REFLECTED BACKWARD STOCHASTIC DIFFERENTIAL EQUATION WITH JUMPS AND RCLL OBSTACLE.

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Abstract. In this paper we study one-dimensional reflected backward stochastic differential equation when the noise is driven by a Brownian motion and an independent Poisson point process when the solution is forced to stay above a right continuous left-hand limited obstacle. We prove existence and uniqueness of the solution by using a penalization method combined with a monotonic limit theorem.

1. Introduction

Let $(B_t)_{0\leq t\leq T}$ be a d-dimensional Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, \mathrm{P})$. Let $(\mathcal{F}_t)_{0\leq t\leq T}$ denote the natural filtration of (B_t) such that $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$, and ξ be an $\mathcal{F}_T$-measurable one dimensional random variable. Let f be an $I\!\!R$-valued function defined on $[0,T]\times\Omega\times I\!\!R\times I\!\!R^d$ such that for all $(y,z)\in I\!\!R\times I\!\!R^d$, the map $(t,\omega)\longrightarrow f(t,\omega,y,z)$ is $\mathcal{F}_t$-progressively measurable. We consider the following backward stochastic differential equation (BSDE for short) associated with the coefficient f and the terminal value ξ

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dB_s, \qquad 0 \leq t \leq T. \tag{1}$$

A solution for such equation is a couple of adapted processes (Y, Z) with values in $I\!\!R\times I\!\!R^d$ which mainly satisfies equation (1). This Kind of equations have been first introduced by Pardoux & Peng [13]. Their aim was to give a probabilistic interpretation of a solution of second order quasi-linear partial differential equation. Since then, those equations have been intensively investigated due to their connections with financial mathematics, optimal

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control and stochastic game, non-linear PDEs and homogenization (see, for example, [5, 6, 7, 8, 14, 13, 2, 3] and the references therein).

The notion of reflected BSDE have been introduced by El Karoui et *al* [6]. A solution of such equation, associated with a coefficient f; terminal value ξ and a barrier S, is a triple of process (Y, Z, K) with values in $I\!R \times I\!R^d \times I\!R_+$ satisfying

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + K_T - K_t - \int_t^T Z_s dB_s, \qquad Y_t \geq S_t \qquad \forall t \leq T. \tag{2}$$

Here the additional process K is continuous nondecreasing and its role is to push upwards the process Y in order to keep it above the barrier S and moreover it satisfies $\int_0^T (Y_s - S_s)dK_s = 0$, this means that the process K acts only when the process reaches the barrier S in a minimal way. The authors have proved that equation (2) has a unique solution when ξ is square integrable, f is uniformly Lipschitz with respect to (y, z) and S is continuous.

The extension to the case of reflected BSDE with jumps, which is a standard reflected BSDE driven by a Brownian motion and an independent Poisson point process, have been established by Hamadène & Ouknine [9]. A solution for such equation, associated with a coefficient f; terminal value ξ and a barrier S, is a quadruple of process (Y, Z, K, V) of adapted solutions which satisfy the following equation

$$\begin{cases} (i)\ Y_t = \xi + \int_t^T f(s, Y_s, Z_s, V_s)ds + K_T - K_t - \int_t^T Z_s dB_s - \int_t^T\int_U V_s(e)\tilde{\mu}(ds, de), \\ (ii)\ \forall t \leq T; Y_t \geq S_t, \\ (iii)\ \int_0^T (Y_t - S_t)dK_t = 0. \end{cases} \tag{3}$$

Using two methods: the first one is based on the penalization argument and the second one on the snell envelope theory, the authors have shown the existence and uniqueness of solutions if ξ is square integrable, f is uniformly lipschitz with respect to y and z and the barrier S is right continuous left-hand limited (*rcll* for short) whose jumping times are inaccessible stopping times. Note that this later condition played a crucial role in their proofs. It is worth nothing also that, in this case, the jumping times of the process Y come only from those of its Poisson process and then they are inaccessible.

The problem of existence and uniqueness of reflected BSDE when the noise is driven only by a Brownian motion and the reflecting barrier S is *rcll* has been studied, first, by Hamadène [7] using the snell envelope method and later by Lepeltier and Xu [12] using a monotonic limit theorem initially introduced by Peng [15].

In this work, we study the problem of existence and uniqueness of solution to equation (3) when the barrier S is just *rcll* and the jumping times of process Y come not only from those of its Poisson process (inaccessible jumps) but also from those of the process S (predictable jumps), which means that the process Y have two types of jumps: inaccessible and predicatable ones. The difficulty here lies in the fact that since the barrier S is allowed to have predictable jumps then the process Y and then the reflecting process K are no longer continuous but just *rcll*. Roughly speaking, we consider the following reflected BSDE with jumps

$$\begin{cases} (i)\, Y_t = \xi + \int_t^T f(s, Y_s, Z_s, V_s) ds + K_T - K_t - \int_t^T Z_s dB_s - \int_t^T\int_U V_s(e)\tilde{\mu}(ds, de), \\ (ii)\, \forall t \leq T;\ \ Y_t \geq S_t, \\ (iii)\ \int_0^T (Y_{t^-} - S_{t^-}) dK_t = 0. \end{cases} \tag{4}$$

Note that the difference between equation (3) and (4) is in the Skorohod condition (iii).

In order to state the existence of solution for our reflected BSDE with jumps (4), we consider the following penalized equation

$$Y_t^n = $$
$$= \xi + \int_t^T f(s, Y_s^n, Z_s^n, V_s^n) ds + K_T^n - K_t^n - \int_t^T Z_s^n dB_s - \int_t^T\int_U V_s^n(e)\tilde{\mu}(ds, de), t \leq T,$$

where $K_t^n = n\int_0^t (Y_s^n - S_s)^- ds$. We prove that (Y^n, Z^n, K^n, V^n) has, in some sense, a limit (Y, Z, K, V) which satisfies our reflected BSDE with jumps (4). To get this convergence we need to state a monotonic limit theorem, in the framework of filtration generated by a Brownian motion and Poisson point processs, which generalizes a useful tool initially introduced by Peng [15].

At the same time, Hamadène & Ouknine [10] studied the same problem of existence and uniqueness of reflected BSDE with jumps and *rcll* barrier using another proof based on a combination of penalization and the snell envelope theorey.

Let us describe our plan. First of all, most of the material used in this paper is defined in Section 2, uniqueness of solutions for our reflected BSDE with jumps is also given. A monotonic limit theorem is proved in Section 3. In Section 4, we use the monotonic limit theorem in order to prove the convergence of a our penalized equation. The proof of existence result of our reflected BSDE with jumps is stated is Section 5.

2. Problem formulation, assumptions and uniqueness of the solution for reflected BSDEs with jumps

2.1. **Problem formulation and assumptions.** Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\leq 1}))$ be a stochastic basis such that $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$, $\mathcal{F}_{t+} = \bigcap\limits_{\epsilon>0} \mathcal{F}_{t+\epsilon} = \mathcal{F}_t$, $\forall t \leq 1$, and suppose that the filtration is generated by the two following mutually independent processes :

• a d-dimensional Brownian motion $(B_t)_{t\leq 1}$,

• a Poisson random measure μ on $I\!R^+ \times U$, where $U := I\!R^l \setminus \{0\}$ is equipped with its Borel fields $\mathcal{U}$, with compensator $\nu(dt, de) = dt\lambda(de)$, such that $\{\tilde{\mu}([0,t] \times A) = (\mu - \nu)([0,t] \times A)\}_{t\leq 1}$ is a martingale for every $A \in \mathcal{U}$ satisfying $\lambda(A) < \infty$. λ is assumed to be a σ-finite mesure on $(U,\mathcal{U})$ satisfying

$$\int_U (1 \wedge |e|^2)\lambda(de) < \infty.$$

We will need the following notations :

• $\mathcal{P}$ be the sigma algebra of $\mathcal{F}_t$-progressively measurable sets on $\Omega \times [0, 1]$.

• $\widetilde{\mathcal{P}}$ be the sigma algebra of predictable sets on $\Omega \times [0, 1]$.

• $\mathcal{S}^2$ be the set of $\mathcal{F}_t$-adapted *rcll* processes $(Y_t)_{t\leq 1}$ with values in $I\!R$ and $I\!E[\sup\limits_{t\leq 1} |Y_t|^2] < \infty$.

• $\mathcal{H}^{2,k}$ be the set of $\mathcal{P}$-measurable processes with values in $I\!R^k$ such that

$$I\!E[\int_0^1 |Z_s|^2 ds] < \infty.$$

• $\mathcal{L}^2$ be the set of mappings $V : \Omega \times [0, 1] \times U \to I\!R$ which are $\widetilde{\mathcal{P}} \otimes \mathcal{U}$-measurable such that

$$I\!E[\int_0^1 ds \int_U (V_s(e))^2 \lambda(de)] < \infty.$$

• $\mathcal{K}^2$ be the set of $\mathcal{F}_t$-adapted *rcll* increasing processes K such that $K(0) = 0$ et $I\!E(K_1^2) < \infty$.

• For a given *rcll* process $(w_t)_{t\leq 1}$, $w_{t^-} = \lim_{s\nearrow t} w_s, t \leq 1$ $(w_{0^-} = w_0)$; $w_- := (w_{t^-})_{t\leq 1}$ and $\Delta_s w = w_s - w_{s^-}$ $\Box$

Let ξ be an $\mathcal{F}_1$-measurable one dimensional random variable and a function $f : \Omega \times [0,1] \times I\!R^{1+d} \times L^2(U,\mathcal{U},\lambda; I\!R) \longrightarrow I\!R$ which to (t,ω,y,z,v) associates $f(t,\omega,y,z,v)$ which is $\mathcal{P} \otimes \mathcal{B}(I\!R^{1+d}) \otimes \mathcal{B}(L^2(U,\mathcal{U},\lambda; I\!R))$-measurable and a real valued barrier $\{S_t,\ 0 \le t \le 1\}$ which is $\mathcal{P}$-measurable process. For the problem of existence and uniqueness of solution for reflected BSDE with jumps, we introduce the following assumptions:

(**A.1**) The terminal value ξ is square integrable, *i.e.* $\xi \in L^2(\Omega, F_1, P)$

(**A.2**) The function f satisfies the following conditions :

(i) the process $(f(t,0,0,0))_{t\le 1}$ belongs to $L^2(\Omega \times [0,1], dP \otimes dt)$

(ii) f is uniformly Lipschitz with respect to (y,z), *i.e.*, there exists a constant $0 < k < \infty$ such that for any $y,\ y',\ z,\ z' \in I\!R$ and $v,\ v' \in L^2(U,\mathcal{U},\lambda; I\!R)$,

$$P-a.s.,\ |f(\omega,t,y,z,v) - f(\omega,t,y',z',v')| \le k(|y-y'| + |z-z'|).$$

(iii) there exist two constants $-1 < C_1 \le 0$ and $C_2 \ge 0$ such that $\forall y \in I\!R, \forall z \in I\!R^d, \forall v, v' \in L^2(U,\mathcal{U},\lambda; I\!R)$, we have

$$f(\omega,t,y,z,v) - f(\omega,t,y,z,v') \le \int_U (v(e) - v'(e))\gamma_t^{y,z,v,v'}(e)\lambda(de),$$

where $\gamma_t^{y,z,v,v'} : \Omega \times [0,T] \times U \longrightarrow I\!R$ is $\mathcal{P} \times \mathcal{U}-$measurable and satisfies $C_1(1 \wedge x) \le \gamma_t(e) \le C_2(1 \wedge x)$.

(**A.3**) The barrier process $\{S_t,\ 0 \le t \le 1\}$, is right continuous left-hand limited and satisfying

$$I\!E[\sup_{0\le t\le 1} (S_t)^2] < +\infty \quad \text{and} \quad S_1 \le \xi,\ a.s.$$

Remark 2.1. *Note that, under condition **(A.2)**(iii), the function f is Lipschitz with respect to v, i.e., there exists a constant $0 < \Gamma < \infty$ such that for any $y,\ z \in I\!R$ and $v,\ v' \in L^2(U,\mathcal{U},\lambda; I\!R)$, P-a.s.,*

$$\mid f(\omega,t,y,z,v) - f(\omega,t,y,z,v') \mid \le \Gamma\bigg(\int_U \mid v(e) - v'(e) \mid^2 \lambda(de)\bigg)^{\frac{1}{2}}.$$

Now we introduce the definition of our reflected BSDE with jumps with a single lower obstacle S.

Definition 2.1. *A solution for such an equation is a quadruple* $(Y,Z,K,V) := (Y_t, Z_t, K_t, V_t)_{t\leq 1}$ *of processes with values in* $I\!R^{1+d} \times I\!R^+ \times L^2(U,\mathcal{U},\lambda; I\!R)$ *and which satisfies :*

$$\left\{\begin{array}{ll} (i) & Y \in \mathcal{S}^2;\ Z \in H^{2,d}; V \in \mathcal{L}^2 \ \text{and}\ K \in \mathcal{K}^2, \\ (ii) & Y_t = \xi + \int_t^1 f(s, Y_s, Z_s, V_s)ds + K_1 - K_t - \int_t^1 Z_s dB_s - \\ & -\int_t^1 \int_U V_s(e)\tilde{\mu}(ds, de)\,, t \leq 1, \\ (iii) & Y \text{ dominates } S,\ i.e.\ \forall t \leq 1,\ Y_t \geq S_t, \\ (iv) & \text{the Skorohod condition holds :} \\ & \int_0^1 (Y_{t^-} - S_{t^-})dK_t = 0,\ a.s. \end{array}\right. \tag{5}$$

In our definition, the jumping times of process Y come not only from those of its Poisson process (inaccessible jumps) but also from those of the process S (predictable jumps).

Remark 2.2. *It's worth nothing that condition* (iv) *is equivalent to the following condition :*
If $K = K^c + K^d$, *where* K^c *(resp.* K^d*) is the continuous (resp. the discontinuous) part of* K, *then*
$\int_0^1 (Y_t - S_t)dK_t^c = 0$ *and for every predictable stopping time* $\tau \leq 1$
$\Delta_\tau Y = Y_\tau - Y_{\tau^-}$
$= -(S_{\tau^-} - Y_\tau)^+ 1_{\{S_{\tau^-} = Y_{\tau^-}\}}$. *Moreover, since the jumping times of the Poisson process are inaccessible, for every predictable stopping time* $\tau \leq 1$, $\Delta_\tau Y = -\Delta_\tau K = -(S_{\tau^-} - Y_\tau)^+ 1_{\{S_{\tau^-} = Y_{\tau^-}\}}$.

Now let us recall the Itô formula for *rcll* semimartingales.

2.2. **Itô's formula for *rcll* semi-martingales.** Let $X = \{X_t : t \in [0,T]\}$ be a *rcll* semimartingale, its quadratic variation is denoted by $[X] = \{[X]_t : t \in [0,T]\}$ and let F be a $\mathcal{C}^2$ real valued function, then $F(X)$ is also a semimartingale, and the following formula holds:

$$\begin{aligned} F(X_t) &= F(X_0) + \int_0^t F'(X_{s-})dX_s + \frac{1}{2}\int_0^t F''(X_s)d[X]_s^c \\ &+ \sum_{0<s\leq t} \{F(X_s) - F(X_{s-}) - F'(X_{s-})\Delta X_s\}. \end{aligned} \tag{6}$$

where $[X]^c$ (sometimes denoted by $\langle X \rangle$) is the continuous part of the quadratic variation $[X]$. We also note that in the case where $F(x) = x^2$, the formula (6) takes the form

$$X_t^2 = X_0^2 + \int_0^t 2X_{s-}dX_s + \int_0^t d[X]_s. \quad (7)$$

Moreover if X and Y are two càdlàg semimartingales then we have

$$X_tY_t = X_0Y_0 + \int_0^t X_{s-}dY_s + \int_0^t Y_{s-}dX_s + \int_0^t d[X,Y]_s.$$

where $[X,Y]$ stands for the quadratic covariation of X , Y also called the bracket process. For a complete survey in this topic we refer to Protter [17].

After these preliminaries, we are going to show the uniqueness of the solution for the reflected BSDE with jumps (5) under the above assumptions on f, ξ and S.

2.3. **Uniqueness of the solution for reflected BSDE with jumps.**

Proposition 2.1. *Assume that assumptions (**A.1**), (**A.2**) and (**A.3**) on f, ξ and $(S_t)_{t\leq 1}$ are satisfied. Then the reflected BSDE (5) associated with (f,ξ,S) has a unique solution.*

Proof: Assume (Y,Z,K,V) and (Y',Z',K',V') are two solutions of equation (5). Using Itô's formula (7) with the discontinuous semi-martingale $Y-Y'$ yields

$$\begin{aligned}
&|Y_t - Y_t'|^2 + \int_t^1 |Z_s - Z_s'|^2 ds + \int_t^1\int_U \left(V_s\left(e\right) - V_s'\left(e\right)\right)^2 \lambda(de)ds \\
&= 2\int_t^1 (Y_s - Y_s')(f(s,Y_s,Z_s,V_s) - f(s,Y_s',Z_s',V_s'))ds \\
&+2\int_t^1 (Y_{s-} - Y_{s-}')(dK_s - dK_s') - 2\int_t^1 (Y_s - Y_s')(Z_s - Z_s')dB_s \\
&-2\int_t^1\int_U [(Y_{s-} - Y_{s-}' + V_s(e) - V_s'(e))^2 - (Y_{s-} - Y_{s-}')^2]\tilde{\mu}(ds,de).
\end{aligned} \quad (8)$$

Thanks to the Skorohod condition (iv), we obtain

$$\begin{aligned}
&\int_t^1 \left(Y_{t^-} - Y_{t^-}'\right)\left(dK_t - dK_t'\right) \\
&= \int_t^1 \left(Y_{t^-} - S_{t^-}\right)dK_t + \int_t^1 \left(S_{t^-} - Y_{t^-}'\right)dK_t + \\
&\qquad + \int_t^1 \left(Y_{t^-}' - S_{t^-}'\right)dK_t' + \int_t^1 \left(S_{t^-}' - Y_{t^-}\right)dK_t' \leq 0.
\end{aligned}$$

Now since $\int_t^1\int_U[(Y_{s-} - Y'_{s-} + V_s(e) - V'_s(e))^2 - (Y_{s-} - Y'_{s-})^2]\tilde{\mu}(ds,de)$ and $\int_0^{\cdot} (Y_s - Y'_s)(Z_s - Z'_s)\,dB_s$ are $(\mathcal{F}_t, P)$-martingales, then taking the expectation in both sides of equality (8) yields, for any $t \le 1$,

$$\begin{aligned}
&I\!\!E\left[|Y_t - Y'_t|^2 + \int_t^1 |Z_s - Z'_s|^2 ds + \int_t^1\int_U (V_s(e) - V'_s(e))^2 \lambda(de)\,ds\right] \\
&\le 2I\!\!E\int_t^1 (Y_s - Y'_s)(f(s,Y_s,Z_s,V_s) - f(s,Y'_s,Z'_s,V'_s))\,ds \\
&= 2I\!\!E\int_t^1 (Y_s - Y'_s)(f(s,Y_s,Z_s,V_s) - f(s,Y'_s,Z'_s,V_s))\,ds \\
&+2I\!\!E\int_t^1 (Y_s - Y'_s)(f(s,Y'_s,Z'_s,V_s) - f(s,Y'_s,Z'_s,V'_s))\,ds
\end{aligned}$$

Using assumptions **(A.2)**$(ii) - (iii)$ we have

$$\begin{aligned}
&I\!\!E\left[|Y_t - Y'_t|^2 + \int_t^1 |Z_s - Z'_s|^2 ds + \int_t^1\int_U (V_s(e) - V'_s(e))^2 \lambda(de)\,ds\right] \\
&\le 2I\!\!E\int_t^1 \mid Y_s - Y'_s \mid (k \mid Y_s - Y'_s \mid + k \mid Z_s - Z'_s \mid + \Gamma\|V_s - V'_s\|)\,ds \\
&\le (2k + k\alpha^2 + \Gamma\beta^2)I\!\!E\int_t^1 \mid Y_s - Y'_s \mid^2 ds + \frac{k}{\alpha^2}I\!\!E\int_t^1 \mid Z_s - Z'_s \mid^2 ds \\
&+\frac{\Gamma}{\beta^2}I\!\!E\int_t^1 \|V_s - V'_s\|^2 ds,
\end{aligned}$$

where α and β are two constants. Now, if we choose $\frac{k}{\alpha^2} = \frac{1}{2} = \frac{\Gamma}{\beta^2}$ it follows that

$$\begin{aligned}
I\!\!E\left[|Y_t - Y'_t|^2 + \frac{1}{2}\int_t^1 |Z_s - Z'_s|^2 ds + \frac{1}{2}\int_t^1\int_U (V_s(e) - V'_s(e))^2 \lambda(de)\,ds\right] \\
\le (2k + 2k^2 + 2\Gamma^2)I\!\!E[\int_t^1 (Y_s - Y'_s)^2\,ds].
\end{aligned}$$

With this estimate and using Gronwall's lemma and the right continuity of $(Y_t - Y'_t)_{t\le 1}$, we get $Y = Y'$.
Consequently $(Y, Z, V, K) = (Y', Z', V', K')$ whence the uniqueness of the solution of (5). ■

We now make more precise the dependence of the norm of the solution (Y, Z, K, V) upon the data (ξ, f, S). Using the same technique as in the proof of uniqueness we have also the following estimate :

Proposition 2.2. *Under the above assumption, there exists a constant C which depends only on k and Γ such that*

$$\mathbb{E}\sup_{0\leq t\leq 1}|Y_t|^2+\mathbb{E}\sup_{0\leq t\leq 1}|K_t|^2+\mathbb{E}\int_0^1| Z_t |^2\, dt+\mathbb{E}\int_t^1\int_U (V_s(e))^2\,\lambda(de)\,ds$$
$$\leq C\mathbb{E}\left(| \xi |^2+\int_0^1 | f(t,0,0,0) |^2\, dt+\sup_{0\leq t\leq 1}(S_t)^2\right).$$

3. Monotonic limit theorem for reflected BSDE with jumps.

In this section, we will prove a convergence theorem for a monotonic sequence of processes. It is a generalized version, in the framework of filtration generated by a Brownian motion and Poisson point processs, of a monotonic limit theorem obtained in [15]. This theorem is the following :

Theorem 3.1. *We assume that f satisfies condition **(A.2)**, $\xi\in L^2(\Omega,F_1,P)$ and K^n is continuous process with $\mathbb{E}(K_1^n)^2<\infty$ and $K_0^n=0$, for any $n\in\mathbb{N}$. Let (Y^n,Z^n,V^n) be the solution of the following BSDE*

$$Y_t^n=$$
$$=\xi+\int_t^1 f(s,Y_s^n,Z_s^n,V_s^n)ds+K_1^n-K_t^n-\int_t^1 Z_s^n dB_s-\int_t^1\int_U V_s^n(e)\tilde{\mu}(ds,de)\,,t\leq 1,$$

such that $\mathbb{E}\int_0^1|Z_s^n|^2ds<\infty$ and $\mathbb{E}\int_0^1\int_U (V_s^n(e))^2\lambda(de)ds<\infty$, for all $n\in\mathbb{N}$. If (Y^n) converges increasingly to Y with $\mathbb{E}\sup_{0\leq t\leq 1}|Y_t|^2<\infty$, then there exist $Z\in\mathcal{H}^{2,d}$, $K\in\mathcal{K}^2$ and $V\in\mathcal{L}^2$, such that, the triple (Z,K,V) satisfies the following equation

$$Y_t=\xi+\int_t^1 f(s,Y_s,Z_s,V_s)ds+K_1-K_t-\int_t^1 Z_s dB_s-\int_t^1\int_U V_s(e)\tilde{\mu}(ds,de)\,,t\leq 1.$$

Here Z is the weak limit in $\mathcal{H}^{2,d}$, K is the weak limit of (K_t^n) in $L^2(\mathcal{F}_t)$ and V is the weak limit in $\mathcal{L}^2$. Moreover, for every $p\in[1,2[$, the following strong convergence hold

$$\mathbb{E}[\int_0^1|Y_s^n-Y_s|^2ds]+\mathbb{E}[\int_0^1|Z_s^n-Z_s|^p ds+\int_0^1\left(\int_U|V_s^n-V_s|^2\lambda(de)\right)^{\frac{p}{2}}ds]\to 0.$$

Proof. From the hypothesis, the sequences $(Z^n)_{n\geq 0},(V^n)_{n\geq 0}$ and $(f(.,Y^n,Z^n,V^n)_{n\geq 0}$ are bounded in the respective Hilbert spaces $\mathcal{H}^{2,d}$, $\mathcal{L}^2$ and $L^2([0,1]\times\Omega)$. Then we can extract sequences which weakly converge in the related spaces. We call Z,V and g the respective weak limits. Thanks to the martingale representation theorem, for every stopping time $\tau \le 1$, the following weak convergence hold in $L^2(\mathcal{F}_\tau)$

$$\int_0^\tau f(s, Y_s^n, Z_s^n, V_s^n)ds \rightharpoonup \int_0^\tau g(s)ds, \quad \int_0^\tau Z_s^n dB_s \rightharpoonup \int_0^\tau Z_s dB_s,$$

and

$$\int_0^\tau \int_U V_s^n(e)\tilde{\mu}(ds, de) \rightharpoonup \int_0^\tau \int_U V_s(e)\tilde{\mu}(ds, de), \quad \text{when} \quad n \to +\infty.$$

Since

$$K_\tau^n = Y_0^n - Y_\tau^n - \int_0^\tau f(s, Y_s^n, Z_s^n, V_s^n)ds + \int_0^\tau Z_s^n dB_s + \int_0^\tau \int_U V_s^n(e)\tilde{\mu}(ds, de),$$

thus we have also the following weak convergence in $L^2(\mathcal{F}_\tau)$

$$K_\tau^n \rightharpoonup K_\tau = Y_0 - Y_\tau - \int_0^\tau g(s)ds + \int_0^\tau Z_s dB_s + \int_0^\tau \int_U V_s(e)\tilde{\mu}(ds, de).$$

Since the process $(K_t^n)_{0\le t\le 1}$ is increasing predictable process $K_0^n = 0$ the limit process K remains an increasing predictable (K is equal to its dual predictable projection) process with $\mathbb{E}(K_1)^2 < \infty$ and $K_0 = 0$. Moreover the processes K and Y are *rcll* processes (see Lemma 2.2 in [15]) and then Y has the form

$$Y_t = \xi + \int_t^1 g(s)ds + K_1 - K_t - \int_t^1 Z_s dB_s - \int_t^1 \int_U V_s(e)\tilde{\mu}(ds, de)\,, t \le 1.$$

It remains to prove that, for all $p \in [1, 2[$,

$$\int_0^t g(s)ds = \int_0^t f(s, Y_s, Z_s, V_s)ds, \quad \text{and}$$

$$\mathbb{E}[\int_0^1 |Z_s^n - Z_s|^p ds + \int_0^1 \left(\int_U |V_s^n - V_s|^2 \lambda(de)\right)^{\frac{p}{2}} ds] \to 0.$$

Let $N_t = \int_0^t \int_U V_s(e)\tilde{\mu}(ds, de)$ and $N_t^n = \int_0^t \int_U V_s^n(e)\tilde{\mu}(ds, de)$, then $\Delta_s(Y^n - Y) = \Delta_s(N^n - N + K)$. Applying Itô's formula to $(Y_t^n - Y_t)^2$ on each given subinterval $]\sigma, \tau]$, here $0 \le \sigma \le \tau \le 1$ are two predictable stopping times, we obtain

$$
\begin{aligned}
&(Y^n_\sigma - Y_\sigma)^2 + \int_\sigma^\tau |Z^n_s - Z_s|^2 ds + \sum_{\sigma<s\le\tau} \Big(\Delta_s(N^n - N + K)\Big)^2 \\
&= (Y^n_\tau - Y_\tau)^2 + 2\int_\sigma^\tau (Y^n_s - Y_s)dK^n_s - 2\int_\sigma^\tau (Y^n_{s^-} - Y_{s^-})dK_s \\
&+2\int_t^1 (Y^n_s - Y_s)\left(f(s,Y^n_s,Z^n_s,V^n_s) - f(s,Y_s,Z_s,V_s)\right) ds \\
&-2\int_\sigma^\tau (Y^n_{s^-} - Y_{s^-})(Z^n_s - Z_s)dB_s - 2\int_\sigma^\tau\int_U (Y^n_{s^-} - Y_{s^-})(V^n_s(e) - V_s(e))\tilde{\mu}(ds,de) \\
&= (Y^n_\tau - Y_\tau)^2 + 2\int_\sigma^\tau (Y^n_s - Y_s)dK^n_s + 2\int_\sigma^\tau \Delta_s(N^n - N)dK_s + 2\int_\sigma^\tau \Delta_s K dK_s \\
&-2\int_\sigma^\tau (Y^n_s - Y_s)dK_s + 2\int_t^1 (Y^n_s - Y_s)\left(f(s,Y^n_s,Z^n_s,V^n_s) - f(s,Y_s,Z_s,V_s)\right) ds \\
&-2\int_\sigma^\tau (Y^n_{s^-} - Y_{s^-})(Z^n_s - Z_s)dB_s - 2\int_\sigma^\tau\int_U (Y^n_{s^-} - Y_{s^-})(V^n_s(e) - V_s(e))\tilde{\mu}(ds,de)
\end{aligned}
$$

Taking the expectation and using the fact that $I\!E \sum_{\sigma<s\le\tau} \Delta_s(N^n - N)\Delta_s K = I\!E\int_\sigma^\tau \Delta_s(N^n - N)dK_s$ and $I\!E\sum_{\sigma<s\le\tau}(\Delta_s K)^2 = I\!E\int_\sigma^\tau \Delta_s K dK_s$ (see Lemma A.1 in M. Royer [16]), we get

$$
\begin{aligned}
&I\!E(Y^n_\sigma - Y_\sigma)^2 + I\!E\int_\sigma^\tau |Z^n_s - Z_s|^2 ds + I\!E\sum_{\sigma<s\le\tau}\Big(\Delta_s(N^n - N)\Big)^2 \\
&= I\!E(Y^n_\tau - Y_\tau)^2 + 2I\!E\int_\sigma^\tau (Y^n_s - Y_s)dK^n_s + I\!E\sum_{\sigma<s\le\tau}(\Delta_s K)^2 - 2I\!E\int_\sigma^\tau (Y^n_s - Y_s)dK_s \\
&+2I\!E\int_\sigma^\tau (Y^n_s - Y_s)\left(f(s,Y^n_s,Z^n_s,V^n_s) - g(s)\right) ds.
\end{aligned}
$$

Since $\int_\sigma^\tau (Y^n_s - Y_s)dK^n_s \le 0$ and $I\!E\sum_{\sigma<s\le\tau}\Big(\Delta_s(N^n - N)\Big)^2 = I\!E\int_\sigma^\tau ds\int_U |V^n_s(e) - V_s(e)|^2\lambda(de)$, we obtain

$$
\begin{aligned}
&I\!E(Y^n_\sigma - Y_\sigma)^2 + I\!E\int_\sigma^\tau |Z^n_s - Z_s|^2 ds + I\!E\int_\sigma^\tau ds\int_U |V^n_s(e) - V_s(e)|^2\lambda(de) \\
&\le I\!E(Y^n_\tau - Y_\tau)^2 + I\!E\sum_{\sigma<s\le\tau}(\Delta_s K)^2 - 2I\!E\int_\sigma^\tau (Y^n_s - Y_s)dK_s \\
&+2I\!E\int_\sigma^\tau (Y^n_s - Y_s)\left(f(s,Y^n_s,Z^n_s,V^n_s) - g(s)\right) ds. \qquad (9)
\end{aligned}
$$

Fix a nonnegative constants ε, δ, thanks to Appendix in [16] there exist predictable times $\sigma_k, \tau_k, k = 0, 1, ...N$ such that $]\sigma_j, \tau_j] \cap]\sigma_i, \tau_i] = \emptyset$, $\forall j \neq i$ and

$i) \mathbb{E} \sum_{k=0}^{N} (\tau_k - \sigma_k)(\omega) \geq 1 - \frac{\varepsilon}{2}$,

$ii) \sum_{k=0}^{N} \mathbb{E} \sum_{\sigma_k < t \leq \tau_k} \mid \Delta_s K \mid^2 \leq \frac{\varepsilon \delta}{3}$.

Now for each $\sigma = \sigma_k$ and $\tau = \tau_k$ we apply estimate (9) and then take the sum, it follows that

$$\begin{aligned}
&\mathbb{E} \sum_{k=0}^{N} \int_{\sigma_k}^{\tau_k} |Z_s^n - Z_s|^2 ds + \mathbb{E} \sum_{k=0}^{N} \int_{\sigma_k}^{\tau_k} ds \int_U |V_s^n(e) - V_s(e)|^2 \lambda(de) \\
&\leq \mathbb{E} \sum_{k=0}^{N} (Y_{\tau_k}^n - Y_{\tau_k})^2 + \mathbb{E} \sum_{k=0}^{N} \sum_{\sigma_k < s \leq \tau_k} (\Delta_s K)^2 + 2\mathbb{E} \int_0^1 \mid Y_s^n - Y_s \mid dK_s \\
&+ 2 \mid \mathbb{E} \int_0^1 (Y_s^n - Y_s) \left(f(s, Y_s^n, Z_s^n, V_s^n) - g(s) \right) ds \mid .
\end{aligned} \tag{10}$$

For the last term of the right hand of equation (10) we have

$$\begin{aligned}
&\mid \mathbb{E} \int_0^1 (Y_s^n - Y_s) \left(f(s, Y_s^n, Z_s^n, V_s^n) - g(s) \right) ds \mid \\
&\leq \left(\mathbb{E} \int_0^1 \mid f(s, Y_s^n, Z_s^n, V_s^n) - g(s) \mid^2 ds \right)^{\frac{1}{2}} \left(\mathbb{E} \int_0^1 (Y_s^n - Y_s)^2 ds \right)^{\frac{1}{2}} \\
&\leq C \left(\mathbb{E} \int_0^1 (Y_s^n - Y_s)^2 ds \right)^{\frac{1}{2}} \longrightarrow 0.
\end{aligned} \tag{11}$$

It follows also from dominated convergence theorem that

$$\mathbb{E} \int_{]0,1]} \mid Y_s^n - Y_s \mid dK_s \longrightarrow 0. \tag{12}$$

Taking into account the convergence results (11) and (12) we obtain from estimate (10) that

$$\begin{aligned}
&\limsup_{n \to \infty} \left(\mathbb{E} \sum_{k=0}^{N} \int_{\sigma_k}^{\tau_k} |Z_s^n - Z_s|^2 ds + \mathbb{E} \sum_{k=0}^{N} \int_{\sigma_k}^{\tau_k} ds \int_U |V_s^n(e) - V_s(e)|^2 \lambda(de) \right) \\
&\leq \mathbb{E} \sum_{k=0}^{N} \sum_{\sigma_k < s \leq \tau_k} (\Delta_s K)^2 \leq \frac{\varepsilon \delta}{3}.
\end{aligned}$$

Hence, there exists $N(\varepsilon, \delta) \in \mathbb{N}$ such that $\forall n \geq N(\varepsilon, \delta)$ we obtain

$$\mathbb{E}\sum_{k=0}^{N}\int_{\sigma_k}^{\tau_k}|Z_s^n-Z_s|^2ds+\mathbb{E}\sum_{k=0}^{N}\int_{\sigma_k}^{\tau_k}ds\int_U|V_s^n(e)-V_s(e)|^2\lambda(de)\le\frac{\varepsilon\delta}{3}.$$

Denoting by m the Lebesgue measure on $[0,1]$ one can prove that

$$m\times P\{(\omega,s)\in\Omega\times\bigcup_{0\le k\le N}]\sigma_k(\omega),\tau_k(\omega)]/\mid Z_s^n(\omega)-Z_s(\omega)\mid^2\ge\delta\}\le\frac{\varepsilon}{2},$$

$$m\times P\{(\omega,s)\in\Omega\times\bigcup_{0\le k\le N}]\sigma_k(\omega),\tau_k(\omega)]/\int_U|V_s^n(\omega,e)-V_s(\omega,e)|^2\lambda(de)\ge\delta\}\le\frac{\varepsilon}{2},$$

and then

$$\lim_{n\to\infty}m\times P\{(\omega,s)\in\Omega\times[0,1]/\mid Z_s^n-Z_s\mid^2\ge\delta\}=0,$$

$$\lim_{n\to\infty}m\times P\{(\omega,s)\in\Omega\times[0,1]/\int_U|V_s^n(e)-V_s(e)|^2\lambda(de)\ge\delta\}=0.$$

Thus, on $[0,1]\times\Omega$ (resp. $[0,1]\times\Omega\times U$), the sequence $(Z^n)_{n\ge0}$ (resp. $(V^n)_{n\ge0}$) converges in measure to Z (resp. V). Since $(Z^n)_{n\ge0}$ and $(V^n)_{n\ge0}$ are also bounded in $\mathcal{H}^{2,d}$ and $\mathcal{L}^2$ respectively. Then, for any $p\in[1,2[$, the uniform integrability give us

$$\mathbb{E}[\int_0^1|Z_s^n-Z_s|^pds+\int_t^1\Big(\int_U|V_s^n-V_s|^2\lambda(de)\Big)^{\frac{p}{2}}ds]\to0,\forall p\in[1,2[.$$

Moreover we have also, for any $p\in[1,2[$

$$\begin{aligned}&\mathbb{E}\Big(\int_0^1\mid f(s,Y_s^n,Z_s^n,V_s^n)-f(s,Y_s,Z_s,V_s)\mid^p ds\Big)^{\frac{1}{p}}\\&\le\mathbb{E}\Big(\int_0^1\Big(\Big(k\mid Y_s^n-Y_s\mid+k\mid Z_s^n-Z_s\mid+\\&\qquad+\Gamma(\int_U|V_s^n(e)-V_s(e)|^2\lambda(de))^{\frac{1}{2}}\Big)^p\Big)ds\Big)^{\frac{1}{p}}\\&\le k\mathbb{E}\Big(\int_0^1|Y_s^n-Y_s|^pds\Big)^{\frac{1}{p}}+k\mathbb{E}\Big(\int_0^1|Z_s^n-Z_s|^pds\Big)^{\frac{1}{p}}+\\&\qquad+\Gamma\mathbb{E}\Big(\int_0^1\Big(\int_U|V_s^n-V_s|^2\lambda(de)\Big)^{\frac{p}{2}}ds\Big)^{\frac{1}{p}}\to0.\end{aligned}\tag{13}$$

Henceforth

$$\int_0^t g(s)ds=\int_0^t f(s,Y_s,Z_s,V_s)ds.$$

The proof of Theorem 3.1 is completed. ∎

4. The penalization method for reflected BSDE with jumps

Before giving the main result of this section, it is worth nothing that, in general, we do not have a comparison theorem for solution of BSDE driven by Brownain motion and an independent Poisson process. However, if we consider the following BSDE with jumps

$$Y_t = \xi + \int_t^1 f(s, Y_s, Z_s, V_s)ds - \int_t^1 Z_s dB_s - \int_t^1 \int_U V_s(e)\tilde{\mu}(ds, de)\,, t \le 1, \tag{14}$$

and suppose, in addition, that f satisfies **(A.2)**(iii) we have the following

Theorem 4.1. *(see M. Royer* [16]*) Let us give two pair* (f^1, ξ^1) *and* (f^2, ξ^2) *where* $\xi^1, \xi^2 \in L^2(\Omega, \mathcal{F}_1, P)$*. Denote by* (Y^1, Z^1, V^1) *and* (Y^2, Z^2, V^2) *the solutions of BSDEs with jumps (14) associated respectively with* (ξ^1, f^1) *and* (ξ^2, f^2)*. Assume that **(A .2)** is fulfilled for* f^1 *and* f^2*,* $\xi^1 \le \xi^2$ *a.s. and* $f^1\left(t, Y_t^1, Z_t^1, V_t^1\right) \le f^2\left(t, Y_t^1, Z_t^1, V_t^1\right)$ $dt \times dP$ *a.e. Then* $Y_t^1 \le Y_t^2$*, for* $t \in [0, 1]$*, a.s.*

Note that this comparison theorem will be used only in Step 2 of the proof of Theorem 4.1 below.
Now let us introduce the following penalized equation

$$Y_t^n = \xi + \int_t^1 f(s, Y_s^n, Z_s^n, V_s^n)ds + K_1^n - K_t^n - \int_t^1 Z_s^n dB_s - \int_t^1\int_U V_s^n(e)\tilde{\mu}(ds, de), t \le 1,$$

where $K_t^n = n\displaystyle\int_0^t (Y_s^n - S_s)^- ds$. Note that this equation has a unique solution (see, for example, Hamadène & Ouknine [9] or Barles et *al* [1] or Tang & Li [18]).
We have the following theorem :

Theorem 4.2. *The sequence of processes* $(Y^n, Z^n, V^n), n \in N$ *has a limit* (Y, Z, V) *such that* Y^n *converges to* $Y \in \mathcal{S}^2$ *and* Z *is the weak (resp. strong) limit in* $\mathcal{H}^{2,d}$ *(resp.* $\mathcal{H}^{p,d}$*,* $1 \le p < 2$*),* K *is the weak limit of* (K_t^n) *in* $L^2(\mathcal{F}_t)$ *and* V *is the weak (resp. strong) limit in* $\mathcal{L}^2$ *(resp.* $\mathcal{L}^p$*,* $1 \le p < 2$*).*

Proof. First, let us prove that there exists a constant $C \ge 0$ such that

$$\forall n \ge 0 \text{ and } t \le 1,\ I\!\!E[|Y_t^n|^2 + \int_0^1 |Z_s^n|^2 ds + \int_0^1 ds \int_U (V_s^n(e))^2 \lambda(de) + (K_1^n)^2] \le C. \tag{15}$$

By Itô's formula we obtain,

$$Y_t^{n2} + \int_t^1 |Z_s^n|^2 ds + \sum_{t<s\le 1} (\Delta_s Y^n)^2 = \xi^2 + 2\int_{]t,1]} Y_s^n f(s, Y_s^n, Z_s^n, V_s^n) ds$$
$$+2\int_{]t,1]} nY_s^n (Y_s^n - S_s)^- ds - 2\int_{]t,1]} Y_{s-}^n Z_s^n dB_s - 2\int_{]t,1]} Y_{s-}^n \int_U V_s^n(e)\tilde{\mu}(ds, de), t \le 1.$$

Then, by taking the expectation in both sides and since $I\!\!E \sum_{t<s\le 1} (\Delta_s Y^n)^2 = I\!\!E \int_{]t,1]} ds \int_U (V_s^n(e))^2 \lambda(de)$, we get

$$I\!\!E[|Y_t^n|^2 + \int_t^1 |Z_s^n|^2 ds + \int_{]t,1]} ds \int_U (V_s^n(e))^2 \lambda(de)]$$
$$\le I\!\!E[\xi^2] + 2I\!\!E[\int_{]t,1]} Y_s^n f(s, Y_s^n, Z_s^n, V_s^n) ds] + 2I\!\!E[\int_{]t,1]} nY_s^n (Y_s^n - S_s)^- ds]$$
$$\le I\!\!E[\xi^2] + I\!\!E[\int_{]t,1]} (Y_s^n)^2 ds] + I\!\!E[\int_{]t,1]} (f(s,0,0,0))^2 ds] + I\!\!E[\int_{]t,1]} \Big[k|Y_s^n|(| Y_s^n |$$
$$+ | Z_s^n |) + \Gamma |Y_s^n| \|V_s^n\| \Big] ds + \gamma^{-1} I\!\!E[\sup_{t\le s\le 1} (S_s)^2] + \gamma I\!\!E[(K_1^n - K_t^n)^2];$$

where γ is a universal non-negative real constant. But for any $t \le 1$ we have,

$$I\!\!E[(K_1^n - K_t^n)^2]$$
$$\le C\{I\!\!E[\xi^2 + |Y_t^n|^2 + (\int_t^1 |f(s, Y_s^n, Z_s^n, U_s^n)| ds)^2 + (\int_t^1 Z_s^n dB_s)^2$$
$$+(\int_{]t,1]} \int_U V_s^n(e)\tilde{\mu}(ds, de))^2]\}$$
$$\le C\{I\!\!E[\xi^2 + |Y_t^n|^2 + (\int_t^1 |f(s,0,0,0)| ds)^2 + \int_t^1 |Z_s^n|^2 ds)$$
$$+ \int_{]t,1]} ds \int_U |V_s^n(e)|^2 \lambda(de)]\}$$

where C is a constant. Now plugging this inequality in the previous one yields,

$$\begin{aligned}
&I\!\!E[|Y_t^n|^2 + \int_t^1 |Z_s^n|^2 ds + \int_{]t,1]} ds \int_U (V_s^n(e))^2 \lambda(de)] \\
&\leq (1+\gamma C) I\!\!E[\xi^2] + \gamma C I\!\!E[|Y_t^n|^2] + (1 + k + k\alpha^2 + \Gamma\beta^2) I\!\!E[\int_{]t,1]} (Y_s^n)^2 ds] \\
&+(1+\gamma C) I\!\!E[\int_{]t,1]} (f(s,0,0,0))^2 ds] + \gamma^{-1} I\!\!E[\sup_{t\leq s\leq 1} (S_s)^2] \\
&+(\gamma C + \tfrac{k}{\alpha^2}) I\!\!E[\int_t^1 |Z_s^n|^2 ds) + (\gamma C + \frac{\Gamma}{\beta^2}) \int_{]t,1]} ds \int_U (V_s^n(e))^2 \lambda(de)], t \leq 1.
\end{aligned}$$

Choosing $\gamma C = 1/4 = \frac{\Gamma}{\beta^2} = \frac{k}{\alpha^2}$ we obtain

$$I\!\!E[|Y_t^n|^2 + \frac{1}{2}\int_t^1 |Z_s^n|^2 ds + \frac{1}{2}\int_{]t,1]} ds \int_U (V_s^n(e))^2 \lambda(de)] \leq \widetilde{C}(1 + I\!\!E[\int_t^1 (Y_s^n)^2 ds]), t \leq 1,$$

where $\widetilde{C}$ is positive real constant. Finally applying Gronwall's inequality we get the desired result for $I\!\!E[|Y_t^n|^2]$ and then also for $I\!\!E[\int_0^1 |Z_s^n|^2 ds]$, $I\!\!E[\int_0^1 ds \int_U (V_s^n(e))^2 \lambda(de)]$ and $I\!\!E[(K_1^n)^2]$.

Second, we prove that there exists a constant $C \geq 0$ such that for any $n \geq 0$ we have $I\!\!E[\sup_{0\leq t\leq 1} |Y_t^n|^2] \leq C$.

Indeed for $n \geq 0$, using Itô's formula we have,

$$\begin{aligned}
&Y_t^{n2} + \int_t^1 |Z_s^n|^2 ds + \sum_{t<s\leq 1} (\Delta_s Y^n)^2 \\
&= \xi^2 + 2\int_{]t,1]} Y_s^n f(s, Y_s^n, Z_s^n, U_s^n) ds + 2\int_{]t,1]} n Y_s^n (Y_s^n - S_s)^- ds \qquad (16) \\
&-2\int_{]t,1]} Y_{s-}^n Z_s^n dB_s - 2\int_{]t,1]} Y_{s-}^n \int_U V_s^n(e) \tilde{\mu}(ds, de), t \leq 1.
\end{aligned}$$

But

$$\begin{aligned}
&|\int_t^1 Y_s^n f(s, Y_s^n, Z_s^n, U_s^n) ds| \\
&\qquad \leq (k + k\alpha + \Gamma\beta) \int_t^1 |Y_s^n|^2 ds + \frac{k}{\alpha}\int_t^1 |Z_s^n|^2 ds \\
&\qquad + \tfrac{\Gamma}{\beta}\int_t^1 \|V_s^n\|^2 ds + \int_t^1 \{C_1 |Y_s^n|^2 + C_1^{-1} |f(s,0,0,0)|^2\} ds,
\end{aligned}$$

and

$$\int_t^1 Y_s^n dK_s^n \leq C_2 \sup_{t\leq s\leq 1} |S_s|^2 + C_2^{-1}(K_1^n - K_t^n)^2.$$

On the other hand using Burkholder-Davis-Gundy's inequality ([4],p.304) we get,

$$I\!E[\sup_{t\leq s\leq 1} |\int_{]s,1]} Y_{r-}^n Z_r^n dB_r|] \leq C_3 I\!E[\sup_{t\leq s\leq 1} |Y_s^n|^2] + C_3^{-1} I\!E[\int_t^1 |Z_r^n|^2 dr]$$

and

$$\begin{aligned} I\!E[\sup_{t\leq s\leq 1} |\int_{]s,1]}\int_U Y_{r-}^n V_r^n(e)\tilde{\mu}(dr,de)|] \\ &\leq C I\!E[\{\int_{]t,1]} dr \int_U |Y_{r-}^n V_r^n(e)|^2 \lambda(de)\}^{1/2}] \\ &\leq C_4 I\!E[\sup_{t\leq s\leq 1} |Y_s^n|^2] + \frac{1}{C_4} I\!E[\int_t^1 ds \int_U (V_s^n(e))^2 \lambda(de)]. \end{aligned}$$

Here α, β, C_1, C_2, C_3 and C_4 are universal non-negative real constants. Now combining these inequalities with (16) yields,

$$\begin{aligned} &I\!E[\sup_{t\leq s\leq 1} |Y_s^n|^2 + \int_t^1 |Z_s^n|^2 ds + \int_{]t,1]} ds \int_U (V_s^n(e))^2 \lambda(de)] \\ &\leq I\!E[\xi^2] + 2(k + k\alpha + \Gamma\beta) I\!E[\int_t^1 |Y_s^n|^2 ds + \frac{k}{\alpha}\int_t^1 |Z_s^n|^2 ds + \frac{\Gamma}{\beta}\int_t^1 \|V_s^n\|^2 ds \\ &+ \int_t^1 \{C_1 |Y_s^n|^2 + C_1^{-1}|f(s,0,0,0)|^2\} ds + 2C_2 I\!E[\sup_{t\leq s\leq 1} |S_s|^2] \\ &+ 2C_2^{-1} I\!E[(K_1^n - K_t^n)^2] + 2C_3 I\!E[\sup_{t\leq s\leq 1} |Y_s^n|^2] + 2C_3^{-1} I\!E[\int_t^1 |Z_r^n|^2 dr] \\ &+ 2C_4 I\!E[\sup_{t\leq s\leq 1} |Y_s^n|^2] + 2C_4^{-1} I\!E[\int_t^1 ds \int_U (V_s^n(e))^2 \lambda(de)], \forall t \leq 1. \end{aligned}$$

Finally for choosing suitable constants we obtain $I\!E[\sup_{t\leq 1} |Y_t^n|^2] \leq C$.
Now let $Y_t = \liminf_{n\to\infty} Y_t^n$, $t \leq 1$. Since $f_n(s,y,z,v) = f(s,y,z,v) + n(y - S_s)^-$ satisfies condition **(A.2)**(iii), it follows from comparison theorem (see Theorem 4.1) that for any $n \geq 0$, $Y^n \leq Y^{n+1}$ then, using Fatou's lemma, $I\!E[Y_t^1] \leq I\!E[Y_t] \leq \liminf_{n\to\infty} I\!E[Y_t^n] \leq C$. It follows that for any $t \leq 1$, $Y_t < \infty$ and then P-$a.s.$, $Y_t^n \uparrow Y_t$ as $n \to \infty$.
The proof of Theorem 4.2 will be finished by using Theorem 3.1. ■

5. Existence of the solution for reflected BSDE with jumps

Now we are in position to show the main result of this paper.

Theorem 5.1. *The limit* $(Y_t, Z_t, K_t, V_t)_{t\leq 1}$ *of* $(Y^n_t, Z^n_t, K_t, V^n_t)_{t\leq 1}$ *is the unique solution of the reflected BSDE with jumps (5).*

Proof. The uniqueness result is proved in Section 2. Let us now focus on the existence. We have already proved that (Y, Z, K, V) satisfy (i) and (ii) of equation (5). It remains to prove (iii) and (iv). First observe that for each n, (Y^n, Z^n, V^n) is the solution of the reflected BSDE with jumps and lower barrier $Y^n_t \wedge S_t$. We get from Hamadène & Ouknine [10], that

$$Y^n_t = ess\sup_{v\in\mathcal{T}_t} I\!E\left[\xi 1_{\{v=1\}} + Y^n_\nu \wedge S_\nu 1_{\{v<1\}} + \int_t^v f(s, Y^n_s, Z^n_s, V^n_s)ds\,|\mathcal{F}_t\right], \tag{17}$$

where $\mathcal{T}_t$ is the set of all stopping times valued between t and 1.
Therefore

$$Y^n_t \leq ess\sup_{v\in\mathcal{T}_t} I\!E\left[\xi 1_{\{v=1\}} + S_v 1_{\{v<1\}} + \int_t^v f(s, Y_s, Z_s, V_s)ds\,|\mathcal{F}_t\right]$$
$$+I\!E\left[\int_0^1\left(k|Y^n_s - Y_s| + k|Z^n_s - Z_s| + \Gamma(\int_U | V^n_s(e) - V_s(e)|^2 \lambda(de))^{\frac{1}{2}}\right)ds \mid \mathcal{F}_t\right]$$

By the convergence in Theorem 3.1, we can choose a subsequence such that the last term converges to 0, a.s. It follows that

$$Y_t \leq ess\sup_{v\in\mathcal{T}_t} I\!E\left[\xi 1_{\{v=1\}} + S_v 1_{\{v<1\}} + \int_t^v f(s, Y_s, Z_s, V_s)ds\,|\mathcal{F}_t\right] \tag{18}$$

On the other hand, from Hamadène & Ouknine [9], we deduce that for every stopping time $\tau \leq 1$, $Y_\tau \geq S_\tau 1_{\{\tau<1\}} + \xi 1_{\{\tau=1\}}$. From that and the section theorem (see [4]), we deduce that, $Y_t \geq S_t 1_{\{t<1\}} + \xi 1_{\{t=1\}}$, $\forall t \leq 1$ $P-a.s.$ Moreover, since $Y_t + \int_0^t f(s, Y_s, Z_s, V_s)ds$ is a supermartingale then

$$Y_t \geq ess\sup_{v\in\mathcal{T}_t} I\!E\left[\xi 1_{\{v=1\}} + S_v 1_{\{v<1\}} + \int_t^v f(s, Y_s, Z_s, V_s)ds\,|\mathcal{F}_t\right]. \tag{19}$$

Combining (18) and (19) we obtain

$$Y_t = ess\sup_{v\in\mathcal{T}_t} I\!E\left[\xi 1_{\{v=1\}} + S_v 1_{\{v<1\}} + \int_t^v f(s, Y_s, Z_s, V_s)ds\,|\mathcal{F}_t\right] \tag{20}$$

Let $\eta := (\eta_t)_{t\leq 1}$ be the process defined as follows :

$$\eta_t = \xi 1_{\{t=1\}} + S_t 1_{\{t<1\}} + \int_0^t f(s, Y_s, Z_s, V_s)ds - I\!E\left[\xi + \int_0^1 f(s, Y_s, Z_s, V_s)ds\,|\mathcal{F}_t\right],$$

such that $\eta_1 = 0$. Observe that η is *rcll*. Moreover

$$\sup_{0\le t\le 1} |\eta_t| \in L^2(\Omega). \tag{21}$$

The Snell envelope of η is the smallest *rcll* supermartingale which dominates the process η, it is given by :

$$\mathcal{S}_t(\eta) = ess \sup_{\nu\in\mathcal{T}_t} I\!E\left[\eta_\nu | F_t\right].$$

Now, by assumptions **(A.1)** and **(A.2)**, we have $I\!E[\sup_{t\le 1}|\mathcal{S}_t|^2] < \infty$ and then $(\mathcal{S}_t(\eta))_{t\le 1}$ is of class [D], i.e. the set of random variables $\{\mathcal{S}_\tau(\eta), \tau \in \mathcal{T}_0\}$ is uniformly integrable. Henceforth it has the following Doob-Meyer decomposition

$$\mathcal{S}_t(\eta) = Y_t - I\!E[\xi + \int_t^1 f(s, Y_s, Z_s, V_s)ds|F_t] = M_t^1 + K_t^1,$$

where M^1 is an $\mathcal{F}_t$ martingale and $(K^1 = K^{1,c} + K^{1,d})_{t\le 1}$ is a predictable *rcll* non-decreasing process such that $I\!E(K_1^1)^2 < \infty$ and $K_0^1 = 0$. Through the representation theorem of martingales with respect to $(\mathcal{F}_t)_{t\le 1}$, applied to the martingale

$$M_t^1 + I\!E[\xi + \int_0^1 f(s, Y_s, Z_s, V_s)ds|F_t]$$

there exist two processes $Z^1 = (Z_t^1)_{t\le 1}$ and $V^1 = (V_t^1)_{t\le 1}$ which belong respectively to $H^{2,d}$ and $\mathcal{L}^2$ such that,

$$Y_t = Y_0 - \int_0^t f(s, Y_s, Z_s, V_s)ds + \int_0^t Z_s^1 dW_s + \int_0^t \int_U V_s^1(e)\tilde{\mu}(ds, de) - K_t^1.$$

By identification with (ii) we see that $K^1 = K, Z^1 = Z$ and $V^1 = V$.

Now let us show that $\int_0^1 (Y_{t^-} - S_{t^-})dK_t = 0$, a.s.. We have $\mathcal{S}_t(\eta) = M_t^1 - K_t^{1,c} - K_t^{1,d}$. Since the filtration is generated by a Brownian motion and an independent Poisson measure, the jumping times of $(M_t^1)_{t\le 1}$ are those of the poisson part and then there are inaccessible. Therefore, when $K^{1,d}$ jumps, the process S has the same jump. Then $\{\Delta K^{1,d} > 0\} \subset \{S_-(\eta) = \eta_-\}$ and $\Delta_t K^{1,d} = (\eta_{t^-} - S_t(\eta))^+ 1_{\{\eta_{t^-} = S_{t^-}(\eta)\}}$. Henceforth $\int_0^1 (Y_{t^-} - S_{t^-})dK_t^{1,d} = 0$. Now, since the supermartingale $(S_t(\eta) + K_t^d)_{t\le 1}$ is regular, *i.e.* ${}^p(S(\eta) + K^d) = S_-(\eta) + K_-^d = (S(\eta) + K^d)_-$, where ${}^p(S(\eta) + K^d)$ denote the predictable projection of $(S(\eta) + K^d)$, using the same argument as in Lepeltier & Xu [12] (see also [7]) we have also $\int_0^1 (Y_{t^-} - S_{t^-})dK_t^{1,c} = 0$.

Finally
$$\int_0^1 (Y_{t^-} - S_{t^-}) dK_t = 0.$$
The process (Y, Z, K, V) is then the solution of our reflected BSDE with jumps. ■

Now, let us give a comparison theorem for reflected BSDE with jumps. Let (Y^i, Z^i, K^i, V^i) $(i = 1, 2)$ be two solutions of our equation with jumps (5) associated respectively with (ξ^1, f^1) and (ξ^2, f^2), then we have the following

Theorem 5.2. *Assume that* ***(A .1)***, ***(A .2)*** *and* ***(A .3)*** *are fulfilled for* ξ^1, ξ^2, f^1, f^2 *and* S, $\xi^1 \leq \xi^2$ *a.s. and* $f^1\big(t, Y_t^1, Z_t^1, V_t^1\big) \leq f^2\left(t, Y_t^1, Z_t^1, V_t^1\right) dt \times dP$ *a.e. Then* $Y_t^1 \leq Y_t^2$, *for* $t \in [0, 1]$, *a.s.*

Proof. Consider the two penalized equations
$$\begin{aligned} Y_t^{1,n} &= \xi^1 + \int_t^1 f(s, Y_s^{1,n}, Z_s^{1,n}, V_s^{1,n}) ds + K_1^{1,n} - K_t^{1,n} - \\ &\quad - \int_t^1 Z_s^{1,n} dB_s - \int_t^1 \int_U V_s^{1,n}(e) \tilde{\mu}(ds, de), \\ Y_t^{2,n} &= \xi^2 + \int_t^1 f(s, Y_s^{2,n}, Z_s^{2,n}, V_s^{2,n}) ds + K_2^{1,n} - K_t^{2,n} - \\ &\quad - \int_t^1 Z_s^{2,n} dB_s - \int_t^1 \int_U V_s^{2,n}(e) \tilde{\mu}(ds, de), \end{aligned}$$
where $K_t^{1,n} = n \int_0^t (Y_s^{1,n} - S_s)^- ds$ and $K_t^{2,n} = n \int_0^t (Y_s^{2,n} - S_s)^- ds$. Since $f_n^1(s, y, z, v) = f^1(s, y, z, v) + n(y - S_s)^-$ and $f_n^2(s, y, z, v) = f^2(s, y, z, v) + n(y - S_s)^-$ satisfy condition **(A.2)**, $f_n^1(s, y, z, v) \leq f_n^2(s, y, z, v)$ and $\xi^1 \leq \xi^2$, then by Theorem 4.1 we get $Y_t^{1,n} \leq Y_t^{2,n}$, for $t \in [0, 1]$. Passing to the limit, by Theorem 3.1 we have that $Y_t^1 \leq Y_t^2$, for $t \in [0, 1]$, a.s. ■

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