PERIODIC ORBITS IN COMPLEX ABEL EQUATIONS

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Abstract. This paper is devoted to prove two unexpected properties of the Abel equation $dz/dt = z^3 + B(t)z^2 + C(t)z$, where B and C are smooth, $2\pi-$periodic complex valuated functions, $t \in \mathbb{R}$ and $z \in \mathbb{C}$. The first one is that there is no upper bound for its number of isolated $2\pi-$periodic solutions. In contrast, recall that if the functions B and C are real valuated then the number of complex $2\pi-$periodic solutions is at most three. The second property is that there are examples of the above equation with B and C being low degree trigonometric polynomials such that the center variety is formed by infinitely many connected components in the space of coefficients of B and C. This result is also in contrast with the characterization of the center variety for the examples of Abel equations $dz/dt = A(t)z^3 + B(t)z^2$ studied in the literature, where the center variety is located in a finite number of connected components.

1. Introduction

Differential equations of the form

$$\frac{dx}{dt} = x^m + a_{m-1}(t)x^{n-1} + \cdots + a_1(t)x + a_0(t), \tag{1}$$

where $x \in \mathbb{R}$, $t \in \mathbb{R}$ and $a_0, a_1, \ldots, a_{m-1} : \mathbb{R} \to \mathbb{R}$, are smooth 2π-periodic functions and $m \geq 1$, are widely studied, see for instance [11, 14]. More concretely, the problem considered in most papers is the control of the number of solutions of (1) satisfying $x(0) = x(2\pi)$. These solutions are usually called *periodic orbits* of (1). This problem is motivated because the study of the number of periodic orbits of some planar differential equations can be reduced to the study of the number of periodic orbits of an equation of the form (1), sometimes with a coefficient $a_m(t)x^m$ instead of x^m, see for instance [6, 7, 8]. Hence results on the number of periodic orbits of (1) also contribute to advance in the solution of the Hilbert's sixteenth problem.

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It is not difficult to see that when $m = 1, 2$ the maximum number of periodic orbits of equation (1) is m. Indeed, in the case $m = 1$, equation (1) is a linear equation and can be easily integrated, while in the case $m = 2$ it is a Riccati equation. See [13, 15] for some detailed studies of this later case. The fact that equation (1) with $m = 3$ has at most 3 periodic solution was, as far as we know, firstly proved by Pliss in [18, Thm. 9.7] and later extended to more general cases in [2, 10, 13, 16]. Also in the book of Pliss [18] appears a first surprising example: there are equations of the form (1) with $m = 4$ having more than m periodic orbits (he presents an example with at least 5 periodic orbits). At this point it is clear that the polynomial differential equation starts to behave in a different way than the usual polynomial equations. Later on it was proved by Lins [13] that given any $\ell \in \mathbb{N}$ and any $m \geq 4$, there are equations of the form (1) having at least ℓ isolated periodic orbits, see also [9, 17].

The case where x, instead of being a real variable, is considered as a complex one has also been studied in the literature. To avoid misunderstandings we write again equation (1) with $z \in \mathbb{C}$,

$$\frac{dz}{dt} = z^m + a_{m-1}(t)z^{m-1} + \cdots + a_1(t)z + a_0(t), \tag{2}$$

where again $t \in \mathbb{R}$ and $a_0, a_1, \ldots, a_{m-1} : \mathbb{R} \to \mathbb{R}$, are smooth 2π-periodic functions and $m \geq 1$. Curiously enough, for any $m \in \mathbb{N}$ the same results as above hold. Namely, for $m = 1, 2, 3$ the maximum number of periodic orbits of equation (2) is m and there is no upper bound for $m \geq 4$. All the proofs essentially work equal except the one of case $m = 3$, which is much more complicated, see the interesting paper of Lloyd [14].

The first two main results of our paper concern with a second (for us) surprise related now with the case $m = 3$, for which equations (1) or (2) are called Abel equations. We consider the problem of knowing what happens when the functions $a_j(t)$, $j = 1, 2 \ldots, m-1$ appearing in (2) are complex valued. We consider this problem mainly motivated by the Problem stated in the Introduction of [11].

In this new framework, the results given above can also be proved for $m = 1$ or $m \geq 4$. When $m = 2$ it is known that either the Riccati equation has a continuum of periodic orbits or there are at most two of them, and all cases are possible. On the other hand in the case $m = 3$ we obtain some unexpected properties of the complex Abel equation. Before stating our first result we recall that isolated periodic orbits of the equation are also called *limit cycles*.

Theorem 1. *Given any $\ell \in \mathbb{N}$, there are Abel equations*

$$\frac{dz}{dt} = z^3 + B(t)\,z^2 + C(t)\,z + D(t), \tag{3}$$

with $t \in \mathbb{R}$ and B, C and D, 2π-periodic complex valued trigonometric polynomials, having at least ℓ limit cycles.

Our proof of the above Theorem is a consequence of a more general result, see Theorem 6. There we study how many periodic orbits appear from a special perturbation of the integrable complex equation $\dot{z} = z^n(z - i/n)$, for $n \geq 2$. This technique is standard and is already used in [13]. The point is to reduce the study of the number of limit cycles to the study of the number of simple zeros of a one variable function, namely $W(\rho)$, obtained from the variational equations of the perturbed system. In [13] this function is studied only in a neighborhood of the origin $\rho = 0$. Here we have obtained a global expression of $W(\rho)$. In other words the limit cycles of the Abel equations obtained in [13] live in a small neighborhood of $\rho = 0$ while the limit cycles that we obtain appear in almost arbitrary points of the complex plane of the values of ρ. More concretely, Theorem 6 gives a simultaneous bifurcation from $n + 1$ centers. Furthermore it also allows to give lower bounds for the number of complex limit cycles of equations of the form (2) with all the functions $a_j(t)$, $j = 0, 1 \ldots, m-1$, being complex trigonometric polynomials, in terms of their degrees. We recall again that this problem together with the problem of giving realistic upper bounds is proposed in [11, 12].

By using similar methods we have also proved the following result:

Theorem 2. *Given any $\ell \in \mathbb{N}$, there are Abel equations*

$$\frac{dz}{dt} = A(t)\, z^3 + z^2 + C(t)\, z, \tag{4}$$

with $t \in \mathbb{R}$ and $A(t)$ and $C(t)$, 2π-periodic complex valued trigonometrical polynomials, having at least ℓ limit cycles.

This result is also interesting in the light of Theorem B of [10], where it is proved that equation (4), with $A(t)$ and $C(t)$, 2π-periodic real valued functions has at most three real periodic orbits. As far as we know, the study of the number of complex periodic orbits of the problem studied in [10] is not considered in the literature.

The last part of the paper is devoted to study some properties of the structure of the set of Abel equations (3) having a center. Recall that if there exists an open (real) set of initial conditions for equation (1) such that all the solutions starting at it are periodic orbits of the equation it is said that this equation has *a center* at each of these solutions. If instead of (1) we consider (2) with functions $a_j(t)$, $j = 1, 2, \ldots, m-1$ real or complex valued, the definition of a differential equation having a center is the same but changing the real neighborhood by a complex one. Note that equation (1) for $m = 1, 2, 3$ has no centers (in fact its number of periodic orbits is at most

m). On the other hand equation (2) has centers for any $m \geq 2$. For instance it suffices to consider the autonomous equations $\dot{z} = z(z^{m-1} - i/(m-1))$, see Lemma 5.

In the following result we solve the characterization of the centers for a simple family of Abel equations. This result is a corollary of more general results presented in Section 3.

Theorem 3. *The periodic orbit $z = 0$ of the equation*

$$\dot{z} = z^3 + \left(C_{-1}\, e^{-t\,i} + C_0 + C_1\, e^{t\,i}\right) z, \qquad \text{with} \qquad C_1 C_{-1} \neq 0, \tag{5}$$

is a center if and only if $C_0 = k\,i$ for some $k \in \mathbb{Z}$, and

$$J_{2|k|}\left(4\,i\sqrt{C_{-1}\,C_1}\right) = 0,$$

where $J_{2|k|}$ is the Bessel function of order $2\,|k|$.

Note that if we consider the space $\{(C_{-1}, C_0, C_1) \in \mathbb{C}^3 \,:\, C_1 C_{-1} \neq 0\}$ of all the equations of the form (5) then the first center condition $C_0 = ki$ gives a numerable number of hyperplanes of $\mathbb{C}^3$, one for each $k \in \mathbb{Z}$. Inside each one of these hyperplanes the center condition reads as follows: the quantity $4\,i\sqrt{C_{-1}\,C_1}$ is a zero of $J_{2|k|}$. It is well known that there are only a numerable number of zeros of these functions, namely $\{p_k^m\}_{m=1}^{\infty}$ and that all them are real, see [19]. Thus on each of these hyperplanes there are infinitely many different connected components where the Abel equation (5) has a center and on each of them the condition writes as $4\,i\sqrt{C_{-1}\,C_1} = p_k^m$, for $m \in \mathbb{N}$ and $k \in \mathbb{Z}$. As far as we know this is a new structure of the center variety of a family of equations. Usually this variety has finitely many connected components, see for instance [1, 5]. As we will see in Section 3 the reason for which infinitely many components appear for an equation with $a_0(t) \equiv 0$ is the dependence of $a_1(t)$ with respect to t. It is not difficult to see that even for a real Abel equation of the form (1) this phenomenon will appear if $a_1(t)$ is not a constant. Maybe a reason for which it has not been detected before is that the Abel equations that come from planar differential equation have $a_1(t) \equiv a_1 \in \mathbb{R}$, and $a_0(t) \equiv 0$, and the first center condition is $a_1 = 0$. Thus the center problem is usually studied for Abel equations of the form $\dot{x} = A(t)x^3 + B(t)x^2$.

The paper is structured as follows: Theorems 1 and 2 are proved in Section 2, while our results on the center problem and the proof of Theorem 3 are presented in Section 3.

2. Proof of Theorems 1 and 2

Let $\Phi_\epsilon : U \longrightarrow \mathbb{C}$ be a uniparametric family of analytic functions defined in a open U of the complex plain. Assume also that $\Phi_0 = Id$. We will say that $z_0 \in U$ is a *persistent* fixed point if for all ϵ small enough there exists $z(\epsilon) \in U$ fixed point of Φ_ϵ such that

$$\lim_{\epsilon \to 0} z(\epsilon) = z_0$$

Recall that a fixed point of a differentiable map is called *hyperbolic* if the derivative of the map at the fixed point has all its eigenvalues with modulus different from 1. In the one-dimensional complex case this condition is equivalent to show that the complex derivative of the map at the point has modulus diferent from 1. We begin by proving a result that gives sufficient conditions to assure that fixed points obtained from analytic perturbations of the identity are hyperbolic.

Lemma 4. *Let $\Phi_\epsilon : U \longrightarrow \mathbb{C}$ be a uniparametric family of analytic functions defined in an open subset U of the complex plain. Assume also that*

$$\Phi_\epsilon(z) = z + W(z)\epsilon + O(\epsilon^2). \tag{6}$$

Then the simple zeroes of $W(z)$ are persistent fixed points. Furthermore if z_0 is a simple zero of $W(z)$ and $Re(W'(z_0)) \neq 0$ then for ϵ small enough the fixed point $z(\epsilon)$ is hyperbolic.

Proof: For $\epsilon \neq 0$, a fixed point of Φ_ϵ must verify

$$0 = \frac{\Phi_\epsilon(z) - z}{\epsilon} = W(z) + O(\epsilon).$$

Then if $W(z_0) = 0$ and $W'(z_0) \neq 0$ using the implicit function theorem we get that z_0 is persistent.

Now assume that $W(z_0) = 0$ and $W'(z_0) = a + bi$. Denote by $z(\epsilon)$ the fixed point of Φ_ϵ that persists from z_0. Then we have

$$\begin{aligned}\Phi'_\epsilon(z(\epsilon)) &= 1 + W'(z(\epsilon))\epsilon + O(\epsilon^2)\\ &= 1 + W'\left(z_0 + z'(0)\epsilon + O(\epsilon^2)\right)\,\epsilon + O(\epsilon^2)\\ &= 1 + W'(z_0)\epsilon + O(\epsilon^2).\end{aligned}$$

Thus we obtain,

$$\|\Phi'_\epsilon(z(\epsilon))\| = 1 + a\epsilon + O(\epsilon^2).$$

Hence, if $a \neq 0$ and ϵ is small enough, we get that $z(\epsilon)$ is a hyperbolic fixed point of $\Phi_\epsilon(z)$. ∎

Lemma 5. *Consider the following autonomous differential equation with complex coefficients:*

$$\dot{z} \,=\, z\left(z^n - \frac{i}{n}\right). \tag{7}$$

Then, the following statements hold:

(i) *For each* $\rho\in\mathbb{C}$ *equation (7) has the solution*

$$\varphi(t,\rho)=\rho\,\Delta(t)^{\frac{-1}{n}}\quad\text{where}\quad \Delta(t)=(1+in\rho^n)e^{it}-in\rho^n, \tag{8}$$

and $\Delta(t)^{\frac{1}{n}}$ *stands for the continuous determination of the* n*-root of* $\Delta(t)$ *that begins by* $\Delta(0)^{\frac{1}{n}}=1$.

(ii) *System* (7) *has* $n+1$ *centers determined by* $z=0$ *and the* n *roots of* $z^n=i/n$. *All the periodic orbits on the period annulus of* $z=0$ *have period* $2\,n\,\pi$ *while the periodic orbits on the period annulus of the other* n *centers have period* $2\,\pi$.

(iii) *The set of initial conditions* $\{\rho\in\mathbb{C}:\,Im\,(\rho^n)\,=\,1/2n\}$ *is a curve with* n *connected components which are orbits of (7). For this set of initial conditions, the flow of (7) is only defined for* t *belonging to a bounded interval of* $\mathbb{R}$.

Proof: Part (i) follows straightforward. In order to prove (ii) recall the following result: Let $\dot z=f(z)$ a holomorphic differential equation. If $f(z_0)=0$ and $f'(z_0)=\alpha i$, with $\alpha\in\mathbb{R}$ then there is an holomorphic change of variables $w=h(z)$, defined in a neighborhood of z_0, such that the differential equation linearizes as $\dot w=\alpha i w$, see [4]. Hence, since the derivative of $f(z)=z\,(z^n-i/n)$ takes the value $-i/n$ at $z=0$ and the value i at the other critical points the result follows.

(iii) Let us prove that $\mathrm{Im}(\rho^n)=\frac{1}{2n}$ if and only if $\frac{ni\rho^n}{1+ni\rho^n}$ has modulus one. Take ρ such that $\mathrm{Im}(\rho^n)=\frac{1}{2n}$. Since $\mathrm{Re}(ni\rho^n)=-n\,\mathrm{Im}(\rho^n)=-1/2$ we have that

$$\frac{ni\rho^n}{1+ni\rho^n}=\frac{\frac{-1}{2}+bi}{\frac{1}{2}+bi}=\frac{4b^2-1}{4b^2+1}+\frac{4b}{4b^2+1}\,i.$$

So, equation

$$e^{i\alpha}=\frac{4b^2-1}{4b^2+1}+\frac{4b}{4b^2+1}\,i$$

has exactly one solution α in $(0,2\pi)$. Since for $t=\alpha$, $\Delta(\alpha)$ is zero, from (8) we get that the solution beginning at ρ is only definite for $t\in(\alpha-2\pi,\alpha)$.

On the other hand, a simple computation shows that $\mathrm{Im}(\rho^n)=\frac{1}{2n}$ implies that $\mathrm{Im}\left(\frac{\rho^n}{\Delta(t)}\right)=\frac{1}{2n}$, *i.e.*, the curve $\mathrm{Im}(\rho^n)=\frac{1}{2n}$ is formed by solutions of (7). Furthermore, denoting by $\rho=x+iy$, the equation $\mathrm{Im}(\rho^n)=\frac{1}{2n}$ writes as $P_n(x,y)=\frac{1}{2n}$ where $P_n(x,y)$ is a homogeneous polynomial of degree n which decomposes as a product of n different real lines. It is to say the algebraic curve $P_n(x,y)=\frac{1}{2n}$ meets the equator of the Poincaré sphere at $2n$ points. Since in the plane there are not attractor points, not repulsor ones, the α and the ω limit on the points lying on $\mathrm{Im}(\rho^n)=\frac{1}{2n}$ are in the equator of the sphere. All together implies that $\mathrm{Im}(\rho^n) = \frac{1}{2n}$ has exactly n connected components in the complex plane. ∎

Theorem 6. *Consider the following differential equation:*

$$\dot{z} = z\left(z^n - \frac{i}{n}\right) + \varepsilon F(t,z), \quad n \geq 2. \tag{9}$$

Then, for each $k \in \mathbb{N}$ and ε small enough there exists a $F(t,z) = \sum_{j=0}^{n} A_j(t)\, z^j$, where each $A_j(t)$ is a trigonometric polynomial of degree k, such that equation (9) has at least $n(k+1)$ hyperbolic 2π-periodic orbits and n hyperbolic $2n\pi$-periodic orbits

Proof: By Lemma 5, for each $\rho \in \mathbb{C}$ equation (9) with $\varepsilon = 0$ has the solution

$$\varphi(t,\rho) = \rho\,\Delta(t)^{\frac{-1}{n}} \quad \text{where} \quad \Delta(t) = (1 + in\rho^n)e^{it} - in\rho^n,$$

and the solution of (9) can be written as

$$\psi(t,\rho,\varepsilon) = \varphi(t,\rho) + \varepsilon\, W(t,\rho) + O(\varepsilon^2).$$

The function $W(t,\rho)$ satisfies the linear differential equation:

$$\frac{\partial W(t,\rho)}{\partial t} = \left((n+1)\varphi(t,\rho)^n - \frac{i}{n}\right) W(t,\rho) + F(t,\varphi(t,\rho))$$

with $W(0,\rho) = 0$. Hence,

$$W(t,\rho) = \frac{e^{it}}{\Delta(t)^{\frac{n+1}{n}}} \int_0^t \frac{F(s,\varphi(s,\rho))\,\Delta(s)^{\frac{n+1}{n}}}{e^{is}}\, ds,$$

and for $t = 2\pi$ we get

$$W(2\pi,\rho) = \int_0^{2\pi} \frac{F(t,\varphi(t,\rho))\,\Delta(t)^{\frac{n+1}{n}}}{e^{it}}\, dt. \tag{10}$$

Let $U \subset \mathbb{C}$ be defined by

$$U = \{\rho \in \mathbb{C} : \ \mathrm{Im}(\rho^n) > \frac{1}{2n}\}.$$

Then for $\rho \in U$ we get

$$\begin{aligned} \psi(2\pi,\rho,\varepsilon) &= \varphi(2\pi,\rho) + \varepsilon\, W(2\pi,\rho) + O(\varepsilon^2) \\ &= \rho + \varepsilon\, W(2\pi,\rho) + O(\varepsilon^2). \end{aligned}$$

From Lemma 4 the simple zeroes in U of $W(\rho) := W(2\pi,\rho)$, determines 2π-periodic orbits of equation (9). To compute these zeroes note that if $\rho \in U$, then $\Delta(s)$ is a circle not surrounding the origin. Hence, the function which assigns to each z in the interior of the unit circle, the value $(1 + in\rho^n)z - in\rho^n$ is an analytic function in the interior of this circle. So to calculate the above integral we can use the residue theorem. For this, we do the substitution $z = e^{it}$ and we get

$$W(\rho) = \int_\gamma \frac{\left(F(t,\varphi(t,\rho))\,\Delta(t)^{\frac{n+1}{n}} \right)\Big|_{e^{it}=z}}{iz^2}\,dz, \tag{11}$$

where γ is the unit circle.

To do the forthcoming computations we write $F(t,z) = \sum_{j=0}^{n} A_j(t) z^j$ where $A_j(t) = \sum_{l=-k}^{k} a_l^j e^{ilt}$ and $A = -ni\rho^n$. Taking into account the expressions of $\varphi(t,\rho)$ and $\Delta(t)$ we have

$$\begin{aligned} W(\rho) &= \sum_{j=0}^{n} \rho^j \int_\gamma \frac{(A_j(t)\Delta(t)^{\frac{n+1-j}{n}})_{|e^{it}=z}}{iz^2}\,dz \\ &= \sum_{j=0}^{n} \rho^j \int_\gamma \frac{(\sum_{l=-k}^{k} a_l^j z^l)(A+(1-A)z)^{\frac{n+1-j}{n}}}{iz^2}\,dz. \end{aligned}$$

For each $j = 0,\dots,n$ we can write

$$\begin{aligned} (A+(1-A)\,z)^{\frac{n+1-j}{n}} &= A^{\frac{n+1-j}{n}} \left(1 + \tfrac{1-A}{A}\,z\right)^{\frac{n+1-j}{n}} \\ &= A^{\frac{n+1-j}{n}} \left[\sum_{s=0}^{\infty} c_s^j \left(\tfrac{1-A}{A}\right)^s z^s\right], \end{aligned} \tag{12}$$

where $c_s^j = \binom{(n+1-j)/n}{s}$. Taking into account that $A = -ni\rho^n$ we obtain

$$W(\rho) = \rho^{n+1} \sum_{j=0}^{n} (-ni)^{\frac{n+1-j}{n}} \int_\gamma \frac{(\sum_{l=-k}^{k} a_l^j z^l)(1+\frac{1-A}{A}z)^{\frac{n+1-j}{n}}}{iz^2}\,dz$$

Now putting $r_s^j = (-ni)^{\frac{n+1-j}{n}} c_s^j$ and applying the residue theorem we get

$$\begin{aligned} W(\rho) &= 2\pi\rho^{n+1} \sum_{s=0}^{k+1} \left(\sum_{j=0}^{n} r_s^j a_{1-s}^j\right) \left(\tfrac{1-A}{A}\right)^s \\ &= \rho^{n+1} \sum_{s=0}^{k+1} \alpha_s \left(\tfrac{1-A}{A}\right)^s, \end{aligned}$$

where $\alpha_s = 2\pi \sum_{j=0}^{n} r_s^j a_{1-s}^j$. Notice that since the coefficients a_s^j are free the same is true for α_s.

Indeed, we can write $W(\rho)$ as

$$W(\rho) = \frac{\rho^{n+1}}{A^{k+1}} \sum_{s=0}^{k+1} \alpha_s\,(1-A)^s\,A^{k+1-s} = \frac{\rho^{n+1}}{A^{k+1}} \sum_{s=0}^{k+1} \tilde{\alpha}_s\,A^s. \tag{13}$$

Clearly we can choose the coefficients $\tilde{\alpha}_s$ for $s = 1,2,\dots,k+1$ in such a way that $W(\rho) = 0$ has $k+1$ different roots in the variable A. One can choose these roots in such a way that its real part is greater than $1/2$. And since $A = -ni\rho^n$, for each root of A we get n roots of ρ belonging to U. To obtain hyperbolicity it is enough to note that by multiplying all the original coefficients of the polynomial by a complex constant, we obtain a polynomial with the same simple roots but the derivative of the polynomial at each root will be multiplied by the complex constant.

Now set

$$V = \{\rho \in \mathbb{C} : \ \mathrm{Im}\rho^n \ < \ \frac{1}{2n}\}.$$

Arguing as above we get that the $2n\pi$-periodic orbits that persists from the unperturbed system comes from the simple zeros in V of $\bar{W}(\rho) := W(2n\pi, \rho)$. We have for $\rho \in V$,

$$\bar{W}(\rho) = \int_0^{2n\pi} \frac{F(s, \varphi(s, \rho))\, \Delta(s)^{\frac{n+1}{n}}}{e^{is}}\, ds.$$

Taking $s = nt$ we get

$$\bar{W}(\rho) = n \int_0^{2\pi} \frac{F(nt, \varphi(nt, \rho))\, \Delta(nt)^{\frac{n+1}{n}}}{e^{int}}\, dt.$$

Taking into account that $F(t, z) = \sum_{j=0}^{n} A_j(t)\, z^j$ we have

$$\bar{W}(\rho) = n \sum_{j=0}^{n} \rho^j \int_0^{2\pi} \frac{A_j(nt)\Delta(nt)^{\frac{n+1-j}{n}}}{e^{int}}\, dt.$$

Notice that $\Delta(nt) = (1 - A)e^{nit} + A = e^{int}((1 - A) + Ae^{-int})$. Thus we obtain

$$\bar{W}(\rho) = n \sum_{j=0}^{n} \rho^j \int_0^{2\pi} \frac{A_j(nt)e^{(n+1-j)it}((1 - A) + Ae^{-int})^{\frac{n+1-j}{n}}}{e^{int}}\, dt.$$

Now if we take $r = -t$, then we get

$$\bar{W}(\rho) = n \sum_{j=0}^{n} \rho^j \int_0^{2\pi} A_j(-nr)e^{-(1-j)ir}((1 - A) + Ae^{inr})^{\frac{n+1-j}{n}}\, dr.$$

As in the previous analysis the term $(1 - A) + Ae^{inr}$, $r \in [0, 2\pi]$ is a circle not surrounding the origin, so that the function $((1 - A) + Az^n)^{\frac{n+1-j}{n}}$ is analytic in the unit disc.

Taking $A_j(r) = \sum_{l=-k}^{k} a_l^j e^{ilr}$ and $z = e^{ir}$ we have

$$\bar{W}(\rho) = -in \sum_{j=0}^{n} \rho^j \int_\gamma (\sum_{l=-k}^{k} a_l^j z^{-nl}) z^{j-2}((1 - A) + Az^n)^{\frac{n+1-j}{n}}\, dz$$

where γ denotes the unit circle.

Easy computations show that the only non zero integral appears when $j = 1$ and the residue is equal to $a_0^1(1 - A) + a_1^1 A$. Thus

$$\bar{W}(\rho) = 2n\pi\rho(a_0^1(1 - A) + a_1^1 A)$$

which is a linear function on A. Choosing a_0^1, a_1^1 in such a way that this function has one zero with real part less than $1/2$ and taking in account that $A = -ni\rho^n$, we obtain n roots of $\bar{W}(\rho)$ that belongs to V. We note that this election fixes the coefficients a_0^1 and a_1^1 of the perturbation but the coefficients of the polynomial in A appearing in (13) and used to analyze the 2π-periodic orbits can still be chosen arbitrarily. The hyperbolicity of the periodic orbits can be ensured as in the previous case. ∎

Remark 1. *Note that if ρ_0 is an initial condition of a $2n\pi$-persistent periodic orbit then the points $\varphi(2j\pi, \rho_0)$ for $j = 0, 1, \ldots, n-1$ are also persistent initial conditions for a persistent $2n\pi$-periodic orbit. Then the n persistent $2n\pi$-periodic orbits obtained in the previous theorem are in fact equivalent via a time translation.*

Theorem 7. *Consider the following differential equation:*

$$\dot{z} = z\ (z - i) + \varepsilon F(t, z). \tag{14}$$

Then, for each $k \in \mathbb{N}$ and ε small enough there exists $F(t, z) = A(t)z^3$ with $A(t) = \sum_{j=-k}^{k} A_j\, e^{ijt}$ such that it has $2k-1$ non-zero hyperbolic 2π-periodic orbits.

Proof: The proof is similar to the proof of Theorem 6. Now, $n = 1$, $\varphi(t, \rho) = \rho\,/\Delta(t)$ with $\Delta(t) = (1 + i\rho)e^{it} - i\rho$, and

$$F(s, \varphi(s, \rho)) = \frac{\rho^3}{\Delta(s)^3} \sum_{j=-k}^{k} A_j\, e^{sji}.$$

Then, equation (10) becomes

$$W(\rho) = \rho^3 \sum_{j=-k}^{k} A_j \int_0^{2\pi} \frac{e^{(j-1)si}}{\Delta(s)}\, ds$$

and equation (11) writes as

$$\begin{aligned} W(\rho) &= \rho^3 \sum_{j=-k}^{k} A_j \int_\gamma \frac{z^{j-2}}{(i - \rho)z + \rho}\, dz \\ &= \frac{\rho^3}{i - \rho} \sum_{j=-k}^{k} A_j \int_\gamma \frac{z^{j-2}}{z - \dfrac{\rho}{\rho - i}}\, dz. \end{aligned}$$

Also, as in the proof of the previous theorem, set

$$\begin{array}{ll} U & = \{\rho \in \mathbb{C} :\ \operatorname{Im}(\rho) > \frac{1}{2}\} = \{\rho \in \mathbb{C} :\ \|\frac{\rho}{\rho-i}\| > 1\} \\ V & = \{\rho \in \mathbb{C} :\ \operatorname{Im}(\rho) < \frac{1}{2}\} = \{\rho \in \mathbb{C} :\ \|\frac{\rho}{\rho-i}\| < 1\} \end{array}.$$

We notice that when $\rho \in U$ then the only residue that has to be calculated in the above integral is at $z = 0$, while when $\rho \in V$ the pole $z = \frac{\rho}{\rho-1}$ also must be considered. Easy and tedious computations show that

$$W(p) = \begin{cases} 2\pi i \frac{\rho^3}{\rho-i} \sum_{j=-k}^{1} A_j \left(\frac{\rho}{\rho-i}\right)^{j-2}, & \text{if } \rho \in U \\ -2\pi i \frac{\rho^3}{\rho-i} \sum_{j=2}^{k} A_j \left(\frac{\rho}{\rho-i}\right)^{j-2}, & \text{if } \rho \in V \end{cases}$$

Thus since the coefficients A_j are free we can obtain $k+1$ simple roots of $W(\rho)$ in U and $k-2$ roots in V. The hyperbolicity can be ensured as in the previous theorem. ∎

3. Center conditions

The flow of the differential equation (2) with $a_0(t) \equiv 0$ induces a Poincaré map, Π, between the planes $t = 0$ and $t = 2\pi$, which is well defined in a neighbourhood of the solution $z = 0$. This map is holomorphic at the origin and writes as $\Pi(z) - z = \sum_{k=1}^{\infty} V_k z^k$. The expressions $V_k, k \geq 1$ are called called the Lyapunov constants of the solution $z = 0$. Note that a necessary and sufficient condition for the solution $z = 0$ to be a center is that $V_k = 0$ for all $k \geq 1$. For this reason, often a Lyapunov constant V_m is only used when $V_1 = V_2 = \cdots = V_{m-1} = 0$. Following the computations given in [3, pp. 249-50] it is not difficult to prove the following result:

Lemma 8. *Consider the equation*

$$\frac{dz}{dt} = A(t)\, z^3 + B(t)\, z^2 + C(t)\, z, \tag{15}$$

where $A(t)$, $B(t)$ and $B(t)$ are $2\pi-$periodic complex analytic functions. Then the first three Lyapunov constants of $z = 0$ are:

$$\begin{aligned} V_1 &= \exp\left(\int_0^{2\pi} C(t)\, dt\right) - 1, \\ V_2 &= \int_0^{2\pi} B(t) \exp\left(\int_0^t C(\psi)\, d\psi\right) dt, \\ V_3 &= \int_0^{2\pi} A(t) \exp\left(2\int_0^t C(\psi)\, d\psi\right) dt. \end{aligned}$$

To state next result we use the Bessel functions, see [19]. Recall that the Bessel differential equation of order n is

$$z^2\,u''(z)+z\,u'(z)+(z^2-n^2)\,u(z)\;=\;0,$$

and one of its solutions is the Bessel function of order n,

$$J_n(z)=\sum_{l=0}^{\infty}\frac{(-1)^l\,\left(\frac{z}{2}\right)^{n+2l}}{l!\,(n+l)!}.$$

Proposition 9. *Consider equation (15) with*

$$A(t)=\sum_{j=-n}^{n}A_j\,e^{ij\,t}\,,\;B(t)=\sum_{j=-n}^{n}B_j\,e^{i\,j\,t}$$

and

$$C(t)=C_{-1}\,e^{-it}+C_0+C_1\,e^{it}\qquad\textit{with}\qquad C_1C_{-1}\neq 0.$$

Then its first Lyapunov constant is $V_1=\exp\left(2\,\pi\,C_0\right)-1$ *and it vanishes if and only if there exists* $k\in\mathbb{Z}$ *such that* $C_0=k\,i$. *Furthermore when* $C_0=k\,i$ *the next Lyapunov constants are*

$$\begin{aligned}
V_2 &= 2\pi e^{(C_1-C_{-1})\,i}\sum_{j=-n}^{n}B_j\left(\delta_{k,j}\sqrt{\frac{C_{-1}}{C_1}}\right)^{k+j}J_{|k+j|}(2i\sqrt{C_1\,C_{-1}}),\\
V_3 &= 2\pi e^{2(C_1-C_{-1})i}\sum_{j=-n}^{n}A_j\left(\delta_{2k,j}\sqrt{\frac{C_{-1}}{C_1}}\right)^{2k+j}J_{|2k+j|}(4i\sqrt{C_1\,C_{-1}}),
\end{aligned}$$

where $\delta_{p,\,q}=\begin{cases}1 & \text{if } p+q\geq 0,\\ -1 & \text{if } p+q<0,\end{cases}$ *and the determinations of the square roots are taken such that* $\sqrt{\frac{C_{-1}}{C_1}}\sqrt{C_1C_{-1}}=C_{-1}$.

Proof: The expression of V_1 follows from Lemma 8. Clearly it vanishes if and only if $C_0=k\,i$ for $k\in\mathbb{Z}$.

Then, in order to find V_2 we can assume that $C_0=k\,i$. First notice that

$$\int_0^t C(\psi)\,d\psi\;=\;i\,\left(kt-C_{-1}+C_1+C_{-1}\,e^{-\,it}-C_1\,e^{i\,t}\right).$$

Again, from Lemma 8 we have that

$$\begin{aligned}
V_2 &= \int_0^{2\pi}\sum_{j=-n}^{n}B_j\,e^{jti}\,e^{kti}\,e^{(C_1-C_{-1})\,i}\,e^{(C_{-1}\,e^{-t\,i}-C_1e^{t\,i})\,i}\,dt=\\
&= \sum_{j=-n}^{n}B_j\,e^{(C_1-C_{-1})\,i}\int_0^{2\pi}e^{(k+j)ti}\sum_{m=0}^{\infty}\frac{(i)^m\,(C_{-1}\,e^{-t\,i}-C_1e^{t\,i})^m}{m!}\,dt.
\end{aligned}$$

Developing $(C_{-1}\,e^{-t\,i}-C_1e^{t\,i})^m$ we get

$$(C_{-1}\,e^{-t\,i}-C_1e^{t\,i})^m=\sum_{l=0}^{m}\frac{m!}{l!\,(m-l)!}\,(C_{-1}\,e^{-t\,i})^{m-l}\,(-C_1e^{t\,i})^l,$$

and so

$$V_2=e^{(C_1-C_{-1})\,i}\sum_{j=-n}^{n}B_j\sum_{m=0}^{\infty}\sum_{l=0}^{m}\frac{(-1)^l(i)^m}{l!(m-l)!}C_1^lC_{-1}^{m-l}\int_0^{2\pi}e^{(k+j+2l-m)ti}dt.$$

Since the integral of $e^{n\,i}$ between 0 and $2\,\pi$ is zero except when $n=0$, we have that for each j and k, m must satisfy $m=k+j+2l$ and thus the above expression writes as

$$V_2 = 2\,\pi\,e^{(C_1-C_{-1})i}\sum_{j=-n}^{n}B_j\sum_{l=0}^{\infty}\frac{(-1)^l\,(i)^{k+j+2l}}{l!\,(k+j+l)!}\,C_{-1}^{k+j+l}\,C_1^l. \tag{16}$$

Now we distinguish between two cases. First consider $k+j\geq 0$. We notice that

$$C_1^l\,C_{-1}^{k+j+l}=\left(\sqrt{\frac{C_{-1}}{C_1}}\right)^{k+j}\left(\sqrt{C_1\,C_{-1}}\right)^{k+j+2l},$$

where we take the determinations of the square roots as in the statement. Taking into account the expression of J_{k+j} we get that

$$\sum_{l=0}^{\infty}\frac{(-1)^l\,(i)^{k+j+2l}}{l!\,(k+j+l)!}\,C_{-1}^{k+j+l}\,C_1^l=\left(\sqrt{\frac{C_{-1}}{C_1}}\right)^{k+j}J_{k+j}(2\,i\,\sqrt{C_1\,C_{-1}}). \tag{17}$$

If $k+j<0$ we have to consider $l\geq -(k+j)$, because $m-l=k+j+l$ must be non negative. In this case,

$$V_2 = 2\,\pi\,e^{(C_1-C_{-1})i}\sum_{j=-n}^{n}B_j\sum_{l\geq -(k+j)}^{\infty}\frac{(-1)^l\,(i)^{k+j+2l}}{l!\,(k+j+l)!}\,C_{-1}^{k+j+l}\,C_1^l.$$

Let L be defined trough $L=l+k+j$. Then,

$$\begin{aligned}&\sum_{l\geq -(k+j)}^{\infty}\frac{(-1)^l\,(i)^{k+j+2l}}{l!\,(k+j+l)!}\,C_{-1}^{k+j+l}\,C_1^l\\ &=\sum_{L=0}^{\infty}\frac{(-1)^{L-(k+j)}(i)^{2L-(k+j)}}{L!(L-(k+j))!}C_{-1}^LC_1^{L-(k+j)},\end{aligned}$$

which can be written as

$$\sum_{L=0}^{\infty} \frac{(-1)^L(-1)^{|k+j|}(i)^{2L+|k+j|}}{L!(L+|k+j|)!} C_{-1}^L C_1^{L+|k+j|}. \tag{18}$$

Now observe that

$$C_{-1}^L C_1^{L+|k+j|} = \left(\sqrt{\frac{C_1}{C_{-1}}}\right)^{|k+j|} \left(\sqrt{C_1 C_{-1}}\right)^{|k+j|+2L},$$

taking the determinations of the square roots as above. Hence, equation (18) becomes

$$(-1)^{|k+j|} \sum_{L=0}^{\infty} \frac{(-1)^L (i)^{2L+|k+j|}}{L!(L+|k+j|)!} \left(\sqrt{\frac{C_1}{C_{-1}}}\right)^{|k+j|} \left(\sqrt{C_1 C_{-1}}\right)^{|k+j|+2L}$$

$$= (-1)^{k+j} \left(\sqrt{\frac{C_{-1}}{C_1}}\right)^{k+j} J_{|k+j|}(2\, i\, \sqrt{C_1\, C_{-1}}).$$

By using the above formula, (16) and (17), the expression of V_2 follows.

Finally, notice that by Lemma 3 the expression of V_3 follows from the above computations changing $B(t)$ and $C(t)$ by $A(t)$ and $2C(t)$, respectively. ∎

Proposition 10. *The periodic orbit $z = 0$ of the equation*

$$\frac{dz}{dt} = z^3 + C(t)\, z, \tag{19}$$

is a center if and only if $V_1 = 0$, $V_2 = 0$ and $V_3 = 0$.

Proof: Since $B(t) \equiv 0$, from Lemma 8 we know that $V_2 = 0$. By using the transformation $u = z^{-2}$, equation (19) becomes

$$\frac{du}{dt} = -2 - 2\, C(t)\, u,$$

and hence,

$$u(t, u_0) = \left[u_0 - 2\int_0^t e^{2\int_0^{\psi} C(\alpha) d\alpha}\, d\psi\right] e^{-2\int_0^t C(\psi) d\psi}.$$

That is,

$$z(t, z_0) = \frac{z_0\, e^{\int_0^t C(\psi) d\psi}}{\sqrt{1 - 2\, z_0^2 \int_0^t e^{2\int_0^{\psi} C(\alpha) d\alpha} d\psi}}.$$

From Lemma 8, the above expression at $t = 2\pi$ can be written as

$$z(2\pi, z_0) \; = \; \frac{z_0\,(1 + V_1)}{\sqrt{1 - 2\,z_0^2\,V_3}}. \tag{20}$$

Hence, $z(2\pi, z_0) \equiv z_0$ near zero, if and only if $V_1 = V_3 = 0$, as we wanted to prove. ∎

Theorem 3 is an easy consequence of Propositions 9 and 10.

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