangle$. If $K \leq \langle K, x\rangle$ is not algebraic, then $x \notin K$ and $K \leq_{\mathsf{ff}} \langle K, x\rangle$ by Proposition 3.13. It follows that $H \leq \langle H, x\rangle \cap K \leq_{\mathsf{ff}} \langle H, x\rangle \cap \langle K, x\rangle = \langle H, x\rangle$, which forces either $\langle H, x\rangle \cap K = H$ or $\langle H, x\rangle \cap K = \langle H, x\rangle$. The first possibility implies $H = \langle H, x\rangle \cap K \leq_{\mathsf{ff}} \langle H, x\rangle$ contradicting the hypothesis, while the second possibility contradicts $x \notin K$. □

Corollary 3.24. *Let $H \leq F$ be an extension of free groups and let $H \leq_{\mathsf{ealg}} K_i$ ($i = 1, \ldots, n$) be a finite family of e-algebraic extensions of H. Then $K_i \leq_{\mathsf{ealg}} \langle \bigcup_j K_j\rangle$ for each i.*

In particular, if H is finitely generated, then H admits a greatest e-algebraic extension in F.

Proof. It suffices to prove the first statement for $n = 2$. Let us assume that $H = H_0 \leq_{\mathsf{alg}} H_1 \leq_{\mathsf{alg}} \cdots \leq_{\mathsf{alg}} H_p = K_1$ and that $x_1, \ldots, x_p$ are such that $H_i = \langle H_{i-1}, x_i \rangle$ for each $1 \leq i \leq p$. Then a repeated application of Lemma 3.23 shows that $K_2 \leq_{\mathsf{ealg}} \langle K_2, x_1, \ldots, x_p \rangle = \langle K_1 \cup K_2 \rangle$.

If $H \leq_{\mathsf{fg}} F$, H has finitely many algebraic extensions, and among them finitely many e-algebraic extensions. The join of these extensions is again an e-algebraic extension and this concludes the proof. □

The greatest e-algebraic extension of a subgroup $H \leq F$, whose existence is asserted in Corollary 3.24, is called its *e-algebraic closure*. We say that H is *e-algebraically closed* if it is equal to its e-algebraic closure. Proposition 3.13 immediately implies the following characterization.

Corollary 3.25. *Let $H \leq F$ be an extension of free groups. Then H is e-algebraically closed if and only if $\langle H, x \rangle = H * \langle x \rangle$ for each $x \notin H$.*

Example 3.26. Let $x \in F$ be an element of a free group not being a proper power. Then, for every $y \in F$, either $\langle x \rangle = \langle x, y \rangle$ or $\mathsf{rk}(\langle x, y \rangle) = 2$. In other words, maximal cyclic subgroups of free groups are e-algebraically closed.

A subgroup $H \leq F$ is said to be *strictly compressed* if $\mathsf{rk}(H) < \mathsf{rk}(K)$ for each proper extension $H < K \leq F$. It is immediate that strictly compressed subgroups form a natural class of e-algebraically closed subgroups.

By Example 3.5, we know that if F has rank two, then $\langle x \rangle \leq F$ is algebraic if and only if x is not a power of a primitive element of F. Hence, situations like $H = \langle [a, b] \rangle < \langle a, b \rangle$ are examples of algebraic extensions where the base group H is e-algebraically closed. This is a behavior significantly different from what happens in field theory. □

Corollary 3.27. *Let $H \leq F(A)$ be an extension of free groups. If H is finitely generated, it is decidable whether H is e-algebraically closed.*

Proof. Let $x \notin H$, viewed as a reduced word on the alphabet A, let p be the longest prefix of x labeling a path starting at the designated vertex 1 in $\Gamma_A(H)$, and let s be the longest suffix of x labeling a path to 1 in $\Gamma_A(H)$. We denote by $1 \cdot p$ and $1 \cdot s^{-1}$ the end vertices of these two paths.

First assume that the sum of the length of p and s is less than the length of x, that is, if $x = pys$ for some non-empty word y. Then $\Gamma_A(\langle H, x \rangle)$ is obtained from $\Gamma_A(H)$ by gluing a path (made of new vertices and new edges) from $1 \cdot p$ to $1 \cdot s^{-1}$, labeled y. In particular, $\mathsf{rk}(\langle H, x \rangle) = \mathsf{rk}(H) + 1$.

We now assume that the sum of the lengths of p and s is greater than or equal to the length of x, and we let t be the longest suffix of p which is also a prefix of s. That is, $p = p't$, $s = ts'$ and $x = p'ts'$. Let $1 \cdot p'$ be the end vertex of the path starting at 1 and labeled p' in $\Gamma_A(H)$. If $1 \cdot p' = 1 \cdot s^{-1}$, then $x = p's$ labels in fact a loop at 1, that is, $x \in H$, a contradiction. So the labeled graph $\Gamma_A(\langle H, x \rangle)$ is the quotient of $\Gamma_A(H)$ by the congruence generated by the pair $(1 \cdot p', 1 \cdot s^{-1})$ (see the end of Section 2.2).

Thus, in view of Corollary 3.25, H is e-algebraically closed if and only if the following holds: for each pair of distinct vertices (v, w) in $\Gamma_A(H)$, the subgroup represented by the quotient of $\Gamma_A(H)$ by the congruence generated by (v, w) has rank at most $\mathsf{rk}(H)$. This is decidable, and concludes the proof. $\square$

4. Abstract properties of subgroups

Let F be a free group. An abstract *property* of subgroups of F is a set $\mathcal{P}$ of subgroups of F containing at least the total group F itself. For simplicity, if $H \in \mathcal{P}$, we will say that the subgroup H *satisfies property* $\mathcal{P}$.

We say that the property $\mathcal{P}$ is *(finite) intersection closed* if the intersection of any (finite) family of subgroups of F satisfying $\mathcal{P}$ also satisfies $\mathcal{P}$, and that it is *free factor closed* if every free factor of a subgroup of F satisfying $\mathcal{P}$ also satisfies $\mathcal{P}$. Finally, we say that the property $\mathcal{P}$ is *decidable* if there exists an algorithm to decide whether a given finitely generated subgroup $H \leq F(A)$ satisfies $\mathcal{P}$.

4.1. **$\mathcal{P}$-closure of a subgroup.** Let F be a free group, $\mathcal{P}$ be an abstract property of subgroups of F, and let $H \leq F$. If there exists a unique minimal subgroup of F satisfying $\mathcal{P}$ and containing H, it is called the *$\mathcal{P}$-closure* of H, denoted by $\mathsf{cl}_{\mathcal{P}}(H)$; in this situation, we say that H admits a *well defined* $\mathcal{P}$-closure.

Proposition 4.1. *Let F be a free group and let $\mathcal{P}$ be an abstract property of subgroups of F.*

(i) *If $\mathcal{P}$ is intersection closed, then every subgroup $H \leq F$ admits a well defined $\mathcal{P}$-closure.*
(ii) *If $\mathcal{P}$ is finite intersection closed and free factor closed then every finitely generated subgroup $H \leq_{\mathsf{fg}} F$ admits a well defined $\mathcal{P}$-closure.*
(iii) *If $\mathcal{P}$-closures are well defined and $\mathcal{P}$ is free factor closed, then for every subgroup $H \leq F$, we have $H \leq_{\mathsf{alg}} \mathsf{cl}_{\mathcal{P}}(H)$. In particular, if H is finitely generated, then so is $\mathsf{cl}_{\mathcal{P}}(H)$.*

Proof. Statement (i) is immediate: it suffices to consider the intersection of all the extensions of H satisfying $\mathcal{P}$ (there is at least one, namely F itself).

If $\mathcal{P}$ is only finite intersection closed, but is also free factor closed, we use Theorem 3.16: since every extension of a finitely generated subgroup H is a free multiple of an algebraic extension of H, then every extension of H in $\mathcal{P}$ contains an algebraic extension of H in $\mathcal{P}$. It follows that the intersection of all extensions of H in $\mathcal{P}$ is equal to the intersection of the algebraic extensions of H in $\mathcal{P}$. But the latter intersection is finite, and hence it satisfies $\mathcal{P}$ as well, which concludes the proof of (ii).

Finally, if $\mathcal{P}$ is free factor closed, then H is not contained in any proper free factor of its $\mathcal{P}$-closure, that is, $H \leq_{\mathsf{alg}} \mathsf{cl}_{\mathcal{P}}(H)$. $\square$

It would be interesting to produce an example of an abstract property $\mathcal{P}$ that is closed under free factors and finite intersections, not closed under intersections, and non-trivial for finitely generated subgroups (note that the property *to be finitely generated* satisfies the required closure and non-closure properties, but it is trivial for finitely generated subgroups).

Remark 4.2. It is well known that the property of being normal in F is closed under intersections and not under free factors, and that given a subgroup $H \leq F$, the normal closure of H is well-defined, and is not in general finitely generated, even if H is. $\square$

Proposition 4.3. *Let $\mathcal{P}$ be an abstract property for subgroups of $F(A)$ for which $\mathcal{P}$-closures are well defined. If $\mathcal{P}$-closures of finitely generated subgroups of $F(A)$ are computable, then $\mathcal{P}$ is decidable. The converse holds if, additionally, $\mathcal{P}$ is free factor closed.*

Proof. Let us assume that $\mathcal{P}$-closures are computable. Then, in order to decide whether a given $H \leq_{\mathsf{fg}} F(A)$ satisfies $\mathcal{P}$, it suffices to compute $\mathsf{cl}_{\mathcal{P}}(H)$, and to verify whether $H = \mathsf{cl}_{\mathcal{P}}(H)$.

Conversely, suppose that $\mathcal{P}$ is free factor closed and decidable. Then, given $H \leq_{\mathsf{fg}} F(A)$, one can compute the set $\mathsf{AE}(H)$, check which algebraic extensions of H satisfy $\mathcal{P}$ and identify the minimal one(s). By Proposition 4.1, only one of them is minimal, and that one must be $\mathsf{cl}_{\mathcal{P}}(H)$. $\square$

Remark 4.4. Proposition 4.1 states that every property of subgroups that is closed under (finite) intersections and under free factors yields a well-defined closure operator for (finitely generated) subgroups of F, that can be obtained by looking exclusively at algebraic extensions.

A form of converse holds too: if $K \leq_{\mathsf{fg}} F$, let $\mathcal{P}_K$ be the following property. A subgroup L satisfies $\mathcal{P}_K$ if and only if L is a free factor of an extension of K. Clearly, F satisfies this property, and one can verify that $\mathcal{P}_K$ is intersection and free factor closed. Moreover, one can use Proposition 3.16 to verify that the $\mathcal{P}_K$-closure of a subgroup $H \leq K$ is exactly the K-algebraic closure of H. In particular, for every algebraic extension $H \leq_{\mathsf{alg}} K$, K is the $\mathcal{P}$-closure of H for some well-chosen property $\mathcal{P}$. $\square$

4.2. **Some algebraic properties.** Let us recall the definition of certain properties of subgroups, that have been discussed in the literature. Let $H \leq F$ be an extension of free groups. We say that H is

- *malnormal* if $H^g \cap H = 1$ for all $g \in F \setminus H$;

- *pure* if $x^n \in H$, $n \neq 0$ implies $x \in H$ (this property is also called being closed under radical, or being isolated);
- *p-pure* (for a prime p) if $x^n \in H$, $(n,p) = 1$ implies $x \in H$;

The following results on malnormal and pure closure were first shown in [7, Section 13]. The proof given here, while not fundamentally different, is simpler and more general. Corollary 4.14 below gives further properties of these closures.

Proposition 4.5. *Let $F(A)$ be a free group. The properties (of subgroups) defined by malnormal, pure, p-pure (p a prime), retract and e-algebraically closed subgroups are intersection and free factor closed, and decidable for finitely generated subgroups.*

For each of these properties $\mathcal{P}$, each subgroup $H \leq F(A)$ admits a well-defined $\mathcal{P}$-closure $\mathsf{cl}_{\mathcal{P}}(H)$, which is an algebraic extension of H. Finally, if $H \leq_{\mathsf{fg}} F(A)$, the $\mathcal{P}$-closure of H has finite rank and is computable.

Proof. The closure under intersections and free factors of malnormality is immediate from the definition. The decidability of malnormality was established in [1], with a simple algorithm given in [7, Corollary 9.11].

The closure under intersections of the properties of purity and p-purity is immediate. Now, assume that K is pure, $H \leq_{\mathsf{ff}} K$, and let x be such that $x^n \in H$ with $n \neq 0$. Since K is pure, we have $x \in K$, and we simply need to show that a free factor of a free group F is pure, which was established in Example 3.1 above. Thus purity is free factor closed. The proof of the same property for p-purity is identical. The decidability of purity and p-purity was proved in [3, 4].

It is shown in [2, Lemma 18] that an arbitrary intersection of retracts of F is again a retract of F. Moreover, it follows from the definition of retracts that a retract of a retract is a retract, and that a free factor is a retract. Thus the property of being a retract of F is free factor closed. The decidability of this property was established by Turner, but as no proof seems to have been published, we give his in Proposition 4.6 below.

Suppose that $H \leq_{\mathsf{ff}} K \leq F$, K is e-algebraically closed and $x \notin H$. If $x \notin K$, then $\langle K, x\rangle = K * \langle x\rangle$, so $\langle H, x\rangle = H * \langle x\rangle$. If $x \in K \setminus H$, then we have that H is a free factor of $\langle H, x\rangle \leq K$ and so, by Proposition 3.13, we also conclude that $\langle H, x\rangle = H * \langle x\rangle$. Thus the property of being e-algebraically closed is closed under free factors. Next, let $(H_i)_{i\in I}$ be a family of e-algebraically closed subgroups, let $H = \bigcap_i H_i$, and let $x \notin H$. There exists $i \in I$ such that $x \notin H_i$, so $\langle H_i, x\rangle = H_i * \langle x\rangle$. Using Lemma 3.23, we conclude that $\langle H, x\rangle = H * \langle x\rangle$. Thus the property of being e-algebraically closed is also closed under intersections. Finally, this property is decidable by Corollary 3.27.

The last part of the statement follows from Proposition 4.1. □

As announced in the proof of Proposition 4.5, we prove the decidability of retracts, that was established by Turner [19].

Proposition 4.6. *Let $H \leq F(A)$ be an extension of finitely generated free groups. It is decidable whether H is a retract of $F(A)$.*

Proof. (Turner) Suppose that $A = \{a_1, \ldots, a_n\}$ and let $u_1, \ldots, u_r$ be a basis of H. Then H is a retract of $F(A)$ if and only if there exist $x_1, \ldots, x_n \in H$ such that the endomorphism φ of $F(A)$ defined by $\varphi(a_i) = x_i$ maps H identically to itself. That is, if $u_i(x_1, \ldots, x_n) = u_i$ for $i = 1, \ldots, r$. This can be expressed in terms of systems of equations.

Let e_i be the word on alphabet $\{X_1, \ldots, X_n\}$ obtained from the word u_i (on alphabet A) by substituting X_j for a_j for each j. Then H is a retract of F if and only if the system of equations $e_i(X_1, \ldots, X_n) = u_i$, $i = 1, \ldots, r$ (where u_i are viewed as constants in H) admits a solution in H. This is decidable by Makanin's algorithm [10] (note that the form of the system (i.e. the words e_i) depends on the way H is embedded in F, but once this form is established, the system itself is entirely set within H, so Makanin's algorithm works, applied to this system over H). □

Let $H \leq F$ be an extension of free groups. Recall that H is *compressed* if $\mathsf{rk}(H) \leq \mathsf{rk}(K)$ for every $K \leq F$ containing H (see Section 3.4), and say that H is *inert* if $\mathsf{rk}(H \cap K) \leq \mathsf{rk}(K)$ for every $K \leq F$. Both these properties were introduced by Dicks and Ventura [5] in the context of the study of subgroups of free groups that are fixed by sets of endomorphisms or automorphisms (see also [21]).

It is clear that an inert or compressed subgroup is finitely generated, with rank at most $\mathsf{rk}(F)$. It is also clear that inert subgroups (and retracts) of F are compressed. On the other hand, we do not know whether all compressed subgroups are inert, nor whether retracts are inert (both these facts are conjectured in [21] and related to other conjectures about fixed subgroups in free groups).

Proposition 4.7. *Let F be a free group. The properties of inertness and compressedness are closed under free factors. In addition, inertness is closed under intersections.*

Each subgroup $H \leq F$ admits an inert closure, which is an algebraic extension of H.

Proof. The closure of inertness under intersections is shown in [5, Corollary I.4.13]. Free factors of F are trivially inert. Moreover, if $H \leq K \leq F$, H is inert in K and K is inert in F, then H is inert in F. So inertness is also closed under free factors.

Now suppose that $H = L * M \leq F$ is compressed, and let $L \leq K \leq F$. Since $H \leq \langle K, M \rangle$, we have

$$\mathsf{rk}(L) + \mathsf{rk}(M) = \mathsf{rk}(H) \leq \mathsf{rk}(\langle K, M \rangle) \leq \mathsf{rk}(K) + \mathsf{rk}(M).$$

It follows that $\mathsf{rk}(L) \leq \mathsf{rk}(K)$, and hence L is compressed. Thus, compressedness is closed under free factors. The last statement is a direct application of Proposition 4.1. $\square$

Note that, even though a finitely generated subgroup H admits an inert closure, which is one of its (finitely many) algebraic extensions of H, we do not know how to compute this closure, nor how to decide whether a subgroup is inert.

It is not known either whether compressedness is closed under intersections, or even finite intersections, so we don't know whether each subgroup admits a *compressed closure*. However it is decidable whether a finitely generated subgroup of F is compressed [20]. Indeed if $H \leq_{\mathsf{fg}} F$, then H is compressed if and only if $\mathsf{rk}(H) \leq \mathsf{rk}(K)$ for every algebraic extension $H \leq_{\mathsf{alg}} K \leq F$, which reduces the verification to a finite number of rank comparisons.

4.3. **On certain topological closures.** Let $\mathcal{T}$ be a topology on a free group F. The abstract property of subgroups consisting of the subgroups that are closed in $\mathcal{T}$ is trivially closed under intersections. This property becomes more interesting when the topology is related to the algebraic structure of F. This is the case of the pro-$\mathbf{V}$ topologies that we analyze now.

A *pseudovariety of groups* $\mathbf{V}$ is a class of finite groups that is closed under taking subgroups, quotients and finite direct products. $\mathbf{V}$ is called *non-trivial* if it contains some non-trivial finite group. Additionally, if for every short exact sequence of finite groups, $1 \to G_1 \to G_2 \to G_3 \to 1$, with G_1 and G_3 in $\mathbf{V}$, one always has $G_2 \in \mathbf{V}$, we say that $\mathbf{V}$ is *extension-closed*.

For every non-trivial pseudovariety of groups $\mathbf{V}$, the *pro-$\boldsymbol{V}$* topology on a free group F is the initial topology of the collection of morphisms from F into groups in $\mathbf{V}$, or equivalently, the topology for which the normal subgroups N such that $F/N \in \mathbf{V}$ form a basis of neighborhoods of the unit. We refer the readers to [11, 22] for a survey of results concerning these topologies with regard to finitely generated subgroups of free groups. In particular, Ribes and Zalesskiĭ showed that if $\mathbf{V}$ is extension-closed then every free factor of a closed subgroup is closed [12]. The following observation then follows from Proposition 4.1.

Fact 4.8. *Let $\boldsymbol{V}$ be a non-trivial extension-closed pseudovariety of groups. Then the pro-$\boldsymbol{V}$ closure of a finitely generated subgroup H is finitely generated, and an algebraic extension of H.*

In the case of the pro-p topology (p is a prime and the pseudovariety $\mathbf{V}$ is that of finite p-groups, which is closed under extensions), Ribes and Zalesskiĭ [12] showed that one can compute the closure of a given finitely generated subgroup of $F(A)$. A polynomial time algorithm was later given by Margolis, Sapir and Weil [11], based on the finiteness of the number of principal overgroups of H, that is, essentially on the spirit of Fact 4.8. Moreover, they showed that one can simultaneously compute the pro-p closures of H, for all primes p, using the fact that they are all algebraic extensions, and hence that they take only finitely many values. This was also used to show the computability of the pro-nilpotent closure of a finitely generated subgroup: even though the pseudovariety of finite nilpotent groups is not closed under extensions, it still holds that the pro-nilpotent closure of a finitely generated subgroup is finitely generated and computable.

At this point, several remarks are in order. First, Ribes and Zalesskiĭ [12] proved that if $\mathbf{V}$ is extension-closed and if $\bar{H}$ is the pro-$\mathbf{V}$ closure of H, then $\mathsf{rk}(\bar{H}) \leq \mathsf{rk}(H)$. The proof of this fact can be reduced to dimension considerations in appropriate vector spaces. This proof does not seem related with the idea of e-algebraic extensions, which also lowers the rank (Corollary 3.15).

Next, not every algebraic extension arises as a pro-$\mathbf{V}$ closure for some $\mathbf{V}$. This is clear if $H \leq_{\mathsf{alg}} K$ and $\mathsf{rk}(K) > \mathsf{rk}(H)$ by the result of Ribes and Zalesskiĭ cited above, but rank is not the only obstacle. Consider indeed $H = \langle a, bab^{-1} \rangle \leq F(a,b)$. Then $H \leq_{\mathsf{alg}} F$ (Example 3.5) and $\mathsf{AE}(H) = \mathcal{O}_A(H) = \{H, F\}$. We now verify that H is $\mathbf{V}$-closed for each non-trivial extension-closed pseudovariety $\mathbf{V}$, so F is never the $\mathbf{V}$-closure of H. Since $\mathbf{V}$ is non-trivial, the cyclic p-element group $C_p = \langle c \mid c^p \rangle$ sits in $\mathbf{V}$ for some prime p. Let $\varphi_p \colon F \to C_p$ be the morphism defined by $\varphi_p(a) = 1$ and $\varphi_p(b) = c$, and let $N_p = \ker \varphi_p$. Then $H \leq N_p$ and N_p is $\mathbf{V}$-closed, so H is not topologically dense in F. Since the $\mathbf{V}$-closure of H is in $\mathsf{AE}(H)$, it follows that H is closed in the pro-$\mathbf{V}$ topology.

Solvable groups form an extension-closed pseudovariety, so the above results apply to it: in particular, given an extension $H \leq_{\mathsf{fg}} F(A)$, we can compute a finite list of candidates for being the pro-solvable closure of H, namely $\mathsf{AE}(H)$ (or even this list, restricted to the extensions of rank at most $\mathsf{rk}(H)$). However, it is a wide open problem to compute this closure.

Finally, let us consider the (uncountable) collection of extension-closed pseudovarieties of finite groups $\mathbf{V}$ as above. For each finitely generated subgroup $H \leq F$, the pro-$\mathbf{V}$ closures of H are among the (finitely many) algebraic extensions of H, so each finitely generated subgroup H naturally induces a finite index equivalence relation on the collection of the $\mathbf{V}$'s. It would be interesting to investigate the properties of these equivalence relations. In particular, the intersection of these equivalence relations, as H runs over all the (countably many) finitely generated subgroups of $F(a,b)$, has countably many classes, so there are pseudovarieties $\mathbf{V}$ that are indistinguishable in this way.

4.4. **Equations over a subgroup.** In this section we use equations over free groups to define abstract properties of subgroups. Let $H \leq F$ be an extension of free groups. A *(one variable) H-equation* (or *equation over H*) is an element $e = e(X)$ of the free group $H * \langle X\rangle$, where X is a new free letter, called the *variable*. An element $x \in F$ is a *solution of* $e(X)$ if $e(x) = 1$ in F (technically: if the morphism $H * \langle X\rangle \to F$ mapping H identically to itself and X to x, maps e to 1).

Example 4.9. If $H = \langle a^2\rangle$, the H-equation $e(X) = Xa^2X^{-1}a^{-2}$ admits a as a solution. So does the H-equation X^2a^{-2}.

If e does not involve X, that is, $e \in H$, then e has no solution unless it is the *trivial* equation $e = 1$, in which case every element of F is a solution. □

We immediately observe the following.

Lemma 4.10. *Let $H \leq F$ be an extension of free groups and let $x \in F$. The element x is a solution of some non-trivial H-equation if and only if the elementary extension $H \leq \langle H, x\rangle$ is algebraic.*

Proof. Let X be a new free generator and let $\varphi\colon H * \langle X\rangle \to F$ be the morphism that maps H identically to itself and X to x. By definition, x is a solution of some non-trivial equation over H if and only if φ is not injective, and we conclude by Proposition 3.13 and Corollary 3.14 that this is equivalent to $H \leq_{\mathsf{alg}} \langle H, x\rangle$. □

In order to make this natural definition of equations independent on the choice of the subgroup H, we consider a countable set $X, Y_1, Y_2, \ldots$ of variables and we call *equation* any element e of the free group on these variables. If $H \leq F$ is an extension of free groups, a *particularization of e over H* is the H-equation $e(X, h_1, h_2, \ldots)$ obtained by substituting elements $h_1, h_2, \ldots \in H$ for the variables $Y_1, Y_2, \ldots$ (and having X as variable).

A *solution of the equation e over H* is a solution of some non-trivial particularization of e over H, that is, an element $x \in F$ such that, for some $h_1, h_2 \ldots \in H$, $e(X, h_1, h_2, \ldots) \neq 1$ but $e(x, h_1, h_2, \ldots) = 1$. (Note that even when X occurs in e, some particularizations of e over H can be trivial).

Let $\mathcal{E}$ be an arbitrary set of equations. We say that a subgroup $H \leq F$ is *$\mathcal{E}$-closed* if H contains every solution over H of every equation in $\mathcal{E}$. Note that, when looking for solutions, the set $\mathcal{E}$ is not considered as a system of equations, but as a set of mutually unrelated equations. In particular, a larger set $\mathcal{E}$ yields a larger set of solutions.

Proposition 4.11. *Let F be a free group and let $\mathcal{E}$ be a set of equations. Then the property of being $\mathcal{E}$-closed is closed under intersections and under free factors.*

Proof. The closure under intersections follows directly from the definition. Now assume that $K \leq F$ is $\mathcal{E}$-closed and let $H \leq_{\mathsf{ff}} K$. Let x be a solution of an equation of $\mathcal{E}$ over H. Then x is also a solution over K, and hence $x \in K$. Now, by Lemma 4.10, $H \leq_{\mathsf{alg}} \langle H, x\rangle \leq K$. This contradicts $H \leq_{\mathsf{ff}} K$ unless $H = \langle H, x\rangle$, and hence $x \in H$. $\square$

Corollary 4.12. *Let $H \leq F$ and let $\mathcal{E}$ be a set of equations. There exists a least $\mathcal{E}$-closed extension of H, denoted by $\mathsf{cl}_{\mathcal{E}}(H)$ and called the $\mathcal{E}$-closure of H. Moreover, $H \leq_{\mathsf{alg}} \mathsf{cl}_{\mathcal{E}}(H)$.*

If in addition H is finitely generated, then $H \leq_{\mathsf{ealg}} \mathsf{cl}_{\mathcal{E}}(H)$, $\mathsf{rk}(\mathsf{cl}_{\mathcal{E}}(H)) \leq \mathsf{rk}(H)$ and there exists a finite subset $\mathcal{E}_0$ of $\mathcal{E}$ such that $\mathsf{cl}_{\mathcal{E}_0}(H) = \mathsf{cl}_{\mathcal{E}}(H)$.

Proof. Propositions 4.1 and 4.11 directly prove the first part of the statement.

We now suppose that $H \leq_{\mathsf{fg}} F$ and we let $H_0 = H$ and suppose that we have constructed distinct extensions $H_0 \leq_{\mathsf{ealg}} H_1 \leq_{\mathsf{ealg}} \cdots \leq_{\mathsf{ealg}} H_n$ ($n \geq 0$), elements $x_1, \ldots, x_n \in F$, and equations $e_1, \ldots, e_n \in \mathcal{E}$ such that $H_i = \langle H_{i-1}, x_i\rangle$ and x_i is a solution of e_i over H_{i-1}. If H_n is not $\mathcal{E}$-closed, then there exists an equation $e_{n+1} \in \mathcal{E}$, and an element $x_{n+1} \notin H_n$ such that x_{n+1} is a solution of a non-trivial particularization of e_{n+1} over H_n. Then $H_{n+1} = \langle H_n, x_{n+1}\rangle$ is a proper elementary algebraic extension of H_n by Lemma 4.10. Since H has only a finite number of algebraic extensions, this construction must stop, that is, for some n, H_n is $\mathcal{E}$-closed. It follows easily that H_n is the $\mathcal{E}$-closure of H, whose existence was already established. In particular $H \leq_{\mathsf{ealg}} \mathsf{cl}_{\mathcal{E}}(H)$, and $\mathsf{rk}(\mathsf{cl}_{\mathcal{E}}(H)) \leq \mathsf{rk}(H)$ by Corollary 3.15.

Finally, let $\mathcal{E}_0 = \{e_1, \ldots, e_n\}$. Any $\mathcal{E}$-closed subgroup is also $\mathcal{E}_0$-closed, and the $\mathcal{E}_0$-closure of H must contain $H_1, \ldots, H_n$. Thus $\mathsf{cl}_{\mathcal{E}}(H) = \mathsf{cl}_{\mathcal{E}_0}(H)$. $\square$

We conclude with the observation that some of the properties discussed in Section 4.2 can be expressed in terms of equations. Let p be a prime number and let

$$\begin{aligned}
\mathcal{E}_{mal} &= \{X^{-1}Y_1XY_2\}, \\
\mathcal{E}_p &= \{X^nY_1 \mid (n,p) = 1\}, \\
\mathcal{E}_{\mathbb{Z}} &= \{X^nY_1 \mid n \neq 0\} = \bigcup_p \mathcal{E}_p,
\end{aligned}$$

$$\mathcal{E}_{com} \quad = \quad \{X^{-1}Y_1^{-1}XY_1\}.$$

Proposition 4.13. *Let $H \leq F$ be an extension of free groups. The subgroup H is*

(i) *malnormal if and only if it is $\mathcal{E}_{mal}$-closed;*
(ii) *p-pure if and only if it is $\mathcal{E}_p$-closed;*
(iii) *pure if and only if it is $\mathcal{E}_{\mathbb{Z}}$-closed, and if and only if it is $\mathcal{E}_{com}$-closed.*

Proof. H is $\mathcal{E}_{mal}$-closed if and only if, for all $h_1, h_2 \in H$, not simultaneously trivial, every solution of the equation $X^{-1}h_1Xh_2 = 1$ belongs to H. That is, if and only if $x^{-1}Hx \cap H \neq 1$ implies $x \in H$. This is precisely the malnormality property for H. This proves (i).

H is $\mathcal{E}_p$-closed if and only if H contains the n-th roots of every one of its elements, for all n such that $(n, p) = 1$. Again, this is exactly the definition of p-purity, showing (ii).

Similarly, H is $\mathcal{E}_{\mathbb{Z}}$-closed if and only if H is pure. Finally, we recall that two elements x and y in F commute if and only if they are powers of a common $z \in F$. Thus the subgroup generated by H and all the roots of its elements is exactly the $\mathcal{E}_{com}$-closure of H. □

Corollary 4.12 immediately implies the following.

Corollary 4.14. *Let $H \leq_{\mathsf{fg}} F$ and let K be the malnormal (resp. pure, p-pure) closure of H. Then $H \leq_{\mathsf{ealg}} K$ and $\mathsf{rk}(K) \leq \mathsf{rk}(H)$.*

5. Some open questions

To conclude this paper, we would like to draw the readers' attention to a few of the questions it raises.

(1) We believe that the algebraic extensions of a finitely generated subgroup $H \leq_{\mathsf{fg}} F$ are precisely the extensions which occur as principal overgroups of H for every choice of an ambient basis. That is, we conjecture that $\mathsf{AE}(H) = \bigcap_A \mathcal{O}_A(H)$, where A runs over all the bases of F. As noticed in section 3.1, this is the case when $H \leq_{\mathsf{fi}} F$ or $H \leq_{\mathsf{ff}} F$, but nothing is known in general.
(2) With reference to Corollary 3.15, we would like to find an algebraic extension $H \leq_{\mathsf{alg}} K$ of finitely generated groups, where $\mathsf{rk}(K) \leq \mathsf{rk}(H)$, yet the extension is not e-algebraic. It would be appropriate to look for such an extension where H is e-algebraically closed in K, that is, $\langle H, x\rangle = H * \langle x\rangle$ for each $x \in K \setminus H$ (Corollary 3.25).

(3) Even though a finitely generated subgroup H admits an inert closure, which is one of the finitely many (computable) algebraic extensions of H, we do not know how to compute this closure. Equivalently, it would be interesting to find an algorithm to decide whether a subgroup is inert (see section 4.2).
(4) It is not known whether an intersection, even a finite intersection, of (strictly) compressed subgroups is again (strictly) compressed. In other words, does a finitely generated subgroup admit a *(strictly) compressed closure*? If the answer was affirmative, then these closures would be computable, as indicated in section 4.2.
(5) As pointed out in section 4.3, we know that if $\mathbf{V}$ is a non-trivial extension-closed pseudovariety of groups and $H \leq_{\mathsf{fg}} F$, then $\bar{H}$, the pro-$\mathbf{V}$ closure of H, is an algebraic extension of H with rank at most $\mathsf{rk}(H)$. However the known proof of this fact does not rely on the notion of e-algebraic extensions. We would like to find an example of such a subgroup H and a pseudovariety $\mathbf{V}$ such that the extension $H \leq \bar{H}$ is not e-algebraic – or alternately to give a new proof of Ribes and Zalesskiĭ's result (that in this situation, $\mathsf{rk}(\bar{H}) \leq \mathsf{rk}(H)$), by showing that $H \leq_{\mathsf{ealg}} \bar{H}$.
(6) As indicated at the end of section 4.3, it would be interesting to find and investigate explicit examples of pseudovarieties $\mathbf{V}_1$ and $\mathbf{V}_2$, such that the pro-$\mathbf{V}_1$ and pro-$\mathbf{V}_2$ closures of H do coincide, for every $H \leq_{\mathsf{fg}} F$. As argued above, there are uncountably many such pairs being indistinguishable by means of closures of finitely generated subgroups.
(7) Finally, Corollary 4.12 shows that for every set of equations $\mathcal{E}$ and every $H \leq_{\mathsf{fg}} F$, there exists a finite subset $\mathcal{E}_0 \subseteq \mathcal{E}$ such that $\mathsf{cl}_{\mathcal{E}_0}(H) = \mathsf{cl}_{\mathcal{E}}(H)$. Is it true that such a finite set always exists satisfying the previous equality for all finitely generated subgroups of F at the same time (showing a kind of noetherian behavior)?

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