NORMAL FORMS FOR RATIONAL DIFFERENCE EQUATIONS WITH APPLICATIONS TO THE GLOBAL PERIODICITY PROBLEM.

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Abstract. We propose a classification and derive the associated normal forms for rational difference equations with complex coefficients. As an application, we study the global periodicity problem for second order rational difference equations with complex coefficients. We find new necessary conditions as well as some new examples of globally periodic equations.

1. Introduction

Rational difference equations (linear fractional) have been proposed as a paradigmatic family of equations in order to study the dynamical issues of discrete systems in $\mathbb{R}^n$ or $\mathbb{C}^n$, [16]. In this paper we deal with equations of the type

$$x_{n+k} = \frac{a_0 + a_1 x_n + a_2 x_{n+1} + \cdots + a_k x_{n+k-1}}{b_0 + b_1 x_n + b_2 x_{n+1} + \cdots + b_k x_{n+k-1}}, \qquad (1)$$

where $x_{n+j} \in \mathbb{C}$, $j = 0, \ldots, k-1$ and $a_i, b_i \in \mathbb{C}$ for all $i = 0, \ldots, k$, and we assume that

(I)

$$\operatorname{rank}\begin{pmatrix} a_0 & a_1 & \ldots & a_k \\ b_0 & b_1 & \ldots & b_k \end{pmatrix} = 2,$$

since otherwise equation (1) would be trivial.

(II) $|a_1| + |b_1| > 0$, to ensure that it is properly of order k.

(III) $\sum_{i=1}^{k} |b_i| \neq 0$, since otherwise equation (1) would be linear.

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A difference equation is said to be globally periodic of period p if $x_{n+p} = x_n$ for any given initial condition in the *domain of definition* (that is, the complementary of the *forbidden set*, see [15, page 2]). The study of globally periodic difference equations is a current subject of research, (see for instance [1, 2, 3, 4, 5, 7, 9, 10, 12, 14, 17, 19, 20]), and it is a part of a classical problem of the theory of functional equations which is the resolution of the Babbage equation $F^p = \mathrm{Id}$, see for instance [13] and references therein.

The complete characterization of periodic rational difference equations of order k (either with real or complex coefficients) is still an open problem ([7, 10, 14] or [15]). To our knowledge, all the known periodic rational difference equations of type (1) satisfying (I)–(III), (modulo a conjugation of their associated maps via an homotecy, that is, an *equivalence* of the equations in the terminology used in [7, 10]) are :

(i) The *Möbius–generated* transformations $x_{n+r} = \dfrac{a_0 + a_1 x_n}{b_0 + x_n}$, with a_0, a_1, and b_0, such that $\Delta := (b_0 - a_1)^2 + 4a_0 \neq 0$, and $(b_0 + a_1 - \sqrt{\Delta})/(b_0 + a_1 + \sqrt{\Delta})$ is a p–root of the unity. These equations have period pr.

(ii) Those generated by the Lyness one with $a = 0$ and $a = 1$ respectively: that is $x_{n+2r} = \dfrac{x_{n+r}}{x_n}$, which has period $6r$, and $x_{n+2r} = \dfrac{1 + x_{n+r}}{x_n}$ which has period $5r$.

(iii) Those generated by the Todd's equation: $x_{n+3r} = \dfrac{1+x_{n+r}+x_{n+2r}}{x_n}$, with period $8r$.

(iv) Equations $x_{n+3r} = \dfrac{-1 + x_{n+r} - x_{n+2r}}{x_n}$, with period $8r$.

More precisely, in [10] it is proved that if $a_0, a_2, \ldots, a_k \in \mathbb{C}$, and $a_2, \ldots, a_k$ are non–zero, then the difference equation

$$x_{n+k} = \frac{a_0 + a_2 x_{n+1} + \cdots + a_k x_{n+k-1}}{x_n},$$

is periodic if and only if it is equivalent to the Möbius one $x_{n+1} = 1/x_n$, and the ones of cases (ii)–(iv) of above with $r = 1$. Notice that the complete characterization of globally periodic rational equations with strictly positive coefficients follows from this result and the considerations in Example 4.1 of [2].

In [7] it is proved that if $a_i \geq 0$, $b_i \geq 0$, $\sum_{i=0}^{k} a_i \neq 0$ and $\sum_{i=0}^{k} b_i \neq 0$, there are not periodic difference equations of type (1) satisfying (I)–(III) for

$k = 1, 2, 3, 4, 5, 7, 9$ and 11 than those equivalent to either the Möbius–generated one $x_{n+k} = 1/x_n$, or the Lyness–generated ones (the cases in (ii) of the above list), or the Todd-generated (case (iii) of the above list).

It is conjectured ([6]) that if the coefficients in (1) are real, then any rational periodic equation of type (1) is equivalent to one of the above list. In this paper we show that if the coefficients are complex this is not true (see Proposition 20). For instance equation

$$x_{n+2r} = \frac{ax_n + (1-a)x_{n+r}}{\lambda + x_n},$$

where $a = (1-\sqrt{2}/2)(1+i)$ and $\lambda = (\sqrt{2}-1)\,i$, is an $8r$–periodic equation[1], which is not equivalent to any equation in the above list.

We believe that the success in the study of the problem of global periodicity for equations of type (1) depends on having good normal forms. In fact, a reduced form is known for globally periodic equations with nonnegative coefficients, see [7, Theorem 2.4] or [2, Example 4.1]. In this paper we propose a classification of the equations of type (1) in classes which are invariant by conjugations given by certain affine maps. For each class it is given the corresponding normal form. We have used this classification and the derived normal forms to characterize the necessary conditions that have to fulfill any second order equation of type (1) to be globally periodic. These necessary conditions have allowed us to obtain some globally periodic equations which (to our knowledge) have been not reported until now (see Proposition 20). We believe that the classification and the derived normal forms could also be applied to study other dynamical issues.

Our main result (together with Theorems 5–7) is the following:

Theorem 1. *Consider the second order difference equation*

$$x_{n+2} = \frac{a_0 + a_1 x_n + a_2 x_{n+1}}{b_0 + b_1 x_n + b_2 x_{n+1}}, \tag{2}$$

where the coefficients a_i and b_i satisfy assumptions (I)–(III). If (2) is globally periodic of period p, then either

(i) $b_1 \neq 0$, $b_2 = 0$, $b_0 + a_1 = 0$ *and* $a_0 b_1 - b_0(a_1 + a_2) = 0$. *In this case the equation is globally periodic of period* 6, *but affine conjugated*[2] *with the Lyness type equation* $x_{n+2} = x_{n+1}/x_n$.

Or

[1]This equation is the generalization of the second order one stated in Proposition 20 (i). Recall that a p–periodic difference equation of order k: $x_{n+k} = f(x_n, x_{n+1}, x_{n+2}, \ldots, x_{n+k-1})$ *generates* a rp–periodic difference equation of order rk given by $x_{n+rk} = f(x_n, x_{n+r}, x_{n+2r}, \ldots, x_{n+(k-1)r})$.

[2]See Definition 2 below.

(ii) $b_1 \neq 0, b_2 = 0$, $a_0 b_1 - b_0(a_1 + a_2) \neq 0$. *Moreover setting*

$$\lambda := \frac{b_0 + a_1 + a_2 - \delta}{b_0 + a_1 + a_2 + \delta},$$
$$a := \frac{b_0 + a_1 - a_2 + \delta}{b_0 + a_1 + a_2 + \delta},$$

where $\delta := \sqrt{\Delta} := \sqrt{(b_0 + a_1 + a_2)^2 + 4\,(a_0 b_1 - b_0(a_1 + a_2))}$. *It must be satisfied that* $0 < |\lambda| < 1$ *(or* $|\lambda| = 1$ *and* $\mathrm{Im}(\lambda) \geq 0$*), and it must hold one of the following four cases:*

(a) $a = 1$, $\lambda \neq 1$, p *is even and* $\lambda^{\frac{p}{2}} = 1$. *In this case the equation is a globally periodic Möbius–generated equation of period* p.

(b) $a = \frac{3-\sqrt{5}}{2}$ *and* $\lambda = -\frac{3-\sqrt{5}}{2}$. *In this case the equation is globally periodic of period* 5*, and affine conjugated with the Lyness–type equation* $x_{n+2} = (1 + x_{n+1})/x_n$.

(c) $\lambda = 1$, $a \neq \pm 1$, $(-a)^p = 1$, $\left|\frac{a^2-4a+1}{2a^2-2a+1}\right| \leq 2$ *and the zeros of polynomial* $P(z) = z^2 + \frac{a^2-4a+1}{2a^2-2a+1} z + \frac{a^2-2a+2}{2a^2-2a+1}$ *are* p*–roots of the unity.*

(d) $\lambda = -\bar{a}\frac{1-a}{1-\bar{a}}$ *(where* $\bar{a}$ *denotes the conjugate of* a*),* $0 < |a| < 1$, $\left|1 - \frac{1}{a}\right| < 2$, $a \notin \mathbb{R}$ *and the zeros of polynomials* $P_0(z) = z^2 + \frac{1-\bar{a}}{\bar{a}} z + \frac{a}{\bar{a}}\frac{1-\bar{a}}{1-a}$ *and* $P_1(z) = z^2 - (1-a)z + \frac{1-a}{1-\bar{a}}$*, are* p*–roots of the unity.*

The paper is organized as follows: Section 2 is devoted to present the affine classification and the corresponding normal forms of the difference equations of type (1). The main results are Theorems 5–7, stated in Section 2.2. Some definitions and preliminary results are stated in Section 2.1.

Section 3 is devoted to prove Theorem 1. To do this we study the global periodicity problem using the normal form classification obtained in the previous section. The main result is Proposition 9, which is proved in Section 3.2. All the results in Section 3 strongly rely on the classification given by Corollary 8 in Section 2.2. Finally in Section 3.3 we present new examples of globally periodic difference equations (Proposition 20).

The main results of the paper are related in the following way (where the symbol "→" means "which is related with"):

Theorems 5, 6, 7	⇒	Corollary 8	→	Lemmas 12–18
				⇓
		Theorem 1	⇐	Proposition 9
				↓
				Proposition 20

2. Normal forms for rational difference equations

2.1. **Preliminary results.** Set $\mathbf{x}^t := (x_1, \ldots, x_k) \in \mathbb{C}^k$, equation (1) has the associated dynamical system given by the map

$$F(\mathbf{x}) = (x_2, \ldots, x_k, f(\mathbf{x})), \tag{3}$$

with $f(\mathbf{x}) = (a_0 + \mathbf{a}^t \cdot \mathbf{x})/(b_0 + \mathbf{b}^t \cdot \mathbf{x})$, where $\mathbf{a}^t := (a_1, \ldots, a_k)$, and $\mathbf{b}^t := (b_1, \ldots, b_k)$. We also will say that *equation (1) is given by* f.

Definition 2. (a) *We say that a map F is affine conjugated to G, and write $F \sim G$, if $G = \varphi F \varphi^{-1}$, where*

$$\varphi(\mathbf{x}) = \alpha(\mathbf{x} + \beta\mathbf{u}) \text{ with } \alpha \in \mathbb{C} \setminus \{0\},\ \beta \in \mathbb{C}, \text{ and } \mathbf{u}^t = (1, \ldots, 1) \in \mathbb{C}^k, \tag{4}$$

that is: if their associated dynamical systems are conjugated under an affine transformation of the form (4) (notice that affine changes of the form (4) are the only affine changes which preserve the form (3) of the maps associated to the difference equation).

(b) *Two difference equations are affine conjugated if their associated maps are affine conjugated.*

(c) *If two rational difference equations given by $f(\mathbf{x}) = (a_0+\mathbf{a}^t\cdot\mathbf{x})/(b_0+\mathbf{b}^t \cdot \mathbf{x})$ and $g(\mathbf{x}) = (\tilde{a}_0 + \tilde{\mathbf{a}}^t \cdot \mathbf{x})/(\tilde{b}_0 + \tilde{\mathbf{b}}^t \cdot \mathbf{x})$, are affine conjugated we also denote $f \sim g$.*

Notice that the affine conjugation is an equivalence relation since the affine maps form a group with the "composition" operation.

Next lemma characterizes all the difference equations which are affine conjugated to (1).

Lemma 3. (a) *If two difference equations of type (1) given by $f(\mathbf{x}) = (a_0 + \mathbf{a}^t \cdot \mathbf{x})/(b_0 + \mathbf{b}^t \cdot \mathbf{x})$ and $g(\mathbf{x}) = (\tilde{a}_0 + \tilde{\mathbf{a}}^t \cdot \mathbf{x})/(\tilde{b}_0 + \tilde{\mathbf{b}}^t \cdot \mathbf{x})$, are affine conjugated through (4), then the coefficients of f and g are related by:*

$$\tilde{a}_0 = C\,\alpha^2 \left(a_0 + \beta \left(b_0 - \sum_{i=1}^{k} a_i \right) - \beta^2 \sum_{i=1}^{k} b_i \right), \tag{5}$$

$$\tilde{\mathbf{a}} = C\,\alpha\,(\mathbf{a} + \beta\mathbf{b}), \tag{6}$$

$$\tilde{b}_0 = C\,\alpha\,(b_0 - \beta \sum_{i=1}^{k} b_i) \text{ and} \tag{7}$$

$$\tilde{\mathbf{b}} = C\,\mathbf{b}, \tag{8}$$

where $C \neq 0$.

(b) *If the equation given by f satisfies assumptions (I)–(III), then also does the equation given by g.*

Proof. Statement (a) follows from elementary computations and the fact that two difference equations of type (1) are the same if their coefficients are multiplied by the same constant $C \neq 0$. (b) Consider a difference equation given by the function $f(\mathbf{x}) = (a_0 + \mathbf{a}^t \cdot \mathbf{x})/(b_0 + \mathbf{b}^t \cdot \mathbf{x})$, satisfying hypothesis (I)–(III), and the affine conjugated equation given by the function $g(\mathbf{x}) = (\tilde{a}_0 + \tilde{\mathbf{a}}^t \cdot \mathbf{x})/(\tilde{b}_0 + \tilde{\mathbf{b}}^t \cdot \mathbf{x})$, such that the coefficients of f and g are related via equations (5)–(8). Since $\alpha \neq 0$ and $C \neq 0$, then

$$R = \operatorname{rank}\begin{pmatrix} \tilde{a}_0 & \tilde{a}_1 & \dots & \tilde{a}_k \\ \tilde{b}_0 & \tilde{b}_1 & \dots & \tilde{b}_k \end{pmatrix} =$$

$$= \operatorname{rank}\begin{pmatrix} \alpha^2\left(a_0+\beta\left(b_0-\sum\limits_{i=1}^k a_i\right)-\beta^2\sum\limits_{i=1}^k b_i\right) & \alpha(a_1+\beta b_1) & \dots & \alpha(a_k+\beta b_k) \\ \alpha(b_0-\beta\sum\limits_{i=1}^k b_i) & b_1 & \dots & b_k \end{pmatrix}.$$

Since $\alpha^2(a_0+\beta b_0) = \alpha^2\left[a_0+\beta\left(b_0-\sum\limits_{i=1}^k a_i\right)-\beta^2\sum\limits_{i=1}^k b_i\right] + \sum\limits_{i=1}^k \alpha^2\beta(a_i + \beta b_i)$, we have

$$R = \operatorname{rank}\begin{pmatrix} \alpha^2(a_0+\beta b_0) & \alpha(a_1+\beta b_1) & \dots & \alpha(a_k+\beta b_k) \\ \alpha b_0 & b_1 & \dots & b_k \end{pmatrix} =$$

$$= \operatorname{rank}\begin{pmatrix} \alpha^2 a_0 & \alpha a_1 & \dots & \alpha a_k \\ \alpha b_0 & b_1 & \dots & b_k \end{pmatrix} = \operatorname{rank}\begin{pmatrix} a_0 & a_1 & \dots & a_k \\ b_0 & b_1 & \dots & b_k \end{pmatrix} = 2.$$

Hence the difference equation given by g must satisfy (I).

Suppose that $|\tilde{a}_1| + |\tilde{b}_1| = 0$. This implies that $\tilde{b}_1 = 0$, hence using equation (8) we have $b_1 = 0$; and using (6) we have $a_1 + \beta b_1 = 0$, so $a_1 = 0$ which is a contradiction. Therefore the difference equation given by g must satisfy (II).

Since $\mathbf{b} \neq \mathbf{0}$ and from equation (8) we obtain that $\tilde{\mathbf{b}} \neq \mathbf{0}$, hence the difference equation given by g also satisfies (III). ∎

Lemma 4. *If two rational difference equations given by* $f(\mathbf{x}) = (a_0 + \mathbf{a}^t \cdot \mathbf{x})/(b_0 + \mathbf{b}^t \cdot \mathbf{x})$ *and* $g(\mathbf{x}) = (\tilde{a}_0 + \tilde{\mathbf{a}}^t \cdot \mathbf{x})/(\tilde{b}_0 + \tilde{\mathbf{b}}^t \cdot \mathbf{x})$, *are affine conjugated, then:*

(a) $\sum_{i=1}^k b_i = 0$ *if and only if* $\sum_{i=1}^k \tilde{b}_i = 0$. *Moreover if* $\sum_{i=1}^k b_i = 0$ *then*
  - (i) $\sum_{i=1}^k a_i = 0$ *if and only if* $\sum_{i=1}^k \tilde{a}_i = 0$.
  - (ii) $b_0 - \sum_{i=1}^k a_i = 0$ *if and only if* $\tilde{b}_0 - \sum_{i=1}^k \tilde{a}_i = 0$.
  - (iii) $b_0 = 0$ *if and only if* $\tilde{b}_0 = 0$.

(b) $a_0 \sum_{i=1}^k b_i - b_0 \sum_{i=1}^k a_i = 0$ *if and only if* $\tilde{a}_0 \sum_{i=1}^k \tilde{b}_i - \tilde{b}_0 \sum_{i=1}^k \tilde{a}_i = 0$.

(c) $b_0 + \sum_{i=1}^k a_i = 0$ *if and only if* $\tilde{b}_0 + \sum_{i=1}^k \tilde{a}_i = 0$.

Proof. The proof follows from straightforward elementary computations. As an example we prove statement (b), indeed

$$
\begin{aligned}
&\tilde{a}_0 \sum_{i=1}^{k} \tilde{b}_i - \tilde{b}_0 \sum_{i=1}^{k} \tilde{a}_i = \\
= \quad & C^2\alpha^2 \left(a_0 + \beta(b_0 - \sum_{i=1}^{k} a_i) - \beta^2 \sum_{i=1}^{k} b_i \right) \left(\sum_{i=1}^{k} b_i \right) - \\
& C^2\alpha^2 \left(b_0 - \beta \sum_{i=1}^{k} b_i \right) \left(\sum_{i=1}^{k} a_i + \beta \sum_{i=1}^{k} b_i \right) = \\
= \quad & C^2\alpha^2 \left(a_0 \sum_{i=1}^{k} b_i - b_0 \sum_{i=1}^{k} a_i \right).
\end{aligned}
$$

∎

2.2. **Normal forms for rational equations.** Next we use Lemma 4 to classify all the difference equations of type (1) satisfying (I)–(III), into three families which are invariant under affine equivalence:
Class A: Equations of type (1) such that $\sum_{i=1}^{k} b_i \neq 0$.
Class B: Equations of type (1) such that $\sum_{i=1}^{k} b_i = 0$, and $b_0 - \sum_{i=1}^{k} a_i \neq 0$.
Class C: Equations of type (1) such that $\sum_{i=1}^{k} b_i = 0$, and $b_0 - \sum_{i=1}^{k} a_i = 0$.

The following results give our proposed affine normal forms for the classes A–C.

Theorem 5 (Affine normal forms for the class A)**.** *Consider a k–order rational equation of type (1), satisfying assumptions (I)–(III), and such that $\sum_{i=1}^{k} b_i \neq 0$. Then:*

(A_1) *If $a_0 \sum_{i=1}^{k} b_i - b_0 \sum_{i=1}^{k} a_i = 0$, then $f \sim f_{A_1}$, where*

$$f_{A_1}(\mathbf{x}) = \frac{x_{i_0} + \tilde{a}_{i_0+1} x_{i_0+1} + \cdots + \tilde{a}_k x_k}{\tilde{b}_0 + \tilde{b}_1 x_1 + \cdots + \tilde{b}_k x_k}, \tag{9}$$

and where $\tilde{b}_0 \in \mathbb{C}$, $\sum_{i=1}^{k} \tilde{b}_i = 1$, $\sum_{i=i_0+1}^{k} \tilde{a}_i = -1$, $1 \leq i_0 \leq k-1$, and $i_0 = 1$ if $\tilde{b}_1 = 0$. Moreover this normal form is unique in the sense that two affine conjugated difference equations given by functions of the form of f_{A_1}, must have the same coefficients.

(A_2) *If $a_0 \sum_{i=1}^{k} b_i - b_0 \sum_{i=1}^{k} a_i \neq 0$, then $f \sim f_{A_2}$, where*

$$f_{A_2}(\mathbf{x}) = \frac{\tilde{a}_1 x_1 + \tilde{a}_2 x_2 + \cdots + \tilde{a}_k x_k}{\tilde{b}_0 + \tilde{b}_1 x_1 + \cdots + \tilde{b}_k x_k} = \frac{\tilde{\mathbf{a}}^t \mathbf{x}}{\tilde{b}_0 + \tilde{\mathbf{b}}^t \mathbf{x}}, \tag{10}$$

where $0 < |\tilde{b}_0| < 1$, *(or* $|\tilde{b}_0| = 1$ *and* $\mathrm{Im}(\tilde{b}_0) \geq 0$*),* $\sum_{i=1}^k \tilde{a}_i = \sum_{i=1}^k \tilde{b}_i = 1$, *and* $|\tilde{a}_1| + |\tilde{b}_1| > 0$. *This normal form is unique in the sense that two affine conjugated difference equations given by functions of the form of* f_{A_2}*, must have the same coefficients, except in the case when* $\tilde{b}_0 = -1$, *and* $\tilde{\mathbf{a}} \neq \tilde{\mathbf{b}}$. *In such case we have two possible normal forms:*

$$f_{A_2}(\mathbf{x}) = \frac{\tilde{\mathbf{a}}^t\mathbf{x}}{-1 + \tilde{\mathbf{b}}^t\mathbf{x}}, \ \textit{and}\ f_{A_2}(\mathbf{x}) = \frac{(-\tilde{\mathbf{a}}^t + 2\tilde{\mathbf{b}}^t)\mathbf{x}}{-1 + \tilde{\mathbf{b}}^t\mathbf{x}}$$

Proof. (A_1) The result is a direct consequence of Lemmas 3 and 4. Indeed, using that $\sum_{i=1}^k b_i \neq 0$ and $a_0 \sum_{i=1}^k b_i - b_0 \sum_{i=1}^k a_i = 0$, and setting $C = (\sum_{i=1}^k b_i)^{-1}$ and $\beta = -(\sum_{i=1}^k a_i)/(\sum_{i=1}^k b_i)$, we get directly from equations (8) and (5) that $\sum_{i=1}^k \tilde{b}_i = 1$ and $\tilde{a}_0 = 0$ respectively. Observe that $\tilde{\mathbf{a}} \neq \mathbf{0}$ since otherwise the difference equation associated to f_{A_1} would not satisfy assumption (I), in contradiction with Lemma 3 (b). Now, we can choose α such that equation (6) gives $\tilde{\mathbf{a}}^t = (0, \ldots, 1, \tilde{a}_{i_0+1}, \ldots, \tilde{a}_k)$. This means that through the affine conjugation $\varphi(\mathbf{x}) = \alpha(\mathbf{x} + \beta\mathbf{u})$, we obtain $f \sim f_{A_1}$.

Observe that with the above values of C and β, from equation (6) we have $\sum_{i=1}^k \tilde{a}_i = 0$ (hence $\sum_{i=i_0+1}^k \tilde{a}_i = -1$). This implies that $1 \leq i_0 \leq k-1$ since otherwise if $i_0 = k$ then $0 = \sum_{i=1}^k \tilde{a}_i = \tilde{a}_k = 1$.

Finally if $\tilde{b}_1 = 0$ then $i_0 = 1$, since otherwise f_{A_1} would not satisfy assumption (II).

This normal form is unique in the sense that two affine conjugated difference equations given by functions of the form of f_{A_1}, must have the same coefficients. Indeed, consider two equations of the class A_1:

$$\frac{x_{i_0} + a_{i_0+1}x_{i_0+1} + \cdots + a_k x_k}{b_0 + b_1 x_1 + \cdots + b_k x_k} \sim \frac{x_{j_0} + \tilde{a}_{j_0+1}x_{j_0+1} + \cdots + \tilde{a}_k x_k}{\tilde{b}_0 + \tilde{b}_1 x_1 + \cdots + \tilde{b}_k x_k},$$

such that $\sum_{i=1}^k b_i = \sum_{i=1}^k \tilde{b}_i = 1$; $\sum_{i=i_0+1}^k a_i = \sum_{i=j_0+1}^k \tilde{a}_i = -1$; $1 \leq i_0 \leq k-1$, and $i_0 = 1$ if $b_1 = 0$; $1 \leq j_0 \leq k-1$, and $j_0 = 1$ if $\tilde{b}_1 = 0$.

Using equation (8) in Lemma 3, we have that $1 = \sum_{i=1}^k \tilde{b}_i = C\sum_{i=1}^k b_i = C$. Using (6) we have $0 = \sum_{i=1}^k \tilde{a}_i = C\alpha\left(\sum_{i=1}^k a_i + \beta\sum_{i=1}^k b_i\right)$ hence $0 = \alpha\beta$, so $\beta = 0$. Again using equation (6) we have that $i_0 = j_0$, hence $\alpha = 1$. So the coefficients of both maps are the same.

(A_2) Since $\sum_{i=1}^k b_i \neq 0$, we can choose C in equation (8) such that $\sum_{i=1}^k \tilde{b}_i = 1$, and β in (5) giving $\tilde{a}_0 = 0$. Since $a_0 \sum_{i=1}^k b_i - b_0 \sum_{i=1}^k a_i \neq 0$, from Lemma 4 (b) we have $\tilde{b}_0 \sum_{i=1}^k \tilde{a}_i \neq 0$ hence $\sum_{i=1}^k \tilde{a}_i \neq 0$ so we can find α such that (6) gives $\sum_{i=1}^k \tilde{a}_i = 1$. This means that through the affine conjugation $\varphi(\mathbf{x}) = \alpha(\mathbf{x} + \beta\mathbf{u})$, we obtain $f \sim f_{A_2}$. Observe that, in particular Lemma 4 (b) implies that $\tilde{b}_0 \neq 0$. Assumptions (I)–(III) are directly fulfilled by Lemma 3 (b), in particular $|\tilde{a}_1| + |\tilde{b}_1| > 0$.

Observe that if we don't fix additional hypothesis on the normal form f_{A2}, at this step it is not unique in the sense of the statement. Indeed, let

$$f(\mathbf{x}) = \frac{\mathbf{a}^t \mathbf{x}}{b_0 + \mathbf{b}^t \mathbf{x}}, \text{ and } g(\mathbf{x}) = \frac{\tilde{\mathbf{a}}^t \mathbf{x}}{\tilde{b}_0 + \tilde{\mathbf{b}}^t \mathbf{x}},$$

be with $b_0, \tilde{b}_0 \neq 0$, and $\sum_{i=1}^k a_i = \sum_{i=1}^k \tilde{a}_i = \sum_{i=1}^k b_i = \sum_{i=1}^k \tilde{b}_i = 1$. Straightforward computations using equations (5)–(8) in Lemma 3 show that $f \sim g$ if and only if either they have the same coefficients or their coefficients are related via:

$$\tilde{\mathbf{a}} = \frac{1}{b_0}\mathbf{a} + \left(1 - \frac{1}{b_0}\right)\mathbf{b},\ \tilde{\mathbf{b}} = \mathbf{b}, \text{ and } \tilde{b}_0 = 1/b_0. \tag{11}$$

Since either $|b_0| \leq 1$ or $|1/b_0| \leq 1$, we can fix the representation of f_{A_2} with $|\tilde{b}_0| \leq 1$. After this choice there are only two representations if and only if $|\tilde{b}_0| = 1$. So if we can choose $|\tilde{b}_0| < 1$, then the normal form is unique.

Suppose that $|\tilde{b}_0| = 1$ and $\mathrm{Im}(\tilde{b}_0) \neq 0$. In this case, since $\mathrm{Im}(\tilde{b}_0) > 0$ if and only if $\mathrm{Im}(1/\tilde{b}_0) < 0$, the normal form such that $\mathrm{Im}(\tilde{b}_0) > 0$ is unique.

It only remains to study the case $\tilde{b}_0 \in \{-1, 1\}$. Using the relation (11), we have that if $\tilde{b}_0 = 1$, (or if $\tilde{b}_0 = -1$ and $\tilde{\mathbf{a}} = \tilde{\mathbf{b}}$) the normal form is unique. If $\tilde{b}_0 = -1$ and $\tilde{\mathbf{a}} \neq \tilde{\mathbf{b}}$, we have two representations

$$f_{A_2}(\mathbf{x}) = \frac{\tilde{\mathbf{a}}^t \mathbf{x}}{-1 + \tilde{\mathbf{b}}^t \mathbf{x}}, \text{ and } f_{A_2}(\mathbf{x}) = \frac{(-\tilde{\mathbf{a}}^t + 2\tilde{\mathbf{b}}^t)\mathbf{x}}{-1 + \tilde{\mathbf{b}}^t \mathbf{x}}.$$

∎

The proofs of the next results which state the affine normal forms of the families B and C, are analogous to the above one, and are therefore omitted by reasons of space.

Theorem 6 (Affine normal forms for the class B)**.** *Consider a k–order rational equation of type (1), satisfying assumptions (I)–(III), and such that* $\sum_{i=1}^k b_i = 0$, *and* $b_0 - \sum_{i=1}^k a_i \neq 0$. *Then*

(B_1) *If* $\sum_{i=1}^k a_i = 0$, *then* $f \sim f_{B_1}$, *where*

$$f_{B_1}(\mathbf{x}) = \frac{x_{i_0} + \tilde{a}_{i_0+1} x_{i_0+1} + \cdots + \tilde{a}_k x_k}{\tilde{b}_0 + x_{j_0} + \tilde{b}_{j_0+1} x_{j_0+1} + \cdots + \tilde{b}_k x_k}, \tag{12}$$

and where $\tilde{b}_0 \neq 0$, $\sum_{i=i_0+1}^k \tilde{a}_i = \sum_{i=j_0+1}^k \tilde{b}_i = -1$, $1 \leq i_0 \leq k-1$, $1 \leq j_0 \leq k-1$, *and either* $i_0 = 1$ *or* $j_0 = 1$.

(B_2) *If* $\sum_{i=1}^k a_i \neq 0$, *then* $f \sim f_{B_2}$, *where*

$$f_{B_2}(\mathbf{x}) = \frac{\tilde{a}_1 x_1 + \tilde{a}_2 x_2 + \cdots + \tilde{a}_k x_k}{\tilde{b}_0 + x_{j_0} + \tilde{b}_{j_0+1} x_{j_0+1} + \cdots + \tilde{b}_k x_k}, \tag{13}$$

and where $\tilde{b}_0 \neq 1$, $1 \leq j_0 \leq k-1$, $\sum_{i=1}^k \tilde{a}_i = 1$, $\sum_{i=j_0+1}^k \tilde{b}_i = -1$ *and either* $j_0 = 1$ *or* $\tilde{a}_1 \neq 0$.

These normal forms are unique in the sense that two affine conjugated difference equations given by functions of the form of f_{B_i} $i = 1, 2$, *must have the same coefficients.*

Theorem 7 (Affine normal forms for the class C)**.** *Consider a* k*–order rational equation of type (1), satisfying assumptions (I)–(III), and such that* $\sum_{i=1}^k b_i = 0$, *and* $b_0 - \sum_{i=1}^k a_i = 0$. *Then*

(C_1) *If* $\sum_{i=1}^k a_i = 0$, *then* $f \sim f_{C_1}$, *where:*

(i) *If* $b_1 = 0$, *then*

$$f_{C_1}(\mathbf{x}) = \frac{\tilde{a}_0 + x_1 + \tilde{a}_2 x_2 + \cdots + \tilde{a}_{j_0-1} x_{j_0-1} + \tilde{a}_{j_0+1} x_{j_0+1} + \cdots + \tilde{a}_k x_k}{x_{j_0} + \tilde{b}_{j_0+1} x_{j_0+1} + \cdots + \tilde{b}_k x_k}, \tag{14}$$

where $\tilde{a}_0 \in \mathbb{C}$, $\sum_{i=2}^k \tilde{a}_i = -1$, $\sum_{i=j_0+1}^k \tilde{b}_i = -1$, $2 \leq j_0 \leq k-1$.

(ii) *If* $b_1 \neq 0$ *and* $\mathbf{a} \in \mathrm{span}(\mathbf{b})$ *then*

$$f_{C_1}(\mathbf{x}) = \frac{1}{x_1 + \tilde{b}_2 x_2 + \cdots + \tilde{b}_k x_k}, \tag{15}$$

where $\sum_{i=2}^k \tilde{b}_i = -1$.

(iii) *If* $b_1 \neq 0$ *and* $\mathbf{a} \notin \mathrm{span}(\mathbf{b})$ *then*

$$f_{C_1}(\mathbf{x}) = \frac{\tilde{a}_0 + x_{i_0} + \tilde{a}_{i_0+1} x_{i_0+1} + \cdots + \tilde{a}_k x_k}{x_1 + \tilde{b}_2 x_2 + \cdots + \tilde{b}_k x_k}, \tag{16}$$

where $\tilde{a}_0 \in \mathbb{C}$, $\sum_{i=i_0+1}^k \tilde{a}_i = -1$, $\sum_{i=2}^k \tilde{b}_i = -1$, $2 \leq i_0 \leq k-1$.

(C_2) *If* $\sum_{i=1}^k a_i \neq 0$, *then* $f \sim f_{C_2}$, *where:*

$$f_{C_2}(\mathbf{x}) = \frac{\tilde{a}_0 + \tilde{a}_1 x_1 + \cdots + \tilde{a}_{j_0-1} x_{j_0-1} + \tilde{a}_{j_0+1} x_{j_0+1} + \cdots + \tilde{a}_k x_k}{1 + x_{j_0} + \tilde{b}_{j_0+1} x_{j_0+1} + \cdots + \tilde{b}_k x_k}, \tag{17}$$

where $\tilde{a}_0 \in \mathbb{C}$, $\sum_{i=1}^k \tilde{a}_i = 1$, $\sum_{i=j_0+1}^k \tilde{b}_i = -1$, $1 \leq j_0 \leq k-1$, *and either* $j_0 = 1$ *or* $\tilde{a}_1 \neq 0$.

These normal forms are unique in the sense that two affine conjugated difference equations given by functions of the form of f_{C_i} $i = 1, 2$, *must have the same coefficients.*

The next result is an immediate corollary of Theorems 5–7.

Corollary 8. *Consider the 2nd–order rational difference equations given by the functions*

$$f(x,y)=\frac{a_0+a_1x+a_2y}{b_0+b_1x+b_2y},\ \textit{satisfying hypothesis (I)–(III).}$$

(A) *Suppose that* $b_1+b_2\neq 0$*:*

(A_1) *If* $a_0(b_1+b_2)-b_0(a_1+a_2)=0$*, then* f *is affine conjugated with*

$$f_{A_1}(x,y)=\frac{x-y}{\lambda+bx+(1-b)y},$$

where $\lambda,b\in\mathbb{C}$.

(A_2) *If* $a_0(b_1+b_2)-b_0(a_1+a_2)\neq 0$*, then* f *is affine conjugated with*

$$f_{A_2}(x,y)=\frac{ax+(1-a)y}{\lambda+bx+(1-b)y},$$

where $\lambda,a,b\in\mathbb{C}$, $0<|\lambda|<1$ *(or* $|\lambda|=1$ *and* $\mathrm{Im}(\lambda)\geq 0$*) and* $|a|+|b|>0$.

(B) *Suppose that* $b_1+b_2=0$*, and* $b_0-a_1-a_2\neq 0$*:*

(B_1) *If* $a_1+a_2=0$*, then* f *is affine conjugated with*

$$f_{B_1}(x,y)=\frac{x-y}{\lambda+x-y},$$

where $\lambda\neq 0$ *and* $\lambda\in\mathbb{C}$.

(B_2) *If* $a_1+a_2\neq 0$*, then* f *is affine conjugated with*

$$f_{B_2}(x,y)=\frac{ax+(1-a)y}{\lambda+x-y},$$

where $\lambda\neq 1$, $\lambda,a\in\mathbb{C}$.

(C) *Suppose that* $b_1+b_2=0$*, and* $b_0-a_1-a_2=0$*:*

(C_1) *If* $a_1+a_2=0$*, then* f *is affine conjugated with*

$$f_{C_1}(x,y)=\frac{1}{x-y}.$$

(C_2) *If* $a_1+a_2\neq 0$*, then* f *is affine conjugated with*

$$f_{C_2}(x,y)=\frac{\lambda+y}{1+x-y},$$

where $\lambda\in\mathbb{C}$.

All the above normal forms are unique in the sense that two affine equations of type (2) given by functions of the form f_{A_1}*–*f_{C_2}*, must have the same coefficients, except in the case (*A_2*) with* $\lambda=-1$ *and* $a\neq b$*. In this case there are two possible normal forms.*

Notice that the normal form f_{C_1} corresponds with the case (ii) of (C_1) in Theorem 7. It is easy to see that the cases (i) and (iii) cannot occur.

3. On globally periodic second order rational equations

3.1. **Proof of Theorem 1.** To our knowledge, the only known cases of globally periodic second order equations of type (1) are those affine conjugated with either: (i) the Lyness–type equations:

$$x_{n+2} = \frac{1 + x_{n+1}}{x_n}, \tag{18}$$

(with period 5, and whose normal form is given by f_{A_2} with $b = 1, a = (3 - \sqrt{5})/2$ and $\lambda = -(3 - \sqrt{5})/2$); and

$$x_{n+2} = \frac{x_{n+1}}{x_n}, \tag{19}$$

(which is 6–periodic, and whose normal form is given by f_{A_1} with $b = 1$ and $\lambda = -1$); or (ii) The Möbius–type equations

$$x_{n+2} = \frac{a_0 + a_1 x_n}{b_0 + x_n}, \tag{20}$$

such that $\Delta := (b_0 - a_1)^2 + 4a_0 \neq 0$, being $(b_0 + a_1 - \sqrt{\Delta})/(b_0 + a_1 + \sqrt{\Delta})$ a $p/2$–root of the unity (where p is an even number). In this case the period is p, and the associated affine normal form is given by f_{A_2} with $a = b = 1$ and where λ is a $p/2$–root of the unity, $\lambda \neq 1$ and $\mathrm{Im}(\lambda) \geq 0$.

The next result gives necessary conditions to obtain globally periodic equations of type (2), not affine conjugated with (18), (19) or (20). Observe that Proposition 20 in Section 3.3 guarantees that the statement of Proposition 9 is not empty.

Proposition 9. *Any globally p–periodic second order difference equation*

$$x_{n+2} = \frac{a_0 + a_1 x_n + a_2 x_{n+1}}{b_0 + b_1 x_n + b_2 x_{n+1}} \text{ where } a_i \text{ and } b_i \text{ satisfy (I)–(III)},$$

not affine conjugated with (18), (19) or (20) (if it exists), must be in the class A_2; that is, it must verify $b_1 + b_2 \neq 0$ and $a_0(b_1 + b_2) - b_0(a_1 + a_2) \neq 0$, and its normal form is given by

$$f_{A_2}(x, y) = \frac{ax + (1 - a)y}{\lambda + bx + (1 - b)y}, \tag{21}$$

where $\lambda, a, b \in \mathbb{C}$, $0 < |\lambda| < 1$ (or $|\lambda| = 1$ and $\mathrm{Im}(\lambda) \geq 0$) and $|a| + |b| > 0$. Furthermore, one of the following conditions must hold:

(i) *$b = 1$, $\lambda = 1$, $a \neq \pm 1$, $(-a)^p = 1$, $\left|\frac{a^2-4a+1}{2a^2-2a+1}\right| \leq 2$ and the zeros of polynomial $P(z) = z^2 + \frac{a^2-4a+1}{2a^2-2a+1} z + \frac{a^2-2a+2}{2a^2-2a+1}$ are p–roots of the unity.*

(ii) $b=1$, $\lambda=-\bar{a}\frac{1-a}{1-\bar{a}}$, $0<|a|<1$, $\left|1-\frac{1}{a}\right|<2$, $a\notin\mathbb{R}$ *and the zeros of polynomials* $P_0(z)=z^2+\frac{1-\bar{a}}{\bar{a}}z+\frac{a}{\bar{a}}\frac{1-\bar{a}}{1-a}$ *and* $P_1(z)=z^2-(1-a)z+\frac{1-a}{1-\bar{a}}$ *are p–roots of the unity.*

The proof of the above result is given in Section 3.2. Now we can prove Theorem 1.

Proof of Theorem 1. The result stated in Theorem 1 is a straightforward corollary of Proposition 9. It is only necessary to recall the normal forms of equations (18)–(20), and to characterize those equations whose normal form is given by f_{A_1} or f_{A_2} in Corollary 8. This can be done using relations (5)–(8), and the fact that the proof of Theorem 5 is constructive, and allow us to obtain explicitly the coefficients of the normal forms. We omit here these computations. ∎

3.2. **Proof of Proposition 9.** To prove Proposition 9, we will investigate the global periodicity problem in each of the affine classes stated in Corollary 8, by using their normal form. Therefore the proof follows from several technical lemmas corresponding with each one of the classes A_1–C_2.

In order to prove the results below, we will widely use a couple of results, which for the sake of completeness, are stated in the following. Consider the dynamical system given by the map $F:D\subseteq\mathbb{C}^k\to\mathbb{C}^k$. Set $\mathbf{x}_0\in D$, we say that the recurrence $\mathbf{x}_{n+1}=F(\mathbf{x}_n)$, $n\in\{0\}\cup\mathbb{N}$, is *well defined* if it admits infinite iterates, that is, $\mathbf{x}_n\in D$ for all $n\in\{0\}\cup\mathbb{N}$. Let $U\subseteq D$ be the set of all initial conditions for which the dynamical system is well defined. We call this set the *domain of definition* of the dynamical system defined by F (and its complementary is the *forbidden set*). Observe that U is the biggest invariant subset of D.

We say that F is globally p–periodic (with p not necessarily minimal) if $F^p(\mathbf{x})=\mathbf{x}$ for all $\mathbf{x}\in U$. A fixed point of F^r is a point $\mathbf{x}\in D$ such that $F^r(\mathbf{x})=\mathbf{x}$ and $F^i(\mathbf{x})\in D$, for $i=1,\dots,r-1$. Clearly, if F is globally p–periodic and $\mathbf{x}$ is a fixed point of F^r then $A^p=Id$, where $A=DF^r(\mathbf{x})$. It is well known that the condition $A^p=Id$ is equivalent to the fact that A is a diagonalizable matrix and its eigenvalues are p–roots of the unity, in particular they lie on the unit circle. It is proved in [10], that when F is of the form $F(\mathbf{x})=(x_2,\dots,x_k,f(\mathbf{x}))$ and $F(\mathbf{x}_0)=\mathbf{x}_0$, the eigenvalues of $DF(\mathbf{x}_0)$ are simple.

The next result characterizes the condition for a polynomial to have all its roots on the unit circle ([11],[18]), and therefore, we will use it in order to obtain necessary conditions for a dynamical system to be globally periodic, by imposing them to the characteristic polynomial of the matrices DF^r.

Theorem 10 (Cohn, 1922)**.** *The roots of the polynomial $P(z) = a_0 + a_1 z + \cdots + a_n z^n$, $a_n \neq 0$, lie on the unit circle if and only if the following conditions hold:*

(a) $\overline{a_k} = \xi a_{n-k}$, $k = 0, 1, \ldots, n$, *where ξ is a complex number of modulus* 1.
(b) *All the roots of $P'(z)$ lie in or on the unit circle.*

For $F(\mathbf{x}) = (x_2, \ldots, x_k, f(\mathbf{x}))$ of the form (3) the domain of definition U has full Lebesgue measure in $\mathbb{C}^k$ (that is, $\mathbb{C}^k \setminus U$ has zero Lebesgue measure), see [10]. The rational function

$$G(\mathbf{u}) = \left(\frac{-a_0 - \sum_{i=1}^{k-1} a_{i+1} u_i + b_0 u_k + \left(\sum_{i=1}^{k-1} b_{i+1} u_i \right) u_k}{a_1 - b_1 u_k}, u_1, \ldots, u_{k-1} \right)$$

where $\mathbf{u} = (u_1, \ldots, u_k)$, is the inverse function of F. That is, equations $G(F(\mathbf{x})) = \mathbf{x}$ and $F(G(\mathbf{u})) = \mathbf{u}$ are identities in the variables $x_1, \ldots, x_k$ and $u_1, \ldots, u_k$ respectively. Hence F is birrational. The next result follows from elementary considerations.

Lemma 11. *If F is globally p–periodic (where p is minimal), then $G := F^{-1}$ is also globally periodic with minimal period p. Furthermore the domains of definition of F and G are the same.*

The proofs of Lemmas 12–18 below, follow the same spirit. We will give the significant details of the computations performed in the proof of Lemma 13 (which is the most complex one together with Lemma 16), but for reasons of space we will omit all the computations in the rest of the proofs.

Lemma 12 (Case A_1)**.** *The only globally periodic difference equation in the family given by*

$$f(x, y) = \frac{x - y}{\lambda + bx + (1 - b)y},$$

where $\lambda, b \in \mathbb{C}$, is the one with $b = 1$ and $\lambda = -1$, which is conjugated with equation (19).

Proof. Let F denote the map associated with each difference equation in the family. Then

$$G(u, v) := F^{-1}(u, v) = \left(\frac{u + \lambda v + (1 - b)uv}{1 - bv}, u \right).$$

It can be seen that if $1 + b\lambda \neq 0$ then $(-\lambda, -\lambda)$ is a fixed point of G, but it is not on the domain of definition of F; hence by Lemma 11, F cannot be globally periodic. Assume that $1 + b\lambda = 0$ (and in particular $\lambda \neq 0$). In this case $(0, 0)$ is a fixed point of G and F. Applying Cohn's conditions

(Theorem 10) on the characteristic polynomial of $DG(0,0)$ we obtain that a necessary condition for global periodicity is $\lambda = -1$ and $b = 1$, which corresponds with the normal form of equation (19). ∎

Lemma 13 (Case A_2; part 1)**.** *If a difference equation in the family given by*

$$f(x,y) = \frac{ax + (1-a)y}{\lambda + bx + (1-b)y},$$

where $\lambda, a, b \in \mathbb{C}$, $0 < |\lambda| < 1$ *(or* $|\lambda| = 1$ *and* $\mathrm{Im}(\lambda) \geq 0$*) and* $|a| + |b| > 0$, *is p–periodic, then one of the following conditions must hold:*

(i) $b = 1$, $\lambda = 1$, $a \neq \pm 1$, $(-a)^p = 1$, $\left|\frac{a^2-4a+1}{2a^2-2a+1}\right| \leq 2$ *and the zeros of polynomial* $P(z) = z^2 + \frac{a^2-4a+1}{2a^2-2a+1}z + \frac{a^2-2a+2}{2a^2-2a+1}$ *are p–roots of the unity.*

(ii) $b = 1$, $\lambda = -\bar{a}\frac{1-a}{1-\bar{a}}$, $0 < |a| < 1$, $\left|1 - \frac{1}{a}\right| < 2$, *and the zeros of polynomials* $P_0(z) = z^2 + \frac{1-\bar{a}}{\bar{a}}z + \frac{a}{\bar{a}}\frac{1-\bar{a}}{1-a}$ *and* $P_1(z) = z^2 - (1-a)z + \frac{1-a}{1-\bar{a}}$ *are p–roots of the unity.*

(iii) $a = b = 1$, $\lambda \neq 1$, *p is even and* $\lambda^{\frac{p}{2}} = 1$*. This case corresponds to the normal form of equation (20).*

Proof. Observe that F has the fixed points $(0,0)$ and $(1-\lambda, 1-\lambda)$ (which coincide when $\lambda = 1$). The characteristic polynomials of $DF(0,0)$ and $DF(1-\lambda, 1-\lambda)$ are respectively: $P_0(z) = z^2 + \frac{a-1}{\lambda}z - \frac{a}{\lambda}$ and $P_1(z) = z^2 + (\eta - \lambda)z - \eta$ with $\eta = a - b + b\lambda$. On the other hand F^{-1} is given by

$$G(u,v) = \left(\frac{-(1-a)u + \lambda v + (1-b)uv}{a - bv}, u\right).$$

First observe that when $b = 0$ the domain of definition of G is $\mathbb{C}^2$, which do not coincide with the domain of definition of F since the point $(0, -\lambda)$ does not belong to it. Hence, by Lemma 11 there are no globally periodic equations of the form f_{A_2} with $b = 0$. In the following we will assume $b \neq 0$.

• Case $b \neq 0$, $a = 1$, $\lambda \neq 1$: We will see that in this case we have $b = 1$, $\lambda \neq 1$, p is even and $\lambda^{\frac{p}{2}} = 1$, and therefore it must be conjugated with equation (20). In this case we have that $P_0(z) = z^2 - \frac{1}{\lambda}$ and $P_1(z) = z^2 + (\eta - \lambda)z - \eta$ with $\eta = 1 - b + b\lambda$. Since their zeroes must be p–roots of the unity we have that on one hand (using P_0) p must be even and $\lambda^{\frac{p}{2}} = 1$ (in particular $|\lambda| = 1$).

On the other hand, applying Cohn's conditions to P_1 we obtain that $-\eta = \xi$ with $|\xi| = 1$ and $\eta - \lambda = \xi(\bar{\eta} - \bar{\lambda})$. From these equations we get $\eta - \lambda = -\eta(\bar{\eta} - \bar{\lambda}) = -\eta\bar{\eta} + \eta\bar{\lambda} = -1 + \eta\bar{\lambda}$ (where we have used that $|\eta| = |\xi| = 1$). Hence $\eta = -\frac{1-\lambda}{1-\bar{\lambda}}$, that is $1 - b + b\lambda = -\frac{1-\lambda}{1-\bar{\lambda}}$. Isolating b we reach $b = \frac{2-\lambda-\bar{\lambda}}{1-\lambda-\bar{\lambda}+\lambda\bar{\lambda}}$. Using now that $|\lambda|^2 = 1$ we obtain that $b = \frac{2-\lambda-\bar{\lambda}}{2-\lambda-\bar{\lambda}} = 1$.

• Case $b \neq 0$, $a = 1$, $\lambda = 1$: Now we will prove that in this case we obtain a contradiction with the fact that F is p–periodic. Suppose that $a = \lambda = 1$, $b \neq 0$ and F is globally periodic of period p. The iterates of the points of the form $(r, 0)$ are given by $F^{2k-1}(r, 0) = (0, r/(1 + kbr))$ and $F^{2k}(r, 0) = (r/(1 + kbr), 0)$, with $k \geq 1$. This means that $(r, 0)$ belongs to the domain of definition of F if and only if $1 + kbr \neq 0$ for all $k \geq 1$. On the other hand since we are assuming that $F^p = Id$, we have $F^{2p}(r, 0) = (r, 0)$ if $1 + kbr \neq 0$, $k \geq 1$. But if we set $r = 1$ when b is not a rational number (or we set r to be any non–rational number different from zero when b is rational), we have that $1 + kbr \neq 0$ for all $k \geq 1$ and $F^{2p}(r, 0) = (r, 0) = (r/(1 + pbr), 0)$, which implies that $1 + pbr = 1$, so $b = 0$. In contradiction with the assumptions.

• Case $b \neq 0$, $a \neq 1$, $\lambda \neq 1$: We will prove that under these assumptions the conditions in the statement (ii) are necessary in order to obtain a globally periodic equation. Indeed, applying Cohn's conditions to P_0 we have $-\frac{a}{\lambda} = \xi$ with $|\xi| = 1$ and $\frac{a-1}{\lambda} = \xi \frac{\bar{a}-1}{\bar{\lambda}}$. From these equations we get $|\lambda| = |a|$ (in particular $0 < |a| \leq 1$) and $\frac{a-1}{\lambda} = -\frac{a}{\lambda}\frac{\bar{a}-1}{\bar{\lambda}}$. Hence $(a-1)\bar{\lambda} = -a(\bar{a}-1)$, so

$$\lambda = -\bar{a}\frac{1-a}{1-\bar{a}}\,. \tag{22}$$

Observe that condition $|\lambda| = |a|$ is trivially fulfilled for this value of λ. Moreover $|a| \neq 1$, since otherwise $\lambda = -\bar{a}(1-a)/(1-\bar{a}) = (-\bar{a}+|a|^2)/(1-\bar{a}) = (1-\bar{a})/(1-\bar{a}) = 1$, which is in contradiction with the current assumptions. Observe that after substituting (22) into P_0, we get that P_0 is as in statement (ii).

Cohn's Theorem also establishes that the zeroes of P_0' must be in $|z| \leq 1$. This implies that $\left|1 - \frac{1}{a}\right| \leq 2$. Since the roots of P_0 are simple, its discriminant (say δ_0) must be nonzero, so $\delta_0 = \left(\frac{1-\bar{a}}{\bar{a}}\right)^2 - 4\frac{a(1-\bar{a})}{\bar{a}(1-a)}$ $= \frac{1-\bar{a}}{\bar{a}^2(1-a)}\left(|1-a|^2 - 4|a|^2\right) \neq 0$. This implies that $|1-a|^2 - 4|a|^2 \neq 0$, hence $\left|1 - \frac{1}{a}\right| \neq 2$. In summary a must fulfill the following conditions

$$\left|1 - \frac{1}{a}\right| < 2 \text{ and } 0 < |a| < 1. \tag{23}$$

Applying now Cohn's conditions to P_1 we have $-\eta = \mu$ where $|\mu| = 1$ (so $|\eta|$=1), and $\eta - \lambda = \mu(\bar{\eta} - \bar{\lambda})$. From these equations we have that $\eta = -(1-\lambda)/(1-\bar{\lambda})$. Taking into account equation (22), it is easy to see that $(1-\lambda)/(1-\bar{\lambda}) = (1-a)/(1-\bar{a})$ and that $\eta = a - b + b\lambda = (a - b - a\bar{a} + ba\bar{a})/(1-\bar{a})$. Hence $(a - b - a\bar{a} + ba\bar{a})/(1-\bar{a}) = -(1-a)/(1-\bar{a})$, which implies $b + a\bar{a} - ba\bar{a} = 1$, so $b(1-|a|^2) = 1 - |a|^2$. Since $|a| \neq 1$ we obtain that $b = 1$.

Taking into account that $b = 1$ and equation (22), it is easy to see that $P_1(z) = z^2 - (1-a)z + (1-a)/(1-\bar{a})$ and the roots of P_1 are simple if and only if $|1-a| \neq 2$, and the roots of $P_1'(z)$ are in $|z| \leq 1$ if and only if $|1-a| \leq 2$. Observe that these conditions are trivially satisfied since from (23) and the triangular inequality we have $|1-a| \leq 1 + |a| < 2$.

In summary, a necessary condition for F to be globally p–periodic is $\lambda = -\bar{a}(1-a)/(1-\bar{a})$, $b = 1$, $0 < |a| < 1$ and $\left|1 - \frac{1}{a}\right| < 2$. Moreover the zeroes of P_0 and P_1 must be p–roots of the unity, thus proving statement (ii).

• Case $b \neq 0$, $a \neq 1$, $\lambda = 1$: We will prove that under these assumptions, conditions in statement (i) are necessary in order to obtain a globally p–periodic equation. The unique fixed point of F is $(0,0)$ and the zeroes of P_0 are 1 and $-a$. Hence $(-a)^p = 1$ and $a \neq -1$ (since they must be simple p–roots of the unity). It is easy to see that if $b = 1/2$ or $b = 2a(1-a)$ then there are no fixed points of F^2 different from $(0,0)$. But when $b \neq \frac{1}{2}$ and $b \neq 2a(1-a)$ the point (x_0, y_0) where

$$x_0 := \frac{a-1}{b}\left(1 + \sqrt{\frac{1}{1-2b}}\right) \text{ and } y_0 := \frac{a-1}{b}\left(1 - \sqrt{\frac{1}{1-2b}}\right),$$

is a fixed point of F^2. The characteristic polynomial of $DF^2(x_0, y_0)$ is

$$P(z) = z^2 + \frac{2\mu^2 + (5 - 10a + 3a^2)\mu + (a-1)^2}{\mu\left((2a-1)^2 - \mu\right)}\, z - \frac{(\mu-1)(\mu-(a-1)^2)}{\mu\left((2a-1)^2 - \mu\right)}, \tag{24}$$

where $\mu = 1 - 2b$. Applying Cohn's conditions to P we obtain $-\frac{(\mu-1)(\mu-(a-1)^2)}{\mu((2a-1)^2-\mu)} = \xi$ with $|\xi| = 1$, and $\frac{2\mu^2+(5-10a+3a^2)\mu+(a-1)^2}{\mu((2a-1)^2-\mu)} = \xi\frac{2\bar{\mu}^2+(5-10\bar{a}+3\bar{a}^2)\bar{\mu}+(\bar{a}-1)^2}{\bar{\mu}((2\bar{a}-1)^2-\bar{\mu})}$. From the first equation we have that $\alpha = 0$, where

$$\alpha := \left|(\mu-1)(\mu-(a-1)^2)\right|^2 - \left|\mu\left((2a-1)^2 - \mu\right)\right|^2.$$

Substituting the value of ξ in the second equation, and after some computations we have that

$$\begin{aligned} \beta \;:=\; & \left(2\mu^2 + (5-10a+3a^2)\mu + (a-1)^2\right)\bar{\mu}\left((2\bar{a}-1)^2 - \bar{\mu}\right) \\ & +(\mu-1)(\mu-(a-1)^2)\left(2\bar{\mu}^2 + (5-10\bar{a}+3\bar{a}^2)\bar{\mu} + (\bar{a}-1)^2\right) = 0 \end{aligned}$$

Set $\gamma := \alpha - \beta = 0$. Taking into account that $|a| = 1$ (thus $\bar{a} = 1/a$), and after some computations we have

$$\gamma = \frac{8(a-1)^2\bar{\mu}(a\,|\mu| - \mu - 1 + a)(a\,|\mu| + \mu + 1 - a)}{a^2} = 0.$$

Therefore, there are two possibilities: (a) $a\,|\mu| - \mu - 1 + a = 0$, and (b) $a\,|\mu| + \mu + 1 - a = 0$. We claim that the case (a) is not possible. Indeed, in this case $a = \frac{1+\mu}{1+|\mu|}$, and since $|a| = 1$, we obtain that $1 = \frac{1+\mu}{1+|\mu|}\frac{1+\bar\mu}{1+|\mu|}$ $= \frac{1+\mu+\bar\mu+\mu\bar\mu}{1+2|\mu|+\mu\bar\mu}$. This implies that $1+2|\mu|+\mu\bar\mu = 1+\mu+\bar\mu+\mu\bar\mu$, so $|\mu| = \frac{\mu+\bar\mu}{2}$. Hence $|\mu| = \mathrm{Re}(\mu)$. So μ must be real and positive and thus $a = \frac{1+\mu}{1+|\mu|} = 1$. This contradicts the fact that $a \neq 1$.

This means that $a|\mu| + \mu + 1 - a = 0$. We claim that $|\mu| = 1$. Indeed, if $|\mu| \neq 1$ then $a = \frac{1+\mu}{1-|\mu|}$, so $1 = \frac{1+\mu}{1-|\mu|}\frac{1+\bar\mu}{1-|\mu|} = \frac{1+\mu+\bar\mu+\mu\bar\mu}{1-2|\mu|+\mu\bar\mu}$, and then $1-2|\mu|+\mu\bar\mu$ $= 1+\mu+\bar\mu+\mu\bar\mu$. From here we obtain $-2|\mu| = \mu+\bar\mu$, so $|\mu| = -\mathrm{Re}(\mu)$, hence $\mu < 0$. But this implies that $a = \frac{1+\mu}{1-|\mu|} = 1$, in contradiction with the assumptions. So $|\mu| = 1$, which implies that $\mu = -1$, that is $b = 1$.

Substituting $\mu = -1$ in (24) P becomes as in statement (i), and imposing that the root of P' has modulus less or equal than one we get

$$\left|\frac{a^2-4a+1}{2a^2-2a+1}\right| \leq 2. \tag{25}$$

In summary, if $a \neq 1$, $\lambda = 1$, $b \neq \frac{1}{2}$ and $b \neq 2a(1-a)$ then a necessary condition for global periodicity is $b = 1$, $(-a)^p = 1$, $a \neq \pm 1$, the inequality (25) and that the zeroes of P are p–roots of the unity. Now we need to study the remaining cases: (I) $b = 2a(1-a)$ and (II) $b = 1/2$.

(I) Assume that $b = 2a(1-a)$. The point $(u_0, v_0) = \left(\frac{1}{1-2a}, \frac{a-1}{a(1-2a)}\right)$ is a fixed point of G^2 (thus, in particular belongs to the domain of definition of G). But it does not belong to the domain of definition of F, since $F^2(u_0, v_0)$ is not well defined. Therefore by Lemma 11, F is not globally periodic.

(II) Assume that $b = 1/2$. Neither F^2 nor G^2 have fixed points different from $(0,0)$. But the point (x_0, y_0), where y_0 is a solution of $y_0^3 + 6(a^2-a+1)y_0^2 - 24(a^2-a+1)^2 = 0$, and

$$x_0 = \frac{y_0^2 + 2(3a^2-4a+2)y_0 - 12a(a^2-a+1)}{2(2a-1)},$$

is a fixed point of F^3 (and G^3). Applying Cohn's conditions to the characteristic polynomial of $D\left(F^3\right)(x_0, y_0)$ we obtain that $-\frac{4a^3-6a^2+6a-3}{6a^4-12a^3+12a^2-6a+1} = \xi$ with $|\xi| = 1$, and $\frac{2(6a^4-10a^3+18a^2-12a+7)}{6a^4-12a^3+12a^2-6a+1} = \xi\frac{2(6\bar a^4-10\bar a^3+18\bar a^2-12\bar a+7)}{6\bar a^4-12\bar a^3+12\bar a^2-6\bar a+1}$. Hence

$$\begin{aligned}
\alpha &:= (4a^3-6a^2+6a-3)(4\bar a^3-6\bar a^2+6\bar a-3)\\
&\quad -(6a^4-12a^3+12a^2-6a+1)(6\bar a^4-12\bar a^3+12\bar a^2-6\bar a+1) = 0,\\
\beta &:= (6a^4-10a^3+18a^2-12a+7)(6\bar a^4-12\bar a^3+12\bar a^2-6\bar a+1)\\
&\quad +(4a^3-6a^2+6a-3)(6\bar a^4-10\bar a^3+18\bar a^2-12\bar a+7) = 0.
\end{aligned}$$

Taking into account that $\bar{a} = \frac{1}{a}$, and after some computations we get

$$\begin{aligned} \alpha &= -\frac{6(a^2-a+1)(a^4-3a^3+6a^2-3a+1)(a-1)^2}{a^4}, \\ \beta &= \frac{6(a^2-a+1)(a^6-2a^5+7a^4-12a^3+16a^2-11a+4)}{a^4}, \end{aligned}$$

and therefore $\alpha = \beta = 0$ if and only if $a^2 - a + 1 = 0$, hence $a = a_\pm := 1/2 \pm i\sqrt{3}/2$. Since the both associated difference equations are *complex conjugated*[3] it is sufficient to prove that the equation corresponding to the case $a = 1/2 + i\sqrt{3}/2$ is not globally periodic. Indeed, in this case (using a symbolic computation software) it is easy to see that the point (x_1, y_1), given by

$$\begin{aligned} x_1 &:= \frac{-230-184\sqrt{2}+\left(20\sqrt{3}-19\sqrt{2}\sqrt{3}\right)h}{322} + \frac{\left(46\sqrt{3}-92\sqrt{2}\sqrt{3}+\left(16+17\sqrt{2}\right)h\right)}{322}i \\ y_1 &:= \frac{-230+184\sqrt{2}+\left(22\sqrt{3}+9\sqrt{2}\sqrt{3}\right)h}{322} + \frac{\left(46\sqrt{3}+92\sqrt{2}\sqrt{3}+\left(-30+17\sqrt{2}\right)h\right)}{322}i \end{aligned}$$

with $h = \sqrt{68 + 14\sqrt{2}}$ is a hyperbolic saddle of F^4. ∎

Lemma 14 (Case A_2; part 2)**.** *Under the assumptions of Lemma 13 (ii) and if $a \in \mathbb{R}$, then the only globally periodic difference equation given by f_{A_2} is the one with $a = (3-\sqrt{5})/2$, which is affine conjugated with the Lyness equation (18).*

Proof. Consider a second order equation given by f_{A_2}, satisfying assumptions (ii) of Lemma 13, and assume that a is real. Additionally assume that its associated map F is p–globally periodic. From the assumptions, we have $\lambda = -a$ and $1/3 < a < 1$. It is easy to see that the straight line $S = \left\{(x,y) \in \mathbb{C}^2 \mid x + y = 3a - 1\right\}$ is invariant for F^2. Set $g = h^{-1} \circ F^2 \circ h$ where h is the parametrization of S given by $h(x) = (x, 3a-1-x)$. A brief calculus shows that $g(x) = (-3a^2 + 4a - 1 + (2a-1)x)/(-a+x)$. Hence g is a Möbius transformation. Observe that g is constant if and only if $a = a_\pm := (3 \pm \sqrt{5})/2$, but $a = a_+$ is not allowed since $1/3 < a < 1$. The case $a = a_-$ corresponds to the normal form of equation (18).

Assume now that $a \neq (3-\sqrt{5})/2$. Since $F^{2p} = Id$ we have $g^p = Id$. The dynamic of Möbius transformations is well known (see [8, Corollary 4] for instance), and $g^p = Id$, if and only if $\xi = (a-1-\sqrt{\Delta})/(a-1+\sqrt{\Delta})$ is a p–root of the unity, where $\Delta = (3-a)(3a-1)$. Since $0 < \Delta$, ξ must be real and $\xi \neq 1$, hence $\xi = -1$. This implies that $a = 1$, but recall that $1/3 < a < 1$. So we get a contradiction. ∎

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[3]Applying the change of coordinates $\varphi(\mathbf{x}) = \bar{\mathbf{x}}$ to a difference equation of type (1) we obtain the equation with conjugated coefficients, which we call the *complex conjugated* one. In general, two complex conjugated equations are not affine conjugated.

Notice that statements (i) and (ii) of Proposition 9 give necessary conditions for global periodicity. In order to explore if they could be sufficient for global periodicity, we have studied the fixed points of F^i, $i = 2, 3, 4, 5$ under the assumptions of these statements. We have got that the Cohn's conditions are automatically satisfied; also their fixed points coincide with the ones of G^i, $i = 2, 3, 4, 5$, so we could not make use of Lemma 11 in order to obtain further conditions. We have also tried to reduce the dimension of the dynamics by searching invariant curves, but we have not found any easy one. In Section 3.3 we explain that there are numerical evidences that statement (ii) is not sufficient.

Lemma 15 (Case B_1)**.** *The family of difference equations given by* $f(x, y) = \frac{x-y}{\lambda+x-y}$ *where* $\lambda \in \mathbb{C}\backslash\{0\}$ *does not contain globally periodic cases.*

Proof. Let F denote the map associated with each difference equation in the family. Then, $(0, 0)$ is a fixed point of F. Applying Cohn's conditions on the characteristic polynomial of $DF(0, 0)$ we obtain that $\lambda = -1$ is a necessary condition for global periodicity. But in this case $\big((3 + \sqrt{3})/2, (3 - \sqrt{3})/2\big)$ is a hyperbolic saddle for the dynamical system given by the map F^2. ∎

Lemma 16 (Case B_2)**.** *The family of difference equations given by* $f(x, y) = \frac{ax+(1-a)y}{\lambda+x-y}$ *where* $a, \lambda \in \mathbb{C}$*,* $\lambda \neq 1$ *does not contain globally periodic cases.*

Proof. Let F denote the map associated with each difference equation in the family. The inverse map $G := F^{-1}$ is given by

$$G(u, v) = \left(\frac{-(1-a)u + \lambda v - uv}{a - v}, u \right).$$

If $a \neq 0$ and $\lambda = 0$ then $(0, 0)$ is a fixed point of G but it is not on the domain of definition of F. When $a = 0$ and $\lambda \neq 0$, then $(0, 0)$ is a fixed point of F but it does not belong to the domain of definition of G. Therefore in these cases the difference equations cannot be globally periodic. When $a = \lambda = 0$, then $(u_0, v_0) = (-1/2 + i/2, -1/2 - i/2)$ is a fixed point of G^2 and F^2, which is a hyperbolic repelling node of G^2. Hence in this case F cannot be globally periodic.

Assume that $a \neq 0$ and $\lambda \neq 0$. In this case $(0, 0)$ is a fixed point of F and G. Applying Cohn's conditions to the characteristic polynomial of $DF(0, 0)$ we obtain that the following conditions are necessary to obtain global periodicity:

(i) $a = 1$ and $|\lambda| = 1$, or
(ii) $|a| \neq 1$, $a \neq 0$ and $\lambda = -\bar{a}(1 - a)/(1 - \bar{a})$.

In the case (i), $(1-\lambda, 0)$ is a fixed point of F^2 (and G^2). Cohn's conditions applied to the characteristic polynomial of $DF^2(1 - \lambda, 0)$ leads to $\lambda = 1$, but this value of λ is excluded of the characterization of the family.

Assume (ii), that is $|a| \neq 1$, $a \neq 0$ and

$$\lambda = -\bar{a}\frac{1-a}{1-\bar{a}}\,. \tag{26}$$

Consider the point (x_0, y_0) given by

$$\begin{aligned} x_0 &= \frac{1}{2}\left(2a-1-\lambda+\sqrt{(1-\lambda)(2a-1-\lambda)}\right), \\ y_0 &= \frac{1}{2}\left(2a-1-\lambda-\sqrt{(1-\lambda)(2a-1-\lambda)}\right). \end{aligned}$$

Some computations show that if $\lambda \neq 1-1/(2a)$ and $\lambda \neq (a-1)/(a+1)$ then (x_0, y_0) is a fixed point of F^2 and G^2. Observe that if $\lambda = 1-1/(2a)$ then (x_0, y_0) does not belong to the domain of definition of F; that if $\lambda = (a-1)/(a+1)$ then (x_0, y_0) does not belong to the domain of definition of G; hence if any of the two conditions hold then by Lemma 11 F cannot be globally periodic (observe that when $\lambda = 1-1/(2a)$ and $\lambda = (a-1)/(a+1)$, then $a = 1/3$ and therefore $\lambda = -1/2$, but on the other hand using (26) we obtain $\lambda = -1/3$). Therefore, in the following, we can assume that $\lambda \neq 1-1/(2a)$ and $\lambda \neq (a-1)/(a+1)$.

A tedious computation shows that applying Cohn's conditions to the characteristic polynomial of $DF^2(x_0, y_0)$ we obtain $\frac{a\lambda-a+\lambda+1}{2(2a\lambda-2a+1)} = \xi$ with $|\xi| = 1$ and $\frac{3a\lambda-3a-4\lambda^2-\lambda+1}{2(2a\lambda-2a+1)} = \xi\frac{3\bar{a}\bar{\lambda}-3\bar{a}-4\bar{\lambda}^2-\bar{\lambda}+1}{2(2\bar{a}\bar{\lambda}-2\bar{a}+1)}$. From these conditions, using (26), and after some computations (similar to those in the proof of Lemma 13), we obtain that a necessary condition for F to be globally periodic is that

$$3a\bar{a} - 2a - 2\bar{a} + 1 = 0. \tag{27}$$

Observe that the above condition is equivalent to any of the following ones: (a) $\left|a - \frac{2}{3}\right| = \frac{1}{3}$, or (ii) $\lambda = 2a-1$ (where λ is given by (26)). So in the following we can assume that the difference equation is given by

$$f(x,y) = \frac{ax+(1-a)y}{2a-1+x-y},$$

with $\left|a - \frac{2}{3}\right| = \frac{1}{3}$. Since $|a| \neq 1$ in particular $a \neq 1$.

Now we study the fixed points of F^3 and G^3. It is easy to see that they are given by (x_1, y_1), where

$$x_1 = \frac{3y_1^2 + (a-1)y_1 - 2(7a^2-5a+1)}{4a-1}$$

and y_1 is any solution of

$$3y_1^3 + (-21a^2+15a-3)y_1 + (2a-1)(7a^2-5a+1) = 0$$

Some computations done with the aid of a symbolic computation software show that the Cohn's conditions applied to the characteristic polynomial of $DF^3(x_1,y_1)$ are the following: $\frac{(1-2a)(2a^2+2a-1)}{3(6a^3-12a^2+6a-1)}=\xi$ with $|\xi|=1$ and $\frac{2(8a-7)(7a^2-5a+1)}{3(6a^3-12a^2+6a-1)}=\xi\frac{2(8\bar{a}-7)(7\bar{a}^2-5\bar{a}+1)}{3(6\bar{a}^3-12\bar{a}^2+6\bar{a}-1)}$. After some manipulations taking into account condition (27), we have that a necessary condition for global periodicity is that $a=a_\pm:=5/14\pm i\sqrt{3}/14$ (hence $\lambda=\lambda_\pm:=-2/7\pm i\sqrt{3}/7$). Since the corresponding associated difference equations are complex conjugated we only need to study one case.

Suppose that $a=a_+$ and $\lambda=\lambda_+$. With the aid of a computer algebra software it is easy to obtain the fixed points of F^4 and G^4, and to see that they are hyperbolic saddles, so in this case we have not globally periodic equations. ∎

Lemma 17 (Case C_1)**.** *The difference equation given by $f(x,y)=1/(x-y)$ is not globally periodic.*

Proof. Let F be the map associated with the difference equation. Then $(x_0,y_0):=\left(\sqrt{2}/2,-\sqrt{2}/2\right)$ is a fixed point of F^2. Since $\operatorname{Spec}\left(DF^2(x_0,y_0)\right)=\{-3/8\pm i\sqrt{7}/8\}$,hence this point is an attractor of the system defined by F^2. ∎

Lemma 18 (Case C_2)**.** *The family of difference equations given by $f(x,y)=(\lambda+y)/(1+x-y)$, with $\lambda\in\mathbb{C}$ does not contain globally periodic cases.*

Proof. Let F denote the map associated with each difference equation in the family. First observe that if $\lambda=0$ then $(0,0)$ belongs to the domain of definition of F but not of G, hence by Lemma 11 the corresponding equation cannot be globally periodic. So we can assume $\lambda\neq 0$. If $\lambda\neq 1/2$, then $(x_0,y_0)=\left(-1+\sqrt{2\lambda}/2,-1-\sqrt{2\lambda}/2\right)$ is a fixed point of F^2. Applying Cohn's conditions on the characteristic polynomial of $DF^2(x_0,y_0)$, and after some computations we reach that λ must vanish, so we get a contradiction.

If $\lambda=1/2$, then $(x_0,y_0)=(-1/2,-3/2)$ belongs to the domain of definition of G (since it is a fixed point of G^2), but it does not belong to the domain of definition of F. Therefore this family does not contain globally periodic difference equations. ∎

3.3. **New periodic equations.** Statements (i) and (ii) of Proposition 9 (which correspond with statements (c) and (d) of Theorem 1) give necessary conditions for global periodicity. In this section we give algorithms to find the values of the parameters fulfilling these conditions, thus obtaining the normal forms of those second order rational equations not affine conjugated with (18), (19) or (20) which are "candidate" to be globally periodic. First we establish a preliminary result:

Lemma 19. *The zeroes of a polynomial of the form* $P(z) = z^2 + bz + b/\bar{b}$, *where* $b \in \mathbb{C}$ *and* $\mathrm{Im}(b) > 0$, *are* p*–roots of the unity if and only if* $b = -2\cos(n\pi/p)\exp(i\,m\pi/p)$ *where* n *and* m *are integer numbers with the same parity,* $p/2 < n \leq p$ *and* $0 < m < p$*. Furthermore: (i) The zeroes of* P *are simple if and only if* $n \neq p$*, (ii)* $|1+b| > 1$ *if and only if* $m < n$*, and (iii)* $|1+b| < 1$ *if and only if* $m > n$*.*

Proof. Suppose that the zeroes of P are p–roots of the unity. Set $b = r\exp(i\alpha)$ with $r > 0$, $0 < \alpha < \pi$. From Cohn's theorem, the zeroes of P' are inside the closed unit disc, hence $r \leq 2$. Since the product of the zeroes of P is also a p–root of the unity, then $\left(b/\bar{b}\right)^p = 1$. This implies that $\alpha = m\pi/p$, where $0 < m < p$. On the other hand it is easy to see that the zeroes of P are the numbers $z_\pm = \frac{1}{2}\exp(i\alpha)w_\pm$ with $w_\pm = -r \pm i\sqrt{4-r^2}$. Since $1 = z_+^p = \frac{1}{2^p}\exp(i\,m\pi)w_+^p = \frac{1}{2^p}(-1)^m w_+^p$, we obtain that $w_+^p = (-1)^m 2^p$. Taking into account that $|w_+| = 2$, $\mathrm{Re}(w_+) < 0$ and $\mathrm{Im}(w_+) \geq 0$, we obtain that $w_+ = 2\exp(i\,n\pi/p)$, where n has the same parity of m and $p/2 < n \leq p$. In particular $r = -\mathrm{Re}(w_+) = -2\cos(n\pi/p)$. Assume now that $b = -2\cos(n\pi/p)\exp(i\,m\pi/p)$ where n and m satisfy the hypothesis of the statement. It is easy to see that the zeroes of P are the numbers $z_\pm = \exp(i\,(m \pm n)\pi/p)$. Using that n and m have the same parity, we obtain that the zeroes are p–roots of the unity.

Statements (i), (ii) and (iii) are immediate. We will prove (iii) for instance. Since $|1+b|^2 = (1+b)(1+\bar{b}) = 1 + 2\mathrm{Re}(b) + |b|^2$, we have that the condition $|1+b| < 1$ is equivalent to $-2\mathrm{Re}(b) > |b|^2$, which is equivalent to $4\cos(n\pi/p)\cos(m\pi/p) > 4\cos^2(n\pi/p)$. Since $\cos(n\pi/p) < 0$ the last condition implies that $\cos(m\pi/p) < \cos(n\pi/p)$, so $m > n$. ∎

In the following we will take advantage of the above result to derive two algorithms in order to obtain all the values of a, satisfying the statements of Proposition 9. Notice that if a value of a satisfies the conditions of any of these statements then the conjugate $\bar{a}$ also satisfy them. So, in order to avoid unnecessary computations, in the following we assume that $\mathrm{Im}(a) > 0$.

Assume conditions of Proposition 9 (i) (in particular $b = \lambda = 1$). From the hypothesis $(-a)^p = 1$ and $a \neq \pm 1$, we obtain that $a = \exp(i\,k\pi/p)$ where k is an integer number with the same parity of p, and $0 < k < p$. Setting $b = \frac{a^2-4a+1}{2a^2-2a+1}$ and taking into account that $|a| = 1$, it is immediate to see that $\mathrm{Im}(b) > 0$, and that the polynomial of Proposition 9 (i) takes the form $P(z) = z^2 + bz + b/\bar{b}$. Hence from Lemma 19, we know that $b = -2\cos(n\pi/p)\exp(i\,m\pi/p)$ where n and m are integer numbers with the same parity, such that $p/2 < n \leq p$ and $0 < m < p$. These observations lead us to the following algorithm to determine all the values of a (with $\mathrm{Im}(a) > 0$), satisfying Proposition 9 (i).

Algorithm A Fix $1 \leq p \in \mathbb{N}$. For each value $k = 1, \ldots, p-1$ with the same parity of p:

Step 1. Compute the numbers $a = \exp(i\,k\pi/p)$ and $b = (a^2 - 4a + 1)/(2a^2 - 2a + 1)$.

Step 2. Check if there exist n and m, with the same parity, and such that $0 < m < p$, $p/2 < n \leq p$, satisfying $b = -2\cos(n\pi/p)\exp(i\,m\pi/p)$. If these numbers exist, then the number a defined in Step 1 satisfies Proposition 9 (i). Else, take the next value of k and return to Step 1. □

Assume the conditions of statement (ii) of Proposition 9 (and remember that additionally we assume $\mathrm{Im}(a) > 0$). Setting $b = \frac{1-\bar{a}}{\bar{a}}$ and $c = -(1-a)$ we obtain that $P_0(z) = z^2 + bz + b/\bar{b}$, $P_1(z) = z^2 + cz + c/\bar{c}$, $\mathrm{Im}(b) > 0$ and $\mathrm{Im}(c) > 0$. Furthermore, since $0 < |a| < 1$ also is verified that $|1+b| > 1$ and $|1+c| < 1$. Applying Lemma 19 we have that $b = -2\cos(n\pi/p)\exp(i\,m\pi/p)$ and $c = -2\cos(\ell\pi/p)\exp(i\,k\pi/p)$ where n and m (respectively ℓ and k) are integer numbers with the same parity and such that $p/2 < n < p$ and $0 < m < n$ (respectively $p/2 < \ell < k < p$, and have the same parity). Condition $\left|1 - \frac{1}{a}\right| < 2$ is directly satisfied since $|b| < 2$. On the other hand we have $1 + c = a = 1/(1+\bar{b})$, hence $b = -\frac{\bar{c}}{1+\bar{c}}$. Therefore it must be satisfied the following compatibility condition

$$\cos\left(\frac{n\pi}{p}\right)\exp\left(\frac{m\pi}{p}i\right) = \frac{\cos\left(\frac{\ell\pi}{p}\right)\exp\left(-\frac{k\pi}{p}i\right)}{2\cos\left(\frac{\ell\pi}{p}\right)\exp\left(-\frac{k\pi}{p}i\right) - 1}.$$

Finally, since $a = 1 + c$ we obtain that $a = 1 - 2\cos(\ell\pi/p)\exp(i\,k\pi/p)$. So we can collect all these considerations in the following algorithm:

Algorithm B. Fix $1 \leq p \in \mathbb{N}$. For each couple of numbers ℓ and k with the same parity, and such that $p/2 < \ell < k < p$:

Step 1. Compute the number $\xi = \dfrac{\cos(\ell\pi/p)\exp(-i\,k\pi/p)}{2\cos(\ell\pi/p)\exp(-i\,k\pi/p) - 1}$.

Step 2. Check if there exist n and m with the same parity and such that $p/2 < n < p$, $0 < m < n$, and $\xi = \cos(n\pi/p)\exp(i\,m\pi/p)$. If they exist then the number $a = 1 - 2\cos(\ell\pi/p)\exp(i\,k\pi/p)$ satisfies the conditions of Proposition 9 (ii). Else, take the next values of ℓ and k and return to Step 1. □

Implementing the above algorithms in a computer algebra system it is fast and easy to obtain candidates to be globally periodic equations even for higher values of the period. For instance using Algorithm A we have obtained that there are not globally p–periodic equations satisfying Proposition 9 (i) for $p \leq 100$.

Using the Algorithm B we have obtained that for $p \leq 100$ (except $p = 8, 12, 18$ and 30) there are not equations satisfying the assumptions of Proposition 9 (ii), hence for the above values of p there are not globally periodic equations with *minimal* period p.

When $p = 8$, the only values of a satisfying the necessary conditions are $a_{\pm} = (1 - \sqrt{2}/2)(1 \pm i)$. Some computations with the aid of a symbolic computation software show that the reported equations are globally periodic with minimal period 8. Each couple of equations given by $a_{\pm}$ are complex conjugated. They appear also when the algorithm is applied taking p to be multiple of 8.

Setting $p = 12$, the only possible values of a satisfying the necessary conditions are $a_{\pm} = 1/2 \pm i(1-\sqrt{3}/2)$ and $a_{\pm} = 1-\sqrt{3}/2 \pm i/2$. With the help of a symbolic computation software we have checked that the corresponding equations are globally periodic with minimal period 12. Each couple of equations given by $a_{\pm}$ are complex conjugated, and each equation in any of these couple of families is not affine conjugated to one of the other family. These equations also appear when the algorithm is applied taking p to be multiple of 12.

As far as we know the equations obtained for $p = 8$ and $p = 12$ are not known in the literature.

Applying the algorithm when $p = 18$ we have obtained that the only values of a satisfying the necessary conditions are $a_{\pm} = 1 - 2\cos(7\pi/18)\exp(\pm i\pi/18)$, $a_{\pm} = 1 - \sqrt{3}\cos(5\pi/18) \pm i\cos(5\pi/18)$, and $a_{\pm} = 1 - \sqrt{3}\cos(7\pi/18) \pm i\cos(7\pi/18)$. In all the cases the numerical computations evidence that they seem to be globally periodic. However we have not succeed to check analytically the global periodicity, even with the aid of symbolic software packages.

When $p = 30$ the only values of a satisfying the necessary conditions are:

(a) $a_{\pm} = 1 - 2\cos(2\pi/5)\exp(\pm i\pi/15)$.
(b) $a_{\pm} = 1 - 2\cos(\pi/5)\exp(\pm 2i\pi/15)$.
(c) $a_{\pm} = 1 - \cos(11\pi/30)\sqrt{3} \pm i\cos(11\pi/30)$.
(d) $a_{\pm} = 1 - \cos(13\pi/30)\sqrt{3} \pm i\cos(13\pi/30)$.
(e) $a_{\pm} = 1 - \exp(\pm i\pi/5)$.
(f) $a_{\pm} = 1 - 2\cos(2\pi/5)\exp(\pm i\pi/5)$.

Once again each pair of equations given by $a_{\pm}$ are complex conjugated. The numerical experiments taking the above values of a evidence that cases (a)–(d) should correspond to globally periodic equations of minimal period 30, but we could not check this analytically. However the numeric simulations evidence that cases (e) and (f) give rise to non–globally periodic equations. Therefore this can be considerate as an evidence that statement (ii) in Proposition 9 is not sufficient to guarantee global periodicity.

In summary, from the above considerations we have:

Proposition 20. *If a globally periodic second order difference equation of the form (2), not affine conjugated with (18), (19) or (20), has minimal period* p *with* $1 \leq p \leq 100$*, then* p *must take one of the values* $8, 12, 18$ *or* 30*. Furthermore, the equation is affine conjugated with*

$$x_{n+2} = \frac{ax_n + (1-a)x_{n+1}}{\lambda + x_n}, \text{ where } \lambda = -\bar{a}\frac{1-a}{1-\bar{a}}, \tag{28}$$

and a *must take one of the following values:*

(i) $a_{\pm} = (1 - \sqrt{2}/2)(1 \pm i)$. *In this case we obtain globally periodic equations with minimal period* 8.

(ii) $a_{\pm} = 1/2 \pm i(1 - \sqrt{3}/2)$ *and* $a_{\pm} = 1 - \sqrt{3}/2 \pm i/2$. *In this case we obtain globally periodic equations with minimal period* 12.

(iii) $a_{\pm} = 1 - 2\cos(7\pi/18)\exp(\pm i\pi/18)$, $a_{\pm} = 1 - \sqrt{3}\cos(5\pi/18) \pm i\cos(5\pi/18)$, *and* $a_{\pm} = 1 - \sqrt{3}\cos(7\pi/18) \pm i\cos(7\pi/18)$. *If an equation obtained from one of these values of* a *is globally periodic, then its minimal period is* $p = 18$.

(iv) $a_{\pm} = 1 - 2\cos(2\pi/5)\exp(\pm i\pi/15)$, $a_{\pm} = 1 - 2\cos(\pi/5)\exp(\pm 2i\pi/15)$, $a_{\pm} = 1 - 1\cos(11\pi/30)\sqrt{3} \pm i\cos(11\pi/30)$, $a_{\pm} = 1 - \cos(13\pi/30)\sqrt{3} \pm i\cos(13\pi/30)$, $a_{\pm} = 1 - \exp(\pm i\pi/5)$, *and* $a_{\pm} = 1 - 2\cos(2\pi/5) \cdot \cdot \exp(\pm i\pi/5)$. *If an equation obtained from one of these values of* a *is globally periodic, then its minimal period is* $p = 30$.

Notice that it is already pointed out that the numeric simulations evidence that the equations given by the statement (iii) and the first four ones of statement (iv) could be globally periodic, but this is not the case of the last two cases of statement (iv).

From the above results some open problems arise:

Open problem 1. *Are there (or not) globally periodic equations satisfying the assumptions of Proposition 9 (i), or equivalently the ones of Theorem 1 (ii-c)?*

Open problem 2. *Are there (or not) globally periodic equations with odd minimal period satisfying assumptions of Proposition 9 (ii) (or equivalently the ones of Theorem 1 (ii-d))?*

Open problem 3. *Prove that the cases stated in Proposition 20 (iii) give globally periodic equations with minimal period* $p = 18$.

Open problem 4. *Prove that the first four cases stated in Proposition 20 (iv) give globally periodic equations with minimal period* $p = 30$*, while the last two cases give not–globally periodic equations.*

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